COHOMOLOGICALLY HYPERBOLIC ENDOMORPHISMS OF COMPLEX MANIFOLDS

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Abstract. We show that if a compact Kähler manifold X admits a cohomologically hyperbolic surjective endomorphism then its Kodaira dimension is non-positive. This gives an affirmative answer to a conjecture of Guedj in the holomorphic case.

The main part of the paper is to determine the geometric structure and the fundamental groups (up to finite index) for those X of dimension 3.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers.

Let X be a compact Kähler manifold of dimension $n \geq 2$. A surjective endomorphism $f : X \to X$ is *cohomologically hyperbolic* in the sense of [Gu06], if there is an $\ell \in \{1, 2, \ldots, n\}$ such that the ℓ-th dynamical degree

$$d_\ell(f) > d_i(f) \qquad \text{for all} \quad (\ell \neq)\ i \in \{0, 1, \ldots, n\}$$

(or equivalently, for both $i = \ell \pm 1$, by the Khovanskii - Tessier inequality). Here we refer to, for instance, [Gu05, (1.1)] for the definition of the dynamical degrees; we set $d_0(f) = 1$ and $d_{n+1}(f) = 0$.

In his papers [Gu05] - [Gu06], Guedj assumed that a dominant rational self map $f : X \cdots\to X$ has *large topological degree* (i.e., it is cohomologically hyperbolic with $\ell = \dim X$ in the definition above), and constructed a *unique* f_*-invariant measure μ_f. Further, the measure is proved to be of maximal entropy, ergodic, equidistributive for f-periodic and repulsive points, and with strictly positive Lyapunov exponents. In [Gu06], Guedj classified cohomologically hyperbolic rational self maps of surfaces S and deduced that the Kodaira dimension $\kappa(S) \leq 0$. Then he conjectured that the same should hold in higher dimension.

The result below gives an affirmative answer to the above-mentioned conjecture of Guedj [Gu06] page 7 for holomorphic endomorphisms (see [Zh2, Theorem 1.3] for the case of automorphisms on threefolds). The proof is given very simply by making use of results in [NZ]. It is classification-free and for arbitrary dimension.

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Theorem 1.1. *Let X be a compact complex Kähler manifold and $f : X \to X$ a surjective and cohomologically hyperbolic endomorphism. Then the Kodaira dimension $\kappa(X) \le 0$.*

The main part of this paper is to determine the geometric structure for projective threefolds in Theorem 1.1. The result below is part of the more detailed one in Theorem 2.1.

Theorem 1.2. *Let V be a smooth projective threefold and let $f : V \to V$ be a surjective and cohomologically hyperbolic étale endomorphism. Then one of the following cases occurs; see §3 for some realizations.*

(1) *V is f-equivariantly birational to a Q-torus in the sense of* [Ny].
(2) *V is birational to a weak Calabi-Yau variety, and $f \in \mathrm{Aut}(V)$.*
(3) *V is rationally connected in the sense of* [Cp] *and* [KoMM], *and $f \in \mathrm{Aut}(V)$.*
(4) *The albanese map $V \to \mathrm{Alb}(V)$ is a smooth and surjective morphism onto the elliptic curve $\mathrm{Alb}(V)$ with every fibre a smooth projective rational surface of Picard number ≥ 11. Further, the dynamical degrees satisfy $d_2(f) > d_1(f) \ge \deg(f) \ge 2$.*
(5) *V is f-equivariantly birational to the quotient space of a product* (Elliptic curve)$\times$ *$(K3)$ by a finite and free action. Further, the dynamical degrees satisfy $d_2(f) > d_1(f) \ge \deg(f) \ge 2$.*

We can also determine the topological fundamental groups (up to finite index) for those threefolds admitting a cohomologically hyperbolic étale endomorphism.

Theorem 1.3. *Let X be a smooth projective threefold admitting a surjective and cohomologically hyperbolic étale endomorphism f. Then either $\pi_1(X)$ is finite, or $\pi_1(X)$ contains a finite-index subgroup isomorphic to either one of:*

$$\mathbb{Z}^{\oplus 2}, \quad \mathbb{Z}^{\oplus 6}.$$

Note. Our approach is algebro-geometric in nature; see Fujimoto [Fm], Fujimoto-Nakayama [FN], Oguiso [Og03] - [Og06] for similar approach.

Conventions 1.4. We shall use the conventions of Hartshorne's book, [KMM] and [KM].

(1) A normal projective variety X is *minimal* if it is $\mathbb{Q}$-factorial, has at worst terminal singularities and the canonical divisor K_X is nef.
(2) A minimal projective variety X is a *weak Calabi-Yau variety* if $K_X \sim_{\mathbb{Q}} 0$ and $q^{\max}(X) = 0$. Here

$$q^{\max}(Z) := \max\{q(Y) \mid Y \to Z \text{ finite étale}\}.$$

A minimal projective variety X of dimension n is a *Calabi-Yau variety* if

$$K_X \sim 0, \quad \pi_1(X) = (1), \quad H^i(X, \mathcal{O}_X) = 0 \quad (1 \le i \le n-1).$$

(3) A morphism $\sigma\colon X \to Y$ is *f-equivariant* if there are endomorphisms $f = f|X\colon X \to X$ and $f = f|Y\colon Y \to Y$ such that $\sigma \circ (f|X) = (f|Y) \circ \sigma$.

(4) In this paper, every endomorphism on a compact Kähler manifold (or a projective variety) is assumed to be surjective, so it is also finite, and even étale when the Kodaira dimension $\kappa(X) \ge 0$; see [Fm, Lemma 2.3].

