HEEGNER DIVISORS, L-FUNCTIONS AND HARMONIC WEAK MAASS FORMS

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1. Introduction and Statement of Results

Half-integral weight modular forms play important roles in arithmetic geometry and number theory. Thanks to the theory of theta functions, such forms include important generating functions for the representation numbers of integers by quadratic forms. Among weight 3/2 modular forms, one finds Gauss' function ($q := e^{2\pi i\tau}$ throughout)

$$\sum_{x,y,z\in\mathbb{Z}} q^{x^2+y^2+z^2} = 1 + 6q + 12q^2 + 8q^3 + 6q^4 + 24q^5 + \cdots ,$$

which is essentially the generating function for class numbers of imaginary quadratic fields, as well as Gross's theta functions [Gro2] which enumerate the supersingular reductions of CM elliptic curves.

In the 1980s, Waldspurger [Wa], and Kohnen and Zagier [KZ, K] established that half-integral weight modular forms also serve as generating functions of a different type. Using the Shimura correspondence [Sh], they proved that certain coefficients of half-integral weight cusp forms essentially are square-roots of central values of quadratic twists of modular L-functions. When the weight is 3/2, these results appear prominently in works on the ancient "congruent number problem" [T], as well as the deep works of Gross, Zagier and Kohnen [GZ, GKZ] on the Birch and Swinnerton-Dyer Conjecture.

In analogy with these works, Katok and Sarnak [KS] employed a Shimura correspondence to relate coefficients of weight 1/2 Maass forms to sums of values and sums of line integrals of Maass cusp forms. We investigate the arithmetic properties of the coefficients of a different class of Maass forms, the weight 1/2 harmonic weak Maass forms.

A *harmonic weak Maass form of weight* $k \in \frac{1}{2}\mathbb{Z}$ *on* $\Gamma_0(N)$ (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$) is a smooth function on $\mathbb{H}$, the upper half of the complex plane, which satisfies:

(i) $f \mid_k \gamma = f$ for all $\gamma \in \Gamma_0(N)$;
(ii) $\Delta_k f = 0$, where Δ_k is the weight k hyperbolic Laplacian on $\mathbb{H}$ (see (2.4));
(iii) There is a polynomial $P_f = \sum_{n\leq 0} c^+(n)q^n \in \mathbb{C}[q^{-1}]$ such that $f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$. Analogous conditions are required at all cusps.

Throughout, for $\tau \in \mathbb{H}$, we let $\tau = u + iv$, where $u, v \in \mathbb{R}$, and we let $q := e^{2\pi i\tau}$.

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Remark 1. The polynomial P_f, the *principal part of f at ∞*, is uniquely determined. If P_f is non-constant, then f has exponential growth at the cusp ∞. Similar remarks apply at all of the cusps.

Remark 2. The results in the body of the paper are phrased in terms of vector valued harmonic weak Maass forms. These forms are defined in Section 2.2.

Spaces of harmonic weak Maass forms include *weakly holomorphic modular forms*, those meromorphic modular forms whose poles (if any) are supported at cusps. We are interested in those harmonic weak Maass forms which do not arise in this way. Such forms have been a source of recent interest due to their connection to Ramanujan's mock theta functions (see [BO1, BO2, Zw1, Zw2]). For example, it turns out that

$$M_f(\tau) := q^{-1}f(q^{24}) + 2i\sqrt{3}\cdot N_f(\tau) \tag{1.1}$$

is a weight 1/2 harmonic weak Maass form, where

$$N_f(\tau) := \int_{-24\overline{\tau}}^{i\infty} \frac{\sum_{n=-\infty}^{\infty}\left(n+\frac{1}{6}\right)e^{3\pi i\left(n+\frac{1}{6}\right)^2 z}}{\sqrt{-i(z+24\tau)}}\, dz = \frac{i}{\sqrt{3\pi}}\sum_{n\in\mathbb{Z}}\Gamma(1/2, 4\pi(6n+1)^2 v)q^{-(6n+1)^2}$$

is a period integral of a theta function, $\Gamma(a,x)$ is the incomplete Gamma function, and $f(q)$ is Ramanujan's mock theta function

$$f(q) := 1 + \sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}.$$

This example reveals two important features common to all harmonic weak Maass forms on $\Gamma_0(N)$. Firstly, all such f have Fourier expansions of the form

$$f(\tau) = \sum_{n\gg-\infty} c^+(n)q^n + \sum_{n<0} c^-(n)W(2\pi nv)q^n, \tag{1.2}$$

where $W(x) = W_k(x) := \Gamma(1-k, 2|x|)$. We call $\sum_{n\gg-\infty} c^+(n)q^n$ the *holomorphic part* of f, and we call its complement its *non-holomorphic part*. Secondly, the non-holomorphic parts are period integrals of weight $2-k$ modular forms. Equivalently, $\xi_k(f)$ is a weight $2-k$ modular form on $\Gamma_0(N)$, where ξ_k is a differential operator (see (2.6)) which is essentially the Maass lowering operator.

Every weight $2-k$ cusp form is the image under ξ_k of a weight k harmonic weak Maass form. The mock theta functions correspond to those forms whose images under $\xi_{\frac{1}{2}}$ are weight 3/2 theta functions. We turn our attention to those weight 1/2 harmonic weak Maass forms whose images under $\xi_{\frac{1}{2}}$ are orthogonal to the elementary theta series. Unlike the mock theta functions, whose holomorphic parts are often generating functions in the theory of partitions (for example, see [BO1, BO2]), we show that these other harmonic weak Maass forms can be "generating functions" simultaneously for both the values and central derivatives of quadratic twists of weight 2 modular L-functions.

Although we treat the general case in this paper, to simplify exposition, in the introduction we assume that p is prime and that $G(\tau) = \sum_{n=1}^{\infty} B_G(n)q^n \in S_2^{new}(\Gamma_0(p))$ is a normalized Hecke eigenform with the property that the sign of the functional equation of

$$L(G,s) = \sum_{n=1}^{\infty} \frac{B_G(n)}{n^s}$$

is $\epsilon(G) = -1$. Therefore, we have that $L(G,1) = 0$.

By Kohnen's theory of plus-spaces [K], there is a half-integral weight newform

$$g(\tau) = \sum_{n=1}^{\infty} b_g(n) q^n \in S^+_{\frac{3}{2}}(\Gamma_0(4p)), \tag{1.3}$$

unique up to a multiplicative constant, which lifts to G under the Shimura correspondence. For convenience, we choose g so that its coefficients are in F_G, the totally real number field obtained by adjoining the Fourier coefficients of G to $\mathbb{Q}$. We shall prove that there is a weight $1/2$ harmonic weak Maass form on $\Gamma_0(4p)$ in the plus space, say

$$f_g(\tau) = \sum_{n \gg -\infty} c_g^+(n) q^n + \sum_{n<0} c_g^-(n) W(2\pi n v) q^n, \tag{1.4}$$

whose principal part P_{f_g} has coefficients in F_G, which also enjoys the property that $\xi_{\frac{1}{2}}(f_g) = \|g\|^{-2} g$, where $\|g\|$ denotes the usual Petersson norm.

A calculation shows that if $n > 0$ (see (2.7)), then

$$b_g(n) = -4\sqrt{\pi n}\|g\|^2 \cdot c_g^-(-n). \tag{1.5}$$

The coefficients $c_g^+(n)$ are more mysterious. We show that both types of coefficients are related to L-functions. To make this precise, for fundamental discriminants D let χ_D be the Kronecker character for $\mathbb{Q}(\sqrt{D})$, and let $L(G, \chi_D, s)$ be the quadratic twist of $L(G,s)$ by χ_D. These coefficients are related to these L-functions in the following way.

Theorem 1.1. *Assume that p is prime, and that $G \in S_2^{new}(\Gamma_0(p))$ is a newform. If the sign of the functional equation of $L(G,s)$ is $\epsilon(G) = -1$, then the following are true:*

(1) *If $\Delta < 0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right) = 1$, then*

$$L(G, \chi_\Delta, 1) = 32\|G\|^2\|g\|^2\pi^2\sqrt{|\Delta|} \cdot c_g^-(\Delta)^2.$$

(2) *If $\Delta > 0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right) = 1$, then $L'(G, \chi_\Delta, 1) = 0$ if and only if $c_g^+(\Delta)$ is algebraic.*

Remark 3. In Theorem 1.1 (2), we have that $L(G, \chi_\Delta, 1) = 0$ since the sign of the functional equation of $L(G, \chi_\Delta, s)$ is -1. Therefore it is natural to consider derivatives in these cases.

Remark 4. The f_g are uniquely determined up to the addition of a weight $1/2$ weakly holomorphic modular form with coefficients in F_G. Furthermore, absolute values of the nonvanishing coefficients $c_g^+(n)$ are typically asymptotic to subexponential functions in n. For these reasons, Theorem 1.1 (2) cannot be simply modified to obtain a formula for $L'(G, \chi_\Delta, 1)$. It would be very interesting to obtain a more precise relationship between these derivatives and the coefficients $c_g^+(\Delta)$.

Remark 5. Here we comment on the construction of the weak harmonic Maass forms f_g. Due to the general results in this paper, we discuss the problem in the context of vector valued forms. It is not difficult to see that this problem boils down to the question of producing inverse images of classical Poincaré series under $\xi_{\frac{1}{2}}$. A simple observation establishes that these preimages should be weight $1/2$ Maass-Poincaré series which are explicitly described in Chapter 1 of [Br]. Since standard Weil-type bounds fall short of establishing convergence of these series, we briefly discuss a method for establishing convergence. One may employ a generalization of work of Goldfeld and Sarnak [GS] (for example, see [P]) on Kloosterman-Selberg zeta functions. This theory proves that the relevant zeta functions are holomorphic at $s = 3/4$, the crucial point for the task at hand. One then deduces convergence using standard methods relating the series expansions of Kloosterman-Selberg zeta functions with their integral representations (for example, using Perron-type formulas). The reader may see [FO] where an argument of this type is carried out in detail.

Theorem 1.1 (1) follows from Kohnen's theory (see Corollary 1 on page 242 of [K]) of half-integral newforms, the existence of f_g, and (1.5). The proof of Theorem 1.1 (2) is more difficult, and it involves a detailed study of Heegner divisors. We establish that the algebraicity of the coefficients $c_g^+(\Delta)$ is dictated by the vanishing of certain twisted Heegner divisors in the Jacobian of $X_0(p)$. This result, when combined with the work of Gross and Zagier [GZ], will imply Theorem 1.1 (2).

To make this precise, we first recall some definitions. Let $d < 0$ and $\Delta > 0$ be fundamental discriminants which are both squares modulo p. Let $\mathcal{Q}_{d,p}$ be the set of discriminant $d = b^2 - 4ac$ integral binary quadratic forms $aX^2 + bXY + cY^2$ with the property that $p \mid a$. For these pairs of discriminants, we define the *twisted Heegner divisor* $H_\Delta(d)$ by

$$H_\Delta(d) := \sum_{Q \in \mathcal{Q}_{\Delta d,p}/\Gamma_0(p)} \chi_\Delta(Q) \cdot \frac{\alpha_Q}{w_Q}, \tag{1.6}$$

where χ_Δ denotes the generalized genus character corresponding to the decomposition $\Delta \cdot d$ as in [GKZ], α_Q is the unique root of $Q(x, 1)$ in $\mathbb{H}$, and w_Q denotes the order of the stabilizer of Q in $\Gamma_0(p)$. Then $H_\Delta(d)$ is a divisor on $X_0(p)$ defined over $\mathbb{Q}(\sqrt{\Delta})$ (see Lemma 5.1). We use these twisted Heegner divisors to define the degree 0 divisor

$$y_\Delta(d) := H_\Delta(d) - \deg(H_\Delta(d)) \cdot \infty. \tag{1.7}$$

Finally, we associate a divisor to f_g by letting

$$y_\Delta(f_g) := \sum_{n<0} c_g^+(n) y_\Delta(n) \in \mathrm{Div}^0(X_0(p)) \otimes F_G. \tag{1.8}$$

Recall that we have selected f_g so that the coefficients of P_{f_g} are in F_G.

To state our results, let J be the Jacobian of $X_0(p)$, and let $J(F)$ denote the points of J over a number field F. The Hecke algebra acts on $J(F) \otimes \mathbb{C}$, which by the Mordell-Weil Theorem is a finite dimensional vector space. The main results of Section 7 (see Theorems 7.5 and 7.6) imply the following theorem.

Theorem 1.2. *Assuming the notation and hypotheses above, the point corresponding to* $y_\Delta(f_g)$ *in* $J(\mathbb{Q}(\sqrt{\Delta}))\otimes\mathbb{C}$ *is in its* G*-isotypical component. Moreover, the following are equivalent:*

(i) *The Heegner divisor* $y_\Delta(f_g)$ *vanishes in* $J(\mathbb{Q}(\sqrt{\Delta}))\otimes\mathbb{C}$.
(ii) *The coefficient* $c_g^+(\Delta)$ *is algebraic.*
(iii) *The coefficient* $c_g^+(\Delta)$ *is contained in* F_G.

We then obtain the following generalization of the Gross-Kohnen-Zagier theorem [GKZ] (see Corollary 6.7 and Theorem 7.7).

Theorem 1.3. *Assuming the notation and hypotheses above, we have that*

$$\sum_{n>0} y_\Delta^G(-n)q^n = g(\tau)\otimes y_\Delta(f_g)\in S^+_{\frac{3}{2}}(\Gamma_0(4p))\otimes J(\mathbb{Q}(\sqrt{\Delta})),$$

where $y_\Delta^G(-n)$ *denotes the projection of* $y_\Delta(-n)$ *onto its* G*-isotypical component.*

This result, when combined with the Gross-Zagier theorem [GZ], gives the conclusion (see Theorem 7.8) that the Heegner divisor $y_\Delta(f_g)$ vanishes in $J(\mathbb{Q}(\sqrt{\Delta}))\otimes\mathbb{C}$ if and only if $L'(G,\chi_\Delta,1)=0$, thereby proving Theorem 1.1 (2).

These results arise from the interplay between Heegner divisors, harmonic weak Maass forms and Borcherds products, relations which are of independent interest. We extend the theory of regularized theta lifts of harmonic weak Maass forms, and we apply these results to obtain generalized Borcherds products. In that way harmonic weak Maass forms are placed in the central position which allows us to obtain the main results in this paper.

In view of Theorem 1.1, it is natural to investigate the algebraicity and the nonvanishing of the coefficients of harmonic weak Maass forms, questions which are of particular interest in the context of elliptic curves. As a companion to Goldfeld's famous conjecture for quadratic twists of elliptic curves [G], which asserts that "half" of the quadratic twists of a fixed elliptic curve have rank zero (resp. one), we make the following conjecture.

Conjecture. *Suppose that*

$$f(\tau)=\sum_{n\gg-\infty} c^+(n)q^n+\sum_{n<0} c^-(n)W(2\pi nv)q^n$$

is a weight $1/2$ *harmonic weak Maass form on* $\Gamma_0(N)$ *whose principal parts at cusps are defined over a number field. If* $\xi_{1/2}(f)$ *is non-zero and is not a linear combination of elementary theta series, then*

$$\#\{0<n<X\ :\ c^-(-n)\neq 0\}\gg_f X,$$
$$\#\{0<n<X\ :\ c^+(n)\ \textit{is transcendental}\}\gg_f X.$$

Remark 6. Suppose that G is as in Theorem 1.1. If G corresponds to a modular elliptic curve E, then the truth of a sufficiently precise form of the conjecture for f_g, combined with Kolyvagin's Theorem on the Birch and Swinnerton-Dyer Conjecture, would prove that a "proportion" of quadratic twists of E have Mordell-Weil rank zero (resp. one).

Remark 7. In a recent paper, Sarnak [Sa] characterized those Maass cusp forms whose Fourier coefficients are all integers. He proved that such a Maass cusp form must correspond to even irreducible 2-dimensional Galois representations which are either dihedral or tetrahedral. More generally, a number of authors such as Langlands [L], and Booker, Strömbergsson, and Venkatesh [BSV] have considered the algebraicity of coefficients of Maass cusp forms. It is generally believed that the coefficients of generic Maass cusp forms are transcendental. The conjecture above suggests that a similar phenomenon should also hold for harmonic weak Maass forms. In this setting, we believe that the exceptional harmonic weak Maass forms are those which arise as preimages of elementary theta functions such as those forms associated to the mock theta functions.

In the direction of this conjecture, we combine Theorem 1.1 with works by the second author and Skinner [OSk] and Perelli and Pomykala [PP] to obtain the following result.

Corollary 1.4. *Assuming the notation and hypotheses above, as $X \to +\infty$ we have*

$$\#\{-X < \Delta < 0 \text{ fundamental } : \ c_g^-(\Delta) \neq 0\} \gg_{f_g} \frac{X}{\log X},$$
$$\#\{0 < \Delta < X \text{ fundamental } : \ c_g^+(\Delta) \text{ is transcendental}\} \gg_{f_g,\epsilon} X^{1-\epsilon}.$$

Remark 8. One can typically obtain better estimates for $c_g^-(\Delta)$ using properties of 2-adic Galois representations. For example, if $L(G,s)$ is the Hasse-Weil L-function of an elliptic curve where p is not of the form x^2+64, where x is an integer, then using Theorem 1 of [O] and the theory of Setzer curves [Se], one can find a rational number $0 < \alpha < 1$ for which

$$\#\{-X < \Delta < 0 \text{ fundamental } : \ c_g^-(\Delta) \neq 0\} \gg_{f_g} \frac{X}{(\log X)^{1-\alpha}}.$$

Now we briefly provide an overview of the ideas behind the proofs of our main theorems. They depend on the construction of canonical differentials of the third kind for twisted Heegner divisors. In Section 5 we produce such differentials of the form $\eta_{\Delta,r}(z,f) = -\frac{1}{2}\partial\Phi_{\Delta,r}(z,f)$, where $\Phi_{\Delta,r}(z,f)$ are automorphic Green functions on $X_0(N)$ which are obtained as liftings of weight $1/2$ harmonic weak Maass forms f. To define these liftings, in Section 5 we generalize the regularized theta lift due to Borcherds, Harvey, and Moore (for example, see [Bo1], [Br]). We then employ transcendence results of Waldschmidt and Scholl (see [W], [Sch]), for the periods of differentials, to relate the vanishing of twisted Heegner divisors in the Jacobian to the algebraicity of the corresponding canonical differentials of the third kind. By means of the q-expansion principle, we obtain the connection to the coefficients of harmonic weak Maass forms.

