VARIATIONS ON A CONJECTURE OF HALPERIN

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Abstract. Halperin has conjectured that the Serre spectral sequence of any fibration that has fibre space a certain kind of elliptic space should collapse at the E_2-term. In this paper we obtain an equivalent phrasing of this conjecture, in terms of formality relations between base and total spaces in such a fibration. Also, we obtain results on relations between various numerical invariants of the base, total and fibre spaces in these fibrations. Some of the latter results can be seen as establishing weak versions of the conjecture.

§1 — Introduction

We begin with a description of the conjecture referred to in the title. In this paper, all spaces are simply connected CW complexes and are of finite-type, i.e., have finite-dimensional rational homology groups. A fibration $F \xrightarrow{j} E \xrightarrow{p} B$ is said to be *totally non-cohomologous to zero* (abbreviated TNCZ) if the induced homomorphism $j^* : H^*(E;\mathbb{Q}) \to H^*(F;\mathbb{Q})$ is onto. This is a very strong condition to place on a fibration. It is equivalent to requiring that the Serre spectral sequence (for cohomology with rational coefficients) collapse at the E_2-term. In this case there is an isomorphism $H^*(E;\mathbb{Q}) \cong H^*(B;\mathbb{Q}) \otimes H^*(F;\mathbb{Q})$ of $H^*(B;\mathbb{Q})$-modules. Thus a TNCZ fibration is somewhere between being trivial from the rational homology point of view and being trivial from the rational cohomology algebra point of view (cf. Example 1.2).

In the sequel we focus on certain fibre spaces F that satisfy the following conditions:

(1) $H^*(F;\mathbb{Q})$ is finite-dimensional.
(2) $\pi_*(F) \otimes \mathbb{Q}$ is finite-dimensional.
(3) The Euler characteristic of F, i.e., $\sum_i (-1)^i \dim\big(H^i(F;\mathbb{Q})\big)$, is positive.

Any space that satisfies (1) and (2) is called (rationally) *elliptic*. See [Au] for a discussion of these spaces. It is known that elliptic spaces have non-negative Euler characteristic [Ha$_1$]. So condition (3) further restricts F to being one of two types of elliptic space. We often refer to a space that satisfies conditions (1)–(3) as a *positively elliptic* space. However, we also refer to such spaces in the long-hand, as 'elliptic with positive Euler characteristic', particularly when stating results.

The conjecture of the title, with which we are concerned, is as follows:

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1.1 Conjecture (Halperin). *Let F be elliptic with positive Euler characteristic. Then any fibration $F \to E \to B$ is TNCZ.*

This conjecture has been established in various cases, but in general it remains open. Some results that concern it are mentioned later in this introduction.

We point out that this paper does not resolve Conjecture 1.1, not even in special cases! Rather, as the title suggests, we are concerned here with variations on the theme provided by the conjecture. These variations come about by considering consequences of Conjecture 1.1, assuming it to be true. The motivation is two-fold: First, it is hoped to open up new lines of approach to the conjecture itself. Second, by looking at such consequences one can obtain weak versions of the conjecture. On the one hand, these weak versions might prove more tractable than the original. On the other hand, they should lead to a fuller understanding of the conjecture.

Although we consider consequences of Conjecture 1.1, some of our results are independent of the status of this conjecture and furthermore are interesting in their own right. For instance, in Theorem 4.7 we obtain the following: If $F \to E \to S^{2n+1}$ is a fibration with fibre a positively elliptic space and base an odd sphere, then $\mathrm{cat}_0(E) = \mathrm{cat}_0(F) + 1$. This result establishes a weak form of Conjecture 1.1. But also, for instance, it can be viewed as a strong form of Ganea's conjecture in the (very) restricted circumstances to which it applies.

Next, we outline the contents of the paper. This section continues with a discussion of positively elliptic spaces and some of their properties. We go on to discuss models of fibrations, the basic technical tool that we use. The introduction finishes with a brief summary of some results on Conjecture 1.1 and some notational conventions. In Section 2 we describe some special models of fibrations. This is a somewhat technical section, although in Theorem 2.3 we obtain a very strong consequence of Conjecture 1.1. In Section 3, we relate the formality of E and B, for fibrations $F \to E \to B$ to which Conjecture 1.1 applies. In Proposition 3.2, for example, we show B formal implies E formal, under the hypothesis that Conjecture 1.1 is true. These results rely on the models described in Section 2. We also obtain an equivalent phrasing of Conjecture 1.1, in Theorem 3.4. In Section 4 we consider some numerical rational homotopy invariants. We show that there are inequalities relating the values of these invariants, for base, total and fibre spaces in fibrations to which Conjecture 1.1 applies. We obtain complete results for some invariants, in case the base space is a wedge of odd-dimensional spheres (Theorem 4.7). We also give some weak versions of Halperin's conjecture. In the last section, we discuss extensions of these results and suggest directions for future work.

The spaces F that feature in the conjecture are clearly of a very restricted kind. Here is a discussion of some of their properties. We have characterized them by conditions (1)–(3) above. However, results of Halperin [Ha$_1$] give various equivalent characterizations. Halperin shows that, for an elliptic space F, the three conditions positive Euler characteristic, $\chi_\pi(F) = 0$ and $H^{\mathrm{odd}}(F;\mathbb{Q}) = 0$ are equivalent. Here, $\chi_\pi(F)$ denotes the so-called *homotopy Euler characteristic* of F. This is a number defined for any space that has finite-dimensional rational homotopy as $\chi_\pi(F) := \sum_i (-1)^i \dim\big(\pi_i(F) \otimes \mathbb{Q}\big)$. The cohomology algebra of a positively elliptic space F is zero in odd degrees and has a presentation of the form

$$H^*(F;\mathbb{Q}) \cong \frac{\mathbb{Q}[x_1, \dots , x_n]}{(R_1, \dots , R_n)}$$

with $\{R_1, \dots, R_n\}$ a regular sequence of maximal length in the polynomial algebra $\mathbb{Q}[x_1, \dots, x_n]$. The *minimal model* of such a space is of a particularly restricted form. Recall that the minimal model of a space X is a differential graded algebra $\mathcal{M}_X$, that as a graded algebra is a free graded commutative algebra (polynomial on even degree generators and exterior on odd degree generators). Also, its (degree $+1$) differential is decomposable, in the sense that it induces the trivial differential after passing to the quotient module of indecomposables, i.e., it has zero linear part. See [Gr-Mo], [Ha_3] and [Ta] for the basics of minimal models and their use in rational homotopy theory. The notes by Aubry [Au] contain more recent material and references. Condition (1) above implies that the minimal model has finite-dimensional cohomology, as the cohomology of the minimal model is identified with that of the space. Condition (2) translates into the condition that the minimal model be *finitely-generated* as a free graded algebra, since the algebra generators of the minimal model are identified, as a graded vector space, with the rational homotopy groups of the space. For an elliptic space, condition (3) greatly restricts the form of the minimal model further. It implies, for instance, that up to isomorphism the model is a *pure* model [Ha_1]. This is to say that it has the form

$$\mathcal{M}_F = \Lambda(V^{\mathrm{even}}) \otimes \Lambda(V^{\mathrm{odd}})$$

with $d(V^{\mathrm{even}}) = 0$ and $d(V^{\mathrm{odd}}) \subseteq \Lambda(V^{\mathrm{even}})$. For an elliptic space, condition (3) further implies that the minimal model has the same number of even degree generators as odd degree generators. In symbols, this means $\dim(V^{\mathrm{even}}) = \dim(V^{\mathrm{odd}})$ and this fact corresponds to the condition that $\chi_\pi = 0$. We state one more property of these remarkable spaces. We have said that the cohomology algebra of a positively elliptic space is zero in odd degrees. In fact, more is true. Any cocycle that is in the ideal of $\mathcal{M}_F$ generated by odd-degree generators is a boundary and $H(\mathcal{M}_F) \cong \Lambda(V^{\mathrm{even}})/(dV^{\mathrm{odd}})$ (see [Ha_1,Th.2]).

It is worthwhile remarking that, despite the highly restricted form of F, there are many examples of such spaces, some of which correspond to spaces familiar to topologists and geometers: Even-dimensional spheres, complex projective spaces, Grassmann manifolds and in general homogeneous spaces G/H with G a compact, connected Lie group and H a closed subgroup of maximal rank. Further, given any algebra presented as above, there is some space F, necessarily positively elliptic, that realizes the algebra as its rational cohomology algebra.

