January 2002

Discretized Yang-Mills and Born-Infeld actions on finite group geometries

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Abstract

Discretized nonabelian gauge theories living on finite group spaces G are defined by means of a geometric action $\int Tr\ F \wedge *F$. This technique is extended to obtain discrete versions of the Born-Infeld action. The discretizations are in 1-1 correspondence with differential calculi on finite groups.

A consistency condition for duality invariance of the discretized field equations is derived for discretized $U(1)$ actions $S[F]$ living on a 4-dimensional abelian G. Discretized electromagnetism satisfies this condition and therefore admits duality rotations.

Yang-Mills and Born-Infeld theories are also considered on product spaces $M^D \times G$, and we find the corresponding field theories on M^D after Kaluza-Klein reduction on the G discrete internal spaces. We examine in some detail the case $G = Z_N$, and discuss the limit $N \to \infty$.

A self-contained review on the noncommutative differential geometry of finite groups is included.

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1 Introduction

The differential geometry of finite groups G has proved to be a useful tool in constructing gauge and gravity theories on discrete spaces [1, 2, 3, 4, 5]. These spaces are the finite group "manifolds" associated to every differential calculus (DC) on G, in 1-1 correspondence with unions of its conjugation classes [1, 6, 2]. The construction of DC on finite G is a particular instance of a general procedure yielding DC on Hopf algebras, first studied by Woronowicz [7] in the noncommutative context of quantum groups (see also [8] for a review with applications to field theory).

In this paper we extend the results of [2, 3, 4] and formulate discretized gauge theories by means of a geometric action $\int F \wedge *F$ on a finite group G. Using the same geometrical tools, we also construct a discretized version of Born-Infeld theory.

The continuum Born-Infeld theory [9], in its commutative and noncommutative settings, related by the Seiberg-Witten map, has become relevant in the description of D-brane dynamics, see for ex. [10]. Noncommutative structures ([11] for a review) in string/brane theory have emerged in the last years (see for ex. [12, 10] and ref.s therein), and are the object of intense research , see e.g. [13] and included ref.s. This motivates in part our investigation concerning a particularly simple form of noncommutative Born-Infeld (BI) theory: the one that arises by considering the BI action living on finite group spaces.

Here the noncommutativity is mild, in the sense that fields commute between themselves (in the classical theory), and only the commutations between fields and differentials, and of differentials between themselves are nontrivial. In this framework we obtain a discretized Yang-Mills and BI theory for every differential calculus on a finite group.

We study the issue of duality invariance of a nonlinear electromagnetic theory described by a generic action $S[F]$ on a 4-dimensional abelian finite group. A consistency condition is found, and generalizes the results of the continuum limit.

Considering then Yang-Mills and BI theories as living on the product space (D-Minkowski) $\times$ (finite group space) we obtain, after use of Kaluza-Klein reduction techniques, D-dimensional Yang-Mills and BI theories coupled to scalar fields. The harmonic analysis on the discrete internal spaces is trivial. It is tempting to interpret the product space (D-Minkowski) $\times$ (finite group space) as a bundle of n (D-1)-dimensional branes evolving in time, n being the dimension of the finite group.

The paper is organized as follows. In Section 2, after a short review of the differential geometry of finite groups, we show that the metric is essentially unique (for each real differential calculus) and then define the Hodge dual. Section 3 recalls previously obtained results on gauge theories living on finite group manifolds, with some new observations. In particular the Hodge dual is used to formulate discretized gauge theories on finite G manifolds in purely geometrical terms, as in the continuum case. In Section 4 we address the question of duality invariance. In Section 5 we consider Yang-Mills theory on $M^D \times G$, and reduce it via Kaluza-Klein techniques to a continuum gauge theory in M^D coupled to scalar fields. We then specialize our analysis in Section 6 to the case $G = Z_N$. As in the case $M^D \times Z_2$ (see for ex. [3]), a potential for the scalar field emerges in the D - dimensional action. In Section 7 a finite group lattice action for Born-Infeld theory is presented. Section 8 deals with Kaluza-Klein BI theory on $M^D \times G$, with some explicit results for the case $M^4 \times Z_N$. In the Appendix we discuss in more detail the Hopf algebraic structure of finite groups G and of $Fun(G)$.

2 A brief review of differential calculus on finite groups

Let G be a finite group of order n with generic element g and unit e. Consider $Fun(G)$, the set of complex functions on G. An element f of $Fun(G)$ is specified by its values $f_g \equiv f(g)$ on the group elements g, and can be written as

$$f = \sum_{g \in G} f_g x^g, \quad f_g \in \boldsymbol{C} \tag{2.1}$$

where the functions x^g are defined by

$$x^g(g') = \delta^g_{g'} \tag{2.2}$$

Thus $Fun(G)$ is a n-dimensional vector space, and the n functions x^g provide a basis. $Fun(G)$ is also a commutative algebra, with the usual pointwise sum and product, and unit I defined by $I(g) = 1, \forall g \in G$. In particular:

$$x^g x^{g'} = \delta_{g,g'} x^g, \quad \sum_{g \in G} x^g = I \tag{2.3}$$

The left and right actions of the group G on itself

$$L_g \, g' = g \, g' = R_{g'} \, g \quad \forall g, g' \in G \;, \tag{2.4}$$

induce the left and right actions (pullbacks) $\mathcal{L}_g$, $\mathcal{R}_g$ on $Fun(G)$

$$[\mathcal{L}_g \, f](g') = f(g \, g') = [\mathcal{R}_{g'} \, f](g) \quad \forall f \in Fun(G) \;. \tag{2.5}$$

For the basis functions we find easily:

$$\mathcal{L}_{g_1} x^g = x^{g_1^{-1} g}, \quad \mathcal{R}_{g_1} x^g = x^{g g_1^{-1}} \tag{2.6}$$

Moreover:

$$\mathcal{L}_{g_1} \mathcal{L}_{g_2} = \mathcal{L}_{g_2 g_1}, \quad \mathcal{R}_{g_1} \mathcal{R}_{g_2} = \mathcal{R}_{g_1 g_2}, \quad \mathcal{L}_{g_1} \mathcal{R}_{g_2} = \mathcal{R}_{g_2} \mathcal{L}_{g_1} \tag{2.7}$$

The G group structure induces a Hopf algebra structure on $Fun(G)$, and this allows the construction of differential calculi on $Fun(G)$, according to the techniques of ref. [7, 8]. We summarize here the main properties of these calculi. A detailed treatment can be found in [2], and Hopf algebraic formulas, allowing contact with the general method of [7, 8], are listed in the Appendix.

A (first-order) differential calculus on $Fun(G)$ is defined by a linear map d: $Fun(G) \to \Gamma$, satisfying the Leibniz rule $d(ab) = (da)b + a(db), \quad \forall a, b \in Fun(G)$. The "space of 1-forms" Γ is an appropriate bimodule on $Fun(G)$, which essentially means that its elements can be multiplied on the left and on the right by elements of $Fun(G)$. From the Leibniz rule $da = d(Ia) = (dI)a + Ida$ we deduce $dI = 0$. Consider the differentials of the basis functions x^g. From $0 = dI = d(\sum_{g\in G} x^g) = \sum_{g\in G} dx^g$ we see that in this calculus only $n-1$ differentials are independent.

A differential calculus is *left or right covariant* if the left or right action of G ($\mathcal{L}_g$ or $\mathcal{R}_g$) commutes with the exterior derivative d. Requiring left and right covariance in fact *defines* the action of $\mathcal{L}_g$ and $\mathcal{R}_g$ on differentials: $\mathcal{L}_g db \equiv d(\mathcal{L}_g b), \forall b \in Fun(G)$ and similarly for $\mathcal{R}_g db$. More generally, on elements of Γ (one-forms) we define $\mathcal{L}_g$ as:

$$\mathcal{L}_g(adb) \equiv (\mathcal{L}_g a)\mathcal{L}_g db = (\mathcal{L}_g a)d(\mathcal{L}_g b) \tag{2.8}$$

and similarly for $\mathcal{R}_g$. A differential calculus is called *bicovariant* if it is both left and right covariant.

As in usual Lie group manifolds, we can introduce in Γ the left-invariant one-forms θ^g:

$$\theta^g \equiv \sum_{h\in G} x^{hg^{-1}} dx^h = \sum_{h\in G} x^h dx^{hg}, \tag{2.9}$$

These θ^g correspond to the $\theta^{g^{-1}}$ of ref.s [1, 2, 3, 4].

It is immediate to check that indeed $\mathcal{L}_k \theta^g = \theta^g$. The right action of G on the elements θ^g is given by:

$$\mathcal{R}_h \theta^g = \theta^{ad(h)g}, \quad \forall h \in G \tag{2.10}$$

where ad is the adjoint action of G on itself, i.e. $ad(h)g \equiv hgh^{-1}$. Notice that θ^e is biinvariant, i.e. both left and right invariant.

From $\sum_{g\in G} dx^g = 0$ one finds:

$$\sum_{g\in G} \theta^g = \sum_{g,h\in G} x^h dx^{hg} = \sum_{h\in G} x^h \sum_{g\in G} dx^{hg} = 0 \tag{2.11}$$

Therefore we can take as basis of the cotangent space Γ the $n-1$ linearly independent left-invariant one-forms θ^g with $g \neq e$ (but smaller sets of θ^g can be consistently chosen as basis, see later). Using (2.3) the relations (2.9) can be inverted:

$$dx^h = \sum_{g\in G} x^{hg^{-1}} \theta^g = \sum_{g\neq e} (x^{hg^{-1}} - x^h)\theta^g \tag{2.12}$$

Analogous results hold for right invariant one-forms ζ^g:

$$\zeta^g = \sum_{h\in G} x^{g^{-1}h} dx^h \tag{2.13}$$

Using the definition of θ^g (2.9), the commutations between x and θ are easily obtained:

$$x^h dx^g = x^h \theta^{h^{-1}g} = \theta^{h^{-1}g} x^g \quad (h \neq g) \quad \Rightarrow \theta^g x^h = x^{hg^{-1}} \theta^g \quad (g \neq e) \tag{2.14}$$

and imply the general commutation rule between functions and left-invariant one-forms:

$$\theta^g f = [\mathcal{R}_g f]\theta^g \qquad (g \neq e) \tag{2.15}$$

Thus functions do commute between themselves (i.e. $Fun(G)$ is a commutative algebra) but do not commute with the basis of one-forms θ^g. In this sense the differential geometry of $Fun(G)$ is noncommutative.

The differential of an arbitrary function $f \in Fun(G)$ can be found with the help of (2.12):

$$
\begin{aligned}
df &= \sum_h f_h \, dx^h = \sum_{g,h} f_h \, x^{h\,g^{-1}} \, \theta^g = \sum_{g \neq e} (\sum_h f_h \, x^{h\,g^{-1}} - f) \, \theta^g = \\
&= \sum_{g \neq e} ([\mathcal{R}_g \, f] - f) \, \theta^g = \sum_{g \neq e} (t_g \, f) \, \theta^g \; .
\end{aligned} \tag{2.16}
$$

Here the finite difference operators $t_g = \mathcal{R}_g - 1$ are the analogues of (left-invariant) tangent vectors. They satisfy the Leibniz rule:

$$
t_g(ff') = (t_g f)f' + \mathcal{R}_g(f) t_g f' = (t_g f)\mathcal{R}_g f' + f t_g f' \tag{2.17}
$$

and close on the fusion algebra:

$$
t_g \, t_{g'} = (\mathcal{R}_{gg'} - 1) - (\mathcal{R}_g - 1) - (\mathcal{R}_{g'} - 1) = \sum_h C^h_{g,g'} \, t_h \; , \tag{2.18}
$$

where the structure constants $C^h_{g,g'}$ are

$$
C^h_{g,g'} = \delta^h_{gg'} - \delta^h_g - \delta^h_{g'} \; , \tag{2.19}
$$

The commutation rule (2.15) allows to express the differential of a function $f \in Fun(G)$ as a commutator of f with the biinvariant form $\sum_{g \neq e} \theta^g = -\theta^e$:

$$
df = [\sum_{g \neq e} \theta^g, \, f] = -[\theta^e, f] \; . \tag{2.20}
$$

An *exterior product*, compatible with the left and right actions of G, can be defined as

$$
\begin{aligned}
\theta^g \wedge \theta^{g'} &= \theta^g \otimes \theta^{g'} - \sum_{k,k'} \Lambda^{g\,g'}{}_{k'\,k} \theta^{k'} \otimes \theta^k = \theta^g \otimes \theta^{g'} - \theta^{gg'g^{-1}} \otimes \theta^g = \\
&= \theta^g \otimes \theta^{g'} - [\mathcal{R}_g \theta^{g'}] \otimes \theta^g \; , \quad (g, g' \neq e) \; ,
\end{aligned} \tag{2.21}
$$

where the tensor product between elements $\rho, \rho' \in \Gamma$ is defined to have the properties $\rho a \otimes \rho' = \rho \otimes a\rho'$, $a(\rho \otimes \rho') = (a\rho) \otimes \rho'$ and $(\rho \otimes \rho')a = \rho \otimes (\rho' a)$. The braiding matrix Λ:

$$
\Lambda^{g\,g'}{}_{k'\,k} = \delta^{gg'g^{-1}}_{k'} \, \delta^g_k \; , \quad (g, g' \neq e) \; . \tag{2.22}
$$

satisfies the Yang-Baxter equation $\Lambda^{nm}{}_{ij} \Lambda^{ik}{}_{rp} \Lambda^{js}{}_{kq} = \Lambda^{nk}{}_{ri} \Lambda^{ms}{}_{kj} \Lambda^{ij}{}_{pq}$. With this exterior product we find

$$
\theta^g \wedge \theta^g = 0 \quad (\forall g) \; , \quad \theta^g \wedge \theta^{g'} = -\theta^{g'} \wedge \theta^g \quad (\forall g, g' : \; [g, \, g'] = 0 \; , \; \; g \neq e) \; . \tag{2.23}
$$