In Section 6 we construct generalized Borcherds products for twisted Heegner divisors, and we study their properties and multiplier systems (Theorem 6.1). In particular, we give a necessary and sufficient condition that the character of such a Borcherds product has finite order (Theorem 6.2).

In Section 7, we consider the implications of these results when restricting to Hecke stable components. We obtain the general versions of Theorems 1.1, 1.2, and 1.3, and we prove Corollary 1.4. In particular, Theorem 1.3, the Gross-Kohnen-Zagier theorem for twisted

Heegner divisors, is proved by adapting an argument of Borcherds [Bo2], combined with an analysis of the Hecke action on cusp forms, harmonic weak Maass forms, and the Jacobian.

In the body of the paper we consider weight 2 newforms G of arbitrary level and functional equation. For the regularized theta lift, it is convenient to identify SL_2 with $\mathrm{Spin}(V)$ for a certain rational quadratic space V of signature $(2,1)$, and to view the theta lift as a map from vector valued modular forms for the metaplectic group to modular forms on $\mathrm{Spin}(V)$. We define the basic setup in Section 2, and in Section 3 we recall some results of Scholl on canonical differentials of the third kind. The relevant theta kernels are then studied in Section 4. They can be viewed as vector valued weight 1/2 versions of the weight 3/2 twisted theta kernels studied in [Sk2] in the context of Jacobi forms. The regularized theta lift is studied in Section 5, and Sections 6 and 7 are devoted to the proofs of the general forms of our main results as described in the previous paragraphs.

We conclude the paper with explicit examples. In Section 8 we give examples of relations among Heegner divisors which are *not* given by Borcherds lifts of weakly holomorphic modular forms. They are obtained as lifts of harmonic weak Maass forms. One of these examples is related to a famous example of Gross [Za1], and the other is related to Ramanujan's mock theta function $f(q)$. We also derive the infinite product expansions of Zagier's twisted modular polynomials [Za2].

2. Preliminaries

To ease exposition, the results in the introduction were stated using the classical language of half-integral weight modular forms. To treat the case of general levels and functional equations, it will be more convenient to work with vector valued forms and certain Weil representations. Here we recall this framework, and we discuss important theta functions which will be used to study differentials of the third kind.

We begin by fixing notation. Let (V, Q) be a non-degenerate rational quadratic space of signature (b^+, b^-). Let $L \subset V$ be an even lattice with dual L'. The discriminant group L'/L, together with the $\mathbb{Q}/\mathbb{Z}$-valued quadratic form induced by Q, is called the *discriminant form* of the lattice L.

2.1. **The Weil representation.** Let $\mathbb{H} = \{\tau \in \mathbb{C};\ \Im(\tau) > 0\}$ be the complex upper half plane. We write $\mathrm{Mp}_2(\mathbb{R})$ for the metaplectic two-fold cover of $\mathrm{SL}_2(\mathbb{R})$. The elements of this group are pairs $(M, \phi(\tau))$, where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with $\phi(\tau)^2 = c\tau + d$. The multiplication is defined by

$$(M, \phi(\tau))(M', \phi'(\tau)) = (MM', \phi(M'\tau)\phi'(\tau)).$$

We denote the integral metaplectic group, the inverse image of $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ under the covering map, by $\tilde{\Gamma} := \mathrm{Mp}_2(\mathbb{Z})$. It is well known that $\tilde{\Gamma}$ is generated by $T := \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$, and $S := \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$. One has the relations $S^2 = (ST)^3 = Z$, where $Z := \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i\right)$ is the standard generator of the center of $\tilde{\Gamma}$. We let $\tilde{\Gamma}_\infty := \langle T \rangle \subset \tilde{\Gamma}$.

We now recall the Weil representation associated with the discriminant form L'/L (for example, see [Bo1], [Br]). It is a representation of $\tilde{\Gamma}$ on the group algebra $\mathbb{C}[L'/L]$. We denote the standard basis elements of $\mathbb{C}[L'/L]$ by $\mathfrak{e}_h$, $h \in L'/L$, and write $\langle \cdot, \cdot \rangle$ for the standard scalar product (antilinear in the second entry) such that $\langle \mathfrak{e}_h, \mathfrak{e}_{h'} \rangle = \delta_{h,h'}$. The Weil representation ρ_L associated with the discriminant form L'/L is the unitary representation of $\tilde{\Gamma}$ on $\mathbb{C}[L'/L]$ defined by

$$\rho_L(T)(\mathfrak{e}_h) := e(h^2/2)\mathfrak{e}_h, \tag{2.1}$$

$$\rho_L(S)(\mathfrak{e}_h) := \frac{e((b^- - b^+)/8)}{\sqrt{|L'/L|}} \sum_{h' \in L'/L} e(-(h,h'))\mathfrak{e}_{h'}. \tag{2.2}$$

Note that

$$\rho_L(Z)(\mathfrak{e}_h) = e((b^- - b^+)/4)\mathfrak{e}_{-h}. \tag{2.3}$$

2.2. **Vector valued modular forms.** If $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is a function, we write $f = \sum_{\lambda \in L'/L} f_h \mathfrak{e}_h$ for its decomposition in components with respect to the standard basis of $\mathbb{C}[L'/L]$. Let $k \in \frac{1}{2}\mathbb{Z}$, and let $M^!_{k,\rho_L}$ denote the space of $\mathbb{C}[L'/L]$-valued weakly holomorphic modular forms of weight k and type ρ_L for the group $\tilde{\Gamma}$. The subspaces of holomorphic modular forms (resp. cusp forms) are denoted by M_{k,ρ_L} (resp. S_{k,ρ_L}).

Now assume that $k \leq 1$. A twice continuously differentiable function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is called a *harmonic weak Maass form* (of weight k with respect to $\tilde{\Gamma}$ and ρ_L) if it satisfies:

(i) $f(M\tau) = \phi(\tau)^{2k} \rho_L(M,\phi) f(\tau)$ for all $(M,\phi) \in \tilde{\Gamma}$;
(ii) there is a $C > 0$ such that $f(\tau) = O(e^{Cv})$ as $v \to \infty$;
(iii) $\Delta_k f = 0$.

Here we have that

$$\Delta_k := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \tag{2.4}$$

is the usual weight k hyperbolic Laplace operator, where $\tau = u + iv$ (see [BF]). We denote the vector space of these harmonic weak Maass forms by $\mathcal{N}_{k,\rho_L}$. Moreover, we write N_{k,ρ_L} for the subspace of $f \in \mathcal{N}_{k,\rho_L}$ whose singularity at ∞ is locally given by the pole of a meromorphic function. In particular, this means that f satisfies

$$f(\tau) = P(\tau) + O(e^{-\varepsilon v}), \qquad v \to \infty,$$

for some Fourier polynomial

$$P_f(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Z} + Q(h) \\ -\infty \ll n \leq 0}} c^+(n,h) e(n\tau) \mathfrak{e}_h$$

and some $\varepsilon > 0$. In this situation, P_f is uniquely determined by f. It is called the *principal part* of f. (The space N_{k,ρ_L} was called H^+_{k,ρ_L} in [BF].) We have $M^!_{k,\rho_L} \subset N_{k,\rho_L} \subset \mathcal{N}_{k,\rho_L}$. The Fourier expansion of any $f \in N_{k,\rho_L}$ gives a unique decomposition $f = f^+ + f^-$, where

$$f^+(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c^+(n,h) e(n\tau) \mathfrak{e}_h, \tag{2.5a}$$

$$f^-(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n < 0}} c^-(n,h) W(2\pi n v) e(n\tau) \mathfrak{e}_h, \tag{2.5b}$$

and $W(x) = W_k(x) := \int_{-2x}^{\infty} e^{-t}t^{-k}\,dt = \Gamma(1-k, 2|x|)$ for $x < 0$.

Recall that there is an antilinear differential operator defined by

$$\xi_k : N_{k,\rho_L} \longrightarrow S_{2-k,\bar{\rho}_L}, \qquad f(\tau) \mapsto \xi_k(f)(\tau) := v^{k-2}\overline{L_k f(\tau)}. \tag{2.6}$$

Here $L_k := -2iv^2\frac{\partial}{\partial\bar{\tau}}$ is the usual Maass lowering operator. Note that $\xi_{2-k}\xi_k = \Delta_k$. The Fourier expansion of $\xi_k(f)$ is given by

$$\xi_k(f) = -2\sum_{h\in L'/L}\sum_{\substack{n\in\mathbb{Q}\\ n>0}}(4\pi n)^{1-k}c^-(-n,h)e(n\tau)\mathfrak{e}_h. \tag{2.7}$$

The kernel of ξ_k is equal to $M^!_{k,\rho_L}$. By Corollary 3.8 of [BF], the following sequence is exact

$$0 \longrightarrow M^!_{k,\rho_L} \longrightarrow N_{k,\rho_L} \xrightarrow{\xi_k} S_{2-k,\bar{\rho}_L} \longrightarrow 0\ . \tag{2.8}$$

Moreover, by Proposition 3.11 of [BF], for any given Fourier polynomial of the form

$$Q(\tau) = \sum_{h\in L'/L}\sum_{\substack{n\in\mathbb{Z}+Q(h)\\ n<0}} a(n,h)e(n\tau)\mathfrak{e}_h$$

with $a(n,h) = (-1)^{k-\mathrm{sig}(L)/2}a(n,-h)$, there is an $f \in N_{k,\rho_L}$ with principal part $P_f = Q + \mathfrak{c}$ for some T-invariant constant $\mathfrak{c} \in \mathbb{C}[L'/L]$.

Using the Petersson scalar product, we obtain a bilinear pairing between $M_{2-k,\bar{\rho}_L}$ and N_{k,ρ_L} defined by

$$\{g, f\} = \big(g,\, \xi_k(f)\big)_{2-k} := \int_{\Gamma\backslash\mathbb{H}} \langle g,\, \xi_k(f)\rangle v^{2-k}\frac{du\,dv}{v^2}, \tag{2.9}$$

where $g \in M_{2-k,\bar{\rho}_L}$ and $f \in N_{k,\rho_L}$. If g has the Fourier expansion $g = \sum_{h,n} b(n,h)e(n\tau)\mathfrak{e}_h$, and we denote the expansion of f as in (2.5), then by Proposition 3.5 of [BF] we have

$$\{g, f\} = \sum_{h\in L'/L}\sum_{n\le 0} c^+(n,h)b(-n,h). \tag{2.10}$$

Hence $\{g, f\}$ only depends on the principal part of f. The exactness of (2.8) implies that the induced pairing between $S_{2-k,\bar{\rho}_L}$ and $N_{k,\rho_L}/M^!_{k,\rho_L}$ is non-degenerate. Moreover, the pairing is compatible with the natural $\mathbb{Q}$-structures on $M_{2-k,\bar{\rho}_L}$ and $N_{k,\rho_L}/M^!_{k,\rho_L}$ given by modular forms with rational coefficients and harmonic weak Maass forms with rational principal part, respectively.

We conclude this subsection with a notion which will be used later in the paper. A harmonic weak Maass form $f \in N_{k,\rho_L}$ is said to be *orthogonal to cusp forms of weight k* if for all $s \in S_{k,\rho_L}$ we have

$$(f, s)^{reg} := \int_{\mathcal{F}}^{reg} \langle f(\tau), s(\tau)\rangle v^k\,\frac{du\,dv}{v^2} = 0.$$

Here $\mathcal{F}$ denotes the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, and the integral has been regularized as in [Bo1].

2.3. **Siegel theta functions.** Now we recall some basic properties of theta functions associated to indefinite quadratic forms. Let

$$\mathrm{Gr}(V) := \{z \subset V(\mathbb{R}) : \quad z \text{ is a 2-dimensional subspace with } Q|_z > 0\}$$

be the Grassmannian of 2-dimensional positive definite subspaces of $V(\mathbb{R})$. If $\lambda \in V(\mathbb{R})$ and $z \in \mathrm{Gr}(V)$, we write λ_z and $\lambda_{z^\perp}$ for the orthogonal projection of λ to z and $z^\perp$, respectively.

Let $\alpha, \beta \in V(\mathbb{R})$. For $\tau = u + iv \in \mathbb{H}$ and $z \in \mathrm{Gr}(V)$, we define a theta function by

$$\vartheta_L(\tau, z, \alpha, \beta) := v^{b^-/2} \sum_{\lambda \in L} e\left(\frac{(\lambda+\beta)_z^2}{2}\tau + \frac{(\lambda+\beta)_{z^\perp}^2}{2}\bar{\tau} - (\lambda + \beta/2, \alpha)\right). \tag{2.11}$$

Proposition 2.1. *We have the transformation formula*

$$\vartheta_L(-1/\tau, z, -\beta, \alpha) = \left(\frac{\tau}{i}\right)^{\frac{b^+-b^-}{2}} |L'/L|^{-1/2} \vartheta_{L'}(\tau, z, \alpha, \beta).$$

Proof. This follows by Poisson summation (for example, see Theorem 4.1 of [Bo1]). □

The proposition can be used to define vector valued Siegel theta functions of weight $k = (b^+ - b^-)/2$ and type ρ_L (see Section 4 of [Bo1]).

2.4. **A lattice related to $\Gamma_0(N)$.** Let N be a positive integer. We consider the rational quadratic space

$$V := \{X \in \mathrm{Mat}_2(\mathbb{Q}) : \ \mathrm{tr}(X) = 0\} \tag{2.12}$$

with the quadratic form $Q(X) := -N\det(X)$. The corresponding bilinear form is given by $(X, Y) = N\,\mathrm{tr}(XY)$ for $X, Y \in V$. The signature of V is $(2, 1)$. The even Clifford algebra $C^0(V)$ of V can be identified with $\mathrm{Mat}_2(\mathbb{Q})$. The Clifford norm on $C^0(V)$ is identified with the determinant. The group $\mathrm{GL}_2(\mathbb{Q})$ acts on V by

$$\gamma.X = \gamma X \gamma^{-1}, \qquad \gamma \in \mathrm{GL}_2(\mathbb{Q}),$$

leaving the quadratic form invariant, inducing isomorphisms of algebraic groups over $\mathbb{Q}$

$$\mathrm{GL}_2 \cong \mathrm{GSpin}(V), \qquad \mathrm{SL}_2 \cong \mathrm{Spin}(V).$$

There is an isometry from (V, Q) to the trace zero part of $(C^0(V), -N\det)$. We let L be the lattice

$$L := \left\{ \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix} : \quad a, b, c \in \mathbb{Z} \right\}. \tag{2.13}$$

Then the dual lattice is given by

$$L' := \left\{ \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix} : \quad a, b, c \in \mathbb{Z} \right\}. \tag{2.14}$$

We identify L'/L with $\mathbb{Z}/2N\mathbb{Z}$. Here the quadratic form on L'/L is identified with the quadratic form $x \mapsto x^2$ on $\mathbb{Z}/2N\mathbb{Z}$. The level of L is $4N$.

If $D \in \mathbb{Z}$, let L_D be the set of vectors $X \in L'$ with $Q(X) = D/4N$. Notice that L_D is empty unless D is a square modulo $4N$. For $r \in L'/L$ with $r^2 \equiv D \pmod{4N}$ we define

$$L_{D,r} := \{X \in L' : \quad Q(X) = D/4N \text{ and } X \equiv r \pmod{L}\}.$$

We write L_D^0 be the subset of *primitive* vectors in L_D, and $L_{D,r}^0$ for the primitive vectors in $L_{D,r}$, respectively. If $X = \left(\begin{smallmatrix} b/2N & -a/N \\ c & -b/2N \end{smallmatrix}\right) \in L_{D,r}$, then the matrix

$$\psi(X) := \begin{pmatrix} a & b/2 \\ b/2 & Nc \end{pmatrix} = X \begin{pmatrix} 0 & N \\ -N & 0 \end{pmatrix} \tag{2.15}$$

defines an integral binary quadratic form of discriminant $D = b^2 - 4Nac = 4NQ(X)$ with $b \equiv r \pmod{2N}$.

It is easily seen that the natural homomorphism $\mathrm{SO}(L) \to \mathrm{Aut}(L'/L)$ is surjective. Its kernel is called the discriminant kernel subgroup, which we denote by $\Gamma(L)$. We write $\mathrm{SO}^+(L)$ for the intersection of $\mathrm{SO}(L)$ and the connected component of the identity of $\mathrm{SO}(V)(\mathbb{R})$. The group $\Gamma_0(N) \subset \mathrm{Spin}(V)$ takes L to itself and acts trivially on L'/L.

Proposition 2.2. *The image of* $\Gamma_0(N)$ *in* $\mathrm{SO}(L)$ *is equal to* $\Gamma(L) \cap \mathrm{SO}^+(L)$. *The image in* $\mathrm{SO}(L)$ *of the extension of* $\Gamma_0(N)$ *by all Atkin-Lehner involutions is equal to* $\mathrm{SO}^+(L)$.