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2. Proofs of Theorems 1.1 - 1.3

In this section, we shall prove Theorems 1.1 and 1.3, and also Theorem 2.1 below which implies Theorem 1.2 and determines the geometric structure for projective threefolds in Theorem 1.1.

Theorem 2.1. *Let V be a smooth projective threefold and let $f\colon V \to V$ be a surjective and cohomologically hyperbolic étale endomorphism. Then one of the following cases occurs; see §3 for some realizations.*

(1) $\kappa(V) = 0$ *and* $q^{\max}(V) = \dim V = 3$. *Further, V is f-equivariantly birational to a Q-torus in the sense of* [Ny]. *To be precise, there are an f-equivariant birational morphism $V \to X$ and an f-equivariant étale Galois cover $Y \to X$ from an abelian variety Y.*

(2) $\kappa(V) = 0 = q^{\max}(V)$, *$\pi_1(V)$ is finite, and $f \in \mathrm{Aut}(V)$. Further, V is birational to a weak Calabi-Yau variety.*

(3) $\kappa(V) = -\infty$, $q^{\max}(V) = 0$, $\pi_1(V) = (1)$ *and* $f \in \mathrm{Aut}(V)$. *Further, V is rationally connected in the sense of* [Cp] *and* [KoMM].

(4) $\kappa(V) = -\infty$, $q^{\max}(V) = q(V) = 1$ *and the dynamical degrees satisfy* $d_2(f) > d_1(f) \ge \deg(f) \ge 2$. *Further, the albanese map* $V \to \mathrm{Alb}(V)$ *is smooth and surjective with every fibre F a smooth projective rational surface of Picard number* ≥ 11.

(5) $\kappa(V) = 0$, $q^{\max}(V) = 1$ *and the dynamical degrees satisfy* $d_2(f) > d_1(f) \ge \deg(f) \ge 2$. *Further, V is f-equivariantly birational to the quotient space of a product* (Elliptic curve)$\times$ $(K3)$ *by a finite and free action. To be precise, V has a unique minimal model X and $f|V$ induces a finite étale endomorphism $f|X$ on X. There is an f-equivariant étale Galois cover $Y = E \times S \to X$ with E an elliptic curve and S a (smooth and minimal) $K3$ surface, such that* $f|Y = (f|E) \times (f|S)$

for some isogeny $f|E$ *with* $\deg(f|E) = \deg(f|V)$ *and* $f|S \in \mathrm{Aut}(S)$ *of positive entropy.*

Remark.

(1) See [Fm] and [FN] for the case where $\kappa(V) \ge 0$ and $\deg(f) \ge 2$.
(2) The étaleness of f in Theorems 1.3 and 2.1 above is automatic if either $\deg(f) = 1$, or if $\deg(f) \ge 2$ and $\kappa(X) \ge 0$; see [Fm, Lemma 2.3].
(3) In Theorem 2.1 (1) and (5), we have $d_i(f|V) = d_i(f|Y)$ for all i. In general, we have $d_j(g|V) = d_j(g|W)$ for all j if $V \to W$ is a g-equivariant generically finite morphism; see [NZ, Appendix, Lemma A.8].

2.2. Proof of Theorem 1.1.

We will make use of [NZ, Theorem A, and Appendix]. Suppose the contrary that the Kodaira dimension $\kappa(X) \ge 1$. Then $f\colon X \to X$ is a finite étale morphism (see [Fm, Lemma 2.3]). We choose $m \gg 0$ such that

$$\Phi_m = \Phi_{|mK_X|}\colon X \cdots\to W_m \subseteq \mathbb{P}(H^0(X, mK_X))$$

gives rise to the Iitaka fibring. By [NZ, Theorem A], f induces an automorphism $f_m\colon W_m \to W_m$ of finite order, such that $\Phi_m \circ f = f_m \circ \Phi_m$. Replacing X by an f-equivariant resolution of base locus of $|mK_X|$ due to Hironaka (see also [NZ, §1.4]), we may assume that Φ_m is a well defined morphism. Now the theorem follows from the result below, noting that $\dim W_m = \kappa(X) \ge 1$.

Lemma 2.3. *Let* $\pi\colon X \to Y$ *be a proper holomorphic map from a compact Kähler manifold* X *to a compact complex analytic variety* Y *with general fibres connected, and let* $f\colon X \to X$ *and* $f_Y\colon Y \to Y$ *be surjective endomorphisms such that* $\pi \circ f = f_Y \circ \pi$. *Suppose that* f *is étale,* f_Y *is an automorphism of finite order and* f *is cohomologically hyperbolic. Then* $\dim Y = 0$ *(and* Y *is a single point).*

Proof. Replacing f by its power, we may assume that $f_Y = \mathrm{id}$. Let F be a smooth general fibre of π. We claim that $f|F$ is also cohomologically hyperbolic. Indeed, by the fundamental work of Gromov and Yomdin, the topological entropy $h(g)$ of an endomorphism g of a compact Kähler manifold is the maximum of logarithms $\log d_i(g)$ of dynamical degrees. So suppose that for some $1 \le r \le k := \dim F$, we have:

$$h(f|F) = \log d_r(f|F) = \max_{1\le i\le k} d_i(f|F).$$

By [NZ, Appendix, Proposition A.9 and Theorem D], we have:

$$h(f|F) = \log d_r(f|F) \le \log d_r(f|X) \le h(f|X) = h(f|F),$$

so $d_r(f|F) = d_r(f|X)$. Now for any $i \neq r$, by [ibid.], we have:

$$d_i(f|F) \leq d_i(f|X) < d_r(f|X) = d_r(f|F).$$

Here the strict inequality holds because $f|X$ is cohomologically hyperbolic. This proves the claim.