Moreover

$$\theta^g \wedge \theta^{g'} \; f = (\mathcal{R}_{gg'} f) \theta^g \wedge \theta^{g'} \tag{2.24}$$

Left and right actions on $\Gamma \otimes \Gamma$ are simply defined by:

$$\mathcal{L}_h(\rho \otimes \rho') = \mathcal{L}_h \rho \otimes \mathcal{L}_h \rho', \quad \mathcal{R}_h(\rho \otimes \rho') = \mathcal{R}_h \rho \otimes \mathcal{R}_h \rho' \tag{2.25}$$

Compatibility of the exterior product with $\mathcal{L}$ and $\mathcal{R}$ means that

$$\mathcal{L}(\theta^i \wedge \theta^j) = \mathcal{L}\theta^i \wedge \mathcal{L}\theta^j, \quad \mathcal{R}(\theta^i \wedge \theta^j) = \mathcal{R}\theta^i \wedge \mathcal{R}\theta^j \tag{2.26}$$

Only the second relation is nontrivial and is verified upon use of the definition (2.21). We can generalize the previous definition to exterior products of k left-invariant one-forms:

$$\theta^{i_1} \wedge ... \wedge \theta^{i_k} \equiv A^{i_1..i_k}_{j_1..j_k} \; \theta^{j_1} \otimes ... \otimes \theta^{j_k} \tag{2.27}$$

or in short-hand notation (FRT matrix notations [14]):

$$\theta^1 \wedge ... \wedge \theta^k = A_{1...k} \; \theta^1 \otimes ... \otimes \theta^k \tag{2.28}$$

The labels $1...k$ in A refer to index couples, and $A_{1,...k}$ is the analogue of the antisymmetrizer of k spaces, defined by the recursion relation

$$A_{1...k} = (1 - \Lambda_{k-1,k} + \Lambda_{k-2,k-1}\Lambda_{k-1,k} - \ldots - (-1)^k \Lambda_{12}\Lambda_{23} \cdots \Lambda_{k-1,k}) A_{1...k-1}, \tag{2.29}$$

where $A_{12} = 1 - \Lambda_{12}$. The space of k-forms $\Gamma^{\wedge k}$ is therefore defined as in the usual case but with the new permutation operator Λ, and can be shown to be a bicovariant bimodule (see for ex. [15]), with left and right action defined as for $\Gamma \otimes ... \otimes \Gamma$ with the tensor product replaced by the wedge product. The property (2.24) generalizes to:

$$\theta^{i_1} \wedge ... \wedge \theta^{i_k} \; f = (\mathcal{R}_{i_1...i_k} f) \theta^{i_1} \wedge ... \wedge \theta^{i_k} \tag{2.30}$$

The graded bimodule $\Omega = \sum_k \Gamma^{\wedge k}$ is the exterior algebra of forms. As recalled in the Appendix, this algebra is also a Hopf algebra [16, 17].

The *exterior derivative* is defined as a linear map $d \; : \; \Gamma^{\wedge k} \to \Gamma^{\wedge (k+1)}$ satisfying $d^2 = 0$ and the graded Leibniz rule

$$d(\rho \wedge \rho') = d\rho \wedge \rho' + (-1)^k \rho \wedge d\rho' \tag{2.31}$$

where $\rho \in \Gamma^{\wedge k}$, $\rho' \in \Gamma^{\wedge k'}$, $\Gamma^{\wedge 0} \equiv Fun(G)$. Left and right action is defined as usual:

$$\mathcal{L}_g(d\rho) = d\mathcal{L}_g \rho, \quad \mathcal{R}_g(d\rho) = d\mathcal{R}_g \rho \tag{2.32}$$

In view of relation (2.10), and (A.20), the algebra Ω has natural quotients over the ideals $H_g = \{\theta^{hgh^{-1}}, \forall h\}$, corresponding to the various conjugacy classes of the elements g in G. The different bicovariant calculi on $Fun(G)$ are in 1-1 correspondence with different quotients of Ω by any sum of the ideals $H = \sum H_g$, cf. [1, 6, 2]. In practice one simply sets $\theta^g = 0$ for all $g \neq e$ not belonging to the particular union G' of conjugacy classes characterizing the differential calculus. The dimension of the space of independent 1-forms for each bicovariant calculus on $Fun(G)$ is therefore equal to the dimension of the subspace Γ/H.

The *Cartan-Maurer equation* for the differential forms θ^g (2.9) is obtained by direct calculation, using the definition (2.9), the expression (2.12) of dx^h in terms of θ's, and the commutations (2.14):

$$d\theta^g = -\sum_{h\neq e, h'\neq e} \delta^g_{hh'} \theta^h \wedge \theta^{h'} + \sum_{k\neq e} \theta^k \wedge \theta^g + \sum_{k\neq e} \theta^g \wedge \theta^k = -\sum_{h\neq e} \sum_{h'\neq e} C^g_{h,h'} \theta^h \wedge \theta^{h'} \ , \quad (g \neq e) \tag{2.33}$$

where the structure constants $C^g_{h,h'}$ are given in (2.19). Using the identity:

$$\sum_{h\neq e, h'\neq e} \delta^k_{hh'} \theta^h \wedge \theta^{h'} = \sum_{h\neq e, h'\neq e} \delta^k_{hh'} \left(\theta^h \otimes \theta^{h'} - \theta^{hh'h^{-1}} \otimes \theta^h\right) = 0 \tag{2.34}$$

the Cartan-Maurer equation can be rewritten by means of the anticommutator of θ^g with the biinvariant form θ^e:

$$d\theta^g = -\theta^e \wedge \theta^g - \theta^g \wedge \theta^e \tag{2.35}$$

cf. the case of 0-forms (2.20). Considering now a generic element $\rho = a\theta$ of Γ it is easy to find that $d\rho = -\theta^e \wedge \rho - \rho \wedge \theta^e$. The general rule is

$$d\rho = [-\theta^e, \rho]_{grad} \equiv -\theta^e \wedge \rho + (-1)^{deg(\rho)} \rho \wedge \theta^e \tag{2.36}$$

valid for any k-form, where $[-\theta^e, \rho]_{grad}$ is the graded commutator. Thus $-\theta^e = \sum_{k\neq e} \theta^k$ is the X generator of Woronowicz theory [7], or BRST operator [17]; in our case X belongs to Γ.

There are two (Hopf algebra) *conjugations* on $Fun(G)$ [1, 4]

$$(x^g)^* = x^g \ , \quad (x^g)^\star = x^{g^{-1}} \tag{2.37}$$

These involutions can be extended to the whole exterior (Hopf) algebra Ω:

$$(\theta^g)^* = -\theta^{g^{-1}} \ , \quad (\theta^g)^\star = \zeta^g \tag{2.38}$$

such that $(\rho \wedge \rho')^* = (-1)^{deg(\rho)deg(\rho')} \rho'^* \wedge \rho^*$ etc. In fact the conjugations can be defined directly on the tensor algebra: $(\rho \otimes \rho')^* = (-1)^{deg(\rho)deg(\rho')} \rho'^* \otimes \rho^*$. We'll use the $*$-conjugation in the sequel. Consistency of this conjugation requires that if $\theta^g \neq 0$ then $\theta^{g^{-1}} \neq 0$ as well: we have to include in Γ/H at least the two ideals H_g and $H_{g^{-1}}$ (if they do not coincide). We obtain thus a $*$-differential calculus, i.e. $(df)^* = d(f^*)$.

The fact that both θ^g and $\theta^{g^{-1}}$ are included in the basis of left-invariant 1-forms characterizing the differential calculus also ensures the existence of a unique metric (up to a normalization).

The *metric* is defined as a bimodule pairing, symmetric on left-invariant 1-forms. It maps couples of 1-forms ρ, σ into $Fun(G)$, and satisfies the properties

$$< f\rho, \sigma h >= f < \rho, \sigma > h \ , \quad < \rho f, \sigma >=< \rho, f\sigma > \ . \tag{2.39}$$

where f and h are arbitrary functions belonging to $Fun(G)$. Up to a normalization the above properties determine the metric on the left-invariant 1-forms. Indeed from $< \theta^g, f\theta^h >=< \theta^g, \theta^h > \mathcal{R}_{h^{-1}} f = \mathcal{R}_g f < \theta^g, \theta^h >$ one deduces:

$$g^{rs} \equiv < \theta^r, \theta^s > \equiv \delta^r_{s^{-1}} \tag{2.40}$$

Thus g^{rs} is symmetric and θ^r has nonzero pairing only with $\theta^{r^{-1}}$. The pairing is compatible with the *-conjugation

$$< \rho, \sigma >^* = < \sigma^*, \rho^* > \tag{2.41}$$

We can generalize $< \, , \, >$ to tensor products of left-invariant 1-forms as follows [18]:

$$< \theta^{i_1} \otimes \cdots \otimes \theta^{i_k}, \theta^{j_k} \otimes \cdots \otimes \theta^{j_1} > \equiv g^{i_1 j_1} \cdots g^{i_k j_k} \tag{2.42}$$

The pairing (2.42) is extended to all tensor products by $< f\rho, \sigma h >= f < \rho, \sigma > h$ where now ρ and σ are generic tensor products of same order. Then we prove easily that $< \rho f, \sigma >=< \rho, f\sigma >$ for any function f (the reversed order in the second tensor product in (2.42) is essential for the proof), so that $< \, , \, >$ is a bimodule pairing.

In general for a differential calculus with m independent tangent vectors, there is an integer $p \geq m$ such that the linear space of p-forms is 1-dimensional, and $(p+1)$- forms vanish identically [1]. This means that every product of p basis one-forms $\theta^{g_1} \wedge \theta^{g_2} \wedge \ldots \wedge \theta^{g_p}$ is proportional to one of these products, that can be chosen to define the volume form vol:

$$\theta^{g_1} \wedge \theta^{g_2} \wedge \ldots \wedge \theta^{g_p} = \epsilon^{g_1, g_2, \cdots g_p} vol \tag{2.43}$$

where $\epsilon^{g_1, g_2, \cdots g_p}$ is the proportionality constant, a real number since the braiding matrix Λ is real.

The volume p-form is obviously left invariant. It is also right invariant [2] (the proof is based on the $ad(G)$ invariance of the ϵ tensor: $\epsilon^{ad(g)h_1, \ldots ad(g)h_p} = \epsilon^{h_1, \ldots h_p}$).

Finally, if $vol = \theta^{k_1} \wedge \ldots \wedge \theta^{k_p}$, then

$$vol^* = (-1)^{\frac{p(p+1)}{2}} \epsilon^{k_p^{-1} \ldots k_1^{-1}} vol \tag{2.44}$$

so that vol is either real or imaginary. If $vol^* = -vol$ we can always multiply it by i and obtain a real volume form. In that case comparing $(\theta^{g_1} \wedge \ldots \wedge \theta^{g_p})^* = (-1)^{\frac{p(p+1)}{2}} \theta^{g_p^{-1}} \wedge \ldots \wedge \theta^{g_1^{-1}} = \epsilon^{g_p^{-1} \ldots g_1^{-1}} (-1)^{\frac{p(p+1)}{2}} vol$ with $(\theta^{g_1} \wedge \ldots \wedge \theta^{g_p})^* = \epsilon^{g_1 \cdots g_p} \, (vol)^* = \epsilon^{g_1 \cdots g_p} \, vol$ yields

$$\epsilon^{g_p^{-1} \ldots g_1^{-1}} = (-1)^{\frac{p(p+1)}{2}} \; \epsilon^{g_1 \cdots g_p} \tag{2.45}$$

Computing the pairing of vol with itself yields:

$$< vol, vol >= \mathcal{N}, \quad \mathcal{N} \equiv \sum_{i_1, \ldots i_p} \epsilon^{i_1 \ldots i_p} \epsilon^{i_p^{-1} \ldots i_1^{-1}} \tag{2.46}$$

Clearly the pairing (2.42) or vol can be normalized so that $< vol, vol >= \pm 1$ but we'll use (2.46) in the following.