In particular, $\Gamma_0(N)$ acts on $L_{D,r}$ and $L_{D,r}^0$. By reduction theory, the number of orbits of $L_{D,r}^0$ is finite. The number of orbits of $L_{D,r}$ is finite if $D \neq 0$.

3. Differentials of the third kind

We shall construct differentials of the third kind associated to twisted Heegner divisors using regularized Borcherds products. We begin by recalling some general facts concerning such differentials [Sch] and [Gri]. Let X be a non-singular projective curve over $\mathbb{C}$ of genus g. A differential of the first kind on X is a holomorphic 1-form. A differential of the second kind is a meromorphic 1-form on X whose residues all vanish. A differential of the third kind on X is a meromorphic 1-form on X whose poles are all of first order with residues in $\mathbb{Z}$. Let ψ be a differential of the third kind on X that has poles at the points P_j, with residues c_j, and is holomorphic elsewhere. Then the *residue divisor* of ψ is

$$\mathrm{res}(\psi) := \sum_j c_j P_j.$$

By the residue theorem, the restriction of this divisor to any component of X has degree 0.

Conversely, if $D = \sum_j c_j P_j$ is any divisor on X whose restriction to any component of X has degree 0, then the Riemann-Roch theorem shows that there is a differential ψ_D of the third kind with residue divisor D. Moreover, ψ_D is determined by this condition up to addition of a differential of the first kind. Let $U = X \setminus \{P_j\}$. The canonical homomorphism $H_1(U, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ is surjective and its kernel is spanned by the classes of small circles δ_j around the points P_j. In particular, we have $\int_{\delta_j} \psi_D = 2\pi i c_j$.

Using the Riemann period relations, it can be shown that there is a unique differential of the third kind η_D on X with residue divisor D such that

$$\Re\left(\int_\gamma \eta_D\right) = 0$$

for all $\gamma \in H_1(U, \mathbb{Z})$. It is called the *canonical differential of the third kind* associated with D. For instance, if f is a meromorphic function on X with divisor D, then df/f

is a canonical differential of the third kind on X with residue divisor $\operatorname{div}(f)$. A different characterization of η_D is given in Proposition 1 of [Sch].

Proposition 3.1. *The differential η_D is the unique differential of the third kind with residue divisor D which can be written as $\eta_D = \partial h$, where h is a harmonic function on U.*

Let $\bar{\mathbb{Q}} \subset \mathbb{C}$ be a fixed algebraic closure of $\mathbb{Q}$. We now assume that the curve X and the divisor D are defined over a number field $F \subset \bar{\mathbb{Q}}$. The following theorem by Scholl on the transcendence of canonical differentials of the third kind will be important for us (see Theorem 1 of [Sch]). Its proof is based on results by Waldschmidt on the transcendence of periods of differentials of the third kind (see Section 5.2 of [W], and Theorem 2 of [Sch]).

Theorem 3.2 (Scholl)**.** *If some non-zero multiple of D is a principal divisor, then η_D is defined over F. Otherwise, η_D is not defined over $\bar{\mathbb{Q}}$.*

3.1. **Differentials of the third kind on modular curves.** We consider the modular curve $Y_0(N) := \Gamma_0(N)\backslash\mathbb{H}$. By adding cusps in the usual way, we obtain the compact modular curve $X_0(N)$. It is well known that $X_0(N)$ is defined over $\mathbb{Q}$. The cusps are defined over $\mathbb{Q}(\zeta_N)$, where ζ_N denotes a primitive N-th root of unity. The action of the Galois group on them can described explicitly (for example, see [Ogg]). In particular, it turns out that the cusps are defined over $\mathbb{Q}$ when N or $N/2$ is square-free. Moreover, by the Manin-Drinfeld theorem, any divisor of degree 0 supported on the cusps is a non-zero multiple of a principal divisor. We let J be the Jacobian of $X_0(N)$, and let $J(F)$ denote its points over any number field F. They correspond to divisor classes of degree zero on $X_0(N)$ which are rational over F. By the Mordell-Weil theorem, $J(F)$ is a finitely generated abelian group.

Let ψ be a differential of the third kind on $X_0(N)$. We may write $\psi = 2\pi i f\, dz$, where f is a meromorphic modular form of weight 2 for the group $\Gamma_0(N)$. All poles of f lie on $Y_0(N)$ and are of first order, and they have residues in $\mathbb{Z}$. In a neighborhood of the cusp ∞, the modular form f has a Fourier expansion

$$f(z) = \sum_{n\geq 0}^{\infty} a(n)q^n.$$

The constant coefficient $a(0)$ is the residue of ψ at ∞. We have analogous expansions at the other cusps. According to the q-expansion principle, ψ is defined over a number field F, if and only if all Fourier coefficients $a(n)$ are contained in F. Combining these facts with Theorem 3.2, we obtain the following criterion.

Theorem 3.3. *Let D be a divisor of degree 0 on $X_0(N)$ defined over F. Let η_D be the canonical differential of the third kind associated to D and write $\eta_D = 2\pi i f\, dz$. If some non-zero multiple of D is a principal divisor, then all the coefficients $a(n)$ of f at the cusp ∞ are contained in F. Otherwise, there exists an n such that $a(n)$ is transcendental.*

4. Twisted Siegel theta functions

To define a generalized theta lift in the next section, we first must consider twisted Siegel theta functions. We begin with some notation. Let N be a positive integer, and let L be the lattice defined in Section 2.4. Let $\Delta \in \mathbb{Z}$ be a fundamental discriminant and $r \in \mathbb{Z}$ such that $\Delta \equiv r^2 \pmod{4N}$. Following [GKZ], we define a generalized genus character for $\lambda = \left(\begin{smallmatrix} b/2N & -a/N \\ c & -b/2N \end{smallmatrix}\right) \in L'$ by putting

$$\chi_\Delta(\lambda) = \chi_\Delta([a,b,Nc]) := \begin{cases} \left(\frac{\Delta}{n}\right), & \text{if } \Delta \mid b^2-4Nac \text{ and } (b^2-4Nac)/\Delta \text{ is a} \\ & \text{square modulo } 4N \text{ and } \gcd(a,b,c,\Delta)=1, \\ 0, & \text{otherwise.} \end{cases}$$

Here $[a,b,Nc]$ is the integral binary quadratic form corresponding to λ, and n is any integer prime to Δ represented by one of the quadratic forms $[N_1a, b, N_2c]$ with $N_1N_2 = N$ and $N_1, N_2 > 0$ (see Section 1.2 of [GKZ], and also Section 1 of [Sk2]).

The function χ_Δ is invariant under the action of $\Gamma_0(N)$ and under the action of all Atkin-Lehner involutions. Hence it is invariant under $\mathrm{SO}^+(L)$. It can be computed by the following explicit formula (see Section I.2, Proposition 1 of [GKZ]): If $\Delta = \Delta_1\Delta_2$ is a factorization of Δ into discriminants, and $N = N_1N_2$ is a factorization of N into positive factors such that $(\Delta_1, N_1a) = (\Delta_2, N_2c) = 1$, then

$$\chi_\Delta([a,b,Nc]) = \left(\frac{\Delta_1}{N_1a}\right)\left(\frac{\Delta_2}{N_2c}\right). \tag{4.1}$$

If no such factorizations of Δ and N exist, then we have $\chi_\Delta([a,b,Nc]) = 0$.

We define a twisted variant of the Siegel theta function for L as follows. For a coset $h \in L'/L$, and variables $\tau = u + iv \in \mathbb{H}$, $z \in \mathrm{Gr}(V)$, we put

$$\begin{aligned}\theta_{\Delta,r,h}(\tau,z) &:= v^{1/2} \sum_{\substack{\lambda \in L+rh \\ Q(\lambda)\equiv \Delta Q(h)\ (\Delta)}} \chi_\Delta(\lambda) e\left(\frac{1}{|\Delta|}\frac{\lambda_z^2}{2}\tau + \frac{1}{|\Delta|}\frac{\lambda_{z^\perp}^2}{2}\bar{\tau}\right) \\ &= v^{1/2} \sum_{\substack{\lambda \in L+rh \\ Q(\lambda)\equiv \Delta Q(h)\ (\Delta)}} \chi_\Delta(\lambda) e\left(\frac{1}{|\Delta|}\frac{\lambda^2}{2}u + \frac{1}{|\Delta|}\left(\frac{\lambda_z^2}{2} - \frac{\lambda_{z^\perp}^2}{2}\right) iv\right).\end{aligned} \tag{4.2}$$

Moreover, we define a $\mathbb{C}[L'/L]$-valued theta function by putting

$$\Theta_{\Delta,r}(\tau,z) := \sum_{h \in L'/L} \theta_{\Delta,r,h}(\tau,z)\mathfrak{e}_h. \tag{4.3}$$

We will often omit the dependency on Δ, r from the notation if it is clear from the context. In the variable z, the function $\Theta_{\Delta,r}(\tau,z)$ is invariant under $\Gamma_0(N)$. In the next theorem we consider the transformation behavior in the variable τ.

Theorem 4.1. *The theta function $\Theta_{\Delta,r}(\tau,z)$ is a non-holomorphic $\mathbb{C}[L'/L]$-valued modular form for $\mathrm{Mp}_2(\mathbb{Z})$ of weight $1/2$. It transforms with the representation ρ_L if $\Delta > 0$, and with $\bar{\rho}_L$ if $\Delta < 0$.*

Theorem 4.1 of [Bo1] gives the $\Delta = 1$ case. For general Δ, a similar result for Jacobi forms is contained in [Sk2] (see §2, pp.507). The following is crucial for its proof.

Proposition 4.2. *For $h \in L'/L$ and $\lambda \in L'/\Delta L$, the exponential sum*

$$G_h(\lambda, \Delta, r) = \sum_{\substack{\lambda' \in L'/\Delta L \\ \lambda' \equiv rh \ (L) \\ Q(\lambda') \equiv \Delta Q(h) \ (\Delta)}} \chi_\Delta(\lambda') e\left(-\frac{1}{|\Delta|}(\lambda, \lambda') \right)$$

is equal to

$$\varepsilon |\Delta|^{3/2} \chi_\Delta(\lambda) \sum_{\substack{h' \in L'/L \\ \lambda \equiv rh' \ (L) \\ Q(\lambda) \equiv \Delta Q(h') \ (\Delta)}} e\big(-\operatorname{sgn}(\Delta)(h, h') \big).$$

Here $\varepsilon = 1$ if $\Delta > 0$, and $\varepsilon = i$ if $\Delta < 0$.

Proof. By applying a finite Fourier transform in r' modulo $2N$, the claim follows from identity (3) on page 517 of [Sk2]. □

Proof of Theorem 4.1. We only have to check the transformation behavior under the generators T and S of $\tilde{\Gamma}$. The transformation law under T follows directly from the definition in (4.2). For the transformation law under S we notice that we may write

$$\theta_h(\tau, z) = \sum_{\substack{\alpha \in L'/\Delta L \\ \alpha \equiv rh \ (L) \\ Q(\alpha) \equiv \Delta Q(h) \ (\Delta)}} \chi_\Delta(\alpha) |\Delta|^{-1/2} \vartheta_L(|\Delta|\tau, z, 0, \alpha/|\Delta|),$$

where ϑ_L is the theta function for the lattice L defined in (2.11). Here we have used that $\chi_\Delta(\lambda)$ only depends on $\lambda \in L'$ modulo ΔL. By Proposition 2.1, we find that

$$\begin{aligned}
\theta_h(-1/\tau, z) &= \sqrt{\frac{\tau}{i}} |L'/L|^{-1/2} |\Delta|^{-1} \sum_{\substack{\alpha \in L'/\Delta L \\ \alpha \equiv rh \ (L) \\ Q(\alpha) \equiv \Delta Q(h) \ (\Delta)}} \chi_\Delta(\alpha) \vartheta_{L'}(\tau/|\Delta|, z, \alpha/|\Delta|, 0) \\
&= \sqrt{\frac{\tau}{i}} (2N)^{-1/2} |\Delta|^{-3/2} v^{1/2} \sum_{\lambda \in L'} G_h(\lambda, \Delta, r) e\left(\frac{1}{|\Delta|} \frac{\lambda_z^2}{2} \tau + \frac{1}{|\Delta|} \frac{\lambda_{z^\perp}^2}{2} \bar{\tau} \right).
\end{aligned}$$

By Proposition 4.2, we obtain

$$\theta_h(-1/\tau, z) = \sqrt{\frac{\tau}{i}} \varepsilon (2N)^{-1/2} \sum_{h' \in L'/L} e\big(\operatorname{sgn}(\Delta)(h, h') \big) \theta_{h'}(\tau, z).$$

This completes the proof of the theorem. □

4.1. **Partial Poisson summation.** We now consider the Fourier expansion of $\theta_h(\tau, z)$ in the variable z. Following [Bo1] and [Br], it is obtained by applying a partial Poisson summation to the theta kernel.

Recall that the cusps of $\Gamma_0(N)$ correspond to $\Gamma_0(N)$-classes of primitive isotropic vectors in L. Let $\ell \in L$ be a primitive isotropic vector. Let $\ell' \in L'$ with $(\ell, \ell') = 1$. The 1-dimensional lattice

$$K = L \cap \ell'^\perp \cap \ell^\perp$$

is positive definite. For simplicity we assume that $(\ell, L) = \mathbb{Z}$. In this case we may chose $\ell' \in L$. Then L splits into

$$L = K \oplus \mathbb{Z}\ell' \oplus \mathbb{Z}\ell, \tag{4.4}$$

and $K'/K \cong L'/L$. (If N is squarefree, then any primitive isotropic vector $\ell \in L$ satisfies $(\ell, L) = \mathbb{Z}$ and our assumption is not a restriction. For general N, the results of this section still hold with the appropriate modifications, but the formulas get considerably longer.)

We denote by w the orthogonal complement of ℓ_z in z. Hence

$$V(\mathbb{R}) = z \oplus z^\perp = w \oplus \mathbb{R}\ell_z \oplus \mathbb{R}\ell_{z^\perp}.$$

If $\lambda \in V(\mathbb{R})$, let λ_w be the orthogonal projection of λ to w. We denote by μ the vector

$$\mu = \mu(z) := -\ell' + \frac{\ell_z}{2\ell_z^2} + \frac{\ell_{z^\perp}}{2\ell_{z^\perp}^2}$$

in $V(\mathbb{R}) \cap \ell^\perp$. The Grassmannian of K consists of a single point. Therefore we omit the variable z in the corresponding theta function ϑ_K defined in (2.11).

Let $\alpha, \beta \in \mathbb{Z}$, and let $\mu \in K \otimes_{\mathbb{Z}} \mathbb{R}$. For $h \in K'/K$ and $\tau \in \mathbb{H}$, we let

$$\begin{aligned}\xi_h(\tau, \mu, \alpha, \beta) := \sum_{\lambda \in K + rh} \sum_{\substack{t\ (\Delta) \\ Q(\lambda - \beta\ell' + t\ell) \equiv \Delta Q(h)\ (\Delta)}} \chi_\Delta(\lambda - \beta\ell' + t\ell) e(-\alpha t/|\Delta|) \\ \times e\left(\frac{(\lambda + \beta\mu)^2}{2} \frac{\tau}{|\Delta|} - \frac{1}{|\Delta|}(\lambda + \beta\mu/2, \alpha\mu)\right).\end{aligned} \tag{4.5}$$

Moreover, we define a $\mathbb{C}[K'/K]$-valued theta function by putting

$$\Xi(\tau, \mu, \alpha, \beta) := \sum_{h \in K'/K} \xi_h(\tau, \mu, \alpha, \beta)\mathfrak{e}_h. \tag{4.6}$$

Later we will use the following slightly more explicit formula for $\Xi(\tau, \mu, n, 0)$.

Proposition 4.3. *If n is an integer, then we have*

$$\xi_h(\tau, \mu, n, 0) = \left(\frac{\Delta}{n}\right) \bar{\varepsilon} |\Delta|^{1/2} \sum_{\substack{\lambda \in K + rh \\ Q(\lambda) \equiv \Delta Q(h)\ (\Delta)}} e\left(\frac{\lambda^2}{2} \frac{\tau}{|\Delta|} - \frac{n}{|\Delta|}(\lambda, \mu)\right).$$

Here $\left(\frac{\Delta}{0}\right) = 1$ if $\Delta = 1$, and $\left(\frac{\Delta}{0}\right) = 0$ otherwise.

Proof. By definition we have

$$\xi_h(\tau, \mu, n, 0) = \sum_{\substack{\lambda \in K + rh \\ Q(\lambda) \equiv \Delta Q(h)\ (\Delta)}} \sum_{t\ (\Delta)} \chi_\Delta(\lambda + t\ell) e(-nt/|\Delta|) e\left(\frac{\lambda^2}{2} \frac{\tau}{|\Delta|} - \frac{n}{|\Delta|}(\lambda, \mu)\right).$$

Using the $\mathrm{SO}^+(L)$-invariance of χ_Δ, we find that $\chi_\Delta(\lambda + t\ell) = \chi_\Delta([t, *, 0]) = \left(\frac{\Delta}{t}\right)$ for $\lambda \in K + rh$ with $Q(\lambda) \equiv \Delta Q(h) \pmod{\Delta}$. Inserting the value of the Gauss sum

$$\sum_{t\ (\Delta)} \left(\frac{\Delta}{t}\right) e(nt/|\Delta|) = \left(\frac{\Delta}{n}\right) \varepsilon |\Delta|^{1/2}, \tag{4.7}$$

we obtain the assertion. □

Theorem 4.4. *If $(M,\phi)\in \mathrm{Mp}_2(\mathbb{Z})$ with $M=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$, then we have that*

$$\Xi(M\tau,\mu,a\alpha+b\beta,c\alpha+d\beta)=\phi(\tau)\tilde\rho_K(M,\phi)\cdot\Xi(\tau,\mu,\alpha,\beta).$$

Here $\tilde\rho_K$ is the representation ρ_K when $\Delta>0$, and the representation $\bar\rho_K$ when $\Delta<0$.