On the other hand, note that $\deg(f|X) = \deg(f|F)$. Hence, by [ibid.], we have the following, with $n = \dim X$ and $k = \dim F$:

$$h(f|F) = \log d_r(f|F) \leq \log d_{r+n-k}(f|X) \leq \log d_r(f|X) = h(f|F).$$

Thus all inequalities above become equalities; since $f|X$ is cohomologically hyperbolic, the maximality of d_r implies that $r + n - k = r$. So $n = k$ and Y is a point. This proves the lemma and also Theorem 1.1. $\square$

We need the result below in the proof of Theorem 2.1.

Lemma 2.4. *Let X be a compact Kähler manifold of dimension n and let $f\colon X \to X$ be a surjective endomorphism. Then the dynamical degrees satisfy $d_{n-i}(f) = d_i(f^{-1})(\deg(f))$. Here $d_j(f^{-1})$ denotes the spectral radius of the linear transformation*

$$(f^*)^{-1}\colon H^{j,j}(X, \mathbb{C}) \to H^{j,j}(X, \mathbb{C}).$$

Proof. One can use the fact that $f_*f^* = (\deg(f))\,\mathrm{id}$ on the cohomology ring of X to give a simple proof. Below is another elementary proof. Set $s = h^{i,i}(X, \mathbb{C}) = h^{n-i,n-i}(X, \mathbb{C})$. Let $\{e_1, \ldots, e_s\}$ and $\{\varepsilon_1, \ldots, \varepsilon_s\}$ be dual bases of $H^{i,i}(X, \mathbb{C})$ and $H^{n-i,n-i}(X, \mathbb{C})$ with respect to the perfect pairing below such that $e_i.\varepsilon_j = \delta_{ij}$ (Kronecker's symbol):

$$H^{i,i}(X, \mathbb{C}) \times H^{n-i,n-i}(X, \mathbb{C}) \to \mathbb{C}.$$

Let A (resp. B) be the matrix representation of $(f^*)^{-1}|H^{i,i}(X, \mathbb{C})$ (resp. $f^*|H^{n-i,n-i}(X, \mathbb{C})$). Then a calulation in linear algebra implies that $B = (\deg(f))A^T$. The lemma follows. $\square$

2.5. Proof of Theorem 2.1.

By Theorem 1.1, $\kappa(V) \leq 0$. Our Theorem 2.1 follows from the three lemmas below.

Lemma 2.6. *Theorem 2.1 is true when $\kappa(V) = 0$.*

Proof. We will make use of [NZ, Theorem B]. Let $f\colon V \to V$ be as in the theorem. Let X be a ($\mathbb{Q}$-factorial) minimal model of V with at worst terminal singularities, whence $K_X \sim_{\mathbb{Q}} 0$ (see [Mi], [Ka]). Then $f|V$ induces a dominant rational map $f\colon X \cdots\to X$, which is nearly étale in the sense of [NZ, §3]. By [NZ, Theorem B and its Remark], either an étale cover $\widetilde{X}$ of X is a weak Calabi-Yau variety, or there are an étale cover $\tau\colon F \times A \to X$, an automorphism $\varphi_F\colon F \to F$ and a finite étale endomorphism $\varphi_A\colon A \to$

A with $\deg(\varphi_A) = \deg(f)$ such that $f \circ \tau = \tau \circ (\varphi_F \times \varphi_A)$. Here F is either a point (and hence A is a 3-torus), or K3 or Enriques (and A is an elliptic curve), by the classification of lower dimensional weak Calabi-Yau varieties.

By further étale cover (to the Galois closure), we may assume that τ is Galois. Replacing τ, we may also reduce the Enriques case of F to the $K3$ case.

The case above involving $\widetilde{X}$ fits Theorem 2.1 (2). Indeed, $\pi_1(V) = \pi_1(X)$ (see [Ko, Theorem 7.8] and [Ty, Theorem 1.1]), so $\pi_1(V)$ is finite by [NS, Corollary 1.4].

Consider the case $\dim A = 3$. Then X is a Q-torus in the sense of [Ny]. Both the birational map $V \dashrightarrow X$ and the dominant rational map $f\colon X \dashrightarrow X$ are well defined morphisms by the absence of rational curves on tori and Hironaka's resolution of indeterminancy of a rational map; see also [Un, Lemma 9.11]. So Theorem 2.1(1) occurs.

Consider the case $\dim A = 1$. We shall show that Theorem 2.1 (5) takes place. Note that $F \times A$ is the unique minimal model of its biraitonal class, up to isomorphism. This is because other minimal models are obtained from $F \times A$ by a finite sequence of flops with centre a union of rational curves which must be contained in some fibres of $F \times A \to \mathrm{Alb}(F \times A) = A$, i.e., contained in the $K3$ surfaces F. However, we assert that $F \times A$ admits no flop. Indeed, such a flop induces a non-isomorphic birational automorphism of F, which is an isomorphism away from a few rational curves, and hence is indeed an isomorphism by the uniqueness of a surface minimal model, absurd! So the assertion is true. This assertion also appeared in [Fm, page 66].