[1] with the exception of Z_2, see ref. [3]

Having identified the volume p-form it is natural to define the integral of the generic p-form $f\ vol$ on G

$$\int f\ vol = \sum_{g \in G} f(g) \tag{2.47}$$

the right-hand side being just the Haar measure of the function f.

The *Hodge dual*, an important ingredient for gauge theories, can be defined as the unique map from k-forms σ to $(p-k)$-forms $*\sigma$ such that (see [18] for a similar construction):

$$\rho \wedge *\sigma = <\rho, \sigma> vol \qquad \rho, \sigma\ n\text{-forms} \tag{2.48}$$

The Hodge dual is left linear; if vol is central it is also right linear :

$$*(f\rho\, h) = f(*\rho)h \tag{2.49}$$

with $f, h \in Fun(G)$. Moreover

$$*\,\mathcal{N} = \mathcal{N}\ vol\ \ , \quad *\ vol = \mathcal{N}\ \ , (*\rho)^* = *(\rho^*) \tag{2.50}$$

Note: the "group manifold" of a finite group is simply a collection of points corresponding to the group elements, linked together in various ways, each corresponding to a particular differential calculus on $Fun(G)$ [1, 2]. The links are associated to the tangent vectors $\mathcal{R}_h - 1$ of the differential calculus, or equivalently to the right actions $\mathcal{R}_h$, where h belongs to the union G' of conjugacy classes characterizing the differential calculus. Two points x^g and $x^{g'}$ are linked if $x^{g'} = \mathcal{R}_h x^g$, i.e. if $g' = gh^{-1}$ for some h in G'. The link is oriented from x^g to $x^{g'}$ (unless $h = h^{-1}$ in which case the link is unoriented): the resulting "manifold" is an oriented graph. From every point exactly m (= number of independent 1-forms) links originate. Some examples of finite group manifolds can be found in [2].

3 Gauge theories on finite groups

A natural question arises: is it possible to construct field theories on the discrete spaces provided by finite group manifolds ? The answer is affirmative: exploiting the differential calculus on finite G, gauge and gravity theories have been constructed in ref.s [1, 2, 3, 4, 5]. To prepare the ground for discretized Born-Infeld theory, we recall in this section how to construct $\mathcal{G} = U(N)$ gauge field theory on finite G group spaces and add some new observations. The discretized Yang-Mills action involves only geometric objects: the 2-form field strength F , the $*$-Hodge operator and the invariant integral on G.

The gauge field of a Yang-Mills theory on a finite group G is a matrix-valued one-form $A(x) = A_h(x)\theta^h$. The components A_h are functions on G: they can be considered functions of the "coordinates" x^g, since any element of $Fun(G)$ can be expanded on the basis functions x^g. Moreover they are matrix-valued, i.e. $A = (A_h)^\alpha{}_\beta \theta^h$, $\alpha, \beta = 1, ...N$. In the following matrix multiplication is implicit.

As in the usual case, $\mathcal{G}$ gauge transformations are defined as

$$A' = -(dT)T^{-1} + TAT^{-1} \tag{3.1}$$

where $T(x) = T(x)^\alpha{}_\beta$ is an $N \times N$ representation of a $\mathcal{G}$ group element; its matrix entries belong to $Fun(G)$. In components:

$$A'_h = -(t_h T)\mathcal{R}_h T^{-1} + T A_h \mathcal{R}_h T^{-1} = T\,[1 + A_h]\,[\mathcal{R}_h T^{-1}] - 1 \tag{3.2}$$

Matter fields $\psi_\alpha(x)$ transform in a representation of $\mathcal{G}$ as $\psi \to \psi' = T\psi$, and their covariant derivative, defined by

$$D\psi = d\psi + A\psi, \tag{3.3}$$

transforms homogeneously: $(D\psi)' = T(D\psi)$. The 2-form field strength F arises as usual in the square of the covariant derivative $(D^2)\psi = F\psi$; it is given by the familiar expression $F = dA + A \wedge A$, which in components takes the form:

$$\begin{aligned} F &= dA + A \wedge A = d(A_k\theta^k) + A_h\theta^h \wedge A_k\theta^k \\ &= (t_h A_k)\ \theta^h \wedge \theta^k + A_k d\theta^k + A_h(\mathcal{R}_h A_k)\ \theta^h \wedge \theta^k \\ &= [t_h A_k - A_j C^j{}_{hk} + A_h(\mathcal{R}_h A_k)]\ \theta^h \wedge \theta^k \end{aligned} \tag{3.4}$$

and satisfies the Bianchi identity:

$$dF + A \wedge F - F \wedge A = 0 \tag{3.5}$$

Note that $A \wedge A \neq 0$ even if the gauge group $\mathcal{G}$ is abelian. Thus $U(1)$ gauge theory on a finite group space looks like a nonabelian theory, a situation occurring also in noncommutative $*$-deformed gauge theories.

Under gauge transformations (3.1) F varies homogeneously:

$$F' = TFT^{-1} \tag{3.6}$$

The gauge variation (3.2) suggests the definition of the link fields $U_h(g)$ and link 1-form $U(g)$:

$$U_h = 1 + A_h\ , \quad U = \sum_{h \neq e} U_h\,\theta^h = \sum_{h \neq e} \theta^h + A \tag{3.7}$$

transforming homogeneously:

$$U'_g = T\,U_g\,[\mathcal{R}_g T^{-1}]\ , \quad U' = T\,U\,T^{-1}\ . \tag{3.8}$$

The field strength (3.4) can be expressed in terms of the link fields:

$$F = (U_h\,[\mathcal{R}_h U_k] - U_g \delta^g_{hk})\ \theta^h \wedge \theta^k = U_h\,(\mathcal{R}_h U_k)\,\theta^h \wedge \theta^k = U \wedge U \tag{3.9}$$

where we have used the identity (2.34).

Defining the components $F_{h,g}$ as:

$$F \equiv F_{h,k}\ \theta^h \otimes \theta^k \tag{3.10}$$

eq. (3.9) yields:

$$F_{h,k} = U_h\,(\mathcal{R}_h\,U_k) - U_k(\mathcal{R}_k\,U_{k^{-1}hk})\ . \tag{3.11}$$

The Yang-Mills action is the geometrical action quadratic in F given by

$$A_{YM} = \int Tr(F \wedge *F) = \int Tr < F, F > vol \quad . \tag{3.12}$$

Recalling the pairing properties (2.39) the proof of gauge invariance of $Tr < F, F >$ is immediate:
$Tr < F', F' >= Tr < TFT^{-1}, TFT^{-1} >= TrT < F, F > T^{-1} = Tr < F, F > \quad .$

The metric (2.40) is an euclidean metric (as is easily seen using a real basis of one-forms) and as usual we require (3.12) to be real and positive definite. This restricts the gauge group $\mathcal{G}$ and imposes reality conditions on the gauge potential A. Since (cf. (2.47)) $\int Tr < F, F > vol = \sum_G Tr < F, F >$, positivity of (3.12) requires $(Tr < F, F >) \geq 0$. Explicitly

$$\begin{aligned} &< F, F >=< F_{r,s}\theta^r \otimes \theta^s, F_{m,n}\theta^m \otimes \theta^n >= \\ &F_{r,s} < \theta^r \otimes \theta^s, \theta^m \otimes \theta^n > \mathcal{R}_{n^{-1}m^{-1}} F_{m,n} = F_{r,s}\mathcal{R}_{rs}F_{s^{-1},r^{-1}} \end{aligned} \tag{3.13}$$

and

$$Tr < F, F >= (F_{r,s})^\alpha{}_\beta (\mathcal{R}_{rs}F_{s^{-1},r^{-1}})^\beta{}_\alpha \ . \tag{3.14}$$

We see that (3.14) is positive definite if $(\mathcal{R}_{rs}F_{s^{-1},r^{-1}})^\beta{}_\alpha = (F^*_{r,s})^\alpha{}_\beta$, i.e. if

$$F^\dagger_{r,s} = \mathcal{R}_{rs}F_{s^{-1},r^{-1}} \quad \text{or equivalently:} \quad F^\dagger = -F \ . \tag{3.15}$$

Thus F is antihermitian as a two-form. The $\dagger$ conjugation by definition acts as hermitian conjugation on the matrix structure, and as the *-conjugation (introduced in the previous Section) on the $Fun(G)$ entries of the matrix. In order to derive $F^\dagger = F$ in (3.15) we have used

$$F^\dagger = (F_{h,k}\theta^h \otimes \theta^k)^\dagger = (\theta^h \otimes \theta^k)^* \ F^\dagger_{h,k} = -(\theta^{k^*} \otimes \theta^{h^*})F^\dagger_{h,k} = -(\theta^{k^{-1}} \otimes \theta^{h^{-1}})F^\dagger_{h,k}$$

The gauge action then can be rewritten (sum on the indices h, k, α, β understood)

$$A_{YM} = \sum_G Tr(F_{h,k}F^\dagger_{h,k}) = \sum_G (F_{h,k})^\alpha{}_\beta (F^*_{h,k})^\alpha{}_\beta \ . \tag{3.16}$$

From (3.4) we see that $F^\dagger = -F$ implies that of $A^\dagger = -A$, and equivalently $U^\dagger = -U$ (use $(\sum_{h\neq e}\theta^h)^* = -\sum_{h\neq e}\theta^h$ in (3.7)). Hermitian conjugation on matrix valued one forms A being defined as:

$$A^\dagger = (A_h\theta^h)^\dagger = (\theta^h)^* A^\dagger_h = -\theta^{h^{-1}} A^\dagger_h = -\theta^h A^\dagger_{h^{-1}} \ , \tag{3.17}$$

antihermiticity of A ($A^\dagger = -A$) becomes in components:

$$A^\dagger_h = \mathcal{R}_h A_{h^{-1}} \ . \tag{3.18}$$

Finally gauge trasformations must preserve antihermiticity of A and F and this is the case if the representation T of $\mathcal{G}$ is unitary. We thus conclude that the action (3.12) has maximal gauge group $\mathcal{G} = U(N)$. Notice also from (3.2) that since $t_h T$ is a finite difference of group elements, then A_h must belong to the $\mathcal{G}$ *group algebra*, rather than to the $\mathcal{G}$ Lie algebra.

The action (3.16) can be expressed in terms of the link fields U_h: substituting (3.11) into (3.16) leads to

$$A_{YM} = 2 \sum_G Tr[U_k U_k^\dagger U_h U_h^\dagger - U_k^\dagger U_h (\mathcal{R}_h U_k)(\mathcal{R}_k U_{k^{-1}hk}^\dagger)] \tag{3.19}$$

after use of the cyclic property of Tr, and of

$$U^\dagger = -U \quad \Longrightarrow U_r^\dagger = \mathcal{R}_r U_{r^{-1}} \tag{3.20}$$

Moreover we also needed the obvious properties

$$\sum_G f = \sum_G \mathcal{R}_k f \qquad \forall k,\ \forall f \in Fun(G) \tag{3.21}$$

$$\sum_h f_h = \sum_h f_{k^{-1}hk} \quad \forall k,\ \forall f_h \in Fun(G) \tag{3.22}$$

When the finite group G is abelian, the action (3.19) reduces to the Wilson action. In particular this happens for $G = Z_N \times \cdots \times Z_N$, a result already obtained in ref. [19].

Note 1. The action (3.12) can also be written as

$$A_{YM} = \int Tr(F \wedge *F) = \int Tr(*F \wedge F) \tag{3.23}$$

Proof: Since $vol = vol^*$ we have $(\int f vol)^* = \sum_G f^* = \int f^* vol = \int vol^* f^* = \int (f vol)^*$ where in the penultimate equality we used invariance of the integral under right translations. Then, since A_{YM} is real we have

$$\int Tr(F \wedge *F) = [\int Tr(F \wedge *F)]^\dagger = \int Tr(F \wedge *F)^\dagger = \int Tr((*F)^\dagger \wedge F^\dagger) = \int Tr(*F \wedge F)$$

where compatibility of the Hodge dual with *-conjugation, and therefore with $\dagger$ conjugation, has been used.

Note 2. One can construct the finite group lattice analogues of Wilson loops. Indeed, consider the exterior product U^k :

$$\begin{aligned} U^k &= U_{h_1}\theta^{h_1} \wedge U_{h_2}\theta^{h_2} \wedge \ldots \wedge U_{h_k}\theta^{h_k} = \\ &= U_{h_1}[\mathcal{R}_{h_1} U_{h_2}]\,[\mathcal{R}_{h_1 h_2} U_{h_3}] \ldots [\mathcal{R}_{h_1 \ldots h_{k-1}} U_{h_k}]\theta^{h_1} \wedge \theta^{h_2} \wedge \ \ldots\ \wedge \theta^{h_k} \equiv \\ &\equiv U_{h_1,\ldots,h_k}\theta^{h_1} \wedge\ \ldots\ \wedge \theta^{h_k} \end{aligned} \tag{3.24}$$

(no sum on the indices h_i) such that $h_1 h_2 \cdots h_k = 1$. Then, the trace Tr of the component $U_{h_1,\ldots,h_k}$ gives a gauge invariant object which can be interpreted as the finite group analog of a Wilson loop. When the volume form $vol = \theta^{i_1} \wedge \cdots \wedge \theta^{i_p}$ is central ($[vol, f] = 0, \forall f \in Fun(G)$), then also the p-form U^p is gauge invariant.