Next, we review some material on rational fibrations and minimal models (cf. [Ha_2]). Just as a space has a minimal model, so too does a map $p : E \to B$. Suppose that $\Lambda W, d_B$ is the minimal model of B. Then the minimal model for p is a sequence of DG algebras

$$\Lambda W, d_B \xrightarrow{i} \Lambda W \otimes \Lambda V, D \xrightarrow{q} \Lambda V, d \qquad (*)$$

where the differential D restricts to d_B on ΛW and projects to a minimal differential d on ΛV. In general there is a 'twisting' present in D, which may include non-minimal (linear) terms, even though we use the terminology 'minimal' for such a model. By 'twisting' we mean the following: On generators of V, the differential D takes the form $(D - d) : V \to \Lambda^+ W \otimes \Lambda V$. Here $\Lambda^+ W \otimes \Lambda V$ denotes the ideal in $\Lambda W \otimes \Lambda V$ generated by elements of W. If $\Lambda V, d$ is a minimal model for the (homotopy) fibre of $p : E \to B$, then we call p a *rational fibration*. It is known, for example, that any Serre fibration $F \to E \to B$, with F, E and B simply connected and of finite type, is a rational fibration [Ha$_3$]. Let $F \to E \to B$ be a rational fibration with minimal model $(*)$. Recall that the minimal model functor is contravariant. Then the fibre inclusion $j : F \to E$ is modeled by the projection $q \colon \Lambda W \otimes \Lambda V, D \to \Lambda V, d$ and the fibration $p : E \to B$ by the inclusion $i \colon \Lambda W, d_B \to \Lambda W \otimes \Lambda V, D$. In particular, it follows from basic properties of the minimal model that j^* is surjective if and only if q^* is surjective. More generally, we refer to a sequence of DG algebras $B, d_B \to B \otimes \Lambda V, D \to \Lambda V, d$ as a *model of a fibration*. This is more general than $(*)$ in that B, d_B need not be a minimal DG algebra. Given two such models, we will say they are *quasi-isomorphic models* if there is a commutative diagram

$$\begin{tikzcd}
B, d_B \arrow[r, "i"] \arrow[d, "\phi_1"'] & B \otimes \Lambda V, D \arrow[r, "q"] \arrow[d, "\phi_2"] & \Lambda V, d \arrow[d, "\phi_3"] \\
B', d_{B'} \arrow[r, "i'"'] & B' \otimes \Lambda V', D' \arrow[r, "q'"'] & \Lambda V', d'
\end{tikzcd}$$

in which each ϕ_i is a quasi-isomorphism. Then q' is surjective if and only if q is surjective, so either model is sufficient for our purposes.

We illustrate the fore-going discussion with an example:

1.2 Example There is a fibration $S^2 \to \mathbb{C}P^3 \to S^4$, obtained from the Hopf fibration, which has minimal model

$$\Lambda(w_4, w_7, d_B) \to \Lambda(w_4, w_7) \otimes \Lambda(v_2, v_3), D \to \Lambda(v_2, v_3, d).$$

Here, subscripts on generators indicate their degree and the differentials are given by $d_B(w_4) = 0$, $d_B(w_7) = w_4^2$, $D(v_2) = 0$ and $D(v_3) = v_2^2 - w_4$. We obtain d by projecting onto $\Lambda(v_2, v_3)$. Notice that D has a linear twisting. This simple example turns out to have several interesting features that illustrate quite well the topics of this paper. In particular, we observe that the fibration is TNCZ but the cohomology of E is not isomorphic as an algebra to the tensor product of the cohomology algebras of the base and fibre.

The (minimal) model of a fibration is the main technical tool that is used in this paper. Indeed, all results proved really boil down to algebraic results about models of fibrations and the results are proven by analysing them directly. When we say 'fibration', therefore, we mean rational fibration, as it is to this class of maps that our methods apply.

We now mention two ways to re-phrase Conjecture 1.1 that are used in the sequel. If F is positively elliptic, then its minimal model is pure. We extend the notion of pureness to fibrations with fibre F as follows:

1.3 Definition. *Let $F \to E \to B$ be a fibration in which F is elliptic with positive Euler characteristic. The fibration is pure (as a fibration) if it has a model $\Lambda W, d_B \to \Lambda W \otimes \Lambda V, D \to \Lambda V, d$ in which $D(V^{\text{even}}) = 0$ and $D(V^{\text{odd}}) \subseteq \Lambda W \otimes \Lambda(V^{\text{even}})$.*

Our first rephrasing of Conjecture 1.1 is given in the following:

1.4 Theorem [Th$_1$]. *For a given fibration $F \to E \to B$, in which F is elliptic with positive Euler characteristic, the following are equivalent:*

(1) *The fibration is TNCZ.*
(2) *The fibration is pure.*

Halperin's conjecture is therefore equivalent to the conjecture that each fibration with fibre a positively elliptic space is a pure fibration. Thomas uses his result to show the conjecture is true if the model for F has $\dim(V^{\text{even}}) = 1$ or 2.

Another re-phrasing of Conjecture 1.1 is given by Meier:

1.5 Theorem [Me]. *Let F be elliptic with positive Euler characteristic. Then the following are equivalent:*

(1) *Each fibration $F \to E \to B$, for arbitrary base space B, is TNCZ.*
(2) *Each fibration $F \to E \to S^{2n+1}$, for $n \geq 1$, is TNCZ.*
(3) *The (graded Lie algebra of) negative-degree derivations of the cohomology algebra of F are trivial, $Der^{<0}\, H^*(F;\mathbb{Q}) = 0$.*

Meier then uses his result to establish some special cases of the conjecture.

In addition to these results, various special cases of the conjecture have been established. In [Sh-Te], the conjecture is shown to be true for F a homogeneous space of the form G/H, where G is a compact, connected Lie group and H is a closed subgroup of maximal rank. In [Lu], the result of Thomas is extended to the case when $\dim(V^{\text{even}}) = 3$. In [Ma], it is shown that the class of spaces for which Halperin's conjecture is true is closed under fibrations. This paper of Markl also contains an introduction to Halperin's conjecture and some other interesting results. The conjecture is also closely related with $\mathrm{aut}_1(F)$, the identity component of the monoid of self-equivalences of F. This is an aspect of the conjecture that we are not directly concerned with here. See the articles [Fé-Th] and [Me] for information and references about this.

We end this introductory section with some notation and conventions. In general, we adopt the notation of [Au] or [Ha-St]. We use V or W to denote a positively graded, rational graded vector space. We have already used ΛV to denote the free graded commutative algebra on the graded vector space V. We write $\Lambda^+ V$ to denote the elements of positive degree. In this paper $\Lambda^+ V$ is the augmentation ideal of the canonical augmentation of ΛV. Also, $\Lambda^n V$ denotes the vector space of polynomials of homogeneous degree n, with respect to the grading by word length. If V is zero in even degrees, for example, this agrees with the n'th exterior power of V. Then $\Lambda^{\geq n} V$ denotes the ideal in ΛV generated by $\Lambda^n V$. We use the akronym DG for 'differential graded'. The (co)homology of a DG algebra (A, d) is denoted $H(A, d)$ or just $H(A)$. The elements of positive degree are denoted $H^+(A)$. A map of DG algebras that induces an isomorphism on homology is called a *quasi-isomorphism*. Other notation and definitions will be given as we go along.

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§2 — Some Models of Fibrations

Here we describe convenient models for the fibrations with which we are concerned. This is important for our results, as it allows direct analysis of the model.

Recall that a space is *formal* if its minimal model is determined by its cohomology algebra. Specifically, X is formal if there is a quasi-isomorphism

$$\psi : \mathcal{M}_X, d_X \to H^*(X;\mathbb{Q}), 0.$$

Such a quasi-isomorphism is sometimes referred to as a *formalization*. There are many interesting examples of formal spaces including H-spaces and co-H-spaces, symmetric spaces and simply connected, compact Kähler manifolds (for this last assertion, see [D-G-M-S]). A product or wedge of formal spaces is again a formal space. A positively elliptic space is a formal space. For further discussion of formal spaces, see the next section.

2.1 Lemma. *Let $F \to E \to B$ be a fibration in which B is formal. Up to quasi-isomorphism, the fibration has a model of the form*

$$H(B), 0 \to H(B) \otimes \Lambda V, D \to \Lambda V, d.$$

Proof. The result follows by pushing-out a minimal model over a formalization. Suppose that we have a minimal model $\Lambda W, d_B \xrightarrow{i} \Lambda W \otimes \Lambda V, D \xrightarrow{q} \Lambda V, d$, where $\Lambda W, d_B$ is a minimal model of B. Since B is formal, there is a quasi-isomorphism $\psi : \Lambda W, d_B \to H(B)$. Here, i is a cofibration and ψ a weak-equivalence, each in the sense of Baues. We form the pushout as described in [Ba,p.66], for example. This gives the following pushout diagram:

$$\begin{tikzcd}
\Lambda W, d_B \arrow[r, "i"] \arrow[d, "\psi"'] & \Lambda W \otimes \Lambda V, D \arrow[d, "\bar\psi"] \\
H(B), 0 \arrow[r, "\bar i"'] & H(B) \otimes \Lambda V, \delta
\end{tikzcd}$$

From the properties and construction of the pushout we obtain that $\bar\psi$ is also a quasi-isomorphism and that the new differential δ projects to the original d on ΛV. Hence we have the diagram

$$\begin{tikzcd}
\Lambda W, d_B \arrow[r, "i"] \arrow[d, "\psi"'] & \Lambda W \otimes \Lambda V, D \arrow[r, "q"] \arrow[d, "\bar\psi"] & \Lambda V, d \arrow[d, "\iota"] \\
H(B), 0 \arrow[r, "\bar i"'] & H(B) \otimes \Lambda V, \delta \arrow[r, "q"'] & \Lambda V, d
\end{tikzcd}$$

in which ψ and $\bar\psi$ are quasi-isomorphisms. The bottom sequence in this diagram is the model we want. $\square$

A slightly stronger result than the following was given in [Lu]. Nonetheless we include this here for completeness' sake.

2.2 Lemma. *Let $F \to E \to S^{2n+1}$ be a fibration in which F is elliptic with positive Euler characteristic. Then there is a model of the form*

$$\Lambda(u) \to \Lambda(u) \otimes \Lambda V, D \to \Lambda V, d,$$

in which $D(V^{\text{even}}) \subseteq u \cdot \Lambda(V^{\text{even}})$.