Next we claim that X is the unique minimal model in its birational class, up to isomorphism. This claim appeared in [Fm, page 61]. We prove it for the convenience of the readers. It is enough to show the assertion of the absence of flops from X. Suppose the contrary that $\sigma\colon X \dashrightarrow X'$ is a flop to another minimal model. Then X' is also smooth. Since the fundamental group of a smooth variety will not be changed after a smooth blowup or blowdown and after removing some codimension 2 subsets, the existence of an étale Galois cover $\tau\colon F \times A \to X$ induces an étale Galois cover $\tau'\colon \widetilde{X'} \to X'$ and a birational map $\tilde{\sigma}\colon F \times A \dashrightarrow \widetilde{X'}$ lifting the flop $\sigma\colon X \dashrightarrow X'$ and being isomorphic in codimension one. So $\tilde{\sigma}$ is either an isomorphism or a composition of flops. The absence of such flop as shown in the paragraph above, implies that $\tilde{\sigma}$ is indeed an isomorphism. The consideration of the fundamental group again implies that $\mathrm{Gal}((F \times A)/X)$ and $\mathrm{Gal}(\widetilde{X'}/X')$ are conjugate to each other whose quotients are hence isomorphic via the initial map σ. But σ, being a flop, is not isomorphic. We reach a contradiction. Hence both the assertion and the claim are true.

Applying [NZ, Lemma 3.2] to $f\colon X \dashrightarrow X$, we see that f is the composition of a birational map $\gamma\colon X \dashrightarrow X''$ and a finite étale morphism $X'' \to X$. Thus X'' is also a minimal model and hence γ is either an isomorphism or a composition of flops. The assertion in the paragraph above implies that γ is an isomorphism. So our initial f is indeed a well defined finite étale morphism. Thus Theorem 2.1 (5) takes place, where we set $S := F$, $E := A$ and $Y := F \times A$. Indeed, since $d_i(f|V) = d_i(f|Y)$ by [NZ, Appendix, Lemma A.8] and applying the Künneth formula, we have:

$$d_i(f|V) = \max_{0 \le s \le i}\{d_s(f|S)\ d_{i-s}(f|E)\}.$$

Since $f|V$ is cohomologically hyperbolic, we must have $d_1(f|S) \ge 2$ and $d_1(f|E) \ge 2$, whence the inequalities about the dynamical degrees follow. Also, since $\pi_1(Y) = \pi_1(E)$, we see that $(q^{\max}(V) =)\ q^{\max}(Y) = 1$. This completes the proof of the lemma. $\square$

Next we consider the case $\kappa(V) = -\infty$. The completed good minimal model program for threefolds (see [KMM] or [KM]), implies that V is uniruled. Let $\mathrm{MRC}_V \colon V \dashrightarrow Y'$ be a maximal rationally connected fibration in the sense of [Cp] and [KoMM]. Then Y' is not uniruled by [GHS, (1.4)]. So $\kappa(Y') \ge 0$. By [NZ, Theorem C and its Remark], there are a birational morphism $X \to V$ from a smooth projective variety, and a smooth projective variety Y birational to Y', such that $f|V$ induces a finite étale endomorphism $f \colon X \to X$, the induced maps $\pi := \mathrm{MRC}_X \colon X \to Y$ and $f_Y \colon Y \to Y$ are well defined morphisms, and $\pi \circ f = f_Y \circ \pi$. Further, $\deg(f) = \deg(f_Y)$.

Since a torus contains no rational curves, we have $\mathrm{Alb}(V) = \mathrm{Alb}(X) = \mathrm{Alb}(Y)$. Further, the composition $V \dashrightarrow X \to Y \to \mathrm{Alb}(Y)$ is the well defined albanese morphism alb_V. Note also that $\kappa(Y) = \kappa(Y') \ge 0$ and hence f_Y is finite étale.

If $\dim Y = 0$, then Theorem 2.1 (3) takes place because a rationally connected smooth projective variety is simply-connected (see [Cp]), whence $\deg(f) = 1$. We now consider the cases $\dim Y = 1, 2$ separately.

Lemma 2.7. *Assume that $\kappa(V) = -\infty$ and $\mathrm{MRC}_V(V)$ is a curve. Then Theorem 2.1(4) takes place. Further, for the F there, the anti-canonical divisor $-K_F$ is not big and $K_F^2 < 0$.*

We now prove the lemma. By Lemma 2.3, f_Y is not periodic. So Y is an elliptic curve, noting that $\kappa(Y) \ge 0$. Further, either f_Y is an isogeny of $\deg(f_Y) \ge 2$, or f_Y is a translation of infinite order and hence $\deg(f) = \deg(f_Y) = 1$ (so both f and f_Y are automorphisms).

We claim that $\pi \colon X \to Y$ is a smooth morphism. Indeed, suppose the contrary that we have a non-empty set $D(X/Y)$, the discriminant locus of π, i.e., the subset of Y over which $\pi \colon X \to Y$ is not smooth. Since $f \colon X \to X$ is étale and is the lifting of f_Y, we have $f_Y^{-1}(D(X/Y)) = D(X/Y)$. Replacing f by its power, we may assume that f_Y fixes every point in $D(X/Y)$. This contradicts the description of f_Y above. Therefore, $\pi \colon X \to Y$ is smooth and every fibre of it is a *smooth* rational projective surface.

Note that $\pi = \mathrm{alb}_X$. By the same reason, $\mathrm{alb}_V : V \to \mathrm{Alb}(V) = Y$ is smooth. For the clean-ness of the notation, we replace (V, f) by (X, f).

When f is an automorphism, we have $d_1(f) = d_2(f)$ by [Zh2, Lemma 2.8]. This contradicts the concavity as mentioned in Claim 2.11 below. Therefore, f_Y is an isogeny with $\deg(f) = \deg(f_Y) \ge 2$.