Note 3. The gauge invariant action (3.16) on the finite group lattice is easily generalized to the case of higher order field strengths:

$$B = B_{h_1,\ldots h_k} \theta^{h_1} \otimes \ldots \otimes \theta^{h_k} \;, \tag{3.25}$$

transforming as $B \to T\, B\, T^{-1}$, or

$$B_{h_1,\ldots,h_k} \longrightarrow T\ B_{h_1,\ldots,h_k}\ [\mathcal{R}_{h_1,\ldots,h_k} T^{-1}] \;, \tag{3.26}$$

Then

$$\sum B_{h_1,\ldots,h_k}\, B^\dagger_{h_1,\ldots,h_k} \longrightarrow T \sum B_{h_1,\ldots,h_k}\, B^\dagger_{h_1,\ldots,h_k}\, T^{-1} \;, \tag{3.27}$$

If B satisfies $B^\dagger = -B$, then $B^\dagger_{h_1\ldots h_k} = (-1)^{\frac{k(k+1)}{2}-1} \mathcal{R}_{h_1\ldots h_k} B_{h_k^{-1}\ldots h_1^{-1}}$ and

$$B_{h_1,\ldots,h_k}\, B^\dagger_{h_1,\ldots,h_k} = (-1)^{\frac{k(k+1)}{2}-1}\ < B, B > \tag{3.28}$$

The gauge invariant analogue of (3.16) is

$$A = \sum_G \sum_{h_i \neq e} Tr \left(B_{h_1,\ldots,h_k}\, B^\dagger_{h_1,\ldots,h_k} \right) = (-1)^{\frac{k(k+1)}{2}-1} \sum_G Tr\ < B, B > . \tag{3.29}$$

In particular, setting $B = U^k$, where U is the link 1-form, and taking into account the condition (3.20), the gauge invariant action (3.29) is a sum over special Wilson loops as described in Note 2.

4 Duality Rotations

In this Section we consider an "abelian" $U(1)$ gauge theory on an abelian finite group space. We choose a four dimensional differential calculus on this finite group. The $U(1)$ gauge theory is "abelian" in the sense that the field strength satisfies the Bianchi identities $dF = 0$, in particular this holds if $F = dA$. In this case *infinitesimal* gauge transformations read $\delta A = d\lambda$, where λ is a gauge parameter, see for ex. the second ref. in [5]. As in standard electromagnetism the field strength F is therefore invariant under gauge transformations.

It is remarkable that also for this discrete (noncommutative) version of electromagnetism one can consider electric-magnetic duality rotations (for duality rotations in noncommutative geometry where the noncommutativity is given by a $*$-product we refer to [22] and references therein). Following [20], see also the nice review [21], in this section we obtain a consistency condition for an (in general nonlinear) electromagnetic theory to admit duality rotations; we also show that the equations of motions (EOM) of Maxwell theory admit electric-magnetic rotations. As far as we know this is the first example of duality rotations on a lattice.

Let us consider a 4-dimensional *-calculus on G with one-forms $\theta^u, \theta^{u^{-1}}, \theta^v, \theta^{v^{-1}}$ where u, u^{-1}, v, v^{-1} are the four different elements of G that determine the calculus. It follows that the metric $g^{h,k} =< \theta^h \, , \, \theta^k >= \delta^h_{k^{-1}}$ has determinant $\det g = 1$. This is an Euclidean metric as one can check expressing it in a basis of real one-forms. We also notice that since the finite group is abelian, the epsilon tensor $\varepsilon^{hh'gg'}$ is the usual completely antisymmetric tensor (with $\varepsilon^{uu^{-1}vv^{-1}} = 1$).

Consider an action S that depends on the gauge field A only through the field strength F. The EOM are obtained by varying S with respect to A_g. We have

$$\begin{aligned} \delta_A S &\equiv \sum_{p\in G} \frac{\delta S}{\delta A_g(p)} \delta A_g(p) = \frac{1}{2} \sum_{p,q\in G} \frac{\delta S}{\delta F_{h,h'}(q)} \frac{\delta F_{h,h'}(q)}{\delta A_g(p)} \delta A_g(p) = \sum_{q\in G} \frac{\delta S}{\delta F_{h,h'}(q)} t_h \delta A_{h'}(q) \\ &= \sum_{q\in G} -t_h\Big(\frac{\delta S}{\delta F_{h,h'}(q)}\Big) R_h \delta A_{h'}(q) = \sum_{q\in G} -t_h\Big(R_h^{-1} \frac{\delta S}{\delta F_{h,h'}(q)}\Big) \delta A_{h'}(q) \end{aligned}$$

where in the second equality we used that $F_{h,h'} = t_h A_{h'} - t_{h'} A_h$ is antisymmetric, while in the third we have integrated by parts via the Leibniz rule (2.17) $t_g(fh) = t_g f \, R_g h + f t_g h$ and then used Stokes' theorem: $\sum_{q\in G} t_g(f)(q) = \sum_{q\in G} (R_g f(q) - f(q)) = 0$. The EOM are therefore

$$0 = \frac{\delta S}{\delta A_{h'}} = -t_h R_h^{-1} \frac{\delta S}{\delta F_{h,h'}} \qquad \text{(no sum on } h') \tag{4.1}$$

or, applying $R_{h'}^{-1}$ (so that the argument of t_h is antisymmetric in the indices h, h'):

$$t_h\Big(R_{hh'}^{-1} \frac{\delta S}{\delta F_{h,h'}}\Big) = 0 \; . \qquad \text{(no sum on } h') \tag{4.2}$$

We then define

$$\tilde{G}^{h,h'} \equiv \frac{1}{2} \varepsilon^{hh'gg'} G_{g,g'} \equiv R_{hh'}^{-1} \frac{\delta S[F]}{\delta F_{h,h'}} \tag{4.3}$$

where we have written $S[F]$ in order to stress that this definition does not depend on the relation $F = dA$. Using (4.3) the EOM and the Bianchi identities $dF = 0$ have the same structure

$$t_h \tilde{G}^{h,h'} = 0 \;\; , \tag{4.4}$$

$$t_h \tilde{F}^{h,h'} = 0 \;\; , \tag{4.5}$$

where $\tilde{F}^{h,h'} \equiv \frac{1}{2} \varepsilon^{hh'gg'} F_{g,g'}$. In summary, starting with the action S we obtained the EOM (4.4) with $F = dA$. We now relax the condition $F = dA$ and consider the more general theory given by the EOM (4.4) and (4.5) with $\tilde{G}$ given in (4.3) and where *now* S[F] is seen just as a function of F, with F an arbitrary antisymmetric tensor (not the differential of A). It is this theory that possibly admits duality rotations. More precisely we show that if $S[F]$ satisfies (4.16) then we have $SO(1,1)$ duality rotations. As in [20] ([21]) we observe that under the infinitesimal $GL(2)$ rotation $G \to G + \Delta G, \;\; F \to F + \Delta F$ given by

$$\Delta \begin{pmatrix} G \\ F \end{pmatrix} = \begin{pmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix} \tag{4.6}$$

equations (4.4), (4.5) are mapped into themselves.
For ease of notations, in what follows we'll write F_{hk} instead of $F_{h,k}$, and similarly for $\tilde{F}, G, \tilde{G}$.

Consistency of (4.6) with (4.3), i.e.

$$\tilde{G}^{gg'} + \Delta\tilde{G}^{gg'} = R_{gg'}^{-1} \frac{\delta S[F + \Delta F]}{\delta(F_{gg'} + \Delta F_{gg'})} \tag{4.7}$$

constrains the action $S[F]$ and the rotation parameters in (4.6). In order to simplify the expression of the constraint (4.7) we write

$$\begin{aligned}\frac{\delta S[F + \Delta F]}{\delta(F_{gg'} + \Delta F_{gg'})} &= \sum_{q\in G} \frac{1}{2} \frac{\delta S[F + \Delta F]}{\delta F_{hh'}(q)} \frac{\delta F_{hh'}(q)}{\delta(F_{gg'} + \Delta F_{gg'})} \\ &= \frac{\delta S[F + \Delta F]}{\delta F_{gg'}} - \sum_{q\in G} \frac{1}{2} \frac{\delta S[F]}{\delta F_{hh'}(q)} \frac{\delta(\Delta F_{hh'})(q)}{\delta F_{gg'}}\end{aligned}$$

then, using (4.6) and (4.3), condition (4.7) reads

$$\begin{aligned}&AR_{gg'}\tilde{G}^{gg'} + BR_{gg'}\tilde{F}^{gg'} \\ &\quad= \frac{\delta(S[F + \Delta F] - S[F])}{\delta F_{gg'}} - \frac{1}{2} C \sum_{q\in G} \frac{\delta S[F]}{\delta F_{hh'}(q)} \frac{\delta G_{hh'}(q)}{\delta F_{gg'}} - D\frac{\delta S[F]}{\delta F_{gg'}} \\ &\quad= \frac{\delta(S[F + \Delta F] - S[F])}{\delta F_{gg'}} - \frac{1}{2} C \sum_{q\in G} \tilde{G}^{hh'}(q) R_{hh'}^{-1} \frac{\delta G_{hh'}(q)}{\delta F_{gg'}} - D\frac{\delta S[F]}{\delta F_{gg'}}\end{aligned} \tag{4.8}$$

where in the last line we used the invariance of the integral $\sum_{q\in G}$ under the translation $R_{hh'}^{-1}$. In order to further simplify this expression we observe that

$$\frac{\delta}{\delta F_{gg'}} \sum_{q\in G} F_{hh'}(q) R_{hh'} \tilde{F}^{hh'}(q) = 4R_{gg'}\tilde{F}^{gg'} \tag{4.9}$$

Proof: Use invariance of the integral $\sum_{q\in G}$ under the translation $R_{hh'}^{-1}$ and then notice that $R_{hh'}^{-1}\varepsilon^{hh'gg'} = R_{gg'}\varepsilon^{hh'gg'}$ since h, h, g, g' run over the 4 group elements u, u^{-1}, v, v^{-1}.

Similarly one has $\delta(\sum_{q\in G} G_{hh'}(q) R_{hh'} \tilde{G}^{hh'}(q)) = 2\sum_{q\in G} R_{hh'}\tilde{G}^{hh'}(q)\, \delta G_{hh'}(q)$ and therefore

$$\frac{\delta}{\delta F_{gg'}} \sum_{q\in G} G_{hh'}(q) R_{hh'} \tilde{G}^{hh'}(q) = 2\sum_{q\in G} \tilde{G}^{hh'}(q) R_{hh'}^{-1} \frac{\delta G_{hh'}(q)}{\delta F_{gg'}} \tag{4.10}$$

Next we substitute (4.9) and (4.10) in (4.8) and factorize out the functional derivative $\frac{\delta}{\delta F_{gg'}}$. We thus arrive at the equivalent condition (the constant term $(A + D)S[F = 0]$ is obtained by observing that when $F = 0$ also $G = 0$)

$$S[F + \Delta F] - S[F] - \frac{B}{4} \sum_G F_{hh'} R_{hh'} \tilde{F}^{hh'} - \frac{C}{4} \sum_G G_{hh'} R_{hh'} \tilde{G}^{hh'} = (A + D)(S[F] - S[F = 0]) \tag{4.11}$$

This expression can be further simplified. Use $S[F+\Delta F]-S[F]=\frac{1}{2}\sum_G \frac{\delta S[F]}{\delta F_{hh'}}\Delta F_{hh'} = \frac{1}{2}C\sum_G G_{hh'}R_{hh'}\tilde{G}^{hh'}+\frac{1}{2}D\sum_G F_{hh'}R_{hh'}\tilde{G}^{hh'}$ in order to write

$$\frac{C}{4}\sum_G G_{hh'}R_{hh'}\tilde{G}^{hh'}-\frac{B}{4}\sum_G F_{hh'}R_{hh'}\tilde{F}^{hh'}=(A+D)(S[F]-S[F=0])-\frac{D}{2}\sum_G F_{hh'}R_{hh'}\tilde{G}^{hh'} \tag{4.12}$$

Then, assuming that the Lagrangian is parity even, the parity even and parity odd terms in (4.12) must separately vanish and we obtain

$$C\sum_G G_{hh'}R_{hh'}\tilde{G}^{hh'}=B\sum_G F_{hh'}R_{hh'}\tilde{F}^{hh'} \tag{4.13}$$

$$(A+D)(S[F]-S[F=0])=\frac{D}{2}\sum_G F_{hh'}\frac{\delta S[F]}{\delta F_{hh'}} \tag{4.14}$$

We now show that linear electromagnetism satisfies (4.13) and (4.14) with $B=C$ and $A=D$, i.e. that linear electromagnetism admits the duality rotation group $SO(1,1)$ times the group of scale transformations (corresponding to $A=D\neq 0$).