Proof. Suppose $\Lambda(u) \to \Lambda(u) \otimes \Lambda V, \delta \to \Lambda V, d$ is the minimal model. Let $v \in V^{\text{even}}$ and write

$$\delta(v) = u\chi_0 + u\chi_1, \qquad (*_1)$$

with $\chi_0 \in \Lambda(V^{\text{even}})$ and $\chi_1 \in I(V^{\text{odd}})$, the ideal of $\Lambda(u) \otimes \Lambda V$ generated by V^{odd}. Here χ_0 and χ_1 depend linearly on v. Applying δ to $(*_1)$ gives $0 = \delta^2(v) = -ud(\chi_0) - ud(\chi_1) = -ud(\chi_1)$. Therefore, χ_1 is a d-cocycle in $I(V^{\text{odd}})$. As stated in the introduction, each cocycle in this ideal is a boundary, so there is an element $\eta \in \Lambda V$ with $d\eta = \chi_1$. Hence $\delta(u\eta) = -u\chi_1$ and we can write $\delta(v) = u\chi_0 - \delta(u\eta)$.

For each $v \in V^{\text{even}}$ the previous paragraph obtains a corresponding $\eta(v)$ with $\delta(v) = u\chi_0(v) - \delta(u\eta(v))$. Pick a basis of V^{even} and use this to obtain a linear map $\phi\colon V^{\text{even}} \to \Lambda(u) \otimes \Lambda V$ with $\phi(v) = v + \eta(v)$ on basis elements. Extend ϕ to an isomorphism by setting $\phi = \iota$ on $\Lambda(u) \otimes \Lambda(V^{\text{odd}})$. Now define a differential by $D = \phi^{-1}\delta\phi$ so that we have an isomorphism of models as in the diagram

$$\begin{tikzcd}
\Lambda(u) \arrow[r] \arrow[d, "\iota"'] & \Lambda(u) \otimes \Lambda V, D \arrow[r] \arrow[d, "\phi"] & \Lambda V, d \arrow[d, "\iota"] \\
\Lambda(u) \arrow[r] & \Lambda(u) \otimes \Lambda V, \delta \arrow[r] & \Lambda V, d
\end{tikzcd}$$

Notice that for any $\chi \in \Lambda(V^{\text{even}})$ we can write $\phi(\chi) = \chi + u\chi'$, for some χ'. It follows that for $v \in V^{\text{even}}$, we have $D(v) = \phi^{-1}\delta\phi(v) = \phi^{-1}(u\chi_0(v)) = u\phi^{-1}(\chi_0(v)) = u\chi_0(v)$. □

In the next result we focus on the case in which the base is an odd-dimensional sphere, or more generally a wedge of such spheres. From Meier's result, cited as Theorem 1.5, these fibrations are of particular interest from our point of view. As we see, Halperin's conjecture implies the strongest possible restriction here:

2.3 Theorem. *Let $F \to E \to B$ be a fibration in which F is elliptic with positive Euler characteristic and B is rationally a wedge of odd spheres. If the fibration is TNCZ, then $E \simeq_{\mathbb{Q}} B \times F$ and the fibration (rationally) is trivial.*

Proof. A wedge of odd spheres is a formal space, so by Lemma 2.1 we have a model of the form $H(B) \to H(B) \otimes \Lambda V, D \to \Lambda V, d$. If the fibration is TNCZ, this model can be assumed pure. We see this as follows: Certainly the minimal model $\Lambda W, d_B \to \Lambda W \otimes \Lambda V, D \to \Lambda V, d$ can be assumed pure, from the theorem of Thomas (Theorem 1.4). Now $\Lambda V, d$ is two-stage and pure (as a DG algebra), so the inclusion $i : \Lambda W, d_B \to \Lambda W \otimes \Lambda V, D$ is a composition of two 'elementary cofibrations' $i_1 : \Lambda W, d_B \to \Lambda W \otimes \Lambda V^{\text{even}}, d_B \otimes 1$ and $i_2 : \Lambda W \otimes \Lambda V^{\text{even}}, d_B \otimes 1 \to \Lambda W \otimes \Lambda V, D$. Thus the pushout model of Lemma 2.1 is a composition of two pushouts as in the following diagram:

$$\begin{tikzcd}
\Lambda W, d_B \arrow[r, "i_1"] \arrow[d, "\psi"'] & \Lambda W \otimes \Lambda V^{\text{even}}, d_B \otimes 1 \arrow[r, "i_2"] \arrow[d, "\psi'"] & \Lambda W \otimes \Lambda V, D \arrow[d, "\bar{\psi} = \bar{\psi}'"] \\
H(B), 0 \arrow[r, "\bar{i}_1"'] & H(B) \otimes \Lambda V^{\text{even}}, 0 \arrow[r, "\bar{i}_2"'] & H(B) \otimes \Lambda V, \delta
\end{tikzcd}$$

Note the trivial differential in the intermediate stage of the lower sequence. It follows that $H(B) \otimes \Lambda V, \delta$ is also pure (as a model for the fibration). This observation also follows from a close reading of Thomas' proof of Theorem 1.4.

Now let $v \in V^{\text{odd}}$ and let $\{b_i\}$ be a basis for $H^+(B)$. Write the differential D as

$$D(v) = dv + \sum_i b_i \Omega_i(v). \qquad (*_2)$$

Here, each $\Omega_i(v)$ depends linearly on v and for parity of degree reasons is an odd-degree element of ΛV. Since this model is pure, $D(dv) = 0$. Therefore, applying D to $(*_2)$ gives $0 = D^2(v) = -\sum_i b_i d\Omega_i(v)$. For the last expression, we use the fact that products in $H^+(B)$ are trivial. Now we have $d\Omega_i(v) = 0$ for each i, so each $\Omega_i(v)$ is a d-cocycle of odd degree. But $H^{\text{odd}}(\Lambda V) = 0$, as ΛV is a model for F. Therefore, we can choose elements $\eta_i(v) \in \Lambda V$ for which $-\Omega_i(v) = d\eta_i(v)$. This gives $D\big(\sum_i b_i\eta_i(v)\big) = \sum_i b_i\Omega_i(v)$. So far we have dealt with a single element $v \in V^{\text{odd}}$. Now pick a basis for V^{odd} and use the previous argument to define a linear map $\phi\colon V^{\text{odd}} \to H(B) \otimes \Lambda V$ such that on each basis element we have $\phi(v) = v - \sum_i b_i\eta_i(v)$. Extend ϕ to an isomorphism of algebras by setting $\phi = \iota$ on $H(B) \otimes \Lambda(V^{\text{even}})$. Finally, define a differential Δ on $H(B) \otimes \Lambda V$ by $\Delta = \phi^{-1}D\phi$, so that $\phi\colon H(B) \otimes \Lambda V, \Delta \to H(B) \otimes \Lambda V, D$ is an isomorphism of DG algebras that restricts to the identity on $H(B)$ and projects to the identity on $\Lambda V, d$. By construction we have $\Delta = 1 \otimes d$, as is easily checked. The result follows $\square$

§3 — Formality and TNCZ Fibrations

Halperin's conjecture asserts that a fibration is close to trivial if the fibre is positively elliptic. So suppose a fibration has base a formal space and fibre a (formal) positively elliptic space. The conjecture can be read as saying the total space is close to a product of formal spaces and therefore close to being formal. This point of view turns out to yield an equivalent formulation of the conjecture.

A formal space has minimal model that is more highly structured than an arbitrary minimal model. We recall here some properties of the *bigraded model* that are used in the sequel. See [Ha-St] for a full discussion. The bigraded model of a formal space is a minimal model $\Lambda V, d$ for which the vector space of generators has a second grading, $V = \oplus_{i\geq 0} V_i$. This gives ΛV the structure of a bigraded algebra. Furthermore, the bigraded model is a bigraded DG algebra in the sense that $d\colon \Lambda V \to \Lambda V$ decreases second grading by exactly 1. In particular, $d(V_0) = 0$. The second grading therefore passes to cohomology and, as further properties of the bigraded model, we have $H(\Lambda V, d) = H_0(\Lambda V, d)$ and $H_+(\Lambda V, d) = 0$. There is a quasi-isomorphism — indeed, the bigraded model proper — given by

$$\rho\colon \Lambda V, d \to H(\Lambda V, d) = H_0(\Lambda V, d)$$

which maps V_+ to zero and each $v \in V_0$ to the class it represents in $H(\Lambda V, d)$. A positively elliptic space is formal and its bigraded model has a simple form. We identify $V_0 = V^{\text{even}}$ and $V_1 = V^{\text{odd}}$. There are no generators of second degree 2 or more. Notice that this accords with a fact we stated in the introduction, namely that any cocycle in the ideal that we are now denoting $(\Lambda V)_+$ is a boundary. Following [Ha-St], we adopt the notation $V_{(n)} = \oplus_{i=0}^n V_i$ and $(\Lambda V)_{(n)} = \oplus_{i=0}^n (\Lambda V)_i$.

3.1 Proposition. *Let $F \to E \to B$ be a fibration in which F and B are formal. If the fibration is TNCZ, then there is a model*

$$H(B), 0 \to H(B) \otimes \Lambda V, D \to \Lambda V, d,$$

in which $\Lambda V, d$ is the bigraded model of F and $H(B) \otimes \Lambda V, D$ is filtered in the following sense: Let $V = \oplus_{i \geq 0} V_i$ be the second grading of the bigraded model. Then $D(V_0) = 0$ and for each $i \geq 1$, $D(V_i) \subseteq H(B) \otimes (\Lambda V)_{(i-1)}$.