The proof is based on the following proposition.

Proposition 4.5. *Let $h\in K'/K$. For $\kappa\in K'/\Delta K$, $a\in\mathbb{Z}/\Delta\mathbb{Z}$, and $s\in\mathbb{Z}/\Delta\mathbb{Z}$, the exponential sum*

$$g_h(\kappa,a,s)=\sum_{\substack{\kappa'\in K'/\Delta K\\ \kappa'\equiv rh\ (L)\\ b'\ (\Delta)\\ Q(\kappa'+s\ell'+b'\ell)\equiv\Delta Q(h)\ (\Delta)}}\chi_\Delta(\kappa'+s\ell'+b'\ell)e\left(-\frac{1}{|\Delta|}((\kappa,\kappa')+ab')\right)$$

is equal to

$$\varepsilon|\Delta|^{1/2}\sum_{\substack{h'\in K'/K\\ \kappa\equiv rh'\ (K)}}e\big(-\operatorname{sgn}(\Delta)(h,h')\big)\sum_{\substack{b\ (\Delta)\\ Q(\kappa+a\ell'+b\ell)\equiv\Delta Q(h')\ (\Delta)}}\chi_\Delta(\kappa+a\ell'+b\ell)e(bs/|\Delta|).$$

Here $\varepsilon=1$ if $\Delta>0$, and $\varepsilon=i$ if $\Delta<0$.

Proof. This follows from Proposition 4.2 for $\lambda=\kappa+a\ell'+b\ell$, by applying a finite Fourier transform in b modulo Δ. □

Proof of Theorem 4.4. We only have to check the transformation behavior under the generators T and S of $\tilde\Gamma$. The transformation law under T follows directly from (4.5). For the transformation law under S we notice that we may write

$$\xi_h(\tau,\mu,\alpha,\beta)=\sum_{\substack{\lambda_1\in K'/\Delta K\\ \lambda_1\equiv rh\ (K)\\ t\ (\Delta)\\ Q(\lambda_1-\beta\ell'+t\ell)\equiv\Delta Q(h)\ (\Delta)}}\chi_\Delta(\lambda_1-\beta\ell'+t\ell)e\left(-\frac{\alpha t}{|\Delta|}-\frac{(\lambda_1,\alpha\mu)}{2|\Delta|}\right)$$
$$\times\,\vartheta_K\left(|\Delta|\tau,\alpha\mu,\frac{\lambda_1}{|\Delta|}+\frac{\beta\mu}{|\Delta|}\right),$$

where ϑ_K is the theta function for the lattice K in (2.11). By Proposition 2.1, we have

$$\xi_h(-1/\tau,\mu,-\beta,\alpha)=\sqrt{\frac{\tau}{i}}|K'/K|^{-1/2}|\Delta|^{-1/2}\sum_{\lambda\in K'}g_h(\lambda,-\beta,-\alpha)$$
$$\times\, e\left(\frac{(\lambda+\beta\mu)^2}{2}\frac{\tau}{|\Delta|}-\frac{1}{|\Delta|}(\lambda+\beta\mu/2,\alpha\mu)\right).$$

By Proposition 4.5, we find that

$$\xi_h(-1/\tau,\mu,-\beta,\alpha)=\sqrt{\frac{\tau}{i}}\varepsilon|K'/K|^{-1/2}\sum_{h'\in K'/K}e\big(\operatorname{sgn}(\Delta)(h,h')\big)\xi_h(\tau,\mu,\alpha,\beta).$$

This concludes the proof of the theorem. □

Lemma 4.6. *We have that*

$$\theta_h(\tau,z) = \frac{1}{\sqrt{2|\Delta|\ell_z^2}} \sum_{\lambda\in rh+L/\mathbb{Z}\ell} \sum_{d\in\mathbb{Z}} \sum_{\substack{t\ (\Delta)\\ Q(\lambda+t\ell)\equiv\Delta Q(h)\ (\Delta)}} \chi_\Delta(\lambda+t\ell)e(-dt/|\Delta|)$$
$$\times\, e\left(\frac{\lambda_w^2}{2}\frac{\tau}{|\Delta|} - \frac{d}{|\Delta|}\frac{(\lambda,\ell_z-\ell_{z^\perp})}{2\ell_z^2} - \frac{|d+(\lambda,\ell)\tau|^2}{4i|\Delta|v\ell_z^2}\right).$$

Proof. The proof follows the argument of Lemma 5.1 in [Bo1] (see also Lemma 2.3 in [Br]). In the definition of $\theta_h(\tau,z)$, we rewrite the sum over $\lambda\in rh+L$ as a sum over $\lambda'+d|\Delta|\ell$, where λ' runs through $rh+L/\mathbb{Z}\Delta\ell$ and d runs through $\mathbb{Z}$. We obtain

$$\theta_h(\tau,z) = v^{1/2} \sum_{\substack{\lambda\in rh+L/\mathbb{Z}\Delta\ell\\ Q(\lambda)\equiv\Delta Q(h)\ (\Delta)}} \chi_\Delta(\lambda)\sum_{d\in\mathbb{Z}} g(|\Delta|\tau,z,\lambda/|\Delta|;d),$$

where the function $g(\tau,z,\lambda;d)$ is defined by

$$g(\tau,z,\lambda;d) = e\left(\frac{(\lambda+d\ell)_z^2}{2}\tau + \frac{(\lambda+d\ell)_{z^\perp}^2}{2}\bar\tau\right)$$

for $\tau\in\mathbb{H}$, $z\in\mathrm{Gr}(V)$, $\lambda\in V(\mathbb{R})$, and $d\in\mathbb{R}$. We apply Poisson summation to the sum over d. The Fourier transform of g as a function in d is

$$\hat g(\tau,z,\lambda,d) = \frac{1}{\sqrt{2v\ell_z^2}} e\left(\frac{\lambda_w^2}{2}\tau - \frac{d(\lambda,\ell_z-\ell_{z^\perp})}{2\ell_z^2} - \frac{|d+(\lambda,\ell)\tau|^2}{4iv\ell_z^2}\right)$$

(see [Br], p. 43). We obtain

$$\begin{aligned}\theta_h(\tau,z) &= v^{1/2} \sum_{\substack{\lambda\in rh+L/\mathbb{Z}\Delta\ell\\ Q(\lambda)\equiv\Delta Q(h)\ (\Delta)}} \chi_\Delta(\lambda)\sum_{d\in\mathbb{Z}} \hat g(|\Delta|\tau,z,\lambda/|\Delta|;d)\\ &= \frac{1}{\sqrt{2|\Delta|\ell_z^2}} \sum_{\substack{\lambda\in rh+L/\mathbb{Z}\Delta\ell\\ Q(\lambda)\equiv\Delta Q(h)\ (\Delta)}} \chi_\Delta(\lambda)\sum_{d\in\mathbb{Z}} e\left(\frac{\lambda_w^2}{2}\frac{\tau}{|\Delta|} - \frac{d(\lambda,\ell_z-\ell_{z^\perp})}{2|\Delta|\ell_z^2} - \frac{|d+(\lambda,\ell)\tau|^2}{4iv|\Delta|\ell_z^2}\right).\end{aligned}$$

The claim follows by rewriting the sum over $\lambda\in rh+L/\mathbb{Z}\Delta\ell$ as a sum over $\lambda'+t\ell$, where λ' runs through $rh+L/\mathbb{Z}\ell$ and t runs through $\mathbb{Z}/\Delta\mathbb{Z}$, and by using the facts that $\ell_w=0$ and $(\ell,\ell_z-\ell_{z^\perp})/2\ell_z^2=1$. □

Lemma 4.7. *We have that*

$$\theta_h(\tau,z) = \frac{1}{\sqrt{2|\Delta|\ell_z^2}} \sum_{c,d\in\mathbb{Z}} \exp\left(-\frac{\pi|c\tau+d|^2}{2|\Delta|v\ell_z^2}\right)\xi_h(\tau,\mu,d,-c).$$

Proof. Using $rh + L/\mathbb{Z}\ell = rh + K + \mathbb{Z}\ell'$ and the identities

$$\begin{aligned} \ell'_w &= -\mu_w, \\ -\frac{\mu^2}{2} &= \frac{(\ell', \ell_z - \ell_{z^\perp})}{2\ell_z^2}, \\ (\lambda, \mu) &= \frac{(\lambda, \ell_z - \ell_{z^\perp})}{2\ell_z^2} \end{aligned}$$

for $\lambda \in K \otimes \mathbb{R}$, the formula of Lemma 4.6 can rewritten as

$$\begin{aligned} \theta_h(\tau, z) = \frac{1}{\sqrt{2|\Delta|\ell_z^2}} \sum_{\lambda \in rh+K} \sum_{c,d \in \mathbb{Z}} \sum_{\substack{t\ (\Delta) \\ Q(\lambda + c\ell' + t\ell) \equiv \Delta Q(h)\ (\Delta)}} \chi_\Delta(\lambda + c\ell' + t\ell) e(-dt/|\Delta|) \\ \times e\left(\frac{(\lambda - c\mu)_w^2}{2} \frac{\tau}{|\Delta|} - \frac{1}{|\Delta|}(\lambda - c\mu/2, d\mu) - \frac{|c\tau + d|^2}{4i|\Delta| v \ell_z^2} \right). \end{aligned}$$

Inserting the definition (4.5) of $\xi_h(\tau, \mu, \alpha, \beta)$, we obtain the assertion. □

Theorem 4.8. *We have that*

$$\begin{aligned} \Theta_{\Delta,r}(\tau, z) = &\frac{1}{\sqrt{2|\Delta|\ell_z^2}} \Xi(\tau, 0, 0, 0) \\ &+ \frac{1}{\sqrt{2|\Delta|\ell_z^2}} \sum_{n \geq 1} \sum_{\gamma \in \tilde{\Gamma}_\infty \backslash \tilde{\Gamma}} \left[\exp\left(-\frac{\pi n^2}{2|\Delta| \Im(\tau) \ell_z^2} \right) \Xi(\tau, \mu, n, 0) \right] |_{1/2, \tilde{\rho}_K}\ \gamma. \end{aligned}$$

Proof. According to Lemma 4.7, we have

$$\begin{aligned} \Theta_{\Delta,r}(\tau, z) &= \frac{1}{\sqrt{2|\Delta|\ell_z^2}} \sum_{c,d \in \mathbb{Z}} \exp\left(-\frac{\pi |c\tau + d|^2}{2|\Delta| v \ell_z^2} \right) \Xi(\tau, \mu, d, -c) \\ &= \frac{1}{\sqrt{2|\Delta|\ell_z^2}} \Xi(\tau, \mu, 0, 0) \\ &\quad + \frac{1}{\sqrt{2|\Delta|\ell_z^2}} \sum_{n \geq 1} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \exp\left(-\frac{\pi n^2 |c\tau + d|^2}{2|\Delta| v \ell_z^2} \right) \Xi(\tau, \mu, nd, -nc). \end{aligned}$$

Writing the sum over coprime integers c, d as a sum over $\tilde{\Gamma}_\infty \backslash \tilde{\Gamma}$ and using the transformation law for $\Xi(\tau, \mu, \alpha, \beta)$ of Theorem 4.4, we obtain the assertion. □

According to Proposition 4.3, the function $\Xi(\tau, 0, 0, 0)$ in Theorem 4.8 vanishes when $\Delta \neq 1$. When $\Delta = 1$ we have

$$\Xi(\tau, 0, 0, 0) = \sum_{\lambda \in K'} e(Q(\lambda)\tau) \mathfrak{e}_\lambda,$$

and so $\Xi(\tau, 0, 0, 0)$ is the usual vector valued holomorphic theta function of the one-dimensional positive definite lattice K.

Let k_0 be a basis vector for K. If $y \in K \otimes \mathbb{R}$, we write $y > 0$ if y is a positive multiple of k_0. Let $f \in N_{1/2,\tilde{\rho}_L}$. We define the Weyl vector corresponding to f and ℓ to be the unique $\rho_{f,\ell} \in K' \otimes \mathbb{R}$ such that

$$(\rho_{f,\ell}, y) = \frac{\sqrt{(y,y)}}{8\pi\sqrt{2|\Delta|}} \int_{\mathcal{F}}^{reg} \langle f(\tau), \Xi(\tau, 0, 0, 0)\rangle v^{1/2} \frac{du\, dv}{v^2} \tag{4.8}$$

for all $y \in K \otimes \mathbb{R}$ with $y > 0$. Here $\mathcal{F}$ denotes the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, and the integral has to be regularized as in [Bo1]. We have, $\rho_{f,\ell} = 0$ when $\Delta \neq 1$. (This is also true for cusps given by primitive isotropic vectors ℓ with $(\ell, L) \neq \mathbb{Z}$.) One can show that $\rho_{f,\ell}$ does not depend on the choice of the vector ℓ'. The sign of $\rho_{f,\ell}$ depends on the choice of k_0.

We conclude this section with an important fact on the rationality of Weyl vectors $\rho_{f,\ell}$.

Proposition 4.9. *Let $f \in N_{1/2,\tilde{\rho}_L}$ be a harmonic weak Maass form with coefficients $c^{\pm}(m,h)$ as in (2.5). If $c^{+}(m,h) \in \mathbb{Q}$ for all $m \leq 0$ and f is orthogonal to weight $1/2$ cusp forms, then $\rho_{f,\ell} \in \mathbb{Q}$.*

Proof. The idea of the proof is similar to §9 of [Bo1]. But notice that Lemma 9.5 of [Bo1] is actually only true if N is a prime. Therefore we need some additional care. Since $\rho_{f,\ell} = 0$ when $\Delta \neq 1$, we only need to consider the case $\Delta = 1$. Let $E_{3/2}(\tau)$ be the weight $3/2$ Eisenstein series for $\tilde{\Gamma}$ with representation $\bar{\rho}_L$ normalized to have constant term $\mathfrak{e}_0$. It turns out that

$$\xi_{3/2}(E_{3/2})(\tau) = C\frac{\sqrt{N}}{16\pi}\Xi(\tau, 0, 0, 0) + s(\tau),$$

where C is a non-zero rational constant and $s \in S_{1/2,\rho_L}$ is a cusp form. Hence the integral in (4.8) can be computed by means of (2.10) in terms of the coefficients of the holomorphic part of the Eisenstein series $E_{3/2}$. These coefficients are known to be generalized class numbers and thereby rational. Since f is orthogonal to cusp forms, the cusp form s does not give any contribution to the integral. This concludes the proof of the proposition. □

Remark 9. Proposition 4.9 does not hold without the hypothesis that f is orthogonal to weight $1/2$ cusp forms.

5. Regularized theta lifts of weak Maass forms

In this section we generalize the regularized theta lift of Borcherds, Harvey, and Moore in two ways to construct automorphic forms on modular curves. First, we work with the twisted Siegel theta functions of the previous section as kernel functions, and secondly, we consider the lift for harmonic weak Maass forms.

Such generalizations have been studied previously in other settings. In [Ka], Kawai constructed twisted theta lifts of weakly holomorphic modular forms in a different way. However, his automorphic products are of higher level, and only twists by *even* real Dirichlet characters are considered. In [Br] and [BF], the (untwisted) regularized theta lift was studied on harmonic weak Maass forms and was used to construct automorphic Green functions and harmonic square integrable representatives for the Chern classes of Heegner divisors. However, the Chern class construction only leads to non-trivial information about

Heegner divisors if the modular variety under consideration has dimension ≥ 2 (i.e. for $\mathrm{O}(2,n)$ with $n \geq 2$). Here we consider the $\mathrm{O}(2,1)$-case of modular curves. The Chern class of a divisor on a curve is just its degree, and does not contain much arithmetic information. Hence, the approach of [Br] and [BF] to study Heegner divisors does not apply.

Instead of using automorphic Green functions to construct the Chern classes of Heegner divisors, we employ them to construct canonical differentials of the third kind associated to twisted Heegner divisors. By the results of Waldschmidt and Scholl of Section 3, such differentials carry valuable arithmetic information. We begin by fixing some notation.

As in Section 4, let Δ be a fundamental discriminant and let $r \in \mathbb{Z}$ such that $\Delta \equiv r^2 \pmod{4N}$. We let $\ell, \ell' \in L$ be the isotropic vectors

$$\ell = \begin{pmatrix} 0 & 1/N \\ 0 & 0 \end{pmatrix}, \qquad \ell' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then we have $K = \mathbb{Z}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. For $\lambda \in K \otimes \mathbb{R}$, we write $\lambda > 0$ if λ is a positive multiple of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Following Section 13 of [Bo1], and Section 3.2 of [Br], we identify the complex upper half plane $\mathbb{H}$ with an open subset of $K \otimes \mathbb{C}$ by mapping $t \in \mathbb{H}$ to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \otimes t$. Moreover, we identify $\mathbb{H}$ with the Grassmannian $\mathrm{Gr}(V)$ by mapping $t \in \mathbb{H}$ to the positive definite subspace

$$z(t) = \mathbb{R}\Re \begin{pmatrix} t & -t^2 \\ 1 & -t \end{pmatrix} + \mathbb{R}\Im \begin{pmatrix} t & -t^2 \\ 1 & -t \end{pmatrix}$$

of $V(\mathbb{R})$. Under this identification, the action of $\mathrm{Spin}(V)$ on $\mathbb{H}$ by fractional linear transformations corresponds to the linear action on $\mathrm{Gr}(V)$ through $\mathrm{SO}(V)$. We have that

$$\begin{aligned} \ell_z^2 &= \frac{1}{2N\Im(t)^2}, \\ (\lambda, \mu) &= (\lambda, \Re(t)), \\ \lambda^2/\ell_z^2 &= (\lambda, \Im(t))^2, \end{aligned}$$

for $\lambda \in K \otimes \mathbb{R}$. In the following we will frequently identify t and $z(t)$ and simply write z for this variable. We let $z = x + iy$ be the decomposition into real and imaginary part.

