Generalized connections and characteristic classes

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Abstract. We study derivations on a smooth manifold, its twisted de Rham cohomology, generalized connections on vector bundles and their characteristic classes.

0. INTRODUCTION

On a smooth manifold M the most important differential operator is the exterior differential operator d, and it is the only "natural" first order operator up to a multiplicative scalar [5]. This standard d is well understood, e.g., we have de Rham cohomology and Poincaré lemma. But if we are interested in the *special* geometric properties of M, instead of general or functorial properties, then the ordinary d is not enough, since it is too simple and too general. For instance, Laplace operators or Yang-Mills equations play an imporant role to understand the special object M. Also for a Morse function f on M, the operator $d_t = e^{-tf} \circ d \circ e^{tf}$, a 1-parameter family of connections on a trivial line bundle, is used to obtain the Morse inequalities [7]. In this paper we will study all possible (first order) derivations (1.1) on M, which includes, of course, the Cauchy-Riemann operators, when M admits a holomorphic structure. In section 1 we review basic properties of derivations [3] and its twisted de Rham cohomologies. These derivations are used to define generalized connections (section 2). Then semi-connections [4] are generalized connections when M is a complex manifold. These generalized connections appear also in the study of Higgs bundles (e.g., [6]). When derivations are integrable (1.5), characteristic classes are defined with values in the twisted de Rham cohomology spaces. The main result (2.5) and the following remarks describe the relation between ordinary characteristic classes and the twisted ones.

The original motivation of the study was to understand under what condition on a complex manifold X, the m-th plurigenus

$$P_m(X) = \dim H^0(X, K_X^{\otimes m})$$

is a smooth or deformation invariant. A naive approach is the following. For any topological line bundle L over a smooth manifold M, assign a D-connection $\nabla : A^0(L) \to A^1(L)$ to each derivation D on M, in some canonical way, and study its kernel $H^0_\nabla(M, L)$, like the Riemann-Roch problem. Then the reformulation of the problem is that whether the "D-genus of L"

$$P_D(L) = \dim H^0_\nabla(M, L)$$

is independent of the choice of D, when restricted to Cauchy-Riemann operators (i.e., holomorphic structures). When D is the ordinary exterior derivation d, the canonical choice of the connection on L is the one with the "harmonic curvature" (with respect to some Riemannian metric on M), and when D is a Cauchy-Riemann operator $\bar{\partial}$, then the canonical $\bar{\partial}$-connection on L is a holomorphic structure on L. If M is a (compact, oriented) 4-manifold with b^+, the number of positive eigenvalues of the intersection form, equal to 1, then every holomorphic structure on M has a vanishing geometric genus and consequently every topological line bundle has a holomorphic structure. The uniqueness of such structure is guaranteed if the first Betti number $b_1(M)$ vanishes. The final product was somewhat independent of the original motivation.

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1. Derivations

Let M be a compact connected smooth manifold. The space of *complex* valued smooth differential p-forms on M will be denoted by A^p. T and $T^\vee$ denote the *complexified* tangent and cotangent bundle of M and for any complex vector bundle E, the space of smooth p-forms on M with values in E will be denoted by $A^p(E)$. Then A^0 is the algebra of complex valued smooth functions on M and $A^0(T)$ is the space of (complex) vector fields on M.

1.1. Definition. A *complex derivation of degree 1* or simply a *derivation* on M is a complex linear map

$$D : A^0 \to A^1$$

such that $D(fg) = D(f)g + fD(g)$ for $f, g \in A^0$.

Higher degree derivations are obtained if we consider operators from A^0 to A^p. *Real derivations* can be complexified to get complex derivations. Note that derivations are *local*, i.e.,

$$support(Df) \subset support(f), \qquad f \in A^0$$

and hence they can be interpreted as morphisms of sheaves.

Let $\mathcal{D}$ be the set of all derivations on M. Then it is an A^0-module isomorphic to $A^0(\operatorname{End} T) \simeq A^0(\operatorname{End} T^\vee)$. For, if $F : T \to T$ or its dual $F^\vee : T^\vee \to T^\vee$ is given, then

$$D = d^F := F^\vee \circ d$$

defines a derivation and any derivation is defined in this way uniquely, since

$$F(X)(f) = X \rfloor Df =: D_X f$$

for a function f and a vector field X on M. Under this isomorphism the ordinary exterior derivation d corresponds to the identity endomorphism I. If $F = 1/2(I + \sqrt{-1}J)$, where $J : T \to T$ is an almost complex structure (i.e., a real operator with $J^2 = -I$), then

$$\bar{\partial} = d^F$$

is the Cauchy-Riemann operator, provided J is integrable. In general a derivation $D = d^F$ may be considered as "d twisted by F".