Proof. By Lemma 2.1 we have a model $H(B) \to H(B) \otimes \Lambda V, \delta \to \Lambda V, d$ and we can take $\Lambda V, d$ to be the bigraded model of F. Suppose that the fibration is TNCZ. We begin by showing that up to isomorphism of this model, $\delta(V_0) = 0$.

Let $v \in V_0$, so that v represents a class $[v]$ in $H(\Lambda V)$. Since $q^* : H(H(B) \otimes \Lambda V, \delta) \to H(\Lambda V)$ is onto, there is some δ-cocycle $\chi \in H(B) \otimes \Lambda V$ with with $q^*([\chi]) = [v]$. Write χ as $\chi = \chi_1 + \chi_2 + \beta'$ with $\chi_1 \in V$, $\chi_2 \in \Lambda^{\geq 2} V$ and $\beta' \in H^+(B) \otimes \Lambda V$. Then $q^*([\chi]) = [\chi_1 + \chi_2]$, so there is some $\eta \in \Lambda V$ with $d\eta = \chi_1 + \chi_2 - v$. Since $d\eta$ is decomposable, this implies $v = \chi_1$ and also $d\eta = \chi_2$. It follows that $\delta(\eta) = \chi_2 + \beta''$ for some $\beta'' \in H^+(B) \otimes \Lambda V$. Thus we obtain $\chi = v + \beta + \delta\eta$, where we have written $\beta = \beta' - \beta'' \in H^+(B) \otimes \Lambda V$. So far, for an element $v \in V_0$, we have identified a corresponding δ-cocycle $v + \beta$, where $\beta \in H^+(B) \otimes \Lambda V$ depends linearly on V, that satisfies $q^*([v + \beta(v)]) = [v]$. By choosing a basis of V_0, we obtain a linear map $\psi \colon V_0 \to H(B) \otimes \Lambda V$ with $\psi(v) = v + \beta(v)$ on basis elements. Extend this to an isomorphism by setting $\psi = \iota$ on $H(B) \otimes \Lambda(V_+)$. Then define a new differential $\delta_0 = \psi^{-1} \delta \psi$ so that $\psi \colon H(B) \otimes \Lambda V, \delta_0 \to H(B) \otimes \Lambda V, \delta$ is a DG isomorphism. Indeed, we have an isomorphism of models as in the diagram:

$$\begin{tikzcd} H(B) \arrow[r] \arrow[d, "\iota"'] & H(B) \otimes \Lambda V, \delta_0 \arrow[r] \arrow[d, "\psi"] & \Lambda V, d \arrow[d, "\iota"] \\ H(B) \arrow[r] & H(B) \otimes \Lambda V, \delta \arrow[r] & \Lambda V, d \end{tikzcd}$$

By construction $\delta_0(V_0) = 0$, as is easily checked.

Next we show inductively that for each $n \geq 0$, there is a model isomorphic to the one we just obtained, via a diagram

$$\begin{tikzcd} H(B) \arrow[r] \arrow[d, "\iota"'] & H(B) \otimes \Lambda V, \delta_n \arrow[r] \arrow[d, "\phi_{(n)}"] & \Lambda V, d \arrow[d, "\iota"] \\ H(B) \arrow[r] & H(B) \otimes \Lambda V, \delta_0 \arrow[r] & \Lambda V, d \end{tikzcd}$$

and such that δ_n is a filtered differential on $H(B) \otimes (\Lambda V)_{(n)}$, i.e., $\delta_n(V_i) \subseteq H(B) \otimes (\Lambda V)_{(i-1)}$ for $i = 0, \ldots, n$. Induction starts with $n = 0$, where the assertion is trivial. Suppose inductively that for some n we have such an isomorphism $\phi_{(n)}$ and $H(B) \otimes \Lambda V, \delta_n$ which is filtered on $H(B) \otimes (\Lambda V)_{(n)}$. Let $v \in V_{n+1}$ and write

$$\delta_n(v) = dv + \xi_{(n)} + \xi_+ \qquad (*_3)$$

for elements $\xi_{(n)} \in H^+(B) \otimes (\Lambda V)_{(n)}$ and $\xi_+ \in H^+(B) \otimes (\Lambda V)_{\geq n+1}$.

Claim. There is an element $\eta \in H^+(B) \otimes \Lambda V$, depending linearly on v, for which $\delta_n(v) = dv + \xi'_{(n)} + \delta_n(\eta)$ and $\xi'_{(n)} \in H^+(B) \otimes (\Lambda V)_{(n)}$.

Proof of Claim. Suppose that $\xi_+ = \xi_+^m + \xi_+^{m+1} + \cdots + \xi_+^M$, for some $2 \leq m \leq M$, with each $\xi_+^i \in H^i(B) \otimes (\Lambda V)_{\geq n+1}$. Let $\{b_j\}$ be a basis for $H^m(B)$ and write the 'lowest' term ξ_+^m as $\xi_+^m = \sum_j b_j \otimes \chi_j$ for suitable elements $\chi_j \in (\Lambda V)_{\geq n+1}$. Applying δ_n to $(*_3)$ gives

$$0 = \delta_n^2(v) = \delta_n(dv + \xi_{(n)}) + \delta_n(\xi_+^m) + \sum_{i=m+1}^{M} \delta_n(\xi_+^i). \qquad (*_4)$$

The induction hypothesis on δ_n implies that $\delta_n(dv + \xi_{(n)}) \in H(B) \otimes (\Lambda V)_{(n-1)}$. The ideal in $H(B) \otimes \Lambda V$ generated by elements of $H(B)$ of degree at least $m+1$ is δ_n-stable and contains all terms contributed by the $\delta_n(\xi_+^i)$ for $i \geq m+1$. Therefore, in equation (2), terms of the form $\sum_j (-1)^m b_j \otimes d(\chi_j)$ are the only contributions to $H^m(B) \otimes (\Lambda V)_{\geq n}$. It follows that each $d(\chi_j) = 0$. Since this is the bigraded model, in which $H_+(\Lambda V) = 0$, there are elements $\eta_i \in \Lambda V$ such that $(-1)^m \chi_j = d(\eta_j)$, for each j. Thus we have

$$\delta_n\big(\sum_j b_j \otimes \eta_j\big) = \zeta_{(n)} + \xi_+^m + \zeta_+^{m+1} + \cdots + \zeta_+^{M'},$$

with $\zeta_{(n)} \in H(B) \otimes (\Lambda V)_{(n)}$ and $\zeta_+^j \in H^j(B) \otimes (\Lambda V)_{\geq n+1}$, for $m+1 \leq j \leq M'$. Substituting this into $(*_3)$ above gives

$$\delta_n(v) = dv + \xi'_{(n)} + \xi'_+ + \delta_n\big(\sum_j b_j \otimes \eta_j\big),$$

where $\xi'_{(n)} = \xi_{(n)} - \zeta_{(n)} \in H(B) \otimes (\Lambda V)_{(n)}$ and $\xi'_+ = \xi_+ - \xi_+^m - \sum_{j \geq m+1} \zeta_+^j \in H^{\geq m+1}(B) \otimes (\Lambda V)_{\geq n+1}$. An induction argument on m, repeating this last step as necessary, shows the claim. *End of Proof of Claim.*

As in previous arguments, choose a basis for V_{n+1} and use the claim to obtain a linear map $\phi_{n+1} \colon V_{n+1} \to H(B) \otimes \Lambda V$ that satisfies $\phi_{n+1}(v) = v - \eta(v)$ on basis elements, where $\delta_n(v) = dv + \xi'_{(n)}(v) + \delta_n \eta(v)$. Extend ϕ_{n+1} to an isomorphism by setting $\phi_{n+1} = \iota$ on $H(B)$ and on all generators of V other than those in V_{n+1}. In particular, notice that we take $\phi_{n+1} = \iota$ on $H(B) \otimes (\Lambda V)_{(n)}$. Define a differential δ_{n+1} by $\delta_{n+1} = (\phi_{n+1})^{-1} \delta_n \phi_{n+1}$ so that $\phi_{n+1} \colon H(B) \otimes \Lambda V, \delta_{n+1} \to H(B) \otimes \Lambda V, \delta_n$ is an isomorphism of DG algebras. It is easy to check that this construction gives an isomorphism of models

$$\begin{tikzcd}
H(B) \arrow[r] \arrow[d, "\iota"'] & H(B) \otimes \Lambda V, \delta_{n+1} \arrow[r] \arrow[d, "\phi_{n+1}"] & \Lambda V, d \arrow[d, "\iota"] \\
H(B) \arrow[r] & H(B) \otimes \Lambda V, \delta_n \arrow[r] & \Lambda V, d
\end{tikzcd}$$

and that δ_{n+1} is filtered on $H(B) \otimes (\Lambda V)_{(n+1)}$. The induction is completed by setting $\phi_{(n+1)} = \phi_{(n)} \phi_{n+1}$.