We now define twisted Heegner divisors on the modular curve $X_0(N)$. For any vector $\lambda \in L'$ of negative norm, the orthogonal complement $\lambda^\perp \subset V(\mathbb{R})$ defines a point $H(\lambda)$ in $\mathrm{Gr}(V) \cong \mathbb{H}$. For $h \in L'/L$ and a negative rational number $m \in \mathbb{Z} + \mathrm{sgn}(\Delta)Q(h)$, we consider the twisted Heegner divisor

$$H_{\Delta,r}(m,h) := \sum_{\lambda \in L_{d\Delta,hr}/\Gamma_0(N)} \frac{\chi_\Delta(\lambda)}{w(\lambda)} H(\lambda) \in \mathrm{Div}(X_0(N))_{\mathbb{Q}}, \tag{5.1}$$

where $d := 4Nm\,\mathrm{sgn}(\Delta) \in \mathbb{Z}$. Note that d is a discriminant which is congruent to a square modulo $4N$ and which has the opposite sign as Δ. Here $w(\lambda)$ is the order of the stabilizer of λ in $\Gamma_0(N)$. (So $w(\lambda) \in \{2,4,6\}$, and $w(\lambda) = 2$ when $d\Delta < -4$.) We also consider the degree zero divisor

$$y_{\Delta,r}(m,h) := H_{\Delta,r}(m,h) - \deg(H_{\Delta,r}(m,h)) \cdot \infty. \tag{5.2}$$

We have $y_{\Delta,r}(f) = H_{\Delta,r}(f)$ when $\Delta \neq 1$. By the theory of complex multiplication, the divisor $H_{\Delta,r}(h,m)$ is defined over $\mathbb{Q}(\sqrt{D},\sqrt{\Delta})$ (for example, see §12 of [Gro1]). The following lemma shows that it is defined over $\mathbb{Q}(\sqrt{\Delta})$ and summarizes some further properties.

Lemma 5.1. *Let w_N be the Fricke involution on $X_0(N)$, and let τ denote complex conjugation, and let σ be the non-trivial automorphism of $\mathbb{Q}(\sqrt{D},\sqrt{\Delta})/\mathbb{Q}(\sqrt{D})$. Then the following are true:*

(i) $w_N(H_{\Delta,r}(m,h)) = H_{\Delta,r}(m,-h)$,
(ii) $\tau(H_{\Delta,r}(m,h)) = H_{\Delta,r}(m,-h)$,
(iii) $\sigma(H_{\Delta,r}(m,h)) = -H_{\Delta,r}(m,h)$,
(iv) $H_{\Delta,r}(m,-h) = \operatorname{sgn}(\Delta)H_{\Delta,r}(m,h)$,
(v) $H_{\Delta,r}(m,h)$ *is defined over* $\mathbb{Q}(\sqrt{\Delta})$.

Proof. Properties (i) and (ii) are verified by a straightforward computation, and (iv) immediately follows from the definition of the genus character χ_Δ. Moreover, (iii) follows from the theory of complex multiplication (see p. 15 of [BS], and [Gro1]). Finally, (v) is a consequence of (ii), (iii), (iv). $\square$

Remark 10. Our definition of Heegner divisors differs slightly from [GKZ]. They consider the orthogonal complements of vectors $\lambda = \left(\begin{smallmatrix} b/2N & -a/N \\ c & -b/2N \end{smallmatrix}\right) \in L'$ of negative norm with $a > 0$, or equivalently, zeros in the upper half plane of *positive definite* binary quadratic forms.

Recall that $\tilde\rho_L = \rho_L$ for $\Delta > 0$, and $\tilde\rho_L = \bar\rho_L$ for $\Delta < 0$. Let $f \in N_{1/2,\tilde\rho_L}$ be a harmonic weak Maass form of weight $1/2$ with representation $\tilde\rho_L$. We denote the coefficients of $f = f^+ + f^-$ by $c^\pm(m,h)$ as in (2.5). Note that $c^\pm(m,h) = 0$ unless $m \in \mathbb{Z} + \operatorname{sgn}(\Delta)Q(h)$. Moreover, by means of (2.3) we see that $c^\pm(m,h) = c^\pm(m,-h)$ if $\Delta > 0$, and $c^\pm(m,h) = -c^\pm(m,-h)$ if $\Delta < 0$. Throughout we assume that $c^+(m,h) \in \mathbb{R}$ for all m and h.

Using the Fourier coefficients of the principal part of f, we define the twisted Heegner divisor associated to f by

$$H_{\Delta,r}(f) := \sum_{h\in L'/L}\sum_{m<0} c^+(m,h)H_{\Delta,r}(m,h) \in \operatorname{Div}(X_0(N))_{\mathbb{R}}, \tag{5.3}$$

$$y_{\Delta,r}(f) := \sum_{h\in L'/L}\sum_{m<0} c^+(m,h)y_{\Delta,r}(m,h) \in \operatorname{Div}(X_0(N))_{\mathbb{R}}. \tag{5.4}$$

Notice that $y_{\Delta,r}(f) = H_{\Delta,r}(f)$ when $\Delta \neq 1$. The divisors lie in $\operatorname{Div}(X_0(N))_{\mathbb{Q}}$ if the coefficients of the principal part of f are rational.

We define a regularized theta integral of f by

$$\Phi_{\Delta,r}(z,f) = \int_{\tau\in\mathcal{F}}^{reg} \langle f(\tau), \Theta_{\Delta,r}(\tau,z)\rangle v^{1/2}\,\frac{du\,dv}{v^2}. \tag{5.5}$$

Here $\mathcal{F}$ denotes the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, and the integral has to be regularized as in [Bo1].

Proposition 5.2. *The theta integral $\Phi_{\Delta,r}(z,f)$ defines a $\Gamma_0(N)$-invariant function on $\mathbb{H}\backslash H_{\Delta,r}(f)$ with a logarithmic singularity*[1] *on the divisor $-4H_{\Delta,r}(f)$. If Ω denotes the invariant Laplace operator on $\mathbb{H}$, we have*

$$\Omega\Phi_{\Delta,r}(z,f) = \left(\frac{\Delta}{0}\right) c^+(0,0).$$

Proof. Using the argument of Section 6 of [Bo1], one can show that $\Phi_{\Delta,r}(z,f)$ defines a $\Gamma_0(N)$-invariant function on $\mathbb{H}\backslash H_{\Delta,r}(f)$ with a logarithmic singularity on $-4H_{\Delta,r}(f)$. To prove the claim concerning the Laplacian, one may argue as in Theorem 4.6 of [Br]. □

Remark 11. The fact that the function $\Phi_{\Delta,r}(z,f)$ is subharmonic implies that it is real analytic on $\mathbb{H}\backslash H_{\Delta,r}(f)$ by a standard regularity theorem for elliptic differential operators.

We now describe the Fourier expansion of $\Phi_{\Delta,r}(z,f)$. Recall the definition of the Weyl vector $\rho_{f,\ell}$ corresponding to f and ℓ, see (4.8).

Theorem 5.3. *For $z\in\mathbb{H}$ with $y \gg 0$, we have*

$$\Phi_{\Delta,r}(z,f) = -4\sum_{\substack{\lambda\in K'\\ \lambda>0}}\sum_{b\ (\Delta)}\left(\frac{\Delta}{b}\right) c^+(|\Delta|\lambda^2/2, r\lambda)\log|1-e((\lambda,z)+b/\Delta)| + \begin{cases} 2\sqrt{\Delta}c^+(0,0)L(1,\chi_\Delta), & \text{if } \Delta\neq 1,\\ 8\pi(\rho_{f,\ell},y) - c^+(0,0)(\log(4\pi N y^2)+\Gamma'(1)), & \text{if } \Delta = 1.\end{cases} \tag{5.6}$$

Proof. Here we carry out the proof only in the case $\Delta\neq 1$, for which the regularization is slightly easier and there is no Weyl vector term. We note that when $\Delta = 1$, the proof is similar, and when f is weakly holomorphic it is contained in Theorem 13.3 of [Bo1]. In our proof we essentially follow the argument of Theorem 7.1 of [Bo1], and Theorem 2.15 of [Br]. In particular, all questions regarding convergence can be treated analogously. Inserting the formula of Theorem 4.8 in definition (5.5) and unfolding, we obtain

$$\Phi_{\Delta,r}(z,f) = \frac{\sqrt{2}}{\sqrt{|\Delta|\ell_z^2}}\sum_{n\geq 1}\int_{v=0}^{\infty}\int_{u=0}^{1}\exp\left(-\frac{\pi n^2}{2|\Delta|v\ell_z^2}\right)\langle f(\tau),\Xi(\tau,\mu,n,0)\rangle\, du\,\frac{dv}{v^{3/2}}. \tag{5.7}$$

Here we have also used the fact that $\rho_{f,\ell} = 0$ when $\Delta\neq 1$. We temporarily denote the Fourier expansion of f by

$$f(\tau) = \sum_{h\in L'/L}\sum_{n\in\mathbb{Q}} c(n,h,v)e(n\tau).$$

[1]If X is a normal complex space, $D\subset X$ a Cartier divisor, and f a smooth function on $X\setminus\operatorname{supp}(D)$, then f has a logarithmic singularity along D, if for any local equation g for D on an open subset $U\subset X$, the function $f-\log|g|$ is smooth on U.

Inserting the formula for $\Xi(\tau,\mu,n,0)$ of Proposition 4.3 in (5.7), and carrying out the integration over u, we obtain

$$\Phi_{\Delta,r}(z,f) = \frac{\sqrt{2}\varepsilon}{|\ell_z|}\sum_{h\in K'/K}\sum_{\substack{\lambda\in K+rh\\ Q(\lambda)\equiv\Delta Q(h)\ (\Delta)}}\sum_{n\geq1}\left(\frac{\Delta}{n}\right)e\left(\frac{n}{|\Delta|}(\lambda,\mu)\right)$$
$$\times\int_{v=0}^{\infty}c(Q(\lambda)/|\Delta|,h,v)\exp\left(-\frac{\pi n^2}{2|\Delta|v\ell_z^2}-\frac{2\pi\lambda^2 v}{|\Delta|}\right)\frac{dv}{v^{3/2}}. \tag{5.8}$$

Since Δ is fundamental, the conditions $Q(\lambda)\equiv\Delta Q(h)\pmod{\Delta}$ and $\lambda\equiv rh\pmod{K}$ are equivalent to $\lambda=\Delta\lambda'$ and $r\lambda'\equiv h\pmod{K}$ for some $\lambda'\in K'$. Consequently, we have

$$\Phi_{\Delta,r}(z,f) = \frac{\sqrt{2}\varepsilon}{|\ell_z|}\sum_{\lambda\in K'}\sum_{n\geq1}\left(\frac{\Delta}{n}\right)e\left(\operatorname{sgn}(\Delta)n(\lambda,\mu)\right)$$
$$\times\int_{v=0}^{\infty}c(|\Delta|\lambda^2/2,r\lambda,v)\exp\left(-\frac{\pi n^2}{2|\Delta|v\ell_z^2}-2\pi\lambda^2|\Delta|v\right)\frac{dv}{v^{3/2}}. \tag{5.9}$$

Notice that only the coefficients $c(|\Delta|\lambda^2/2,r\lambda,v)$ where $\lambda\in K'$ occur in the latter formula. Since K is positive definite, the quantity $|\Delta|\lambda^2/2$ is non-negative, and so we have

$$c(|\Delta|\lambda^2/2,r\lambda,v)=c^+(|\Delta|\lambda^2/2,r\lambda),$$

that is, only the coefficients of the "holomorphic part" f^+ of f give a contribution. We now compute the integral over v (for example, using page 77 of [Br]). We obtain

$$\int_{v=0}^{\infty}\exp\left(-\frac{\pi n^2}{2|\Delta|v\ell_z^2}-2\pi\lambda^2|\Delta|v\right)\frac{dv}{v^{3/2}}=\frac{\sqrt{2|\Delta|\ell_z^2}}{n}\exp(-2\pi n|\lambda|/|\ell_z|).$$

Inserting this and separating the contribution of $\lambda=0$, we get

$$\Phi_{\Delta,r}(z,f)=2\sqrt{\Delta}c^+(0,0)\sum_{n\geq1}\left(\frac{\Delta}{n}\right)\frac{1}{n}$$
$$+4\sum_{\substack{\lambda\in K'\\ \lambda>0}}c^+(|\Delta|\lambda^2/2,r\lambda)\Re\left(\sqrt{\Delta}\sum_{n\geq1}\frac{1}{n}\left(\frac{\Delta}{n}\right)e\big(\operatorname{sgn}(\Delta)n(\lambda,\mu)+in|\lambda|/|\ell_z|\big)\right).$$

Using the value of the Gauss sum (4.7), we see that this is equal to

$$\Phi_{\Delta,r}(z,f)=2\sqrt{\Delta}c^+(0,0)L(1,\chi_\Delta)$$
$$-4\sum_{\substack{\lambda\in K'\\ \lambda>0}}\sum_{b\ (\Delta)}\left(\frac{\Delta}{b}\right)c^+(|\Delta|\lambda^2/2,r\lambda)\log\left|1-e\left(\frac{b}{\Delta}+(\lambda,\mu)+i\frac{|\lambda|}{|\ell_z|}\right)\right|.$$

We finally put in the identities $(\lambda,\mu)=(\lambda,x)$ and $|\lambda|/|\ell_z|=|(\lambda,y)|$, to derive the theorem. □

Remark 12. i) Note that for lattices of signature $(2,n)$ with $n\geq 2$, the lattice K is Lorentzian, and one gets a non-trivial contribution from f^- to the theta integral, which is investigated in [Br]. So the above situation is very special.

ii) At the other cusps of $X_0(N)$, the function $\Phi_{\Delta,r}(z,f)$ has similar Fourier expansions as in (5.6).

iii) The function $\Phi_{\Delta,r}(z,f)$ is a Green function for the divisor $H_{\Delta,r}(f)+C_{\Delta,r}(f)$ in the sense of [BKK], [BBK]. Here $C_{\Delta,r}(f)$ is a divisor on $X_0(N)$ supported at the cusps, see also (5.12)

5.1. **Canonical differentials of the third kind for Heegner divisors.** For the rest of this section, we assume that $f\in N_{1/2,\tilde\rho_L}$ and that the coefficients $c^+(m,h)$ are rational for all $m\leq 0$ and $h\in L'/L$. Moreover, we assume that the constant term $c^+(0,0)$ of f vanishes when $\Delta=1$, so that $\Phi_{\Delta,r}(z,f)$ is harmonic. We identify $\mathbb{Z}$ with K' by mapping $n\in\mathbb{Z}$ to $\frac{n}{2N}\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. Then the Fourier expansion of $\Phi_{\Delta,r}(z,f)$ given in Theorem 5.3 becomes

$$\begin{aligned}\Phi_{\Delta,r}(z,f) &= 2\sqrt{\Delta}c^+(0,0)L(1,\chi_\Delta)+8\pi\rho_{f,\ell}y\\ &\quad -4\sum_{n\geq 1}\sum_{b\,(\Delta)}\left(\frac{\Delta}{b}\right)c^+\big(\tfrac{|\Delta|n^2}{4N},\tfrac{rn}{2N}\big)\log|1-e(nz+b/\Delta)|\,.\end{aligned}\tag{5.10}$$

It follows from Proposition 5.2 that

$$\eta_{\Delta,r}(z,f)=-\frac{1}{2}\partial\Phi_{\Delta,r}(z,f)\tag{5.11}$$

is a differential of the third kind on $X_0(N)$. It has the residue divisor

$$\operatorname{res}(\eta_{\Delta,r}(z,f))=H_{\Delta,r}(f)+C_{\Delta,r}(f),\tag{5.12}$$

where $H_{\Delta,r}(f)\in\operatorname{Div}(X_0(N))_\mathbb{Q}$, and $C_{\Delta,r}(f)\in\operatorname{Div}(X_0(N))_\mathbb{R}$ is a divisor on $X_0(N)$ which is supported at the cusps. (Here we have relaxed the condition that the residues be integral, and only require them to be real.) The multiplicity of any cusp ℓ in the divisor $C_{\Delta,r}(f)$ is given by the Weyl vector $\rho_{f,\ell}$. According to Proposition 4.9, if f is orthogonal to the cusp forms in $S_{1/2,\tilde\rho_L}$ then $\rho_{f,\ell}$ is rational. When $\Delta\neq 1$, all Weyl vectors vanish and consequently $C_{\Delta,r}(f)=0$.