Let $0 \ne v_{f^\pm}$ be nef $\mathbb{R}$-divisors such that

$$(f^{\pm})^* v_{f^\pm} = d_1(f^{\pm}) v_{f^\pm},$$

guaranteed by a result of Birkhoff [Bi] generalizing the Perron-Frobenius theorem to (the nef) cone. Let F be a fibre of $\pi = \mathrm{alb}_X : X \to Y$. So F is a smooth projective rational surface.

Claim 2.8. The following are true.

(1) f^*F is a disjoint union of $\deg(f)$ fibres;
$f^*F \equiv \deg(f)F$; $d_i(f) \ge \deg(f)$ for both $i = 1, 2$.
(2) $0 = v_f \cdot F \cdot K_X = (v_f)|F \cdot K_F$.
(3) $d_2(f) = d_1(f^{-1}) \deg(f) \ge \deg(f)$ and $d_1(f^{-1}) \ge 1$.
(4) If $v_f \cdot F = 0$ then $v_f \equiv eF$ for some $e > 0$ and $d_2(f) > d_1(f) = \deg(f)$.
(5) If $v_f \cdot F \ne 0$ then $d_2(f) \ge d_1(f) \deg(f) > d_1(f) \ge \deg(f)$.

Proof. (1) The first two assertions are true because f is étale and F, being rational, is simply connected. In particular, $d_1(f) \ge \deg(f)$. Applying f^* to the non-zero cycle $K_X \cdot F = K_F$, we get $d_2(f) \ge \deg(f)$.

(2) If $c := v_f \cdot F \cdot K_X \ne 0$, then $\deg(f)c = f^*c = d_1(f) \deg(f)c$ and $d_1(f) = 1$. This is absurd because f is of positive entropy.

(3) follows from (2) and Lemma 2.4.

(4) The first part follows from the Lefschetz hyperplane section theorem to reduce to the Hodge index theorem for surfaces (see the proof of [Zh2, Lemma 2.6]), while the second follows from the first by applying f^*, the assertion (1), and f being cohomologically hyperbolic.

(5) is similar to (4) by applying f^*. □

It remains to show the assertion that $-K_F$ is not big, and rank $\mathrm{Pic}(F) \ge 11$ or equivalently $K_F^2 \le -1$. Consider the case where $-K_F$ is big or $K_F^2 \ge 0$, and we shall derive a contradiction. If $K_F^2 \ge 1$, then $-K_F$ is big by the Riemann-Roch theorem applied to $-nK_F$. Thus we assume that either $K_F^2 = 0$ or $-K_F$ is big. This assumption and Claim 2.8 (2) imply $(v_f)|F \equiv \alpha K_F = \alpha K_X|F$ for some $\alpha \ne 0$ (by Claim 2.9 below). Applying f^*, we get $d_1(f) = 1$, absurd. Therefore, the assertion is true. The lemma then follows. Indeed, $q^{\max}(V) = q^{\max}(Y)$ $(= 1)$ because $\pi_1(V) = \pi_1(Y)$ as in the proof of Theorem 2.1 at the end of this section.

Claim 2.9. Suppose $K_F^2 = 0$ or $-K_F$ is big. Then the cohomology class of v_f is not a multiple of that of F, so $(v_f) \cdot F$ is not homologous to zero.

Proof. Suppose the contrary that the claim is false. Applying f^*, we get $d_1(f) = \deg(f)$. Since f is cohomologically hyperbolic and by Claim 2.8 (1), we have $d_2(f) > \deg(f)$, and hence $d_1(f^{-1}) > 1$ by Claim 2.8 (3). The latter and the proof of Claim 2.8 (2) imply that $0 = v_{f^{-1}} \cdot F \cdot K_X = (v_{f^{-1}}|F) \cdot K_F$. Then by the assumption on $-K_F$ and the Hodge index theorem (see [BHPV, IV, Cor. 7.2]), we have $v_{f^{-1}}|F \equiv aK_F = aK_X|F$ for some scalar a. If $a \neq 0$, applying f^* to the equality, we get $d_1(f^{-1}) = 1$, absurd. If $a = 0$, then $v_{f^{-1}}|F \equiv 0$ and hence $v_{f^{-1}} \equiv bF$ for some $b > 0$ by the Lefschetz hyperplane section theorem to reduce to the Hodge index theorem for surfaces. Applying f^*, we get $1 > 1/(d_1(f^{-1})) = \deg(f) > 1$, absurd. This proves the claim and also the lemma. $\square$

Lemma 2.10. *In the situation of Theorem 2.1 it is impossible that $\kappa(V) = -\infty$ and $\mathrm{MRC}_V(V)$ is a surface.*

We now prove the lemma. Consider the case where $(\kappa(X) =)$ $\kappa(V) = -\infty$ and $\mathrm{MRC}_V(V)$ (or eqivalently $Y = \pi(X)$) is a surface. If $\kappa(Y) \geq 1$, then after equivariant modification, we may assume that for some $n > 0$, the map $\Phi_{|nK_Y|} \colon Y \to Z$ is a well defined morphism giving rise to the Iitaka fibring. By [NZ, Theorem A], f_Y descends to an automorphism $f_Z \colon Z \to Z$ of finite order. Note that $\dim Z = \kappa(Y) \geq 1$. This contradicts Lemma 2.3.