Notice that the inductive construction of $\phi_{(n)}$ and δ_n is such that they stabilise, i.e., for $v \in V_n$ we have $\phi_{(N)}(v) = \phi_{(N+1)}(v)$ and $\delta_N(v) = \delta_{N+1}(v)$, for $N \geq n$. Therefore, we obtain an isomorphism of models

$$\begin{tikzcd}
H(B) \arrow[r] \arrow[d, "\iota"'] & H(B) \otimes \Lambda V, D \arrow[r] \arrow[d, "\phi"] & \Lambda V, d \arrow[d, "\iota"] \\
H(B) \arrow[r] & H(B) \otimes \Lambda V, \delta_0 \arrow[r] & \Lambda V, d
\end{tikzcd}$$

by setting $\phi(v) = \phi_{(n)}(v)$ if $v \in V_n$, which gives an isomorphism, then defining $D = \phi^{-1}\delta_0\phi$. For a fixed $v \in V$, we have $D(v) = \delta_N(v)$ for N sufficiently large — depending on the terms that appear in $\delta_0\phi(v)$. So we have $D(v) \in H(B) \otimes (\Lambda V)_{(n-1)}$ and the new model is filtered. □

We now develop the main result of the section. The next proposition is a little more general than we need in the sequel. However, it is interesting in its own right. In it, we only assume that the fibre space is formal and elliptic. Recall Halperin's result [Ha$_1$], that any elliptic space has non-negative Euler characteristic. If a space is positively elliptic, then it is formal. However, a space that is formal and elliptic need not have positive Euler characteristic. For instance, a product of odd-dimensional spheres is elliptic and formal, but has Euler characteristic equal to zero. There are strong restrictions on a space, however, that follow from the hypothesis of formal and elliptic. We do not dwell on this point here, but simply point out that any space that is formal and elliptic has a two-stage bigraded model (cf. [Fé-Ha$_1$]). This is the feature of these spaces that we use in the result.

3.2 Proposition. *Let $F \to E \to B$ be a fibration in which F is formal and elliptic and B is formal. If the fibration is TNCZ, then E is formal also.*

Proof. From Proposition 3.1 we have a model of the fibration

$$H(B), 0 \to H(B) \otimes \Lambda V, D, \to \Lambda V, d$$

in which $D(V_0) = 0$ and for $i \geq 1$, $D(V_i) \subseteq H(B) \otimes (\Lambda V)_{(i-1)}$. This means, in particular, that $D(V_1) \subseteq H(B) \otimes (\Lambda V)_0$. Since the bigraded model for F is two-stage, and thus $V = V_0 \oplus V_1$, the total space $H(B) \otimes \Lambda V, D$ in this model is actually a *bigraded DG algebra*.

We now show that, with respect to this bigrading, $H_+\big(H(B) \otimes \Lambda V\big) = 0$. For let $x \in (H(B) \otimes \Lambda V)_r$ be a D-cocycle, for any $r \geq 1$. We can write x as $x = x_m + x_{m+1} + \cdots + x_M$, for $2 \leq m \leq M$, where each $x_i \in H^i(B) \otimes (\Lambda V)r$. Let $\{b_j\}$ be a basis for $H^m(B)$. Then we can write the 'lowest' term x_m as $x_m = \sum_j b_j \otimes \chi_j$ for appropriate terms $\chi_j \in (\Lambda V)_r$, with $r \geq 1$. Now $(D - d)(\chi_j) \in H^+(B) \otimes \Lambda V$ and also the ideal $H^{\geq m+1}(B) \otimes \Lambda V$ is D-stable. Therefore, as x is a D-cocycle, we have

$$0 = D(x) \equiv (-1)^m \sum_j b_j \otimes d(\chi_j),$$

where the congruence is modulo the ideal $H^{\geq m+1}(B) \otimes \Lambda V$. Therefore, $d\chi_j = 0$ and, since $\Lambda V, d$ is the bigraded model, there are elements η_j with $d\eta_j = (-1)^m \chi_j$

for each j. This yields $D\big(\sum_j b_j \otimes \eta_j\big) \equiv \sum_j b_j \otimes \chi_j$, again modulo $H^{\geq m+1}(B) \otimes \Lambda V$. Now we can write

$$x = D\big(\sum_j b_j \otimes \eta_j\big) + x'_{m+1} + \cdots + x'_{M'},$$

with each $x'_i \in H^i(B) \otimes (\Lambda V)r$. An induction argument repeating this argument as necessary obtains the result that x is D-exact.

We have shown that $H\big(H(B) \otimes \Lambda V, \delta\big) = H_0\big(H(B) \otimes \Lambda V, \delta\big)$. But now the projection

$$p : H(B) \otimes \Lambda V, \delta \to \frac{H(B) \otimes \Lambda V, \delta}{\big(\delta V_1\big)} = H_0\big(H(B) \otimes \Lambda V, \delta\big)$$

is a quasi-isomorphism. Since $H(B) \otimes \Lambda V, \delta$ is a model for E, and p is a quasi-isomorphism from it to its cohomology, it follows that E is formal. □

The next result is something of a converse to Proposition 3.2.

3.3 Proposition. *Let F be be elliptic with positive Euler characteristic and let $F \to E \to S^{2n+1}$ be a fibration. If E is formal, then the fibration is TNCZ.*

Proof. We show the contrapositive. From Lemma 2.2 the fibration has a model

$$\Lambda(u) \to \Lambda(u) \otimes \Lambda V, D \to \Lambda V, d$$

with $D(V_0) \subseteq u \cdot \Lambda V_0$. If $D(V_0) = 0$, then the fibration is TNCZ. This follows easily, because then $q^* \colon H(\Lambda(u) \otimes \Lambda V) \to H(\Lambda V)$ is surjective onto generators of $H(\Lambda V)$. So suppose that $D(V_0) \neq 0$. Choose a 'K-S' vector space basis $V_0 = \langle x_1, \ldots, x_n \rangle$. This is an ordered basis such that the differential of each basis element is contained in the subalgebra generated by u and basis elements of lower order. There is at least one of the x_i with non-zero differential, so let r be the largest subscript with $D(x_r) \neq 0$. Then $D(V_0)$ is contained in the ideal of $\Lambda(u) \otimes \Lambda V$ generated by elements $\{ux_1, \ldots, ux_{r-1}\}$, which we denote I_{r-1}. For parity of degree reasons, $(D - d)(V_1)$ is contained in the ideal generated by $u \cdot V_1$. Also, we have $d(V_1) \subseteq \Lambda^{\geq 2} V_0$. Let J denote the ideal of $\Lambda(u) \otimes \Lambda V$ generated by $u \cdot V_1 + \Lambda^{\geq 2} V_0$. Then the image of D is contained in the ideal $I_{r-1} + J$. Now $D(x_r) = u\chi$ for some χ and hence we obtain a D-cocycle ux_r. This cannot be exact, as it is not in the ideal $I_{r-1} + J$. It follows from [D-G-M-S, Th.4.1] that the total space is not formal. □

Finally, we collect together the preceding results into the main result of the section. As we see in the corollary to this result, we have obtained an equivalent formulation of Halperin's conjecture.

3.4 Theorem. *Let F be elliptic with positive Euler characteristic. Then the following are equivalent:*

(1) *Conjecture 1.1 is true for F.*
(2) *For any fibration $F \to E \to B$ in which B is formal, E is formal also.*

Proof. The implication (1) ⇒ (2) follows from Proposition 3.2. Conversely if (2) is true, then in each fibration $F \to E \to S^{2n+1}$ the total space E is formal. Now Proposition 3.3 implies that each such fibration is TNCZ. From Meier's result, Theorem 1.5, all fibrations $F \to E \to B$ are TNCZ and (1) holds. □

3.5 Corollary. *Let F be elliptic with positive Euler characteristic. Halperin's conjecture is true for F if and only if, in each fibration $F \to E \to S^{2n+1}$, the total space E is formal.* □

§4 — Numerical Invariants

Here we consider some invariants related to the *Lusternik-Schnirelmann category*. Recall that this is a numerical homotopy invariant of a space, defined as one less than the smallest number of open sets required to cover the space, when each is contractible in the space. As is usual in rational homotopy theory, we have 'normalised' so that a sphere has category equal to 1. This invariant and its approximations have been much studied both in ordinary and rational homotopy theory. See [Ja] for a recent survey with many references. Here we focus on four rational homotopy invariants. We define these invariants and include some discussion, before proceeding to the results:

(Rational) Cup-length: This is the nilpotency — as an algebra — of the rational cohomology algebra of a space X. It is denoted here $\mathrm{cup}_0(X)$. For example we have $\mathrm{cup}_0(\mathbb{C}P^n) = n$ for each $n \geq 1$.

(Rational) Toomer's invariant: As in [Fé-Ha$_2$,Rem.9.3] we describe this invariant as follows: Let $\Lambda V, d$ be the minimal model of X. Consider the projection

$$p_n \colon \Lambda V \to \frac{\Lambda V}{\Lambda^{\geq n+1} V}.$$

We obtain our rational invariant, denoted $\mathrm{e}_0(X)$, by setting $\mathrm{e}_0(X) \leq n$ if $(p_n)^*$ is injective. In other words, $\mathrm{e}_0(X)$ is the largest n for which some non-zero class in $H(\Lambda V)$ is represented by a cocycle in $\Lambda^{\geq n} V$.

(Rational) Category: This is the Lusternik-Schnirelmann category of the rationalization of X. We denote it $\mathrm{cat}_0(X)$ and, following [Fé-Ha$_2$,Th.4.7], we describe it in terms of the minimal model of X: Set $\mathrm{cat}_0(X) \leq n$ if the above projection p_n makes ΛV into a retract of the quotient $\Lambda V/\Lambda^{\geq n+1} V$.