Let

$$S_{EM}=-\frac{1}{4}\sum_G F_{hh'}F^{\dagger}_{hh'}=\frac{1}{4}\sum_G F_{hh'}R_{hh'}F_{kk'}g^{hk}g^{h'k'} \ ,$$

then $\tilde{G}^{hh'}=F_{ll'}g^{lh}g^{l'h'}\equiv F^{hh'}$, and therefore $G_{hh'}=\tilde{F}^{ll'}g_{lh}g_{l'h'}\equiv\tilde{F}_{hh'}$.[2] Since $R^{-1}_{hh'}\varepsilon^{hh'gg'}=R_{gg'}\varepsilon^{hh'gg'}$, we have $F_{hh'}R_{hh'}\tilde{F}^{hh'}=\tilde{F}_{hh'}R_{hh'}F^{hh'}$ and we conclude that (4.13) is satisfied iff $C=B$. Similarly $\frac{1}{2}D\sum_G F_{hh'}\frac{\delta S[F]}{\delta F_{hh'}}=2DS_{EM}[F]$ so that (4.13) is satisfied iff $A=D$.

For a nonlinear theory $S[F]$, that in the weak field limit reproduces $S_{EM}[F]$ (i.e. $S[F]=S[F=0]+S_{EM}[F]+O(F^4)$), condition (4.14) is equivalent to

$$D(S[F]-S[F=0]-\frac{1}{4}\sum_G F_{hh'}\frac{\delta S[F]}{\delta F_{hh'}})=0. \tag{4.15}$$

But $S[F]$ is not homogeneous in F and therefore (4.14) is satisfied iff $A=D=0$. We conclude that a nonlinear electromagnetic theory, that has definite parity and that in the weak field limit reproduces linear electromagnetism, admits an $SO(1,1)$ duality rotation group if and only if

$$\sum_G G_{hh'}R_{hh'}\tilde{G}^{hh'}=\sum_G F_{hh'}R_{hh'}\tilde{F}^{hh'} \ . \tag{4.16}$$

Note 1. In Section 6 we present two Born-Infeld type actions on finite groups with four independent tangent vectors (four dimensions). The action (7.5) does not satisfy condition (4.16). Whether the action (7.23) in four dimensions admits $SO(1,1)$ duality rotations remains to be determined.

[2] We recall that $\det g=1$ so that the Hodge operator $\tilde{}$ squares to the identity and not to minus the identity as for Minkowski space. This is why in Euclidean space the duality group is $SO(1,1)$ while in Minkowski space it is $SO(2)$.

5 Kaluza-Klein gauge theory on $M^D \times G$

We extend here the results of Section 3 to include spaces of the type $M^D \times$ finite G. This extension is straightforward (see also [3]), and allows us to apply Kaluza-Klein techniques when the internal space is a discrete finite G "manifold".

We'll use the letter x for the M^D coordinates, and y for the G coordinate functions. A basis of 1-forms on $M^D \times G$ is given by $dx^A = \{dx^\mu, \theta^g\}$, with $x^{\mu *} = x^\mu$ and

$$dx^\mu \wedge \theta^g = -\theta^g \wedge dx^\mu \tag{5.1}$$

$$(dx^\mu \wedge \theta^g)^* = -(\theta^g)^* \wedge (dx^\mu)^* = \theta^{g^{-1}} \wedge dx^\mu = -dx^\mu \wedge \theta^{g^{-1}} \tag{5.2}$$

$$dx^{A_1} \wedge \ldots \wedge dx^{A_{D+p}} \equiv \epsilon^{A_1 \ldots A_{D+p}} \; vol(M^D \times G) \tag{5.3}$$

$$dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_D} \wedge \theta^{g_1} \wedge \ldots \wedge \theta^{g_p} = \epsilon^{\mu_1 \cdots \mu_D} \epsilon^{g_1 \cdots g_p} \; vol(M^D) \wedge vol(G) \tag{5.4}$$

$$\int_{M^D \times G} f(x,y) \equiv \int_{M^D} d^D x \left(\int_G f(x,y) \; vol(G) \right) \tag{5.5}$$

$$< dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}, dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_k} >= (k!)^2 \; \delta^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_k}, \tag{5.6}$$

$$< dx^\mu \wedge \theta^g, dx^\nu \wedge \theta^h >= 2\eta^{\mu\nu} \delta^g_{h^{-1}} \tag{5.7}$$

where $vol(M^D \times G) = vol(M^D) \wedge vol(G) = d^D x \; vol(G)$, $\eta^{\mu\nu}$ is the metric of the M^D space (for simplicity we take here the flat euclidean metric, all the formulas hold also for a Minkowski metric if one inserts η's), $\epsilon^{\mu_1 \cdots \mu_D}$ is the usual Levi-Civita tensor and $\epsilon^{g_1 \cdots g_p}$ has been defined in (2.43). The normalization of exterior products is such that, for example, $dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$ and similar for $dx^\mu \wedge \theta^g$.

The gauge potential 1-form $A(x,y)$ is then expanded as:

$$A(x,y) = A_\mu(x,y) \; dx^\mu + A_g(x,y) \; \theta^g \tag{5.8}$$

The gauge variation $A' = -(dT)T^{-1} + TAT^{-1}$ becomes the usual one for the components A_μ, and the one given in (3.2) for A_g, with T depending on x and y. Thus A_μ belongs to the Lie algebra of $\mathcal{G}$, whereas A_g belongs to the group algebra of $\mathcal{G}$.

Defining as usual the field strength as $F = dA + A \wedge A$ and its components as

$$F \equiv \frac{1}{2} F_{\mu\nu} \; dx^\mu \wedge dx^\nu + F_{\mu k} \; dx^\mu \wedge \theta^k + F_{h,k} \; \theta^h \otimes \theta^k \tag{5.9}$$

we find

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu \tag{5.10}$$

$$F_{\mu k} = \partial_\mu U_k + A_\mu U_k - U_k(\mathcal{R}_k A_\mu) \equiv D_\mu U_k \tag{5.11}$$

$$F_{h,k} = U_h(\mathcal{R}_h U_k) - U_k(\mathcal{R}_k U_{k^{-1}hk}) \tag{5.12}$$

where the link field is defined as (cf. Section 3) $U_g(x,y) = 1 + A_g(x,y)$. The gauge transformation $F'(x,y) = T(x,y)F(x,y)T^{-1}(x,y)$ implies:

$$F'_{\mu\nu}(x,y) = T(x,y) F_{\mu\nu}(x,y) T^{-1}(x,y) \tag{5.13}$$

$$F'_{\mu k}(x,y) = T(x,y) F_{\mu k}(x,y) \mathcal{R}_k T^{-1}(x,y) \tag{5.14}$$

$$F'_{h,k}(x,y) = T(x,y) F_{h,k}(x,y) \mathcal{R}_{hk} T^{-1}(x,y) \tag{5.15}$$

From the antihermiticity $A^\dagger = -A$ (or $U^\dagger = -U$), one finds, as in Section 3, $F^\dagger = -F$. In components:

$$F^\dagger_{\mu\nu} = -F_{\mu\nu}, \quad F^\dagger_{\mu k} = \mathcal{R}_k F_{\mu k^{-1}}, \quad F^\dagger_{h,k} = \mathcal{R}_{hk} F_{k^{-1},h^{-1}} \tag{5.16}$$

Then the Yang-Mills action on $M^D \times G$ can be expanded as follows:

$$A_{YM} = \int_{M^D \times G} Tr\ F \wedge *F = \int_{M^D \times G} Tr\ < F, F > vol(M^D \times G) = \tag{5.17}$$

$$= \int_{M^D} d^D x \sum_G\ Tr\ [F_{\mu\nu} F_{\mu\nu} + 2 F_{\mu k} F^\dagger_{\mu k} + F_{h,k} F^\dagger_{h,k}] =$$

$$= \int_{M^D} d^D x \sum_G\ Tr\ [F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} D_\mu U_k (D_\mu U_k)^\dagger +$$

$$+ 2 U_k U_k^\dagger U_h U_h^\dagger - 2 U_k^\dagger U_h (\mathcal{R}_h U_k)(\mathcal{R}_k U^\dagger_{k^{-1}hk})] \tag{5.18}$$

Note that this action is real, and describes a Yang-Mills theory in D dimensions minimally coupled to the scalar fields U_g.

6 Kaluza-Klein gauge theory on $M^D \times Z_N$

In this section we first study the geometry of Z_N equipped with a bicovariant calculus and a *-conjugation, and consider the limit $N \to \infty$ i.e. $Z_N \to S_1$. It is then easy to generalize the results to $M^D \times Z_N$, consider the Yang-Mills action on this space and understand its $N \to \infty$ limit.

6.1 *-bicovariant calculus on Z_N

Let u: $u^j u^k = u^{j+k}$, $u^N = u^0 = e$ be the generator of the cyclic group Z_N. A basis of functions on Z_N is given by $x^{u^j} = \{x^e, x^u, x^{u^2}, ..., x^{u^{N-1}}\}$. It is convenient to use a basis of functions that reproduce the algebra of the Z_n elements u^j. This basis is given by [1] $y^j \equiv \sum_{k=0}^{N-1} q^{jk} x^{u^k}$, where $q \equiv e^{\frac{2\pi i}{N}}$. Thus $y^j y^k = y^{j+k}$, $y^0 = I$. For example $y^1 = y$ is given by

$$y = x^e + q x^u + q^2 x^{u^2} + ... q^{n-1} x^{u^{n-1}} \tag{6.1}$$

having the values:

$$y(u^k) = q^k = e^{2\pi i \frac{k}{N}} \tag{6.2}$$

on the N points of Z_N. Using $\sum_{j=0}^{N-1} q^{kj} = N\ \delta_{k,0}$ one finds the inverse transformation: $x^{u^j} = \frac{1}{N} \sum_{k=0}^{N-1} q^{-jk} y^k$.

The cyclic group Z_N can be seen as a discrete approximation of the circle S_1 of radius R. Let $0 \le x \le 2\pi R$ be the S_1 coordinate on the circle (not to be confused with the basis functions x^{u^j} of Z_N) : then the points of Z_N have coordinates $x_k = x(u^k) = 2\pi R k/N$ on the circle, and their corresponding y values are $y(x_k) = e^{i \frac{x_k}{R}}$. In the limit $N \to \infty$ these points fill the whole circle, and the x_k become the continuous values of the S_1 coordinate x.

Among the many differential calculi on Z_N the most discussed in the literature [1, 2, 4] is the one where only $\theta^u = -\theta^e \neq 0$. This calculus is not (for $N \neq 2$) a *-calculus: $(df)^* \neq d(f^*)$. The compatibility of a differential calculus with the *-product is essential if we want to consider the calculus on Z_N as a discretization of the *real* differential calculus on S_1. Indeed on a circle of radius R we have $(df)^* = d(f^*)$, where * denotes complex conjugation, and $f = \sum_n f_n y^n(x) = \sum_n f_n e^{inx/R}$, $f_n \in \mathbf{C}$, $x^* = x$, i.e. $y^*(x) = y^{-1}(x)$. In the following we therefore choose the *-bicovariant calculus generated by the 1-forms θ^u and $\theta^{u^{-1}}$. This is the minimal *-bicovariant calculus with θ^u nonzero. In fact, from $y^* = y^{-1}$ and $(hdf)^* = d(f^*)\, h^*$ we obtain $\theta^{u\,*} = -\theta^{u^{-1}}$.[3]

In order to gain a better insight about the geometry of Z_N it is convenient to consider the real and closed forms

$$\alpha = -iRy^{-1}dy \ , \quad \beta = iRydy^{-1} \quad ;$$

in the $N \to \infty$ limit, requiring that the exterior differential d becomes the commutative one, we find $\alpha = \beta = dx$, $0 \leq x \leq 2\pi R$. For finite N, α and β are linearly independent, and recalling (2.12) and (6.1) we obtain

$$\alpha = iR(\zeta - \zeta^*) \ , \quad \beta = iR(q^{-1}\zeta - q\zeta^*) \tag{6.3}$$

where we have defined

$$\zeta = (1-q)\theta^u \ , \quad \zeta^* = (q^{-1} - 1)\theta^{u^{-1}} \ . \tag{6.4}$$

From (6.3) we see that ζ, ζ^* are $\neq 0$ in the $N \to \infty$ limit and therefore θ^u and $\theta^{u^{-1}}$ are ill defined in this limit.