Therefore, $\kappa(Y) = 0$. We may assume that Y is minimal. This can be achieved if $\deg(f_Y)$ $(= \deg(f))$ $= 1$ by equivariant blowdown; on the other hand, if $\deg(f_Y) \geq 2$, then Y has no negative $\mathbb{P}^1$ and hence Y is already minimal, for otherwise, iterating f^{-1} will produce infinitely many disjoint negative $\mathbb{P}^1$ (noting that $\mathbb{P}^1$ is simply connected and f_Y is étale), contradicting the finiteness of the Picard number of Y; see [Fm, page 43]. Thus, Y is abelian, hyperelliptic, $K3$ or Enriques.

Claim 2.11. Y is neither $K3$ nor Enriques.

Proof. The claim is clear when the étale map f_Y has $\deg(f_Y) \geq 2$ (so $|\pi_1^{\mathrm{alg}}(Y)| = \infty$ by iterating f_Y), since $|\pi_1(Y)| \leq 2$ when Y is $K3$ or Enriques. Suppose f_Y (and hence f) are automorphisms. By Lemma 2.3, f_Y is not periodic. If f_Y is of positive entropy, then $d_1(f) = d_2(f)$ as proved in [Zh2, Claim 2.11(1)]; this is absurd since f is cohomologically hyperbolic and by the concavity from the Khovanskii-Tessier inequality as in [Gu05, Proposition 1.2]. Thus f_Y is parabolic. Then there is an elliptic fibration $Y \to \mathbb{P}^1$ such that f_Y descends to a periodic automorphism on $\mathbb{P}^1$; see [Zh2, Lemma 2.19]. However, this contradicts Lemma 2.3. $\square$

If Y is a hyperelliptic surface, then Y is a quotient of a torus Z by a group of order $m = 2, 3, 4$, or 6 (taking m minimal). Our $f_Y : Y = Y_1 \to Y = Y_2$ lifts to an endomorphism f_Z of Z. Indeed, $Z \times_{Y_2} Y_1$ is isomorphic to Z (as the minimal torus cover of Y_1). There is a further lifting $\tilde{f} = f \times f_Z$ on $\widetilde{X} := X \times_Y Z$ so that the projection $\widetilde{X} \to Z$ is just $\mathrm{MRC}_{\widetilde{X}}$. Note that $\tilde{f}$ is also cohomologically hyperbolic by [NZ, Appendix, Lemma A.8]. We will reach a contradiction by the argument below when Y is an abelian surface.

We now consider the case where Y is an abelian surface. By Lemma 2.3, f_Y and its equivariant descents are not periodic. If f is an automorphism and f_Y is of positive entropy, then by [Zh2, Claim 2.11 (1)] we have $d_1(f) = d_2(f)$, which is absurd as mentioned in the proof of Claim 2.11. If f is an automorphism and f_Y is rigidly parabolic in the sense of [Zh2, 2.1], then we will get a contradiction as shown in [Zh2, Claim 3.13].

Thus, we may assume that $(\deg(f_Y) =) \deg(f) \geq 2$.

Claim 2.12. $\pi : X \to Y$ is a smooth morphism, so every fibre is $\mathbb{P}^1$.

Proof. Suppose the contrary that the discriminant locus $D := D(X/Y)$ is not empty. Since f is étale, we have $f_Y^{-1}(D) = D$, whence D does not contain isolated points and D is a disjoint union of curves D_i. We may assume that $f_Y^{-1}(D_i) = D_i$ for all i after replacing f by its power. Further, $\deg(f_Y|D_i) = \deg(f_Y) \geq 2$. Thus D_i is not of general type and hence $\kappa(D_i) = 0$. By [Un, Theorem 10.3], every D_i is an elliptic curve and a subtorus for $i = 1$ (after changing the origin). f_Y induces an endomorphism f_Z of the elliptic curve $Z := Y/D_1$ such that $f_Z^{-1}\{d_i\} = \{d_i\}$ for each d_i: the image of D_i. Thus $\mathrm{ord}(f_Z) \leq 6$. This contradicts Lemma 2.3. So the claim is proved. $\square$

Let $0 \neq v_{f^\pm}$ be nef $\mathbb{R}$-divisors such that $(f^*)^{\pm} v_{f^\pm} = d_1(f^\pm) v_{f^\pm}$. Let F be a fibre of $\pi : X \to Y$. So $F \cong \mathbb{P}^1$ by the claim above.

Claim 2.13. The following are true.

(1) f^*F is a disjoint union of $\deg(f)$ fibres; $f^*F \equiv \deg(f)F$.
(2) $d_1(f^{-1}) \deg(f) = d_2(f) \geq \deg(f)$ and $d_1(f^{-1}) \geq 1$.
(3) $F \cdot v_f = 0$; $v_f \equiv \pi^* H^+$ for some nef divisor $0 \neq H^+$ on Y.
(4) $F \cdot v_{f^{-1}} = 0$; $v_{f^{-1}} \equiv \pi^* H^-$ for some nef divisor $0 \neq H^-$ on Y.
(5) $H^+ \cdot H^- \neq 0$.

Proof. (1) and (2) are as in a claim of the previous lemma.

(3) If $\alpha := F \cdot v_f \neq 0$, then we get a contradiction $d_1(f) = 1$, by the calculation:

$$\deg(f)\alpha = f^*\alpha = (\deg f) d_1(f) \alpha.$$

Hence $F \cdot v_f = 0$. This and $-K_X$ being π-ample from Claim 2.12, imply that $v_f \equiv \pi^* H^+$ (see [KMM, Lemma 3-2-5] or [KM, page 46]). Here H^+ is nef because so is v_f.