(Rational) Cone-length: This is the least number of steps required to build the rational homotopy type of a space X as a succession of cofibration sequences of rational spaces. It is denoted $\mathrm{cl}_0(X)$. Specifically, set $\mathrm{cl}_0(X) = 0$ if $X \simeq_{\mathbb{Q}} *$ and $\mathrm{cl}_0(X) = 1$ if X has the rational homotopy type of a wedge of spheres. In general, set $\mathrm{cl}_0(X) \leq n$ if there are spaces $X_1, A_1, \ldots, A_{n-1}$, each of which has the homotopy type of a wedge of rational spheres, and $n-1$ cofibration sequences $A_i \to X_i \to X_{i+1}$, for $i = 1, \ldots, n-1$, such that $X_n \simeq_{\mathbb{Q}} X$. In [Co] it is shown that $\mathrm{cl}_0(X)$ agrees with the 'homotopical nilpotency' of the minimal model of X, i.e., the least n for which the minimal model is quasi-isomorphic to a DG algebra that is of nilpotency n as an algebra. It is in this latter guise that we meet this invariant here.

We have been a little careless in phrasing the above definitions by implicitly assuming these invariants finite. Although the case when one or other of these is infinite does not seem so interesting in our context, it is allowed for, where appropriate, in the following results.

For these invariants, we always have inequalities as follows:

$$\mathrm{cup}_0(X) \leq \mathrm{e}_0(X) \leq \mathrm{cat}_0(X) \leq \mathrm{cl}_0(X).$$

In the special case that X is a *formal* space, all these invariants agree. In this case we will denote their common value $\mathrm{nil}_0(X)$. This usage accords with Cornea's homotopical nilpotency in this case. In particular, if F is a positively elliptic space, then it is formal and we have $\mathrm{cup}_0(F) = \mathrm{e}_0(F) = \mathrm{cat}_0(F) = \mathrm{cl}_0(F)$, which we denote by $\mathrm{nil}_0(F)$.

We mention some examples to illustrate these invariants:

4.1 Examples If $X = S^n$, then X is formal, our four invariants agree and we have $\mathrm{nil}_0(X) = 1$. If $X = \mathbb{C}P^n$, this is likewise formal and $\mathrm{nil}_0(X) = n$. Next, suppose $X = S^2 \vee S^2 \cup_\alpha e^5$, where $\alpha = [\iota_1, [\iota_1, \iota_2]]$, the triple Whitehead product in $\pi_4(S^2 \vee S^2)$. Then we have $\mathrm{cup}_0(X) = 1$ but $\mathrm{e}_0(X) = \mathrm{cat}_0(X) = \mathrm{cl}_0(X) = 2$. A well-known example of Lemaire-Sigrist, developed by Félix-Halperin (cf. [Fé-Ha$_2$]), is $X = (\mathbb{C}P^2 \vee S^2) \cup_\omega e^7$ for a certain attaching map ω. This space satisfies $\mathrm{cup}_0(X) = \mathrm{e}_0(X) = 2$ whilst $\mathrm{cat}_0(X) = \mathrm{cl}_0(X) = 3$. Furthermore, this example has $\mathrm{e}_0(X^n) = 2n$ and $\mathrm{cat}_0(X^n) = 3n$. This illustrates that $\mathrm{e}_0(X)$ can be smaller than $\mathrm{cat}_0(X)$ by an arbitrarily large amount.

These invariants behave quite well for products of spaces. The product formula $\mathrm{cup}_0(X \times Y) = \mathrm{cup}_0(X) + \mathrm{cup}_0(Y)$ is well-known. It is easy to see that e_0 likewise is additive for products [Fé-Ha$_2$,Rem.9.3]. Recently, cat_0 has been shown to be additive for products [Fé-Ha-Le], and cl_0 to be additive at least for products of rational Poincaré duality spaces. Indeed, it has been shown that $\mathrm{e}_0(X) = \mathrm{cat}_0(X) = \mathrm{cl}_0(X)$ whenever X is a rational Poincaré duality space (see [Fé-Ha-Le] and [Co-Fé-Le]). Now Halperin's conjecture asserts that in certain fibrations the total space is close to being a product of base and fibre spaces. This combines to suggest there might be good relations between these invariants for base, total and fibre spaces in such fibrations. We shall see this is the case.

First we give some results that complement one of Jessup [Je,Prop.3.6]. Let $F \to E \to B$ be a TNCZ fibration with F formal. Then Jessup's result gives $\mathrm{cat}_0(E) \geq \mathrm{cat}_0(B) + \mathrm{nil}_0(F)$. Actually, his result applies a little more generally and was proved for (what was then thought to be) a different numerical invariant, Mcat_0. The conclusion for cat_0 follows by a result of Hess [He], which identified cat_0 with Mcat_0.

We specialize this result to the following:

4.2 Proposition. *Let $F \to E \to B$ be a fibration in which F is elliptic with positive Euler characteristic and B is formal. If the fibration is TNCZ, then E is also formal and we have $nil_0(E) \geq nil_0(B) + nil_0(F)$.*

Proof. The formality of E follows immediately from Proposition 3.2. Since E and B are both formal, $\mathrm{nil}_0 = \mathrm{cat}_0$ for these spaces and the conclusion follows from the result of Jessup mentioned above. □

Next, consider the case in which B is not formal. Notice the following result does not require that F be positively elliptic.

4.3 Proposition. *Let $F \to E \to B$ be a fibration with F formal. If the fibration is TNCZ, then we have the following inequalities:*

(1) *$cup_0(E) \geq cup_0(B) + nil_0(F)$.*
(2) *$e_0(E) \geq e_0(B) + nil_0(F)$.*

Furthermore, the result of Jessup gives $cat_0(E) \geq cat_0(B) + nil_0(F)$.

Proof. Actually, for (1) the hypothesis of formality on F is redundant. Recall from the introduction, that if a fibration $F \to E \to B$ is TNCZ, then $H^*(E; \mathbb{Q}) \cong H^*(B; \mathbb{Q}) \otimes H^*(F; \mathbb{Q})$ as $H^*(B; \mathbb{Q})$-modules. Under this isomorphism, two elements of $1 \otimes H^*(F; \mathbb{Q})$ multiply as $(1 \otimes x)(1 \otimes y) = 1 \otimes xy + \chi$, for some χ in the ideal generated by $H^+(B)$. It follows that $\mathrm{cup}_0(E) \geq \mathrm{cup}_0(B) + \mathrm{cup}_0(F)$.

For (2), we work with a model of the fibration. Suppose that F is formal and the fibration is TNCZ. Then it has a model $\Lambda W, d_B \to \Lambda W \otimes \Lambda V, D \to \Lambda V, d$ in which $D(V_0) = 0$. This follows from the argument in the first two paragraphs of the proof of Proposition 3.1, replacing $H(B)$ there by ΛW. So let $\alpha \in \Lambda^{\geq n} V$ be a cocycle representative for some non-zero class in $H(\Lambda V)$. Since F is formal, we can suppose that $\alpha \in \Lambda^{\geq n} V_0$. In our model we have $D(V_0) = 0$, so $[\alpha] \in H(\Lambda W \otimes \Lambda V)$. Furthermore, α is not D-exact, since it is not d-exact. Let $\beta \in \Lambda^{\geq m} W$ be a cocycle representative for some non-zero class of $H(\Lambda W)$. Since we have $H(\Lambda W \otimes \Lambda V) \cong H(\Lambda W) \otimes H(\Lambda V)$ as $H(\Lambda W)$-modules, the product $[\beta][\alpha] = [\beta\alpha]$ is non-zero in $H(\Lambda W \otimes \Lambda V)$. Now $\alpha\beta \in \Lambda^{\geq m+n}(W \oplus V)$ so $\mathrm{e}_0(E) \geq m + n$. □

Example 1.2 illustrates that inequality, as in Proposition 4.3, is the best that can be hoped for in general. But see below for stronger relations in special cases.

If B is not formal in Proposition 4.3, there is no *a priori* reason why the invariants cup_0, e_0 and cat_0 for B or E should agree. Therefore, it can be thought of as giving three distinct necessary conditions for Halperin's conjecture to be true. We summarise this in the following:

4.4 Corollary. *Let $F \to E \to B$ be any fibration in which F is elliptic with positive Euler characteristic. If B is formal and Halperin's conjecture is true, then E is also formal and $nil_0(E) \geq nil_0(B) + nil_0(F)$. For general B, if the fibration is TNCZ then we have the three inequalities*

$$\begin{aligned} cup_0(E) &\geq cup_0(B) + nil_0(F) \\ e_0(E) &\geq e_0(B) + nil_0(F) \\ cat_0(E) &\geq cat_0(B) + nil_0(F). \end{aligned}$$

These inequalities give necessary conditions for Halperin's conjecture to be true. □

A much stronger consequence of Halperin's conjecture follows if we restrict the base as in Theorem 2.3.