Metric

Using $y\theta^u = q^{-1}\theta^u y$, $y\theta^{u^{-1}} = q\theta^{u^{-1}}y$ it is easy to see that, up to normalization, there exists a unique metric (bimodule pairing) on the space of 1-forms such that on left-invariant 1-forms it is symmetric, and satisfies the properties in (2.39). The normalization, compatible with the $N \to \infty$ limit, is naturally fixed requiring $<\alpha\, , \, \alpha>= 1$. Explicitly we have

$$g^{\zeta\zeta^*} =<\zeta, \zeta^*>= \frac{1}{2R^2} \ , \quad g^{\zeta^*\zeta} =<\zeta^*, \zeta>= \frac{1}{2R^2} \tag{6.5}$$

$$g^{\zeta\zeta} =<\zeta, \zeta>= 0 \ , \quad g^{\zeta^*\zeta^*} =<\zeta^*, \zeta^*>= 0 \tag{6.6}$$

We then also have $<\beta, \beta>= 1$. As in (2.41), this pairing is compatible with the *-conjugation $<\rho, \sigma>^* =<\sigma^*, \rho^*>$. Having the metric g, we can find the 1-form γ that is orthogonal to α and has unit length, $<\alpha\, , \, \gamma>= 0\, , \ \ <\gamma\, , \, \gamma>= 1$. This form is given by

$$\gamma = -R(\zeta + \zeta^*) = i\frac{q+q^{-1}}{q-q^{-1}}\alpha - \frac{2i}{q-q^{-1}}\beta \tag{6.7}$$

[3] If N is odd there is no 1-dimensional $*$-bicovariant calculus and the 2-dimensional $*$-bicovariant calculus defined by θ^u and $\theta^{u^{-1}}$ is the most natural. If N is even on the contrary there exist a unique 1-dimensional $*$-bicovariant calculus, it is generated by the 1-form $\theta^{u^{N/2}}$ that is pure imaginary $\theta^{u^{N/2}\,*} = -\theta^{u^{-N/2}} = -\theta^{u^{N/2}}$; for $N \to \infty$ however this calculus does not lead to the standard calculus on S_1.

Notice that $\gamma^* = \gamma$. In the $N \to \infty$ limit α, β, γ are well defined (α and β become dx, and γ can be checked from (6.7) or from $< \gamma, \gamma >= 1$ to remain finite). Then from (6.7) and (6.3) we see that both ζ and ζ^* are well defined (besides being $\neq 0$) in the $N \to \infty$ limit, while θ^u and $\theta^{u^{-1}}$ diverge as $1/(q^{-1} - 1)$. One also has $t_u/(q^{-1} - 1) \to iR\frac{\partial}{\partial x}$ for $N \to \infty$.

The pairing $< \ , \ >$ can be generalized to act on the space of 2-forms. The space of left-invariant 2-forms is one dimensional because the wedge product for abelian groups is the commutative one (e.g. $\zeta \wedge \zeta^* = \zeta \otimes \zeta^* - \zeta^* \otimes \zeta$). Any 2-form can be written as $f \ vol$ where vol is the volume form associated to the metric:

$$vol = \alpha \wedge \gamma = -2iR^2 \zeta \wedge \zeta^* \ .$$

Up to a normalization the ($*$-bimodule) pairing $< \ \ , \ \ >$ is uniquely defined. As in the commutative (continuous) case we choose here the normalization such that

$$< vol, vol >= 1 \ .$$

Notice that vol is central, therefore the space of 2-forms has no noncommutativity. Notice also that $vol^* = (-1)^{|\alpha||\gamma|} \gamma^* \wedge \alpha^* = vol$.

$*$-Hodge operator
The $*$–Hodge operator is defined as in (2.48). We have

$$*\zeta = i\zeta \quad , \quad * \zeta^* = -i\zeta^* \ . \tag{6.8}$$

Integral
The Haar measure h on Z_N is given by

$$h(f) = \sum_{k=0}^{N-1} f(u^k) \tag{6.9}$$

Integration on 2-forms is then simply given by

$$\int f vol = \frac{2\pi R}{N} h(f) \tag{6.10}$$

the normalization is chosen such that in the $N \to \infty$ limit we obtain the usual integral over the circle S_1 of radius R. It is not difficult to check that the integral (6.10) is cyclic, that it satisfies Stokes' theorem and that it is real. For any p- and $(2-p)$-form $\omega^{(p)}, \omega^{(2-p)}$, we have

$$\int \omega^{(p)} \wedge \omega^{(2-p)} = (-1)^{p(2-p)} \int \omega^{(2-p)} \wedge \omega^{(p)} \qquad \text{(cyclicity)} \tag{6.11}$$

$$\int d\omega^{(1)} = 0 \tag{6.12}$$

$$\left(\int \omega^{(2)} \right)^* = \int \omega^{(2)*} \tag{6.13}$$

6.2 *-bicovariant calculus on $M^D \times Z_N$

It is straightforward to generalize the results of the previous subsection to the $M^D \times Z_N$ case. A basis of 1-forms is given by $\{dx^A\} = \{dx^\mu, \alpha, \gamma\}$, $x^{\mu *} = x^\mu$. The metric g^{AB} is block diagonal $g^{\mu\nu} \neq 0$, $g^{\mu\alpha} = g^{\mu\gamma} = g^{\alpha\gamma} = 0$, $g^{\alpha\alpha} = g^{\gamma\gamma} = 1$. The volume element is $vol = \sqrt{|\det g_{AB}|}\ dx^0 \wedge dx^1 \wedge \ldots dx^{D-1} \wedge \alpha \wedge \gamma$, (where $g^{AC} g_{CB} = \delta^A_B$). In the $\{dx^A\}$ basis the epsilon tensor, defined by

$$dx^{A_1} \wedge dx^{A_2} \wedge \ldots dx^{A_{D+2}} = \varepsilon^{A_1 A_2 \ldots A_{D+2}} vol$$

is the classical one. The integral

$$\int_{M^D \times Z_N} f(x^\mu, y)\, vol \equiv \int_{M^D} \sqrt{|\det g_{\mu\nu}|}\ d^D x \left(\int_{Z_N} f(x^\mu, y) \alpha \wedge \gamma \right)$$

is again cyclic, real and satisfies Stokes' theorem.

The $*$-Hodge operator is still given by (2.48), and satisfies the left- and right-linearity property (2.49) and the compatibility with the *-conjugation (2.50). The normalization of the pairing can be chosen such that the Hodge operator squares to $\pm id$.

6.3 Yang-Mills action

The Yang-Mills action on $M^D \times Z_N$ with gauge group $\mathcal{G} = U(M)$ is given by

$$A_{YM} = \int_{M^D \times Z_N} Tr(F \wedge *F) \tag{6.14}$$

where the trace is over $M \times M$ matrices. Positivity and gauge invariance of this action are shown in Section 4. In order to write the Yang-Mills action in terms of F components we first obtain the commutation relations

$$\alpha y^n = \frac{q^n + q^{-n}}{2} y^n \alpha - i \frac{q^n - q^{-n}}{2} y^n \gamma \equiv \tilde{y}_n \alpha - i \check{y}_n \gamma \tag{6.15}$$

$$\gamma y^n = \frac{q^n + q^{-n}}{2} y^n \gamma + i \frac{q^n - q^{-n}}{2} y^n \alpha = \tilde{y}_n \gamma + i \check{y}_n \alpha \tag{6.16}$$

we then expand $F_{AB}(y)$ as

$$F_{AB}(y) = F_{AB\,(k)} y^k \frac{1}{\sqrt{2\pi R}} \tag{6.17}$$

and define

$$F_{AB}(\tilde{y}) \equiv F_{AB\,(k)} \tilde{y}_k \frac{1}{\sqrt{2\pi R}} \quad , \quad F_{AB}(\check{y}) \equiv F_{AB\,(k)} \check{y}_k \frac{1}{\sqrt{2\pi R}} \quad . \tag{6.18}$$

Finally the action reads

$$\begin{aligned} A_{YM} &= \textstyle\int_{M^D \times Z_N} Tr[F_{\mu\nu} F^{\mu\nu} + F_{\alpha\gamma} F^{\alpha\gamma} + 2 F_{\alpha\mu}(y) F^{\alpha\mu}(\tilde{y}) + \\ &\quad + 2 F_{\gamma\mu}(y) F^{\gamma\mu}(\tilde{y}) - 2i F_{\alpha\mu}(y) F^{\gamma\mu}(\check{y}) + 2i F_{\gamma\mu}(y) F^{\alpha\mu}(\check{y})\,]\, vol \\ &= \textstyle\sum_k \int_{M^D} \sqrt{|\det g_{\mu\nu}|}\, Tr[F_{\mu\nu\,(k)} F^{\mu\nu}{}_{(-k)} + F_{\alpha\gamma\,(k)} F^{\alpha\gamma}{}_{(-k)} + 2 q^k F_{\alpha\mu\,(k)} F^{\alpha\mu}{}_{(-k)} \\ &\quad + 2 q^k F_{\gamma\mu\,(k)} F^{\gamma\mu}{}_{(-k)} - 2i (q^k - q^{-k}) F_{\alpha\mu\,(-k)} F^{\gamma\mu}{}_{(k)}\,] \end{aligned} \tag{6.19}$$

where in the second equality we have integrated on Z_N using (6.10) and $\sum_{j=0}^{N-1} q^{kj} = N\ \delta_{k,0}$.

In the $N \to \infty$ limit $F_{AB(k)}$ becomes the $k-th$ Fourier mode, and the action (6.19) becomes the dimensional reduction (in the direction γ) of the usual Yang-Mills action on $M^D \times S_1 \times S_1$, where the first S_1 is in the α direction and the second S_1 is in the γ direction. It is therefore Yang-Mills theory on $M^D \times S_1$ plus the adjoint scalar $\phi = A_\gamma$.

The interesting feature of this action arises for finite N: in this case we have a nontrivial scalar potential term. Using the link fields U it is given in (5.18). These variables are not convenient in the present section because they are ill defined in the $N \to \infty$ limit.

7 Born - Infeld Theory on finite group spaces

We recall the continuum D-dimensional Born - Infeld action for non-linear electrodynamics [9] in flat space

$$A_{BI} = \int_{M^D} d^D x \sqrt{\det(\delta_{\mu\nu} + F_{\mu\nu})} \tag{7.1}$$

When $D = 4$ this action takes the form (after explicitly computing the determinant):

$$A_{BI} = \int_{M^D} d^D x \ \sqrt{1 + \frac{1}{2} F \cdot F + \left(\frac{1}{4} F \cdot \tilde{F}\right)^2}\ , \tag{7.2}$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\, \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$, the dot in products means complete index contraction (for ex. $F \cdot F \equiv F_{\mu\nu} F_{\mu\nu}$), and we consider the euclidean theory.

The action (7.1) can be generalized to the nonabelian case: then $F_{\mu\nu}$ is $\mathcal{G}$ Lie algebra valued, and the determinant in (7.1) is not a number any more but belongs to the universal enveloping algebra of $\mathcal{G}$. We can define its "absolute value" $|\det|$ (still belonging to the enveloping algebra) as the positive square root in $\sqrt{\det\ \det^\dagger}$. Any square root of $|\det|$ is hermitian, so that an overall trace produces a real action:

$$A_{BI} = \int_{M^D} d^D x \ Tr \sqrt{|\det(\delta_{\mu\nu} + F_{\mu\nu})|} \tag{7.3}$$

This is a gauge invariant action for any choice of the square root. The trace can be symmetrized so to fix ordering ambiguities in products of the Lie valued $F_{\mu\nu}$ elements (see for ex. [23] and included references).

We address now the problem of formulating BI theory (for a nonabelian gauge group $\mathcal{G}$) on finite group spaces.

We first consider the special $D = 4$ case (7.2) on finite groups: since the indices of F run here on 4 values, we take finite groups with four independent tangent vectors (for ex. $Z_N \times Z_N$ with the *-bicovariant calculus of Section 6.1). Later we present a generalization of the action (7.1) on any finite group G.

As we have done for the Yang-Mills action, we replace the term $F \cdot F$ by $< F, F > = F_{rs} F^\dagger_{rs}$. Using the bimodule pairing (2.39), also the quartic term $(F \cdot \tilde{F})^2$ can be replaced by the gauge covariant expression

$$< F^2, F^2 > \equiv < F \wedge F, F \wedge F > \tag{7.4}$$

transforming under gauge variations as $< F'^2, F'^2 >= T < F^2, F^2 > T^{-1}$. Then the whole action:

$$A_{BI} = \sum_G Tr \sqrt{1 + \frac{1}{2} < F, F > + \frac{1}{16} < F^2, F^2 >} \tag{7.5}$$

is gauge invariant, and reproduces in the continuum case the Born-Infeld action for non-abelian gauge fields.