Theorem 5.4. *The differential $\eta_{\Delta,r}(z,f)$ is the canonical differential of the third kind corresponding to $H_{\Delta,r}(f)+C_{\Delta,r}(f)$. It has the Fourier expansion*

$$\eta_{\Delta,r}(z,f)=\left(\rho_{f,\ell}-\operatorname{sgn}(\Delta)\sqrt{\Delta}\sum_{n\geq1}\sum_{d\mid n}\frac{n}{d}\left(\frac{\Delta}{d}\right)c^+\big(\tfrac{|\Delta|n^2}{4Nd^2},\tfrac{rn}{2Nd}\big)e(nz)\right)\cdot 2\pi i\,dz.$$

Proof. Since $\Phi_{\Delta,r}(z,f)$ is harmonic on $\mathbb{H}\backslash H_{\Delta,r}(f)$, Proposition 3.1 implies that $\eta_{\Delta,r}(z,f)$ is the canonical differential of the third kind associated with $H_{\Delta,r}(f)+C_{\Delta,r}(f)$. Differentiating (5.10), we obtain

$$\eta_{\Delta,r}(z,f)=\left(\rho_{f,\ell}-\sum_{n\geq1}\sum_{d\geq1}\sum_{b\,(\Delta)}\left(\frac{\Delta}{b}\right)c^+\big(\tfrac{|\Delta|n^2}{4N},\tfrac{rn}{2N}\big)ne(ndz+bd/\Delta)\right)\cdot 2\pi i\,dz.$$

Inserting the value of the Gauss sum (4.7) and reordering the summation, we get the claimed Fourier expansion. □

Theorem 5.5. *Assume that $\Delta\neq 1$. The following are equivalent.*

(i) *A non-zero multiple of $y_{\Delta,r}(f)$ is the divisor of a rational function on $X_0(N)$.*

(ii) *The coefficients* $c^+(\frac{|\Delta|n^2}{4N}, \frac{rn}{2N})$ *of* f *are algebraic for all positive integers* n.

(iii) *The coefficients* $c^+(\frac{|\Delta|n^2}{4N}, \frac{rn}{2N})$ *of* f *are rational for all positive integers* n.

Proof. Statement (iii) trivially implies (ii). If (ii) holds, then, in view of Theorem 5.4, the canonical differential $\eta_{\Delta,r}(z,f)$ of the divisor $y_{\Delta,r}(f) \in \mathrm{Div}(X_0(N))_{\mathbb{Q}}$ is defined over $\bar{\mathbb{Q}}$. Consequently, Theorem 3.2 implies that a non-zero multiple of $y_{\Delta,r}(f)$ is the divisor of a rational function on $X_0(N)$. Hence (i) holds.

It remains to prove that (i) implies (iii). If $y_{\Delta,r}(z,f)$ is a non-zero multiple of the divisor of a rational function on $X_0(N)$, then Lemma 5.1 and Theorem 3.2 imply that $\eta_{\Delta,r}(z,f)$ is defined over $F = \mathbb{Q}(\sqrt{\Delta})$, the field of definition of $y_{\Delta,r}(f)$. Using the q-expansion principle and Möbius inversion, we deduce from Theorem 5.4, for every positive integer n, that

$$\sqrt{\Delta} n c^+(\tfrac{|\Delta|n^2}{4N}, \tfrac{rn}{2N}) \in F. \tag{5.13}$$

Denote by σ the non-trivial automorphism of $F/\mathbb{Q}$. It follows from Lemma 5.1 that $\sigma(y_{\Delta,r}(f)) = -y_{\Delta,r}(f)$. Hence $\sigma(\eta_{\Delta,r}(z,f)) = -\eta_{\Delta,r}(z,f)$. Using the action of σ on the q-expansion of $\eta_{\Delta,r}(z,f)$, we find that σ fixes the coefficients $c^+(\frac{|\Delta|n^2}{4N}, \frac{rn}{2N})$. Consequently, these coefficients are rational. □

Remark 13. Theorem 5.5 also holds for $\Delta = 1$ when $S_{1/2,\tilde{\rho}_L} = 0$, or more generally when f is orthogonal to the cusp forms in $S_{1/2,\tilde{\rho}_L}$. The latter conditions ensure that the Weyl vectors corresponding to f are rational and thereby $C_{\Delta,r}(f) \in \mathrm{Div}((X_0(N))_{\mathbb{Q}}$. Observe that $H_{\Delta,r}(f) + C_{\Delta,r}(f)$ differs from $y_{\Delta,r}(f)$ only by a divisor of degree 0 supported at the cusps. In particular, by the Manin-Drinfeld theorem, the divisors $H_{\Delta,r}(f) + C_{\Delta,r}(f)$ and $y_{\Delta,r}(f)$ define the same point in $J(\mathbb{Q}) \otimes \mathbb{R}$.

Remark 14. Assume that the equivalent conditions of Theorem 5.5 hold.

i) It is interesting to consider whether the rational coefficients $c^+(\frac{|\Delta|n^2}{4N}, \frac{rn}{2N})$ in (iii) have bounded denominators. This is true if f is weakly holomorphic, since $M^!_{1/2,\tilde{\rho}_L}$ has a basis of modular forms with integral coefficients. However, if f is an honest harmonic weak Maass form, it is not clear at all.

ii) The rational function in Theorem 5.5 (i) has an automorphic product expansion as in Theorem 6.1. It is given by a non-zero power of $\Psi_{\Delta,r}(z,f)$.

6. Generalized Borcherds products

In this section we consider certain automorphic products which arise as liftings of harmonic weak Maass forms and which can be viewed as generalizations of the automorphic products in Theorem 13.3 of [Bo1]. In particular, for any Heegner divisor $H_{\Delta,r}(m,h)$, we obtain a meromorphic automorphic product Ψ whose divisor on $X_0(N)$ is the sum of $H_{\Delta,r}(m,h)$ and a divisor supported at the cusps. But unlike the results in [Bo1], the function Ψ will in general transform with a multiplier system of infinite order under $\Gamma_0(N)$. We then give a criterion when the multiplier system has finite order.

As usual, for complex numbers a and b, we let $a^b = \exp(b \,\mathrm{Log}(a))$, where Log denotes the principal branch of the complex logarithm. In particular, if $|a| < 1$ we have $(1-a)^b = \exp(-b \sum_{n\geq 1} \frac{a^n}{n})$.

Theorem 6.1. *Let $f \in N_{1/2,\tilde{\rho}_L}$ be a harmonic weak Maass form with real coefficients $c^+(m,h)$ for all $m \in \mathbb{Q}$ and $h \in L'/L$. Moreover, assume that $c^+(n,h) \in \mathbb{Z}$ for all $n \leq 0$. The infinite product*

$$\Psi_{\Delta,r}(z,f) = e((\rho_{f,\ell},z)) \prod_{\substack{\lambda \in K' \\ \lambda > 0}} \prod_{b\ (\Delta)} [1 - e((\lambda,z) + b/\Delta)]^{\left(\frac{\Delta}{b}\right) c^+(|\Delta|\lambda^2/2, r\lambda)}$$

converges for y sufficiently large and has a meromorphic continuation to all of $\mathbb{H}$ with the following properties.

(i) *It is a meromorphic modular form for $\Gamma_0(N)$ with a unitary character σ which may have infinite order.*
(ii) *The weight of $\Psi_{\Delta,r}(z,f)$ is $c^+(0,0)$ when $\Delta = 1$, and is 0 when $\Delta \neq 1$.*
(iii) *The divisor of $\Psi_{\Delta,r}(z,f)$ on $X_0(N)$ is given by $H_{\Delta,r}(f) + C_{\Delta,r}(f)$.*
(iv) *We have*

$$\Phi_{\Delta,r}(z,f) = \begin{cases} -c^+(0,0)(\log(4\pi N) + \Gamma'(1)) - 4\log|\Psi_{\Delta,r}(z,f) y^{c^+(0,0)/2}|, & \text{if } \Delta = 1, \\ 2\sqrt{\Delta} c(0,0) L(1,\chi_\Delta) - 4\log|\Psi_{\Delta,r}(z,f)|, & \text{if } \Delta \neq 1. \end{cases}$$

Proof. By means of the same argument as in Section 13 of [Bo1], or Chapter 3 of [Br], it can be deduced from Proposition 5.2, Remark 11, and Theorem 5.3 that $\Psi_{\Delta,r}(z,f)$ has a continuation to a meromorphic function on $\mathbb{H}$ satisfying (iii) and (iv). Moreover, using the $\Gamma_0(N)$-invariance of $\Phi_{\Delta,r}(z,f)$ one finds that it satisfies the transformation law

$$\Psi_{\Delta,r}(\gamma z,f) = \sigma(\gamma)(cz+d)^{c^+(0,0)} \Psi_{\Delta,r}(z,f),$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, where $\sigma : \Gamma_0(N) \to \mathbb{C}^\times$ is a unitary character of $\Gamma_0(N)$. □

Theorem 6.2. *Suppose that $\Delta \neq 1$. Let $f \in N_{1/2,\tilde{\rho}_L}$ be a harmonic weak Maass form with real coefficients $c^+(m,h)$ for all $m \in \mathbb{Q}$ and $h \in L'/L$. Moreover, assume that $c^+(n,h) \in \mathbb{Z}$ for all $n \leq 0$. The following are equivalent.*

(i) *The character σ of the function $\Psi_{\Delta,r}(z,f)$ defined in Theorem 6.1 is of finite order.*
(ii) *The coefficients $c^+(|\Delta|\lambda^2/2, r\lambda)$ are rational for all $\lambda \in K'$.*

Proof. If (i) holds, then there is a positive integer M such that $\Psi_{\Delta,r}(z,f)^M$ is a rational function on $X_0(N)$ with divisor $M \cdot H_{\Delta,r}(f)$. By means of Theorem 5.5 we find that $c^+(|\Delta|\lambda^2/2, r\lambda) \in \mathbb{Q}$ for all $\lambda \in K'$. Conversely, if (ii) holds, using Theorem 5.5 we may conclude that $M \cdot H_{\Delta,r}(f)$ is the divisor of a rational function R on $X_0(N)$ for some positive integer M. But this implies that

$$\log|R| - M \log|\Psi_{\Delta,r}(z,f)|$$

is a harmonic function on $X_0(N)$ (without any singularities). By the maximum principle, it is constant. Hence $R/\Psi_{\Delta,r}(z,f)^M$ is a holomorphic function on $\mathbb{H}$ with constant modulus, which must be constant. Consequently, σ^M is the trivial character. □

Remark 15. Theorem 6.2 also holds for $\Delta = 1$ when $S_{1/2,\tilde{\rho}_L} = 0$, or more generally when f is orthogonal to the cusp forms in $S_{1/2,\tilde{\rho}_L}$. The latter conditions ensure that Theorem 5.5 still applies (see Remark 13). Notice that we may reduce to the case that the constant term $c^+(0,0)$ of f vanishes by adding a suitable rational linear combination of $\mathrm{O}(K'/K)$-translates of the vector valued weight $1/2$ theta series for the lattice K.

The rationality of the coefficients $c^+(m,h)$ of a harmonic weak Maass form is usually not easy to verify. In view of Theorem 6.2, it is related to the vanishing of twisted Heegner divisors in the Jacobian, which is a deep question (see [GZ]). However for special harmonic weak Maass forms, such as the mock theta functions, one can read off the rationality directly from the construction. This leads to explicit relations among certain Heegner divisors on $X_0(N)$ comparable to the relations among cuspidal divisors coming from modular units.

For weakly holomorphic modular forms $f \in M^!_{1/2,\tilde\rho_L}$ the rationality of the Fourier coefficients is essentially dictated by the principal part (with minor complications caused by the presence of cusp forms of weight 1/2). More precisely, one can use the following lemmas.

Lemma 6.3. *Suppose that $f \in M^!_{1/2,\tilde\rho_L}$. If $c^+(m,h) \in \mathbb{Q}$ for all $m \leq 0$, then there exists a cusp form $f' \in S_{1/2,\tilde\rho_L}$ such that $f + f'$ has rational coefficients.*

Proof. This follows from the fact that the spaces $M^!_{1/2,\tilde\rho_L}$ and $S_{1/2,\tilde\rho_L}$ have bases of modular forms with rational coefficients (see [McG]). □

Remark 16. Observe that $M_{1/2,\bar\rho_L}$ is always trivial, by a result of Skoruppa (see Theorem 5.7 of [EZ]). However, $M_{1/2,\rho_L}$ may contain non-zero elements (which are linear combinations of theta series of weight 1/2).

Lemma 6.4. *If $f \in S_{1/2,\rho_L}$, then the coefficient $c^+(m,h)$ of f vanishes unless $m = \lambda^2/2$ for some $\lambda \in K'$.*

Proof. This could be proved using the Serre-Stark basis theorem. Here we give a more indirect proof using the twisted theta lifts of the previous section.

Let $\Delta \neq 1$ be a fundamental discriminant and let r be an integer satisfying $\Delta \equiv r^2 \pmod{4N}$. We consider the canonical differential of the third kind $\eta_{\Delta,r}(z,f)$. Since f is a cusp form and $\Delta \neq 1$, the divisor $H_{\Delta,r}(f)$ vanishes. Consequently, $\eta_{\Delta,r}(z,f) \equiv 0$. By Theorem 5.4, we find that $c^+(|\Delta|\lambda^2/2, r\lambda) = 0$ for all $\lambda \in K'$. This proves the lemma. □

Lemma 6.5. *If $f \in M^!_{1/2,\tilde\rho_L}$, then the following are true:*

(i) *If $c^+(m,h) = 0$ for all $m < 1$, then f vanishes identically.*
(ii) *If $c^+(m,h) \in \mathbb{Q}$ for all $m < 1$, then all coefficients of f are contained in $\mathbb{Q}$.*

Proof. (i) The hypothesis implies that f/Δ is a holomorphic modular form of weight $-23/2$ with representation $\tilde\rho_L$. Hence it has to vanish identically. (ii) The assertion follows from (i) using the Galois action on $M^!_{1/2,\tilde\rho_L}$. □

Remark 17. i) If $f \in M^!_{1/2,\tilde\rho_L}$ is a weakly holomorphic modular form with rational coefficients $c^+(m,h)$ and $c^+(m,h) \in \mathbb{Z}$ for $m \leq 0$, then Theorem 6.2 and Remark 15 show that the automorphic product $\Psi_{\Delta,r}(z,f)$ of Theorem 6.1 is a meromorphic modular form for $\Gamma_0(N)$ with a character of finite order. When $\Delta = 1$, this result is contained in Theorem 13.3 of [Bo1]. Borcherds proved the finiteness of the multiplier system in [Bo3] using the embedding trick. (However, the embedding trick argument does not work for harmonic weak Maass forms.) In the special case that $N = 1$ and $\Delta > 0$, twisted Borcherds products were first constructed by Zagier in a different way (see §7 of [Za2]).

ii) The space of weakly holomorphic modular forms of weight 1/2 with representation ρ_L is isomorphic to the space of weakly skew holomorphic Jacobi forms in the sense of [Sk1]. For these forms, Theorem 6.1 gives the automorphic products $\Psi_{\Delta,r}$ for any *positive* fundamental discriminant Δ. The space of weakly holomorphic modular forms of weight 1/2 with representation $\bar{\rho}_L$ is isomorphic to the space of "classical" weakly holomorphic Jacobi forms in the sense of [EZ]. For these forms, the theorem gives the automorphic products $\Psi_{\Delta,r}$ for any *negative* fundamental discriminant Δ.

Corollary 6.6. *Let F be a number field. If $f \in M^!_{1/2,\tilde{\rho}_L}$ has the property that all of its coefficients lie in F, then the divisor $y_{\Delta,r}(f)$ vanishes in the Jacobian $J(\mathbb{Q}(\sqrt{D})) \otimes_{\mathbb{Z}} F$.*

Proof. The assertion follows from Theorems 6.1 and 6.2, using the Galois action on $M^!_{1/2,\tilde{\rho}_L}$. □

As another corollary, we obtain the following generalization of the Gross-Kohnen-Zagier theorem [GKZ]. It can derived from Corollary 6.6 using Serre duality as in [Bo2].

Corollary 6.7. *Let $\rho = \rho_L$ when $\Delta > 0$, and let $\rho = \bar{\rho}_L$ when $\Delta < 0$. The generating series*

$$A_{\Delta,r}(\tau) = \sum_{h \in L'/L} \sum_{n>0} y_{\Delta,r}(-n,h) q^n \mathfrak{e}_h$$

is a cusp form of weight $3/2$ for $\tilde{\Gamma}$ of type $\bar{\rho}$ with values in $J(\mathbb{Q}(\sqrt{\Delta}))$ (i.e. $A_{\Delta,r} \in S_{3/2,\bar{\rho}} \otimes_{\mathbb{Z}} J(\mathbb{Q}(\sqrt{\Delta}))$). □

7. Hecke eigenforms and isotypical components of the Jacobian

Now we consider the implications of the results of the previous section when the action of the Hecke algebra is introduced. We begin with some notation. Let L be the lattice of discriminant $2N$ defined in Section 2.4. Let $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. The space of vector valued holomorphic modular forms M_{k,ρ_L} is isomorphic to the space of skew holomorphic Jacobi forms $J^{skew}_{k+1/2,N}$ of weight $k + 1/2$ and index N. Moreover, $M_{k,\bar{\rho}_L}$ is isomorphic to the space of holomorphic Jacobi forms $J_{k+1/2,N}$. There is an extensive Hecke theory for Jacobi forms (see [EZ], [Sk1], [SZ]), which gives rise to a Hecke theory on M_{k,ρ_L} and $M_{k,\bar{\rho}_L}$, and which is compatible with the Hecke theory on vector valued modular forms considered in [BrSt]. In particular, there is an Atkin-Lehner theory for these spaces.

The subspace S^{new}_{k,ρ_L} of newforms of S_{k,ρ_L} is isomorphic as a module over the Hecke algebra to the space of newforms $S^{new,+}_{2k-1}(N)$ of weight $2k-1$ for $\Gamma_0(N)$ on which the Fricke involution acts by multiplication with $(-1)^{k-1/2}$. The isomorphism is given by the Shimura correspondence. Similarly, the subspace $S^{new}_{k,\bar{\rho}_L}$ of newforms of $S_{k,\bar{\rho}_L}$ is isomorphic as a module over the Hecke algebra to the space of newforms $S^{new,-}_{2k-1}(N)$ of weight $2k-1$ for $\Gamma_0(N)$ on which the Fricke involution acts by multiplication with $(-1)^{k+1/2}$ (see [SZ], [GKZ], [Sk1]). Observe that the Hecke L-series of any $G \in S^{new,\pm}_{2k-1}(N)$ satisfies a functional equation under $s \mapsto 2k-1-s$ with root number ± 1. If $G \in S^{new,\pm}_{2k-1}(N)$ is a normalized newform (in particular a common eigenform of all Hecke operators), we denote by F_G the number field generated by the Hecke eigenvalues of G. It is well known that we may normalize the preimage of G under the Shimura correspondence such that all its Fourier coefficients are contained in F_G.