4.5 Corollary. *Let $F \to E \to B$ be a fibration in which F is elliptic with positive Euler characteristic and B is rationally a wedge of odd spheres. If Halperin's conjecture is true, then E is formal, the invariants $cup_0(E)$, $e_0(E)$, $cat_0(E)$ and $cl_0(E)$ all agree and their common value, $nil_0(E)$, satisfies $nil_0(E) = 1 + nil_0(F)$.*

Proof. This follows from Theorem 2.3. □

So far, we have collected together some consequences of Halperin's conjecture. These consequences can be read as *weak versions of Halperin's conjecture*. We go on to establish some of these weak versions of the conjecture. So far, we can deal quite well with the case in which the base B is an odd-dimensional sphere.

The next result is proved in [Fé-Ha$_2$], but we prove it here anyway.

4.6 Lemma. *Let $F \to E \to S^{2n+1}$ be a fibration in which F is a Poincaré duality space. Then $e_0(E) \geq 1 + e_0(F)$.*

Proof. Suppose $\Lambda(u) \to \Lambda(u) \otimes \Lambda V, D \to \Lambda V, d$ is a minimal model of the fibration. If $\mathrm{e}_0(F) = m$, then Poincaré duality implies that the fundamental class of F can be represented by a cocycle $\alpha \in \Lambda^{\geq m} V$. Since $d\alpha = 0$, $u\alpha$ is a D-cocycle. We must show that it is not exact. Suppose that $D(\eta_1 + u\eta_2) = \lambda u \alpha$ for $\lambda \in \mathbb{Q}$ and $\eta_i \in \Lambda V$. Then $d\eta_1 = 0$, but η_1 has higher degree than α, the fundamental class of

F (we assume simple connectivity). Thus $\eta_1 = d\zeta_1$ and we have $D\zeta_1 = \eta_1 + u\eta_3$. Now $D\big(u(\eta_2 - \eta_3)\big) = \lambda u\alpha$ which implies $d(\eta_2 - \eta_3) = -\lambda\alpha$. Since α represents the fundamental class of F, we have $\lambda = 0$. This shows $u\alpha$ represents a non-zero class and the result follows. □

Next we give the main result of this section.

4.7 Theorem. *Let $F \to E \to B$ be a fibration in which F is elliptic with positive Euler characteristic and B is a wedge of odd-dimensional spheres. Then E satisfies $e_0(E) = cat_0(E) = cl_0(E) = 1 + nil_0(F)$.*

Proof. In fact the proof will display a simple model for E of homotopical nilpotency equal to $1 + \mathrm{nil}_0(F)^{(\dagger)}$. Let $\rho : \Lambda V, d \to H(F)$ be the bigraded model for F. By construction we have $\rho(V_1) = 0$ and hence $\rho\big((\Lambda V)_+\big) = 0$. Consider the ideal $\ker\rho \subseteq \Lambda V$. This is a differential ideal since all boundaries are in $\ker\rho$. Further, it is an acyclic ideal since ρ is a surjective quasi-isomorphism. Now B is formal, so by Lemma 2.1 the fibration has a model $H(B) \to H(B) \otimes \Lambda V, D \to \Lambda V, d$. We claim that the ideal $H(B) \otimes \ker\rho$ of $H(B) \otimes \Lambda V$ is also a differential acyclic ideal. To see this, we argue as follows:

Let $\{b_i\}$ be a basis for $H^+(B)$. For any $\chi \in \Lambda V$, we can write

$$D(\chi) = d\chi + \sum_i b_i \Omega_i(\chi),$$

and a standard argument shows that this defines derivations Ω_i on ΛV, each of negative even degree according as the degree of the b_i (cf. the result of Meier, cited as Theorem 1.5). Further, since all products in $H^+(B)$ are trivial, each of these derivations commutes with the differential d in ΛV. Now for $w \in V_1$, we have $D(w) = dw + \sum_i b_i\Omega_i(w)$. For parity of degree reasons we must have $\Omega_i(w) \in (\Lambda V)_+$. It follows that $\Omega_i\big((\Lambda V)_+\big) \subseteq (\Lambda V)_+$ for each derivation Ω_i.

Next, if $\chi \in \ker\rho \subseteq \Lambda V$, write $\chi = \chi_0 + \chi_+$ where $\chi_0 \in \Lambda V_0$ and $\chi_+ \in (\Lambda V)_+$. Since $\chi \in \ker\rho$, it follows that $\chi_0 \in \ker\rho$ and hence $\chi_0 = d\eta$ for some $\eta \in (\Lambda V)_1$. Then

$$\begin{aligned} D(\chi) &= d(\chi_+) + \sum_i b_i\Omega_i(\chi_0) + \sum_i b_i\Omega_i(\chi_+) \\ &= d(\chi_+) + \sum_i b_i\Omega_i(d\eta) + \sum_i b_i\Omega_i(\chi_+) \\ &= d(\chi_+) + \sum_i b_i d\big(\Omega_i(\eta)\big) + \sum_i b_i\Omega_i(\chi_+). \end{aligned}$$

This term is in $H(B) \otimes \ker\rho$ because $\ker\rho$ contains all boundaries in ΛV and also, as remarked earlier, $\Omega_i(\chi_+) \in \Omega_i\big((\Lambda V)_+\big) \subseteq (\Lambda V)_+ \subseteq \ker\rho$. We have shown that $D(\ker\rho) \subseteq H(B) \otimes \ker\rho$ and since $D = 0$ on $H(B)$, it follows that $H(B) \otimes \ker\rho$ is D-stable.

Further, suppose $\alpha \in H(B) \otimes \ker\rho$ is a D-cocycle. We can write $\alpha = \alpha' + \sum_i b_i\alpha_i$ with α' and each α_i in $\ker\rho$. Then $D(\alpha) = 0$ implies $d(\alpha') = 0$ and hence $\alpha' = d\eta'$ for some $\eta' \in \ker\rho$, as $\ker\rho$ is acyclic. Without loss of generality, we can choose

(†)The observation that this is what the proof does is due to Octav Cornea, and his observation led to my including cl_0 in these results.

$\eta' \in (\Lambda V)_+$, as $d(V_0) = 0$. Then $D(\eta') = d(\eta') + \sum_i b_i \Omega_i(\eta')$. From an earlier remark, each $\Omega_i(\eta') \in (\Lambda V)_+ \subseteq \ker\rho$. So we have

$$\alpha = D(\eta') + \sum_i b_i\big(\alpha_i - \Omega_i(\eta')\big) = D(\eta') + \sum_i b_i \alpha_i'$$

for elements $\eta', \alpha_i' \in \ker\rho$. As all products in $H^+(B)$ are trivial, $D(\alpha) = 0$ now implies each $d(\alpha_i') = 0$, so that $\alpha_i' = d\eta_i$ for elements $\eta_i \in \ker\rho$. But $D(b_i\eta_i) = -b_i d(\eta_i)$ and hence $\chi = D(\eta' - \sum_i b_i\eta_i)$ with $\eta_0 - \sum_i b_i\eta_i \in H(B)\otimes\ker\rho$. This shows $H(B)\otimes\ker\rho$ is an acyclic ideal of $H(B)\otimes\Lambda V$.

To finish, notice that the projection

$$q : H(B)\otimes\Lambda V \to \frac{H(B)\otimes\Lambda V}{H(B)\otimes\ker\rho}$$

is a quasi-isomorphism, as $H(B)\otimes\ker\rho$ is acyclic. If $\mathrm{nil}_0(F) = n$, then $\Lambda^{>n}V \subseteq \Lambda^{>n}V_0 + (\Lambda V)_+ \subseteq \ker\rho$ and hence the quotient $H(B)\otimes\Lambda V/(H(B)\otimes\ker\rho)$ is a nilpotent DG algebra of length $\leq n+1$. Since q is a quasi-isomorphism, this quotient is therefore a model of E of homotopical nilpotency $\leq n+1$ and it follows from [Co] that $\mathrm{cl}_0(E) \leq n+1$. From Lemma 4.6, we have $n+1 \leq \mathrm{e}_0(E)$. But in general we have $\mathrm{e}_0(E) \leq \mathrm{cat}_0(E) \leq \mathrm{cl}_0(E)$. Hence these three invariants must agree and furthermore equal $n+1$. □

§5 — Discussion and Questions

In this last and somewhat informal section, we discuss the foregoing results and give some additional ones. The intention is to indicate both limits on, and possibilities for, development of the work in Sections 3 and 4. We also offer some specific questions along these lines. All this is collected under three subheadings, according as the topic relates to Section 3, Section 4 or Conjecture 1.1 directly.

5.1 Formality.

One can attempt a quite general study of relations between the formality of F, E and B in a fibration (cf. [Th$_2$] and [Vi]). It is tempting to think that the very restrictive hypotheses of Proposition 3.2 might be relaxed, whilst keeping the conclusion. However, the following sort of example suggests these hypotheses are actually sharp.

Example. There is a rational fibration $S^2 \vee S^2 \vee S^2 \to E \to S^3$ that is TNCZ in which E is not formal. We omit details of this example, as it is similar to [Th$_2$,Ex.III.13] and other examples. We note that the fibre is formal with non-zero cohomology only in even degrees. The fibration we have in mind is actually much closer to being trivial than just TNCZ. It satisfies $H^*(E;\mathbb{Q}) \cong H^*(B;\mathbb{Q})\otimes H^*(F;\mathbb{Q})$ as algebras and admits a rational section.

In case a weaker conclusion is acceptable, then of course there are more possibilities. In Proposition 3.2, we can remove the hypothesis of ellipticity on the fibre space, but so far only at the cost of a greatly weakened conclusion, thus:

Proposition. *Let $F \to E \to B$ be a fibration in which F and B are both formal. If the fibration is TNCZ, then E has spherically-generated cohomology.*

The proof of this proposition is omitted. It can be proved with an argument similar to that of Proposition 3.2. The above example also suggests this might be a sharp conclusion, without much more restrictive hypotheses.