Let us explore in more detail the structure of $< F^2, F^2 >$, and its expression in terms of the link field U_h. The differential 4-form F^2

$$\begin{aligned} F^2 = U^4 = U_{h_1} [\mathcal{R}_{h_1} U_{h_2}] [\mathcal{R}_{h_1 h_2} U_{h_3}] [\mathcal{R}_{h_1 h_2 h_3} U_{h_4}] A^{h_1,h_2,h_3,h_4}_{h'_1,h'_2,h'_3,h'_4} \theta^{h'_1} \otimes \theta^{h'_2} \otimes \theta^{h'_3} \otimes \theta^{h'_4} = \\ = (F^2)_{h_1,h_2,h_3,h_4} \theta^{h_1} \otimes \theta^{h_2} \otimes \theta^{h_3} \otimes \theta^{h_4} , \end{aligned} \tag{7.6}$$

transforms under gauge variations as:

$$F^2 \longrightarrow T\, F^2\, T^{-1} \;\; , \;\; (F^2)_{h_1,h_2,h_3,h_4} \longrightarrow T\ (F^2)_{h_1,h_2,h_3,h_4}\ [\mathcal{R}_{h_1 h_2 h_3 h_4} T^{-1}] \, . \tag{7.7}$$

Consider then the quartic term (sum on the indices h_i understood)

$$(F^2)_{h_1,...,h_4} \, (F^2)^\dagger_{h_1,...,h_4} \tag{7.8}$$

This term is gauge covariant, i.e. it transforms as $T \cdots T^{-1}$, and in fact coincides with $< F^2, F^2 >$ (same proof as for $< F, F >= F_{rs} F^\dagger_{rs}$, in particular use $F^2 = F^{2\dagger} \Rightarrow (F^2)^\dagger_{h_1,h_2,h_3,h_4} = R_{h_1 h_2 h_3 h_4} (F^2)_{h_4^{-1}, h_3^{-1}, h_2^{-1}, h_1^{-1}}$).

Note 1: Formulae (7.6), (7.7), (7.8) hold also if $D > 4$, not only for $D = 4$. In the case $D = 4$ the antisymmetrizer in (7.6) factorizes into a product of two epsilon tensors, so that

$$F^2 = U_{h_1} [\mathcal{R}_{h_1} U_{h_2}] [\mathcal{R}_{h_1 h_2} U_{h_3}] [\mathcal{R}_{h_1 h_2 h_3} U_{h_4}] \, \varepsilon^{h_1 h_2 h_3 h_4} vol \equiv F \tilde{F} vol \tag{7.9}$$

and (7.8) also factorizes as (recall(2.46) and use $vol^* = vol$):

$$(F^2)_{h_1,...,h_4} \, (F^2)^\dagger_{h_1,...,h_4} =< F^2, F^2 >=< F^2, (F^2)^\dagger >= \mathcal{N} F \tilde{F} \, (F \tilde{F})^\dagger \, . \tag{7.10}$$

Born-Infeld action on arbitrary finite groups

The analogue of $\delta_{\mu\nu} + F_{\mu\nu}$ becomes simply $E_{g,h} \equiv \delta_{g,h^{-1}} + F_{g,h}$, cf. (2.40), and transforms under gauge variations in the same way as $F_{g,h}$:

$$E'_{g,h} = \delta_{g,h^{-1}} + T F_{g,h} \mathcal{R}_{gh} T^{-1} = T(\delta_{g,h^{-1}} + F_{g,h}) \mathcal{R}_{gh} T^{-1} = T E_{g,h} \mathcal{R}_{gh} T^{-1} \tag{7.11}$$

We need now a gauge covariant definition of determinant for a matrix transforming as in (7.11), and that possibly reduces to the usual determinant in some limit that recovers the continuum case. This limit exists for $N \to \infty$ in the case $G = Z_N \times Z_N \times ... \times Z_N$, and indeed the definition we propose in the following has this property.

Lemma:

$$\epsilon^{g_1 \ldots g_p} \epsilon^{h_1 \ldots h_p} \ \mathcal{R}_{g_1 \ldots g_p} \mathcal{R}_{h_1 \ldots h_p} = \epsilon^{g_1 \ldots g_p} \epsilon^{h_1 \ldots h_p} \ id \quad \text{(no sums on } g, \, h) \tag{7.12}$$

Proof: for any function f:

$$\begin{aligned}
&\epsilon^{g_1 \ldots g_p} \epsilon^{h_1 \ldots h_p} \ f \ \mathcal{N} = < \theta^{g_1} \wedge \ldots \wedge \theta^{g_p}, \theta^{h_1} \wedge \ldots \wedge \theta^{h_p} > \ f = \\
&= < \theta^{g_1} \wedge \ldots \wedge \theta^{g_p}, (\mathcal{R}_{h_1 \ldots h_p} f) \ \theta^{h_1} \wedge \ldots \wedge \theta^{h_p} > = \\
&(\mathcal{R}_{g_1 \ldots g_p} \mathcal{R}_{h_1 \ldots h_p} f) \ < \theta^{g_1} \wedge \ldots \wedge \theta^{g_p}, \theta^{h_1} \wedge \ldots \wedge \theta^{h_p} > = \\
&\epsilon^{g_1 \ldots g_p} \epsilon^{h_1 \ldots h_p} \ (\mathcal{R}_{g_1 \ldots g_p} \mathcal{R}_{h_1 \ldots h_p} f) \ \mathcal{N}
\end{aligned} \tag{7.13}$$

proving the Lemma ($\mathcal{N}$ is defined as $< vol, vol >$, cf. (2.46)).

Proposition 1: the determinant of the matrix $E_{g,h}$ defined as:

$$\begin{aligned}
&\det{}_G E_{g,h} = \\
&\epsilon^{g_1, \ldots, g_p} \ E_{g_1,h_1} \ (\mathcal{R}_{g_1 h_1} E_{g_2,h_2}) \ (\mathcal{R}_{g_1 h_1 g_2 h_2} E_{g_3,h_3}) \ldots (\mathcal{R}_{g_1 h_1 g_2 h_2 \ldots g_{p-1} h_{p-1}} E_{g_p,h_p}) \ \epsilon^{h'_1, \ldots, h'_p} \equiv \\
&\equiv \epsilon^{g_1, \ldots, g_p} \ E_{g_1,h_1,g_2,h_2,\ldots,g_p,h_p} \ \epsilon^{h'_1, \ldots, h'_p}
\end{aligned} \tag{7.14}$$

where $h'_n = (g_{n+1} g_{n+2} \ldots g_p)^{-1} h_n (g_{n+1} g_{n+2} \ldots g_p)$, transforms covariantly:

$$\det{}_G E'_{g,h} = T \ \det{}_G E_{g,h} \ T^{-1} \tag{7.15}$$

Proof: the quantity defined in the last line of (7.14) transforms as:

$$\begin{aligned}
&E'_{g_1,h_1,g_2,h_2,\ldots,g_p,h_p} = T \ E_{g_1,h_1,g_2,h_2,\ldots,g_p,h_p} \ (\mathcal{R}_{g_1 h_1 g_2 h_2 \ldots g_p h_p} T^{-1}) = \\
&= T \ E_{g_1,h_1,g_2,h_2,\ldots,g_p,h_p} \ (\mathcal{R}_{g_1 g_2 \ldots g_p h'_1 h'_2 \ldots h'_p} T^{-1})
\end{aligned} \tag{7.16}$$

where we used the definition of the h' indices. Then recalling $\mathcal{R}_{g_1 g_2 \ldots g_p h'_1 h'_2 \ldots h'_p} = \mathcal{R}_{g_1 \ldots g_p} \mathcal{R}_{h'_1 \ldots h'_p}$ the gauge variation of $\det_G E_{g,h}$ reads

$$\det{}_G E'_{g,h} = T \ \epsilon^{g_1, \ldots, g_p} \ E_{g_1,h_1,g_2,h_2,\ldots,g_p,h_p} (\mathcal{R}_{g_1 \ldots g_p} \mathcal{R}_{h'_1 \ldots h'_p} T^{-1}) \epsilon^{h'_1, \ldots, h'_p} = T \ \det{}_G E_{g,h} \ T^{-1} \tag{7.17}$$

after using the Lemma (7.12) in the last equality.

For abelian G and constant matrices this determinant coincides with the usual determinant, multiplied by $p!$ since we have defined it by means of two ϵ tensors. We could equally well define $\det_G$ by fixing the order of $g_1 \ldots g_p$ to be the one that defines the volume form, i.e. $vol = \theta^{g_1} \wedge \ldots \wedge \theta^{g_p}$ so that $\epsilon^{g_1 \cdots g_p} = 1$ disappears from the formula and the indices $g_1 \ldots g_p$ are not summed any more.

The definition adopted in (7.14) has the advantage of suggesting a second (inequivalent) way to define a covariant determinant.

Proposition 2: the determinant of the matrix $E_{g,h}$ defined as:

$$\overline{\det}_G E_{g,h} \equiv \epsilon^{g'_1, \ldots, g'_p} \ E_{g_1,h_1,g_2,h_2,\ldots,g_p,h_p} \ \epsilon^{h_1, \ldots, h_p} \tag{7.18}$$

where $g'_n = (h_1 h_2 \ldots h_{n-1}) g_n (h_1 h_2 \ldots h_{n-1})^{-1}$, transforms covariantly:

$$\overline{\det}_G E'_{g,h} = T \; \overline{\det}_G E_{g,h} \; T^{-1} \tag{7.19}$$

The proof is quite similar to the previous one, and uses now the factorization $\mathcal{R}_{g_1 h_1 g_2 h_2 \ldots g_p h_p} = \mathcal{R}_{g'_1 g'_2 \ldots g'_p} \mathcal{R}_{h_1 h_2 \ldots h_p}$. In general $\overline{\det}_G E_{g,h} \neq \det_G E_{g,h}$ (they coincide for abelian G).

Proposition 3: the two inequivalent covariant definitions of determinant given above are related by hermitian conjugation:

$$(\det_G E_{g,h})^\dagger = \overline{\det}_G E_{g,h} \tag{7.20}$$

Proof: by direct computation, using

$$E^\dagger_{g,h} = \mathcal{R}_{gh} E_{h^{-1},g^{-1}} \tag{7.21}$$

and (2.45). Thus for abelian finite groups $\det_G E_{g,h}$ is real.

A covariant *hermitian* "absolute value" of the determinant can be defined as:

$$|\det_G(E_{g,h})| \equiv \sqrt{\det_G E_{g,h} \overline{\det}_G E_{g,h}} \tag{7.22}$$

choosing the positive square root.

We can use $|\det_G|$ to construct a real $\mathcal{G}$-gauge invariant Born-Infeld action on the finite group G:

$$A^G_{BI} = \int_G Tr \; \sqrt{|\det_G \; (\delta_{g,h^{-1}} + F_{g,h})|} \; vol(G) = \sum_G Tr \; \sqrt{|\det_G \; (\delta_{g,h^{-1}} + F_{g,h})|} \tag{7.23}$$

for any choice of square root; a positive definite action is obtained by choosing the positive square root. Note that for finite groups the order of the F factors in the definition of $\det_G$ is fixed by (7.14) and (7.18).

As an example consider $G = Z_N$ with the 2-dimensional differential calculus involving the two left-invariant 1-forms θ^u and $\theta^{u^{-1}}$. The matrix $E_{g,h}$ becomes:

$$\begin{pmatrix} 0 & 1 + F_{+-} \\ 1 - F_{+-} & 0 \end{pmatrix} , \tag{7.24}$$

where $F_{+-} = F_{u,u^{-1}}$. The determinant (7.14) in this case coincides with the usual determinant

$$\det_G E_{g,h} = \det E_{g,h} = -1 + F^2_{+-} \quad . \tag{7.25}$$

A real gauge invariant BI action is then obtained as:

$$A_{BI} = \sum_{Z_N} Tr \; \sqrt{|-1 + F^2_{+-}|} \tag{7.26}$$

the positive square root yielding a positive definite action.