Let ρ be one of the representations ρ_L or $\bar{\rho}_L$. For every positive integer l there is a Hecke operator $T(l)$ on $M_{k,\rho}$ which is self adjoint with respect to the Petersson scalar product. The action on the Fourier expansion $g(\tau) = \sum_{h,n} b(n,h)e(m\tau)\mathfrak{e}_h$ of any $g \in M_{k,\rho}$ can be described explicitly (for example, see §4 of [EZ], or §0 (5) of [SZ]). For instance, if p is a prime coprime to N and we write $g \mid_k T(p) = \sum_{h,n} b^*(n,h)e(n\tau)\mathfrak{e}_h$, we have

$$b^*(n,h) = b(p^2n, ph) + p^{k-3/2}\left(\frac{4N\sigma n}{p}\right) b(n,h) + p^{2k-2}b(n/p^2, h/p), \tag{7.1}$$

where $\sigma = 1$ if $\rho = \rho_L$, and $\sigma = -1$ if $\rho = \bar{\rho}_L$. There are similar formulas for general l.

The Hecke operators act on harmonic weak Maass forms and on weakly holomorphic modular forms in an analogous way. In particular, the formula for the action on Fourier coefficients is the same. In the following we assume that $k \leq 1/2$. Recall from Section 2.2 that there is a bilinear pairing $\{\cdot,\cdot\}$ between the spaces $S_{2-k,\bar{\rho}}$ and $N_{k,\rho}$, which induces a non-degenerate pairing of $S_{2-k,\bar{\rho}}$ and $N_{k,\rho}/M^!_{k,\rho}$.

Proposition 7.1. *The Hecke operator $T(l)$ is (up to a scalar factor) self adjoint with respect to the pairing $\{\cdot,\cdot\}$. More precisely we have*

$$\{g, f \mid_k T(l)\} = l^{2k-2}\{g \mid_{2-k} T(l), f\}$$

for any $g \in S_{2-k,\bar{\rho}}$ and $f \in N_{k,\rho}$.

Proof. From the definition of the Hecke operator or from its action on the Fourier expansion of f one sees that

$$\xi_k(f \mid_k T(l)) = l^{2k-2}(\xi_k f) \mid_{2-k} T(l). \tag{7.2}$$

Using the self-adjointness of $T(l)$ with respect to the Petersson scalar product, we have

$$\begin{aligned}\{g, f \mid_k T(l)\} &= \big(g,\, \xi_k(f \mid T(l))\big)\\ &= l^{2k-2}\big(g,\, (\xi_k f) \mid T(l)\big)\\ &= l^{2k-2}\big(g \mid T(l),\, \xi_k f\big)\\ &= l^{2k-2}\{g \mid T(l),\, f\}.\end{aligned}$$

This proves the proposition. □

Recall that (see [McG]) the space $S_{2-k,\bar{\rho}}$ has a basis of cusp forms with rational coefficients. Let F be any subfield of $\mathbb{C}$. We denote by $S_{2-k,\bar{\rho}}(F)$ the F-vector space of cusp forms in $S_{2-k,\bar{\rho}}$ with Fourier coefficients in F. Moreover, we write $N_{k,\rho}(F)$ for the F-vector space of harmonic weak Maass forms whose principal part has coefficients in F. We write $M^!_{k,\rho}(F)$ for the subspace of weakly holomorphic modular forms whose principal part has coefficients in F. Using the pairing $\{\cdot,\cdot\}$, we identify $N_{k,\rho}(F)/M^!_{k,\rho}(F)$ with the F-dual of $S_{2-k,\bar{\rho}}(F)$.

Lemma 7.2. *Let $g \in S_{2-k,\bar{\rho}}$, and suppose that $f \in N_{k,\rho}$ has the property that $\{g, f\} = 1$, and also satisfies $\{g', f\} = 0$ for all $g' \in S_{2-k,\bar{\rho}}$ orthogonal to g. Then $\xi_k(f) = \|g\|^{-2} g$, where $\|g\|$ denotes the Petersson norm of g.*

Proof. This follows directly from the definition of the pairing. □

Lemma 7.3. *Let F be a subfield of $\mathbb{C}$, and let $g \in S_{2-k,\bar{\rho}}(F)$ be a newform. There is a $f \in N_{k,\rho}(F)$ such that*

$$\xi_k(f) = \|g\|^{-2} g.$$

Proof. Since $g \in S_{2-k,\bar{\rho}}(F)$ is a newform, the orthogonal complement of g with respect to the Petersson scalar product has a basis consisting of cusp forms with coefficients in F. Let $g_2, \dots, g_d \in S_{2-k,\bar{\rho}}(F)$ be a basis of the orthogonal complement of g. Let $f_1, \dots, f_d \in N_{k,\rho}(F)$ be the dual basis of the basis $g, g_2, \dots, g_d$ with respect to $\{\cdot, \cdot\}$. In particular $\{g, f_1\} = 1$, and $\{g, f_j\} = 0$ for all $j = 2, \dots, d$. According to Lemma 7.2 we have that $\xi_k(f_1) = \|g\|^{-2} g$. This completes the proof of the lemma. □

Lemma 7.4. *Let $f \in N_{k,\rho}(F)$ and assume that $\xi_k(f) \mid_{2-k} T(l) = \lambda_l \xi_k(f)$ with $\lambda_l \in F$. Then*

$$f \mid_k T(l) - l^{2k-2} \lambda_l f \in M^!_{k,\rho}(F).$$

Proof. The formula for the action of $T(l)$ on the Fourier expansion implies that $f \mid_k T(l) \in N_{k,\rho}(F)$. Moreover, it follows from (7.2) that

$$\xi_k \left(f \mid_k T(l) - l^{2k-2} \lambda_l f \right) = 0.$$

This proves the lemma. □

We now come to the main results of this section. Let $G \in S_2^{new}(N)$ be a normalized newform of weight 2 and write F_G for the number field generated by the eigenvalues of G. If $G \in S_2^{new,-}(N)$, we put $\rho = \rho_L$ and assume that Δ is a positive fundamental discriminant. If $G \in S_2^{new,+}(N)$, we put $\rho = \bar{\rho}_L$ and assume that Δ is a negative fundamental discriminant. There is a newform $g \in S^{new}_{3/2,\bar{\rho}}$ mapping to G under the Shimura correspondence. We may normalize g such that all its coefficients are contained in F_G. Therefore by Lemma 7.3, there is a harmonic weak Maass form $f \in N_{1/2,\rho}(F_G)$ such that

$$\xi_{1/2}(f) = \|g\|^{-2} g.$$

This form is unique up to addition of a weakly holomorphic form in $M^!_{1/2,\rho}(F_G)$.

Theorem 7.5. *The divisor $y_{\Delta,r}(f) \in \operatorname{Div}(X_0(N)) \otimes F_G$ defines a point in the G-isotypical component of the Jacobian $J(\mathbb{Q}(\sqrt{\Delta})) \otimes \mathbb{C}$.*

Proof. We write λ_l for eigenvalue of the Hecke operator $T(l)$ corresponding to G (where $l \in \mathbb{Z}_{>0}$). Let p be any prime coprime to N. It suffices to show that under the action of the Hecke algebra on the Jacobian we have

$$T(p) y_{\Delta,r}(f) = \lambda_p y_{\Delta,r}(f).$$

It is easily seen that

$$T(p)y_{\Delta,r}(m,h) = y_{\Delta,r}(p^2m,ph) + \left(\frac{4N\sigma m}{p}\right) y_{\Delta,r}(m,h) + py_{\Delta,r}(m/p^2,h/p), \tag{7.3}$$

where $m \in \mathbb{Q}$ is negative, $h \in L'/L$, and $\sigma = \operatorname{sgn}(\Delta)$. (for example, see p. 507 and p. 542 of [GKZ] for the case $\Delta = 1$. The general case is analogous.) Combining this with (7.1), we see that

$$T(p)y_{\Delta,r}(f) = py_{\Delta,r}(f\mid_{1/2} T(p)). \tag{7.4}$$

In view of Lemma 7.4, there is a $f' \in M^!_{1/2,\rho}(F_G)$ such that

$$f\mid_{1/2} T(p) = p^{-1}\lambda_p f + f'.$$

Combining this with (7.4), we find that

$$T(p)y_{\Delta,r}(f) = \lambda_p y_{\Delta,r}(f) + py_{\Delta,r}(f').$$

But Lemma 6.3 and Corollary 6.6 imply that $y_{\Delta,r}(f')$ vanishes in $J(\mathbb{Q}(\sqrt{\Delta}))\otimes\mathbb{C}$. □

Theorem 7.6. *Assume that $\Delta \neq 1$. Let $f \in N_{1/2,\rho}(F_G)$ be a weak Maass form such that $\xi_{1/2}(f)$ is a newform which maps to $G \in S_2^{new}(N)$ under the Shimura correspondence. Denote the Fourier coefficients of f^+ by $c^+(m,h)$. Then the following are equivalent:*

(i) *The Heegner divisor $y_{\Delta,r}(f)$ vanishes in $J(\mathbb{Q}(\sqrt{\Delta}))\otimes\mathbb{C}$.*
(ii) *The coefficient $c^+(\frac{|\Delta|}{4N},\frac{r}{2N})$ is algebraic.*
(iii) *The coefficient $c^+(\frac{|\Delta|}{4N},\frac{r}{2N})$ is contained in F_G.*

Proof. If (i) holds, then Theorem 5.5 implies that the coefficients $c^+(\frac{|\Delta|n^2}{4N},\frac{rn}{2N})$ are in F_G for all positive integers n. Hence (iii) holds. Moreover, it is clear that (iii) implies (ii).

Now we show that (ii) implies (i). If $y_{\Delta,r}(f) \neq 0$ is in $J(\mathbb{Q}(\sqrt{\Delta}))\otimes\mathbb{C}$, then Theorem 5.5 implies that there are positive integers n for which $c^+(\frac{|\Delta|n^2}{4N},\frac{rn}{2N})$ is transcendental. Let n_0 be the smallest of these integers. We need to show that $n_0 = 1$.

Assume that $n_0 \neq 1$, and that $p \mid n_0$ is prime. Let λ_p be the eigenvalue of the Hecke operator $T(p)$ corresponding to G. By Lemma 7.4, there is a $f' \in M^!_{1/2,\rho}(F_G)$ such that

$$f\mid_{1/2} T(p) = p^{-1}\lambda_p f + f'.$$

Using the formula for the action of $T(p)$ on the Fourier expansion of f, we see that $c^+(\frac{|\Delta|n_0^2}{4N},\frac{rn_0}{2N})$ is an algebraic linear combination of Fourier coefficients $c^+(\frac{|\Delta|n_1^2}{4N},\frac{r_1n_1}{2N})$ of f with $n_1 \leq n_0/p$ and coefficients of f'. In view of Lemma 6.3 and Lemma 6.4, this implies that $c^+(\frac{|\Delta|n_0^2}{4N},\frac{rn_0}{2N})$ is algebraic, contradicting our assumption. □

Remark 18. Theorem 7.6 also holds for $\Delta = 1$ when $S_{1/2,\rho} = 0$. More generally, it should also hold for $\Delta = 1$ when f is chosen to be orthogonal to the cusp forms in $S_{1/2,\rho}$. The latter condition ensures that Theorem 5.5 still applies, see Remark 13. However, it remains to show that a weakly holomorphic form $f' \in M^!_{1/2,\rho}(F_G)$ which is orthogonal to cusp forms, automatically has *all* coefficients in F_G.

Using the action of the Hecke algebra on the Jacobian, we may derive a more precise version of Corollary 6.7. Let $y^G_{\Delta,r}(m,h)$ denote the projection of the Heegner divisor $y_{\Delta,r}(m,h)$ onto its G-isotypical component. We consider the generating series

$$A^G_{\Delta,r}(\tau) = \sum_{h\in L'/L}\sum_{n>0} y^G_{\Delta,r}(-n,h)q^n\mathfrak{e}_h \in S_{3/2,\bar\rho}\otimes_{\mathbb{Z}} J(\mathbb{Q}(\sqrt{\Delta})).$$

Theorem 7.7. *We have the identity*

$$A^G_{\Delta,r}(\tau) = g(\tau)\otimes y_{\Delta,r}(f).$$

In particular, the space in $(J(\mathbb{Q}(\sqrt{\Delta}))\otimes\mathbb{C})^G$ spanned by the $y^G_{\Delta,r}(m,h)$ is at most one-dimensional and is generated by $y_{\Delta,r}(f)$.

Proof. We write λ_l for eigenvalue of the Hecke operator $T(l)$ corresponding to G (where $l\in\mathbb{Z}_{>0}$). Let p be any prime coprime to N. By means of (7.1) and (7.3), we see that

$$T(p)A_{\Delta,r} = A_{\Delta,r}\mid_{3/2} T(p),$$

where on the left hand side the Hecke operator acts through the Jacobian, while on the right hand side it acts through $S_{3/2,\bar\rho}$. Consequently, for the G-isotypical part we find

$$A^G_{\Delta,r}\mid_{3/2} T(p) = \lambda_p A^G_{\Delta,r}.$$

Hence $A^G_{\Delta,r}$ is an eigenform of all the $T(p)$ for p coprime to N with the same eigenvalues as g. By "multiplicity one" for $S^{new}_{3/2,\bar\rho}$, we find that $A^G_{\Delta,r} = Cg$ for some constant $C\in(J(\mathbb{Q}(\sqrt{\Delta}))\otimes\mathbb{C})^G$. To compute the constant, we determine the pairing with f. We have

$$\begin{aligned}\{A^G_{\Delta,r}, f\} &= y_{\Delta,r}(f),\\ \{Cg, f\} &= C(g,\xi_{1/2}(f)) = C.\end{aligned}$$

This concluses the proof of the theorem. □

Note that in the case $\Delta = 1$, the theorem (in a different formulation) was proved in a different way in [GKZ]. We may use the Gross-Zagier theorem to relate the vanishing of $y_{\Delta,r}(f)$ to the vanishing of a twisted L-series associated with G.

Theorem 7.8. *Let the hypotheses be as in Theorem 7.6. The following are equivalent.*

(i) *The Heegner divisor $y_{\Delta,r}(f)$ vanishes in $J(\mathbb{Q}(\sqrt{\Delta}))\otimes\mathbb{C}$.*
(ii) *We have $L'(G,\chi_\Delta,1)=0$.*

Proof. We denote the Fourier coefficients of g by $b(n,h)$ for $n\in\mathbb{Z}-\operatorname{sgn}(\Delta)Q(h)$ and $h\in L'/L$. Since g is a newform, by Lemma 3.2 of [SZ], there is a fundamental discriminant d coprime to N such that $\operatorname{sgn}(\Delta)d<0$ and such that $b(n,h)\neq 0$ for $n=-\operatorname{sgn}(\Delta)\frac{d}{4N}$ and some $h\in L'/L$. According to Corollary 1 of Chapter II in [GKZ], and [Sk2], we have the Waldspurger type formula

$$|b(n,h)|^2 = \frac{\|g\|^2}{2\pi\|G\|^2}\sqrt{|d|}L(G,\chi_d,1).$$

In particular, the non-vanishing of $b(n,h)$ implies the non-vanishing of $L(G,\chi_d,1)$.

On the other hand, it follows from the Gross-Zagier formula (Theorem 6.3 in [GZ]) that the global Neron-Tate height on $J(H)$ of $y^G_{\Delta,r}(-n,h)$ is given by

$$\langle y^G_{\Delta,r}(-n,h), y^G_{\Delta,r}(-n,h)\rangle = \frac{h_K u^2}{8\pi^2\|G\|^2}\sqrt{|d\Delta|}L'(G,\chi_\Delta,1)L(G,\chi_d,1).$$

Here H is the Hilbert class field of $K=\mathbb{Q}(\sqrt{d\Delta})$, and $2u$ is the number of roots of unity in K, and h_K denotes the class number of K.

Consequently, the Heegner divisor $y^G_{\Delta,r}(-n,h)$ vanishes if and only if $L'(G,\chi_\Delta,1)$ vanishes. But by Theorem 7.7 we know that

$$y^G_{\Delta,r}(-n,h) = y_{\Delta,r}(f)b(n,h).$$

This concludes the proof of the theorem. □

As described in the introduction, the results in this section imply Theorem 1.1. We conclude this section with the proof of Corollary 1.4.

Proof of Corollary 1.4. By Theorem 1.1, it suffices to show that

$$\#\{0\le\Delta<X \ \text{ fundamental } : \ L(G,\chi_\Delta,1)\neq 0\} \gg_G \frac{X}{\log X}, \tag{7.5}$$

and

$$\#\{-X<\Delta<0 \ \text{ fundamental } : \ L'(G,\chi_\Delta,1)\neq 0\} \gg_{G,\epsilon} X^{1-\epsilon}. \tag{7.6}$$

Corollary 3 of [OSk] implies (7.5), and the proof of Theorem 1 of [PP] implies (7.6). □

8. Examples

Here we give some examples related to the main results in this paper.

8.1. **Twisted modular polynomials.** Here we use Theorems 6.1 and 6.2 to deduce the infinite product expansion of twisted modular polynomials found by Zagier (see §7 of [Za2]).