It would be satisfying to have a converse of Proposition 3.2. In Proposition 2.6, we have such a result but only for a very special base.

Question 1. *Let $F \to E \to B$ be a fibration in which F is formal and elliptic and B is formal. If E is formal, then is the fibration is TNCZ?*

For more results and examples along these lines, see [Th$_2$] and [Vi].

5.2 Numerical Invariants.
The results of Section 4 deal nicely with fibrations $F \to E \to S^{2n+1}$, in which F is positively elliptic, but only for the invariants e_0, cat_0 and cl_0.

Question 2. *For a fibration $F \to E \to S^{2n+1}$ with F elliptic with positive Euler characteristic, is $cup_0(E) = 1 + nil_0(F)$?*

Of course, if Halperin's conjecture is true, it implies an affirmative answer to Question 2. It is curious that Question 2 remains unresolved whilst its counterpart for the invariants e_0, cat_0 and cl_0 is established.

Another 'asymmetry' in the results we have presented is the lack of an inequality for cone-length in Proposition 4.3. This gives our next question:

Question 3. *Let $F \to E \to B$ be a fibration with F formal. If the fibration is TNCZ, then is $cl_0(E) \geq cl_0(B) + nil_0(F)$?*

As regards extending the results presented here, we can show the following:

Proposition. *Let $F \to E \to B$ be a fibration in which F is elliptic with positive Euler characteristic and B has $\mathrm{e}_0(B) = 1$. Then $\mathrm{e}_0(E) \geq 1 + nil_0(F)$.*

Of course, if $\mathrm{e}_0(B) = 1$, then the base space B is rationally a wedge of spheres and $\mathrm{e}_0(B) = \mathrm{cat}_0(B) = \mathrm{cl}_0(B) = 1$. Notice, however, that we allow even dimensional spheres and also that we do not have Poincaré duality in either the base or the total spaces. The current proof uses a rather involved reduction argument, similar to that of [Th$_1$]. The Proposition suggests that the 'next step' might be one of the following questions:

Question 4 (a). *Let $F \to E \to B$ be a fibration in which F is elliptic with positive Euler characteristic and $\mathrm{e}_0(B) = 2$. Is $\mathrm{e}_0(E) \geq 2 + nil_0(F)$?*

Question 4 (b). *Let $F \to E \to B$ be a fibration in which F is elliptic with positive Euler characteristic. Is $\mathrm{e}_0(E) \geq 1 + nil_0(F)$?*

5.3 The Conjecture.
It is possible to try and appply Corollary 3.6, together with an obstruction theory for the formality of E in such a fibration. Such an obstruction theory does exist, and I intend to pursue this line of investigation in future work.

In Section 4 we have given some weak versions of the conjecture. We have another weak version which has rather a classical flavour to it. Since at this time we have no particular results to present on this topic, we offer it here as a question.

Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fibration. The injectivity of p^* is well-known to be implied by, and strictly weaker than, the fibration being TNCZ.

Question 5. *If F is elliptic with positive Euler characteristic is the induced homomorphism $p^* : H^*(B;\mathbb{Q}) \to H^*(E;\mathbb{Q})$ injective?*

This question is interesting, in the light of previous results, as it does not reduce to considering the case in which $B = S^{2n+1}$. Indeed, for this special case a stronger result is easy to show:

Proposition. *Let $F \to E \to S^{2n+1}$ be a fibration in which F is elliptic with positive Euler characteristic. Then the fibration admits a section, rationally, and in particular $p^* : H^*(S^{2n+1};\mathbb{Q}) \to H^*(E;\mathbb{Q})$ is injective.*

Proof. From Lemma 2.2 we have a model $\Lambda(u) \to \Lambda(u) \otimes \Lambda V, D \to \Lambda V, d$ in which $D(V_0) \subseteq u \cdot \Lambda V_0$. An easy argument shows further that $D(V_0) \subseteq u \cdot \Lambda^+ V_0$ (cf. [Lu,Lem.3]). For parity of degree reasons, $D(V_1) \subseteq \Lambda V_0 + u \cdot \Lambda^+ V_1 \cdot \Lambda V_0$ and so $\mathrm{im}(D)$ is contained in the ideal generated by V. Now, define $r : \Lambda(u) \otimes \Lambda V, D \to \Lambda(u)$ by $r(u) = u$ and $r(V) = 0$. This is a DG map as the ideal generated by V is D-stable. Therefore, r is a retraction of models, $ri = \iota$, which corresponds to a section of the fibration. □

In general the existence of a section, if F is positively elliptic, is too much to hope for. Example 1.2 is a fibration that is TNCZ, hence p^* is injective, yet it does not admit a (rational) section. We can speculate that, for those fibrations involved in Halperin's conjecture, there may be a closer link between the injectivity of p^* and TNCZ than there is in general. In any case, this gives another weak version of Halperin's conjecture to study.

References

[Au] Aubry, M., *Homotopy Theory and Models*, DMV Seminar **15** (1995), Birkhäuser, Basel.

[Ba] Baues, H. J., *Algebraic Homotopy Theory*, Cambridge Tracts in Mathematics, vol. 15, Cambridge University Press, Cambridge, 1989.

[Co] Cornea, O., *There is Just One Rational Cone-Length*, Transactions A. M. S. **344** (1994), 835–848.

[Co-Fé-Le] Cornea, O., Félix, Y. and Lemaire, J.-M., *Rational Category and Cone Length of Poincaré Complexes*, Pre-Print.

[D-G-M-S] Deligne, P., Griffiths, P., Morgan, J. and Sullivan, D., *Real Homotopy Theory of Kähler Manifolds*, Invent. Math. **29** (1975), 245–274.

[Fé-Ha$_1$] Félix, Y. and Halperin, S., *Formal Spaces with Finite-Dimensional Rational Homotopy*, Transactions A. M. S. **270** (1982), 575–588.

[Fé-Ha$_2$] Félix, Y. and Halperin, S., *Rational L-S Category and its Applications*, Transactions A. M. S. **273** (1982), 1–37.

[Fé-Ha-Le] Félix, Y., Halperin, S. and Lemaire, J.-M., *The Rational LS Category of Products and of Poincaré Duality Complexes*, Pre-Print.

[Fé-Th] Félix, Y. and Thomas, J.-C., *The Monoid of Self-Homotopy Equivalences of Some Homogeneous Spaces*, Expositiones Math. **12** (1994), 305–322.

[Gr-Mo] Griffiths P. and Morgan, J., *Rational Homotopy Theory and Differential Forms*, Progress in Mathematics, vol. 15, Birkhäuser, Boston, 1983.

[Ha$_1$] Halperin, S., *Finiteness in the Minimal Models of Sullivan*, Transactions A. M. S. **230** (1977), 173–199.

[Ha$_2$] Halperin, S., *Rational Fibrations, Minimal Models and Fibrings of Homogeneous Spaces*, Transactions A. M. S. **244** (1978), 199–223.

[Ha$_3$] Halperin, S., *Lectures on Minimal Models*, Mem. S. M. F. **9/10** (1983).

[Ha-St] Halperin, S. and Stasheff, J., *Obstructions to Homotopy Equivalences*, Advances in Math. **32** (1979), 233-279.

[He] Hess, K., *A Proof of Ganea's Conjecture for Rational Spaces*, Topology **30** (1991), 205–214.

[Ja] James, I., *Lusternik-Schnirelmann Category*, Handbook of Algebraic Topology, Elsevier, 1995, pp. 1293–1310.

[Je] Jessup, B., *Rational L-S Category and a Conjecture of Ganea*, J. of Pure and Appl. Alg. **65** (1990), 57–67.

[Lu] Lupton, G., *Note on a Conjecture of Stephen Halperin*, Springer Lecture Notes in Mathematics, vol. 1440, 1990, pp. 148–163.

[Lu-Op] Lupton, G. and Oprea, J., *Symplectic Manifolds and Formality*, J. of Pure and Appl. Alg. **91** (1994), 193–207.

[Ma] Markl, M., *Towards One Conjecture on Collapsing of the Serre Spectral Sequence*, Rend. Circ. Mat. Palermo (2) Suppl. **22** (1990), 151–159.

[Me] Meier, W., *Rational Universal Fibrations and Flag Manifolds*, Math. Ann. **258** (1983), 329–340.

[Sh-Te] Shiga, H. and Tezuka, M., *Rational Fibrations, Homogeneous Spaces with Positive Euler Characteristic and Jacobians*, Ann. Inst. Fourier **37** (1987), 81–106.

[Ta] Tanré, D., *Homotopie Rationelle: Modèles de Chen, Quillen, Sullivan*, Springer Lecture Notes in Mathematics, vol. 1025, 1983.

[Th_1] Thomas, J.-C., *Rational Homotopy of Serre Fibrations*, Ann. Inst. Fourier **31** (1981), 71–90.

[Th_2] Thomas, J.-C., *Eilenberg-Moore Models for Fibrations*, Transactions A. M. S. **274** (1982), 203–225.

[Vi] Vigué, M., *Réalisation de Morphismes Donnés en Cohomologie et Suite Spectrale d'Eilenberg-Moore*, Transactions A. M. S. **265** (1981), 447–484.

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