Note that the diffeomorphism group Diff(M) acts on $\mathcal{D}$ by

$$\phi \cdot D = \phi^{*-1} \circ D \circ \phi^*$$

and the stabilizer of D is equal to Diff(M) if and only if D is proportional to d [5]. We say that two derivations are *equivalent* if they are in the same orbit.

The space $\mathcal{D}^p$ of derivations of degree p is also an A^0-module isomorphic to $A^p(T)$.

For $D \in \mathcal{D}$, we put

$$H^0_D(M) = ker(D).$$

Note that constant functions are always contained in this kernel and equivalent derivations have the isomorphic kernels.

1.2. Definitions. A derivation D is said to be *regular* if $H^0_D(M) = \mathbb{C}$. If the associated endomorphism $F : T \to T$ of a derivation D is an isomorphism, we say that D is *elliptic*.

The set of regular derivations form an open dense subset of $\mathcal{D}$. Regularity and ellipticity are Diff(M)-invariant notions. It is obvious that elliptic derivations are regular. But there are many non-elliptic regular derivations, e.g., if F is an isomorphism on a proper dense subset and singular outside, then the associated derivation is non-elliptic and regular. Holomorphic structures $\bar\partial$ are also such examples.

1.3. Theorem [3]. *Any derivation D has a unique complex linear extension $D : A^p \to A^{p+1}$ such that*

$$(1.3.1) \qquad Dd + dD = 0$$

and

$$(1.3.2) \qquad D(\xi \wedge \eta) = D(\xi) \wedge \eta + (-1)^p \xi \wedge D\eta$$

for $\xi \in A^p$ and $\eta \in A^q$.

For example, if $f \in A^0$, then $d_f := f \cdot d : A^0 \to A^1$ has the extension

$$(1.4) \qquad d_f | A^p = f \cdot d - p \cdot df$$

where $df : A^p \to A^{p+1}$ is the exterior multiplication.

The "symbol" of $d^F : A^p \to A^{p+1}$ at $\xi \in T^\vee_x$, $x \in M$,

$$\sigma(D, \xi) : \wedge^p T^\vee_x \to \wedge^{p+1} T^\vee_x$$

is the exterior multiplication by $F^\vee(\xi)$. Thus our notion of ellipticity (1.2) is correct.

If D is a derivation *associated to* $F : T \to T$, then $D \circ D : A^0 \to A^2$ is a derivation of degree 2 and hence it defines an element N_F of $A^2(T)$. Then

$$N_F(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y]$$

for vector fields X, Y. We will call N_F the *Nijenhuis tensor* associated to an endomorphism F or a derivation D.

1.5. Definition. A derivation D is *integrable* if $D \circ D = 0$.

1.6. Theorem [3]. *A derivation is integrable if and only if the associated Nijenhuis tensor vanishes.*

PROOF: Note that $D \circ D = 0$ if and only if $D \circ D | A^0 = 0$ by (1.3.2). Now the result follows from the identity

$$D^2 f(X,Y) = N_F(X,Y)f$$

where f is a function and X, Y vector fields. ∎

Note that for any function f on M, d_f is an integrable derivation (1.4). If D is integrable, then for any scalar λ, λD and $D + \lambda d$ are integrable. In particular, if J is an integrable almost complex structure, then $F = 1/2(I + \sqrt{-1}J)$ is also integrable.

When D is integrable, we call

$$H^p_D(M) := ker(D : A^p \to A^{p+1})/D(A^{p-1})$$

the *p-th cohomology space* of M associated to D. By (1.3.2)

$$H^\bullet_D(M) = \sum_{p \geq 0} H^p_D(M)$$

is a graded algebra over $\mathbb{C}$. These "twisted" cohomology spaces are finite dimensional when D is elliptic (and integrable) or is a holomorphic structure.. When $D = d$,

$$H^\bullet_d(M) = H^\bullet(M)$$

is the ordinary de Rham cohomology algebra. When $D = \bar{\partial}$ is a holomorphic structure,

$$H^p_{\bar{\partial}} = \sum_{q+r=p} H^{q,r}_{\bar{\partial}}(M)$$

where $H^{q,r}_{\bar{\partial}}$ denotes the Dolbeault cohomologies.

1.7. Lemma [3]. *For an integrable derivation $D = d^F$, $\wedge^p F^\vee : A^p \to A^p$ is a chain map of the de Rham complex $(A^\bullet, d)$ to the twisted one $(A^\bullet, D)$, i.e., the diagram*

$$\begin{array}{ccc} A^p & \xrightarrow{d} & A^{p+1} \\ {\scriptstyle \wedge^p F^\vee}\downarrow & & \downarrow{\scriptstyle \wedge^{p+1} F^\vee} \\ A^p & \xrightarrow[D]{} & A^{p+1} \end{array}$$

commutes for every p.