(4) If $\beta := F \cdot v_{f^{-1}} \neq 0$, then we get $d_1(f^{-1}) = 1$ by the calculation:

$$\deg(f)\beta = f^*\beta = \beta \deg(f)/d_1(f^{-1}).$$

Thus $d_2(f) = \deg(f)$ by (2). This contradicts [Gu05, (1.2)] as argued in Claim 2.11. The rest of (4) is as in (3).

(5) If (5) is false, then $H^+ \equiv \gamma H^-$ for some $\gamma > 0$ by the Hodge index theorem. Applying $f^*\pi^*$, we get $1 < d_1(f) = 1/d_1(f^{-1}) \leq 1$ by (2), absurd! $\square$

By the claim above, we can write $v_f \cdot v_{f^{-1}} \equiv \delta F$ for some $\delta > 0$. Applying f^*, we have

$$d_1(f)/d_1(f^{-1}) = \deg(f) = d_2(f)/d_1(f^{-1}),$$

by Claim 2.13. Hence $d_1(f) = d_2(f) \geq \deg(f)$, by Claim 2.13. This is impossible because f is cohomologically hyperbolic. This proves the lemma. The proof of Theorem 2.1 is also completed.

2.14. Proof of Theorem 1.3.

In view of Theorem 2.1, we have only to consider the case in Theorem 2.1(4). Since a general fibre (indeed every fibre) F of $\mathrm{alb}_X \colon X \to \mathrm{Alb}(X)$ is a smooth projective rational surface, we have $\pi_1(X) = \pi_1(\mathrm{Alb}(X)) = \mathbb{Z}^{\oplus 2}$ (see [Cp]). This proves the theorem.

3. Examples

In this section we give examples to realize some cases in Theorem 2.1.

Example 3.1. Examples for Theorem 2.1 (4)-(5).

Let Z be a compact complex Kähler surface with an automorphism f_Z of positive entropy. Let E be an elliptic curve and $f_E \colon E \to E$ an isogeny of $\deg(f_E) \geq 2$. Set $X := Z \times E$ and $f := f_Z \times f_E$. Then $d_2(f) > d_1(f) \geq d_3(f)$, because

$$d_i(f) = \max_{0 \leq s \leq i} \{d_s(f_Z)\ d_{i-s}(f_E)\}$$

by the Künneth formula for cohomologies. If we take Z to be K3 or Enriques (resp. rational surface) then (X, f) fits Theorem 2.1 (5) (resp. (4)).

For examples of such (Z, f_Z) of positive entropy, see [Ct], [Mc05].

Example 3.2. Cohomologically hyperbolic endomorphisms on rational varieties.

Let S_i $(1 \leq i \leq r)$ be a smooth projective rational surface and f_i an automorphism of S_i of positive entropy; see [Mc05] for such examples. Set

$$X := S_1 \times \cdots \times S_r, \qquad f := f_1 \times \cdots \times f_r \ \in \ \mathrm{Aut}(X).$$

Then f is cohomologically hyperbolic with $d_r(f) > d_i(f)$ for all $i \neq r$.

Let $f_p\colon \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism of $\deg(f_p) \geq 2$. Set

$$Y := \mathbb{P}^1 \times S_1 \times \cdots \times S_r, \qquad f_Y := f_p \times f_1 \times \cdots \times f_r.$$

Then f_Y is a cohomologically hyperbolic endomorphism with $d_{r+1}(f_Y) > d_i(f_Y)$ for all $i \neq r+1$. However, f_Y is not étale.

Example 3.3. Cohomologically hyperbolic rational self maps on smooth Calabi-Yau.

Denote by $\zeta_s = \exp(2\pi\sqrt{-1}/s)$, a primitive s-th root of 1. Let $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\zeta_3)$ be an elliptic curve admitting a group automorphism f_E of order 3. Set $A_3 = E \times E \times E$ and $f_3 = \mathrm{diag}[f_E, f_E, f_E]$. Then f_3 acts on A with 27 fixed points.

Consider the Klein quartic curve below

$$C := \{X_0X_1^3 + X_1X_2^3 + X_2X_0^3 = 0\} \subset \mathbb{P}^2.$$

which is of genus 3 and with $|\operatorname{Aut}(C)| = 42\deg(K_C)$ (reaching the Hurwitz upper bound). Indeed, $\operatorname{Aut}(C) = L_2(7)$, a simple group of order 168. Let

$$f_C\colon [X_0\colon X_1\colon X_2] \mapsto [\zeta_7 X_0\colon \zeta_7^2 X_1\colon \zeta_7^4 X_2].$$

be an order-7 automorphism of C. Let $A_7 = J(C)$ be the Jacobian abelian threefold and let $f_7 = \mathrm{diag}[\zeta_7, \zeta_7^2, \zeta_7^4]$ be the induced order-7 automorphism on A_7.