8 Born - Infeld Theory on $M^D \times G$

It is convenient to extract the $E = g + F$ matrix from the following expansion of F:

$$F \equiv \frac{1}{2} F_{\mu\nu}\ dx^\mu \wedge dx^\nu + F_{\mu g}\ dx^\mu \wedge \theta^g + F_{g,h}\ \theta^g \otimes \theta^h =$$

$$= dx^\mu \otimes F_{\mu\nu}\ dx^\nu + dx^\mu \otimes F_{\mu h}\ \theta^h - \theta^g \otimes\ \mathcal{R}_{g^{-1}} F_{\nu g}\ dx^\nu + \theta^g \otimes \mathcal{R}_{g^{-1}} F_{g,h}\ \theta^h =$$

$$= (dx^\mu\ ,\ \theta^g) \otimes \begin{pmatrix} F_{\mu\nu} & F_{\mu h} \\ -\ \mathcal{R}_{g^{-1}} F_{\nu g} & \mathcal{R}_{g^{-1}} F_{g,h} \end{pmatrix} \begin{pmatrix} dx^\nu \\ \theta^h \end{pmatrix} \tag{8.1}$$

where we used

$$dx^\mu \wedge \theta^g = dx^\mu \otimes \theta^g - \theta^g \otimes dx^\mu$$

Then the $E_{A,B}$ $(A = \mu, g,\ B = \nu, h)$ matrix is given by:

$$E_{A,B} = \begin{pmatrix} g_{\mu\nu} + F_{\mu\nu}\ , & F_{\mu h} \\ -\ \mathcal{R}_{g^{-1}} F_{\nu g}\ , & \delta_{g,h^{-1}} + \mathcal{R}_{g^{-1}} F_{g,h} \end{pmatrix} \tag{8.2}$$

Next we have to define the determinant of a matrix with mixed indices μ, g. Again we find a definition that ensures gauge invariance, and a correct continuum limit when it exists. Consider the algebraic identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & B D^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - B D^{-1} C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1} C & 1 \end{pmatrix} \tag{8.3}$$

With the usual definition of determinant this implies:

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\ (A - BD^{-1}C)(\det\ D) \tag{8.4}$$

For our purposes we define the modified determinant Det:

$$\left[Det \begin{pmatrix} A_{\mu\nu} & B_{\mu h} \\ C_{g\nu} & D_{g,h} \end{pmatrix} \right]^2 = \overline{\det}_G\ (\mathcal{R}_g D_{g,h})(\det M_{\mu\nu})^\dagger \det M_{\mu\nu}\ \det_G\ (\mathcal{R}_g D_{g,h}) \tag{8.5}$$

with $M_{\mu\nu} \equiv A_{\mu\nu} - B_{\mu h}(D^{-1})^{h,g} C_{g,\nu}$. For the matrix $E_{A,B}$ the modified determinant reads

$$(Det\ E_{A,B})^2 = \overline{\det}_G\ (\delta_{g,h^{-1}} + F_{g,h})\ (\det M_{\mu\nu})^2\ \det_G\ (\delta_{g,h^{-1}} + F_{g,h}) \tag{8.6}$$

where

$$M_{\mu\nu} \equiv g_{\mu\nu} + F_{\mu\nu} + 4 F_{\mu h}\ (H^{-1})^{h,g}\ \mathcal{R}_{g^{-1}} F_{\nu g} \tag{8.7}$$

$$H_{g,h} \equiv \delta_{g,h^{-1}} + \mathcal{R}_{g^{-1}} F_{g,h} \tag{8.8}$$

and we have used $(\det M_{\mu\nu})^\dagger = \det M_{\mu\nu}$. To prove this consider first the hermitian conjugate of H and H^{-1}:

$$H^\dagger_{g,h} = \delta_{g,h^{-1}} + \mathcal{R}_{g^{-1}}\,\mathcal{R}_{gh}\,F_{h^{-1},g^{-1}} = H_{h^{-1},g^{-1}} \Longrightarrow ((H^{-1})^{h,k})^\dagger = (H^{-1})^{k^{-1},h^{-1}} \tag{8.9}$$

Then it follows easily that the matrix $M_{\mu\nu}$ of (8.7) satisfies $M^\dagger_{\mu\nu} = M_{\nu\mu}$: its determinant

$$\det M_{\mu\nu} = \epsilon^{\mu_1 ... \mu_D} M_{\mu_1\nu_1} ... M_{\mu_D\nu_D}\epsilon^{\nu_1...\nu_D} \tag{8.10}$$

remains unchanged under hermitian conjugation. This implies that the whole $(Det\ E_{AB})^2$ of (8.6) is hermitian.

Next we prove that $(Det\ E_{AB})^2$ transforms covariantly. The matrix H and its inverse H^{-1} transform as

$$H'_{g,h} = (\mathcal{R}_{g^{-1}}T)\ H_{g,h}\ (\mathcal{R}_h T^{-1}) \quad \Rightarrow \quad (H^{-1})^{h,g} = (\mathcal{R}_h T)\ (H^{-1})^{h,g}\ (\mathcal{R}_{g^{-1}}T^{-1}) \tag{8.11}$$

so that $M_{\mu\nu}$ transforms as $M'_{\mu\nu} = TM_{\mu\nu}T^{-1}$. As a consequence $(Det\ E_{A,B})^2$ transforms covariantly

$$(Det\ E'_{A,B})^2 = T\ (Det\ E_{A,B})^2\ T^{-1} \tag{8.12}$$

A real gauge invariant BI action on $M^D \times G$ can be constructed by taking twice the square root of $(Det\ E_{AB})^2$:

$$A_{BI} = \int d^D x\ \sum_G$$

$$Tr\ \sqrt{|\det\ [g_{\mu\nu} + F_{\mu\nu} + (D_\mu U_h)\ (H^{-1})^{h,g}\ (D_\nu U_{g^{-1}})^\dagger]|}\ \sqrt{|\det{}_G\ (\delta_{g,h^{-1}} + F_{g,h})|} \tag{8.13}$$

using the U field of (5.11).

As an example we consider $G = Z_N$. Then

$$H_{g,h} = \begin{pmatrix} 0 & 1 + \mathcal{R}_{u^{-1}}F_{+-} \\ 1 - \mathcal{R}_u F_{+-} & 0 \end{pmatrix} \Rightarrow (H^{-1})^{g,h} = \begin{pmatrix} 0 & \frac{1}{1-\mathcal{R}_u F_{+-}} \\ \frac{1}{1+\mathcal{R}_{u^{-1}}F_{+-}} & 0 \end{pmatrix} \tag{8.14}$$

and

$$M_{\mu\nu} = g_{\mu\nu} + F_{\mu\nu} + F_{\mu h}\ (H^{-1})^{h,g}\ \mathcal{R}_{g^{-1}}F_{\nu g} =$$

$$= g_{\mu\nu} + F_{\mu\nu} + F_{\mu +}\ \frac{1}{1 - \mathcal{R}_u F_{+-}}\ \mathcal{R}_u F_{\nu -} + F_{\mu -}\ \frac{1}{1 + \mathcal{R}_{u^{-1}}F_{+-}}\ \mathcal{R}_{u^{-1}}F_{\nu +} =$$

$$= g_{\mu\nu} + F_{\mu\nu} + \sum_{g=u^{\pm 1}}\ (D_\mu U_g)\ \mathcal{R}_g \left(\frac{1 + F_{g,g^{-1}}}{1 - F^2_{g,g^{-1}}}\right)\ (D_\nu U_g)^\dagger \tag{8.15}$$

The BI action becomes:

$$\int d^D x\ \sum_{Z_N} Tr\ \left[\sqrt{|-1 + F^2_{+-}|}\ \sqrt{|\det\ M_{\mu\nu}|}\right]$$

and is explicitly real and gauge invariant.

Acknowledgements

We thank Chiara Pagani for useful comments.
The work of P.A. was supported by a Marie Curie Fellowship of the EC programme IHP, contract number MCFI-2000-01982. The work of P.A. and L.C. was supported in part by EC under RTN project HPRN-CT-2000-00131. The work of A.P.I. was supported in part by the RFBR grant 00-01-00299 and by the INFN-JINR(Dubna) program. P.A and A.P.I. are grateful to the Dipartimento di Fisica Teorica of Torino University, where work on this paper was started, and A.P.I also thanks the Max-Planck-Institut für Mathematik in Bonn, where he continued this work, for their kind hospitality and support.

A Hopf algebraic formulas for the differential calculus on Fun(G)

The G group structure induces a Hopf algebra structure on $Fun(G)$, with coproduct Δ, coinverse κ and counit ε defined by group multiplication, inverse and unit as:

$$\Delta(f)(g,g') = f(gg'), \quad \Delta : Fun(G) \to Fun(G) \otimes Fun(G) \tag{A.1}$$
$$\kappa(f)(g) = f(g^{-1}), \quad \kappa : Fun(G) \to Fun(G) \tag{A.2}$$
$$\varepsilon(f) = f(e), \quad \varepsilon : Fun(G) \to \boldsymbol{C} \tag{A.3}$$

In the first line we have used $Fun(G \times G) \approx Fun(G) \otimes Fun(G)$ [indeed a basis for functions on $G \times G$ is given by $x^{g_1} \otimes x^{g_2}, g_1, g_2 \in G$]. On the basis functions x^g the costructures take the form:

$$\Delta(x^g) = \sum_{h \in G} x^h \otimes x^{h^{-1}g} \equiv x^g_{(1)} \otimes x^g_{(2)}, \quad \kappa(x^g) = x^{g^{-1}}, \quad \varepsilon(x^g) = \delta^g_e \tag{A.4}$$

Left and right coactions on Γ

$$\Delta_L(adb) = a_{(1)}\, b_{(1)} \otimes a_{(2)}\, db_{(2)} \ , \quad \Delta_R(adb) = a_{(1)}\, db_{(1)} \otimes a_{(2)}\, b_{(2)} \ . \tag{A.5}$$

where $\Delta(a) \equiv a_{(1)} \otimes a_{(2)}$.

Left and right invariant forms

$$\theta^g = \kappa(x^g_{(1)})\, d\, x^g_{(2)} = \sum_h x^h\, d\, x^{hg} = \sum_h x^{h\, g^{-1}}\, d\, x^h \tag{A.6}$$
$$\zeta^g = d(x^g_{(1)})\, \kappa(x^g_{(2)}) = \sum_h (dx^{gh})\, x^h = -\sum_h x^{gh} dx^h \ . \tag{A.7}$$

Bimodule relations and ${f^g}_{g'}$ *functionals*

Definition of the ${f^g}_{g'}$ functionals:

$$\theta^g a = ({f^g}_{g'} * a)\ \theta^{g'} \equiv (id \otimes {f^g}_{g'})\Delta(a)\theta^{g'} \tag{A.8}$$

Applying the rule $\theta^g a = (\mathcal{R}_g a)\theta^g$ (2.15) to the basis functions x^g yields

$${f^g}_{g'} = \delta^g_{g'} g, \quad g \neq e,\ g' \neq e \tag{A.9}$$

where we denote as g the functional on $Fun(G)$ dual to the basis function x^g, i.e. $g(x^{g'}) = \delta^{g'}_g$

Left and right coaction on Γ

Left and right coaction of G on Γ are given by the mappings $\Delta_L : \Gamma \to Fun(G) \otimes \Gamma$ and $\Delta_R : \Gamma \to \Gamma \otimes Fun(G)$ that encode the information about all left or right translations:

$$\Delta_L(a\rho b) = \Delta(a)\Delta_L(\rho)\Delta(b), \quad \Delta_L(db) = (id \otimes d)\Delta(b) \quad \forall a, b \in Fun(G),\ \rho \in \Gamma \tag{A.10}$$

$$\Delta_R(a\rho b) = \Delta(a)\Delta_R(\rho)\Delta(b), \quad \Delta_R(db) = (d \otimes id)\Delta(b) \quad \forall a, b \in Fun(G),\ \rho \in \Gamma \tag{A.11}$$

For example their action on the basic terms $x^{g_1} dx^{g_2} \in \Gamma$ is:

$$\Delta_L(x^{g_1} dx^{g_2}) = \Delta(x^{g_1})(id \otimes d)\Delta(x^{g_2}) = \sum_{h \in G} x^h \otimes x^{h^{-1}g_1} dx^{h^{-1}g_2} \tag{A.12}$$

$$\Delta_R(x^{g_1} dx^{g_2}) = \Delta(x^{g_1})(d \otimes id)\Delta(x^{g_2}) = \sum_{h \in G} x^{g_1 h} dx^{g_2 h} \otimes x^{h^{-1}} \tag{A.13}$$

Computing Δ_L and Δ_R on the basic differentials yields:

$$\Delta_L(dx^{g_1}) \equiv (id \otimes d)(\Delta x^{g_1}) = \sum_{h \in G} x^h \otimes dx^{h^{-1}g_1} \tag{A.14}$$

$$\Delta_R(dx^{g_1}) \equiv (d \otimes id)(\Delta x^{g_1}) = \sum_{h \in G} dx^h \otimes x^{h^{-1}g_1} \tag{A.15}$$

Adjoint representation

The adjoint representation matrix ${M_{g'}}^g$ is defined by:

$$\Delta_R(\theta^g) = \theta^{g'} \otimes {M_{g'}}^g \tag{A.16}$$

On the other hand computing the right coaction on θ^g according to the formulas of the previous paragraph yields:

$$\Delta_R(\theta^g) = \sum_h \theta^{hgh^{-1}} \otimes x^h = \sum_k \theta^k \otimes \sum_h \delta^{hgh^{-1}}_k x^h \tag{A.17}$$

so that

$${M_{g'}}^g = \sum_h \delta^{hgh^{-1}}_{g'} x^h \tag{A.18}$$

Braiding matrix Λ

$$\Lambda^{gg'}{}_{k'k} = f^g{}_k(M_{k'}{}^{g'}) = \delta^g_k \delta^{gg'g^{-1}}_{k'}, \quad g \neq e,\ g' \neq e \tag{A.19}$$

Hopf algebra structure of Ω

$$\begin{gathered} \Delta(\theta^g) \equiv \Delta_L(\theta^g) + \Delta_R(\theta^g) = 1 \otimes \theta^g + \sum_h \theta^{hgh^{-1}} \otimes x^h \;, \\ \epsilon(\theta^g) = 0 \;, \quad \kappa(\theta^g) = -\sum_h \theta^{h^{-1}gh}\, x^h \;. \end{gathered} \tag{A.20}$$

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