Let $N=1$. Then we have $L'/L\cong\mathbb{Z}/2\mathbb{Z}$. Moreover, $N_{1/2,\rho_L}=M^!_{1/2,\rho_L}$ and $N_{1/2,\bar\rho_L}=0$. Therefore we consider the case where $\Delta>1$ is a positive fundamental discriminant. Let $r\in\mathbb{Z}$ such that $\Delta\equiv r^2 \pmod 4$. By §5 of [EZ], the space $M^!_{1/2,\rho_L}$ can be identified with the space $M^!_{1/2}$ of scalar valued weakly holomorphic modular forms of weight 1/2 for $\Gamma_0(4)$ satisfying the Kohnen plus space condition. For every negative discriminant d, there is a unique $f_d\in M^!_{1/2}$, whose Fourier expansion at the cusp ∞ has the form

$$f_d = q^d + \sum_{\substack{n\ge 1\\ n\equiv 0,1\ (4)}} c_d(n)q^n.$$

The expansions of the first few f_d are given in [Za2], and one sees that the coefficients are rational. Theorems 6.1 and 6.2 gives a meromorphic modular form $\Psi_\Delta(z,f_d):=\Psi_{\Delta,r}(z,f_d)$ of weight 0 for the group $\Gamma=\mathrm{SL}_2(\mathbb{Z})$ whose divisor on $X(1)$ is given by

$$H_\Delta(d) := H_{\Delta,r}(d/4,d/2) = \sum_{\lambda\in L_{\Delta d}/\Gamma}\frac{\chi_\Delta(\lambda)}{w(\lambda)}\cdot H(\lambda).$$

By (2.15), $L_{\Delta d}/\Gamma$ corresponds to the Γ-classes of integral binary quadratic forms of discriminant Δd. Moreover, for sufficiently large $\Im(z)$, we have the product expansion

$$\Psi_\Delta(z, f_d) = \prod_{n=1}^{\infty} \prod_{b\ (\Delta)} [1 - e(nz + b/\Delta)]^{\left(\frac{\Delta}{b}\right) c_d(\Delta n^2)}. \tag{8.1}$$

From these properties it follows that

$$\Psi_\Delta(z, f_d) = \prod_{\lambda \in L_{\Delta d}/\Gamma} \big(j(z) - j(H(\lambda))\big)^{\chi_\Delta(\lambda)}. \tag{8.2}$$

As an example, let $\Delta := 5$ and $d := -3$. There are two classes of binary quadratic forms of discriminant -15, represented by $[1,1,4]$ and $[2,1,2]$, and their corresponding CM points are $\frac{-1+\sqrt{-15}}{2}$ and $\frac{-1+\sqrt{-15}}{4}$. It is well known that the singular moduli of $j(\tau)$ of these points are $-\frac{191025}{2} - \frac{85995}{2}\sqrt{5}$, and $-\frac{191025}{2} + \frac{85995}{2}\sqrt{5}$. The function f_{-3} has the Fourier expansion

$$f_{-3} = q^{-3} - 248\,q + 26752\,q^4 - 85995\,q^5 + 1707264\,q^8 - 4096248\,q^9 + \dots.$$

Multiplying out the product over b in (8.1), we obtain the infinite product expansion

$$\Psi_5(z, f_{-3}) = \frac{j(z) + \frac{191025}{2} + \frac{85995}{2}\sqrt{5}}{j(z) + \frac{191025}{2} - \frac{85995}{2}\sqrt{5}} = \prod_{n=1}^{\infty} \left(\frac{1 + \frac{1-\sqrt{5}}{2} q^n + q^{2n}}{1 + \frac{1+\sqrt{5}}{2} q^n + q^{2n}} \right)^{c_{-3}(5n^2)}.$$

8.2. **Ramanujan's mock theta functions $f(q)$ and $\omega(q)$.** Here we give an example of a Borcherds product arising from Ramanujan's mock theta functions. We first recall the modular transformation properties of $f(q)$, defined in (1.1), and

(8.3)

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2} = \frac{1}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^3)^2} + \frac{q^{12}}{(1-q)^2(1-q^3)^2(1-q^5)^2} + \cdots.$$

Using these functions, define the vector valued function $F(\tau)$ by

$$F(\tau) = (F_0(\tau),\ F_1(\tau),\ F_2(\tau))^T := (q^{-\frac{1}{24}} f(q),\ 2q^{\frac{1}{3}}\omega(q^{\frac{1}{2}}),\ 2q^{\frac{1}{3}}\omega(-q^{\frac{1}{2}}))^T. \tag{8.4}$$

Similarly, let $G(\tau)$ be the vector valued non-holomorphic function defined by

$$G(\tau) = (G_0(\tau),\ G_1(\tau),\ G_2(\tau))^T := 2i\sqrt{3} \int_{-\bar{\tau}}^{i\infty} \frac{(g_1(z),\ g_0(z),\ -g_2(z))^T}{\sqrt{-i(\tau+z)}}\, dz, \tag{8.5}$$

where the $g_i(\tau)$ are the cuspidal weight 3/2 theta functions

$$\begin{aligned} g_0(\tau) &:= \sum_{n=-\infty}^{\infty} (-1)^n \left(n + \frac{1}{3}\right) e^{3\pi i \left(n+\frac{1}{3}\right)^2 \tau}, \\ g_1(\tau) &:= -\sum_{n=-\infty}^{\infty} \left(n + \frac{1}{6}\right) e^{3\pi i \left(n+\frac{1}{6}\right)^2 \tau}, \\ g_2(\tau) &:= \sum_{n=-\infty}^{\infty} \left(n + \frac{1}{3}\right) e^{3\pi i \left(n+\frac{1}{3}\right)^2 \tau}. \end{aligned} \tag{8.6}$$

Using these vector valued functions, Zwegers [Zw1] let $H(\tau) := F(\tau) - G(\tau)$, and he proved [Zw1] that it is a vector valued weight 1/2 harmonic weak Maass form. In particular, it satisfies

$$H(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(\tau), \tag{8.7}$$

$$H(-1/\tau) = \sqrt{-i\tau} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(\tau). \tag{8.8}$$

Now let $N := 6$. One can check the following lemma which asserts that this representation of $\tilde{\Gamma}$ is an irreducible piece of the Weil representation $\bar{\rho}_L$.

Lemma 8.1. *Assume that $H = (h_0, h_1, h_2)^T : \mathbb{H} \to \mathbb{C}^3$ is a vector valued modular form of weight k for $\tilde{\Gamma}$ transforming with the representation defined by* (8.7) *and* (8.8)*. Then the function*

$$\tilde{H} = (0, h_0, h_2 - h_1, 0, -h_1 - h_2, -h_0, 0, h_0, h_1 + h_2, 0, h_1 - h_2, -h_0)^T \tag{8.9}$$

is a vector valued modular form of weight k for $\tilde{\Gamma}$ with representation $\bar{\rho}_L$. Here we have identified $\mathbb{C}[L'/L]$ with $\mathbb{C}^{12}$ by mapping the standard basis vector of $\mathbb{C}[L'/L]$ corresponding to the coset $j/12 + \mathbb{Z} \in L'/L$ to the standard basis vector e_j of $\mathbb{C}^{12}$ (where $j = 0, \dots, 11$).

This lemma shows that H gives rise to an element $\tilde{H} \in N_{1/2, \bar{\rho}_L}$. Let $c^{\pm}(m, h)$ be the coefficients of $\tilde{H}$. For any fundamental discriminant $\Delta < 0$ and any integer r such that $\Delta \equiv r^2 \pmod{24}$, we obtain a twisted generalized Borcherds lift $\Psi_{\Delta,r}(z, \tilde{H})$. By Theorems 6.1 and 6.2, it is a weight 0 meromorphic modular function on $X_0(6)$ with divisor

$$2H_{\Delta,r}(-\tfrac{1}{24}, \tfrac{1}{12}) - 2H_{\Delta,r}(-\tfrac{1}{24}, \tfrac{5}{12}).$$

Moreover, it has the infinite product expansion

$$\Psi_{\Delta,r}(z, \tilde{H}) = \prod_{n=1}^{\infty} P_\Delta\left(q^n\right)^{c^+(|\Delta| n^2/24,\, rn/12)}, \tag{8.10}$$

where

$$P_\Delta(X) := \prod_{b\ (\Delta)} [1 - e(b/\Delta) X]^{\left(\frac{\Delta}{b}\right)}. \tag{8.11}$$

For instance, let $\Delta := -8$ and $r := 4$. The set $L_{-8,4}/\Gamma_0(6)$ is represented by the binary quadratic forms $Q_1 = [6, 4, 1]$ and $Q_2 = [-6, 4, -1]$, and $L_{-8,-4}/\Gamma_0(6)$ is represented by $-Q_1$ and $-Q_2$. The Heegner points in $\mathbb{H}$ corresponding to Q_1 and Q_2 respectively are

$$\alpha_1 = \frac{-2 + \sqrt{-2}}{6}, \qquad \alpha_2 = \frac{2 + \sqrt{-2}}{6}.$$

Consequently, the divisor of $\Psi_{-8,4}(z, \tilde{H})$ on $X_0(6)$ is given by $2(\alpha_1) - 2(\alpha_2)$. In this case the infinite product expansion (8.10) only involves the coefficients of the components of $\tilde{H}$

of the form $\pm(h_1+h_2)$. To simplify the notation, we put

$$-2q^{1/3}\big(\omega(q^{1/2})+\omega(-q^{1/2})\big) =: \sum_{n\in\mathbb{Z}+1/3} a(n)q^n = -4\,q^{1/3}-12\,q^{4/3}-24\,q^{7/3}-40\,q^{10/3}-\dots.$$

We have

$$P_{-8}(X) := \frac{1+\sqrt{-2}X-X^2}{1-\sqrt{-2}X-X^2},$$

and the infinite product expansion (8.10) can be rewritten as

$$\Psi_{-8,4}(z,\tilde{H}) = \prod_{n=1}^{\infty} P_{-8}\left(q^n\right)^{\left(\frac{n}{3}\right)a(n^2/3)}. \tag{8.12}$$

It is amusing to work out an expression for $\Psi_{-8,4}(z,\tilde{H})$. We use the Hauptmodul for $\Gamma_0^*(6)$, the extension of $\Gamma_0(6)$ by all Atkin-Lehner involutions, which is

$$j_6^*(z) = \left(\frac{\eta(z)\eta(2z)}{\eta(3z)\eta(6z)}\right)^4 + 4 + 3^4\left(\frac{\eta(3z)\eta(6z)}{\eta(z)\eta(2z)}\right)^4 = q^{-1}+79\,q+352\,q^2+1431\,q^3+\dots.$$

Here $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ denotes the Dedekind eta function. We have $j_6^*(\alpha_1) = j_6^*(\alpha_2) = -10$. Hence $j_6^*(z)+10$ is a rational function on $X_0(6)$ whose divisor consists of the 4 cusps with multiplicity -1 and the points α_1, α_2 with multiplicity 2. The unique normalized cusp form of weight 4 for $\Gamma_0^*(6)$ is

$$\delta(z) := \eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2 = q-2\,q^2-3\,q^3+4\,q^4+6\,q^5+6\,q^6-16\,q^7-8\,q^8+\dots.$$

Using these functions, we find that

$$\phi(z) := \Psi_{-8,4}(z,\tilde{H})\cdot(j_6^*(z)+10)\delta(z)$$

is a holomorphic modular form of weight 4 for $\Gamma_0(6)$ with divisor $4(\alpha_1)$. Using the classical Eisenstein series, it turns out that

$$\begin{aligned}450\phi(z) &= (3360-1920\sqrt{-2})\delta(z)+(1-7\sqrt{-2})E_4(z)+(4-28\sqrt{-2})E_4(2z)\\ &\quad+(89+7\sqrt{-2})E_4(3z)+(356+28\sqrt{-2})E_4(6z).\end{aligned}$$

Putting this all together, (8.12) becomes

$$\begin{aligned}\prod_{n=1}^{\infty}\left(\frac{1+\sqrt{-2}q^n-q^{2n}}{1-\sqrt{-2}q^n-q^{2n}}\right)^{\left(\frac{n}{3}\right)a(n^2/3)} &= \frac{\phi(z)}{(j_6^*(z)+10)\delta(z)}\\ &= 1-8\sqrt{-2}q-(64-24\sqrt{-2})q^2+(384+168\sqrt{-2})q^3+(64-1768\sqrt{-2})q^4+\dots.\end{aligned}$$

8.3. **Relations among Heegner points and vanishing derivatives of L-functions.** Let $G\in S_2^{new}(\Gamma_0(N))$ be a newform of weight 2, and let Δ be a fundamental discriminant such that $L(G,\chi_\Delta,s)$ has an odd functional equation. By the Gross-Zagier formula, the vanishing of $L'(G,\chi_\Delta,1)$ is equivalent to the vanishing of a certain Heegner divisor in the Jacobian. More precisely, let f be a harmonic weak Maass form of weight 1/2 corresponding to G as in Section 7. Then $L'(G,\chi_\Delta,1)$ vanishes if and only if the divisor $y_{\Delta,r}(f)$ vanishes. If G is defined over $\mathbb{Q}$, we may consider the generalized regularized theta lift of f in Section 6. It gives rise to a rational function $\Psi_{\Delta,r}(z,f)$ on $X_0(N)$ with divisor $y_{\Delta,r}(f)$. (If G is not defined over $\mathbb{Q}$, one also has to consider the Galois conjugates.) Such relations among Heegner divisors cannot be obtained as the Borcherds lift of a weakly holomorphic form. They are given by the generalized regularized theta lift of a harmonic weak Maass form.

As an example, we consider the the relation for Heegner points of discriminant -139 on $X_0(37)$ found by Gross (see §4 of [Za1]). Let $N := 37$, $\Delta := -139$, and $r := 3$. In our notation, $L_{-139,3}/\Gamma_0(37)$ can be represented by the quadratic forms

$$\begin{aligned} Q_1 &= [37,3,1], & Q_2 &= [185,151,31], & Q_3 &= [185,-71,7], \\ Q_1' &= [-37,3,-1], & Q_2' &= [-185,151,-31], & Q_3' &= [-185,-71,-7]. \end{aligned}$$

Denote the corresponding points on $X_0(37)$ by α_1, α_2, α_3 and α_1', α_2', α_3'. Hence we have

$$\begin{aligned} H_{1,1}(-\tfrac{139}{4\cdot 37}, \tfrac{3}{2\cdot 37}) &= \alpha_1+\alpha_2+\alpha_3+\alpha_1'+\alpha_2'+\alpha_3', \\ H_{-139,3}(-\tfrac{1}{4\cdot 37}, \tfrac{1}{2\cdot 37}) &= \alpha_1+\alpha_2+\alpha_3-\alpha_1'-\alpha_2'-\alpha_3'. \end{aligned}$$

Gross proved that the function

$$r(z) = \frac{\eta(z)^2}{\eta(37z)^2} - \frac{3+\sqrt{-139}}{2}$$

on $X_0(37)$ has the divisor $(\alpha_1)+(\alpha_2)+(\alpha_3)-3(\infty)$. This easily implies that $r'(z)$, the image of $r(z)$ under complex conjugation, has the divisor $(\alpha_1')+(\alpha_2')+(\alpha_3')-3(\infty)$.

We show how the function $r(z)$ can be obtained as a regularized theta lift. Let $f_1 \in N_{1/2,\rho_L}$ be the unique harmonic weak Maass form whose Fourier expansion is of the form

$$f_1 = e(-\tfrac{139}{4\cdot 37}\tau)\mathfrak{e}_3 + e(-\tfrac{139}{4\cdot 37}\tau)\mathfrak{e}_{-3} + O(e^{-\varepsilon v}), \qquad v \to \infty.$$

It is known that the dual space $S_{3/2,\bar\rho_L}$ is one-dimensional. Moreover, any element has the property that the coefficients with index $\frac{139}{4\cdot 37}$ vanish (see [EZ], p.145). It follows from (2.10) that $\xi_{1/2}(f_1) = 0$, and so f_1 is weakly holomorphic. Its Borcherds lift is equal to

$$\Psi_{1,1}(z,f_1) = r(z)\cdot r'(z)\cdot \frac{\eta(37z)^2}{\eta(z)^2}. \tag{8.13}$$

On the other hand, we consider the unique harmonic weak Maass form $f_2 \in N_{1/2,\bar\rho_L}$ whose Fourier expansion is of the form

$$f_2 = e(-\tfrac{1}{4\cdot 37}\tau)\mathfrak{e}_1 - e(-\tfrac{1}{4\cdot 37}\tau)\mathfrak{e}_{-1} + O(e^{-\varepsilon v}), \qquad v \to \infty.$$

It is known that the dual space $S_{3/2,\rho_L}$ is one-dimensional. Let g_2 be a generator of this space with rational coefficients. The Shimura lift of g_2 is the newform $G_2 \in S_2^+(\Gamma_0(37))$. Its L-function has an even functional equation, and it is known that $L(G_2,1) \neq 0$. By the Waldspurger type formula for skew holomorphic Jacobi forms, see [Sk2], this implies that the coefficients of g_2 with index $\frac{1}{4\cdot 37}$ do not vanish. In view of (2.10), we find that $\xi_{1/2}(f_2)$ is a non-zero multiple of g_2. So f_2 is not weakly holomorphic. Nevertheless, we may look at the twisted generalized Borcherds lift of f_2. We obtain that

$$\Psi_{-139,3}(z,f_2) = r(z)/r'(z) = \frac{\eta(z)^2 - \frac{3+\sqrt{-139}}{2}\eta(37z)^2}{\eta(z)^2 - \frac{3-\sqrt{-139}}{2}\eta(37z)^2}. \tag{8.14}$$

Notice that one can use this formula to compute the exponents in the product expansion and therefore some coefficients of the holomorphic part of f_2.

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