For A_n ($n = 3, 7$), let $\overline{X}_n = A_n/\langle f_n\rangle$. Thanks to the work of Oguiso-Sakurai [OS, Theorem 3.4], there is a crepant desingularization $X_n \to \overline{X}_n$, and X_n satisfies the following:

$$K_{X_n} \sim 0, \qquad \pi_1(X_n) = (1).$$

Note that $K_{\overline{X}_n} \sim 0$. By [Ko, Theorem 7.8], $\pi_1(\overline{X}_n) = \pi_1(X_n) = (1)$. Thus by the Serre duality, X_n is a smooth Calabi-Yau variety, while $\overline{X}_n$ is a Calabi-Yau variety but with isolated *canonical* singularities. For $m \geq 2$, let $m_n\colon A_n \to A_n$, $a \mapsto m.a$, be an endomorphism of degree m^6. Then $\operatorname{Ker}(m_{A_n}) = (\mathbb{Z}/(m))^{\oplus 6}$. The group below of order $n.m^6$ acts on A_n faithfully

$$G_m := ((\mathbb{Z}/(m))^{\oplus 6}) \rtimes \langle f_n\rangle.$$

m_n induces an endomorphism $\overline{m}_n\colon \overline{X}_n \to \overline{X}_n$ of degree m^6. Note that m_n is cohomologically hyperbolic, and hence so is $\overline{m}_n$ by [NZ, Appendix, Lemma A.8]. The pairs $(\overline{X}_n, \overline{m}_n)$ with $n = 3, 7$ and $m \geq 2$, are close to the situation in Theorem 2.1 (2), though each $\overline{X}_n$ here has isolated singularities, and the map $\overline{m}_n$ may not be étale. $\overline{m}_n$ induces a cohomologically hyperbolic dominant rational map $\widetilde{m}_n\colon X_n \dashrightarrow X_n$ which may not be holomorphic just like the similar construction on smooth Kummer surfaces.

References

[BHPV] W. P. Barth, K. Hulek, C. A. M. Peters and A. Van de Ven, Compact complex surfaces. Second edition. Springer-Verlag, Berlin, 2004.

[Bi] G. Birkhoff, Linear transformations with invariant cones, Amer. Math. Monthly, 74 (1967), 274–276.

[Cp] F. Campana, Connexité rationnelle des variétés de Fano, Ann. Sci. cole Norm. Sup. (4) 25 (1992), no. 5, 539–545.

[Ct] S. Cantat, Dynamique des automorphismes des surfaces $K3$. Acta Math. 187 (2001), no. 1, 1–57.

[Fm] Y. Fujimoto, Endomorphisms of smooth projective 3-folds with non-negative Kodaira dimension. Publ. Res. Inst. Math. Sci. 38 (2002), no. 1, 33–92.

[FN] Y. Fujimoto and N. Nakayama, Endomorphisms of smooth projective 3-folds with non-negative Kodaira dimension, II, preprint RIMS-1566, Res. Inst. Math. Sci. Kyoto Univ. 2006.

[GHS] T. Graber, J. Harris and J. Starr, Families of rationally connected varieties, J. Amer. Math. Soc. 16 (2003), 57–67.

[Gu05] V. Guedj, Ergodic properties of rational mappings with large topological degree. Ann. of Math. (2) 161 (2005), no. 3, 1589–1607.

[Gu06] V. Guedj, Propriétés ergodiques des applications rationnelles, math.CV/0611302.

[Ka] Y. Kawamata, Abundance theorem for minimal threefolds, Invent. Math., 108 (1992), 229–246.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the minimal model problem. Algebraic geometry, Sendai, 1985, 283–360, Adv. Stud. Pure Math., 10, 1987.

[Ko] J. Kollár, Shafarevich maps and plurigenera of algebraic varieties. Invent. Math. 113 (1993), no. 1, 177–215.

[KM] J. Kollár and S. Mori, Birational geometry of algebraic varieties. Cambridge Tracts in Mathematics, 134. Cambridge University Press, Cambridge, 1998.

[KoMM] J. Kollár, Y. Miyaoka and S. Mori, Rational connectedness and boundedness of Fano manifolds.

[Mc05] C. McMullen, Dynamics on blowups of the projective plane, Preprint 2005.

[Mi] Y. Miyaoka, Abundance conjecture for 3-folds: case $\nu = 1$, Comp. Math., 68 (1988), 203–220.

[Ny] N. Nakayama, Compact Kähler manifolds whose universal covering spaces are biholomorphic to $\mathbb{C}^n$, (a modified version, but in preparation); the original is RIMS preprint 1230, Res. Inst. Math. Sci. Kyoto Univ. 1999.

[NZ] N. Nakayama and D. -Q. Zhang, Building blocks of étale endomorphisms of complex projective manifolds. RIMS preprint 1577, Res. Inst. Math. Sci. Kyoto Univ. 2007. http://www.kurims.kyoto-u.ac.jp/preprint/index.html

[NS] Yo. Namikawa and J. H. M. Steenbrink, Global smoothing of Calabi-Yau threefolds, Invent. Math., 122 (1995), 403–419.

[Og03] K. Oguiso, Bimeromorphic automorphism groups of non-projective hyperkähler manifolds - a note inspired by C. T. McMullen, math.AG/0312515; to appear in Journal of Differential Geometry.

[Og06] K. Oguiso, Tits alternative in hyperkähler manifolds, Math. Res. Lett. 13 (2006), 307 – 316.

[OS] K. Oguiso and J. Sakurai, Calabi-Yau threefolds of quotient type. Asian J. Math. 5 (2001), no. 1, 43–77.

[Ty] S. Takayama, Local simple connectedness of resolutions of log-terminal singularities, Internat. J. Math. (14) (2003), no. 8, 825–836.

[Un] K. Ueno, Classification theory of algebraic varieties and compact complex spaces. Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin-New York, 1975.

[Zh1] D. -Q. Zhang, Automorphisms groups and anti-pluricanonical curves, preprint 2006. www.math.nus.edu.sg/∼matzdq

[Zh2] D. -Q. Zhang, Dynamics of automorphisms on projective complex manifolds, preprint 2006; also Max Planck Institute for Mathematics, preprint 019 (year 2007) at: www.mpim-bonn.mpg.de/Research/MPIM+Preprint+Series/

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