We write the above relation simply by $D \circ F^\bullet = F^\bullet \circ d$. As a corollary,

1.8. Theorem. *For an integrable derivation $D = d^F$, there is a canonical homomorphism*

$$F^\bullet := \wedge^\bullet F^\vee : H^\bullet(M) \to H^\bullet_D(M),$$

between graded **C**-*algebras, which is an isomorphism when D is elliptic.*

Remark. When $D = \bar\partial$, a holomorphic structure on M, $F^p(H^p(M)) \subset H^{0,p}_{\bar\partial}(M)$, associated to the projection $A^p \to A^{0,p}_{\bar\partial}$.

2. Generalized connections and characteristic classes

Let E be a smooth complex vector bundle over M.

2.1. Definition. For a derivation D on M, a *D-connection* on E is a **C**-linear map

$$\nabla : A^0(E) \to A^1(E)$$

such that $\nabla(fs) = Df \cdot s + f\nabla(s)$ for $f \in A^0$ and $s \in A^0(E)$.

There always exists a D-connection and the set $\mathrm{Con}_D(E)$ of all D-connections is an affine space with the associated vector space $A^1(\mathrm{End}\, E)$. When $D = d$, we put $\mathrm{Con}(E) = \mathrm{Con}_d(E)$ and obtain ordinary connections. When $D = 0$, we obtain an element of $A^1(\mathrm{End}\, E)$. When $D = \bar\partial$, a holomorphic structure on M, such $\bar\partial$-connections are considered in the study of Higgs bundles (see e.g. [**6**]). Semi-connections in [4] are all $\bar\partial$-connections.

We put

$$H^0_\nabla(E) = ker(\nabla : A^0(E) \to A^1(E)). \tag{2.2}$$

The following propositions are trivial.

2.3. Proposition. (1) *Two generalized connections can be added, i.e., there is a map*

$$\mathrm{Con}_{D_1}(E) \times \mathrm{Con}_{D_2}(E) \to \mathrm{Con}_{D_1+D_2}(E)$$

for any derivations D_1 and D_2 on M. This map is surjective.

(2) *Let $D = d^F$ be a deriviation on M. Then the canonical affine map*

$$F^\vee : \mathrm{Con}(E) \to \mathrm{Con}_D(E), \qquad \nabla \mapsto F^\vee \circ \nabla =: \nabla^F$$

is an isomorphism, when D is elliptic.

2.4. Proposition. (1) *Let ∇_1 and ∇_2 be D-connections on vector bundles E_1 and E_2, respectively. Then they induce, in a standard way, D-connections $\nabla_1 \oplus \nabla_2$, $\nabla_1 \otimes \nabla_2$, $\nabla_1^\vee$ on $E_1 \oplus E_2$, $E_1 \otimes E_2$ and $E_1^\vee$, respectively.*

(2) *Let ∇ be a D-connection on E. Then there exists a unique linear extension $\nabla : A^p(E) \to A^{p+1}(E)$ of ∇ such that $\nabla(\xi \cdot s) = D\xi \cdot s + (-1)^p \xi \wedge \nabla s$ for $\xi \in A^p$, $s \in A^0(E)$. When D is integrable, $R^\nabla = \nabla \circ \nabla$ is an element of $A^2(\mathrm{End}\, E)$, called the* curvature tensor *of ∇, satisfying the Bianchi identity $\nabla_{\mathrm{End}\, E}(R^\nabla) = 0$.*

(3) *Let $D = d^F$ be integrable and let ∇ be an ordinary d-connection on E so that $\nabla^F = F^\vee \circ \nabla : A^0(E) \to A^1(E)$ is a D-connection on E. Then the diagram*

$$\begin{array}{ccc} A^p(E) & \xrightarrow{\nabla} & A^{p+1}(E) \\ {\scriptstyle \wedge^p F^\vee}\downarrow & & \downarrow{\scriptstyle \wedge^{p+1} F^\vee} \\ A^p(E) & \xrightarrow{\nabla^F} & A^{p+1}(E) \end{array}$$

commutes for every p. In particular, if R denotes the curvature of ∇, the the curvature of ∇^F is $F^\bullet R := \wedge^2 F^\vee(R)$.

From now on we will assume that D is integrable. Then for any D-connection ∇ on E, the curvature tensor $R^\nabla \in A^2(\operatorname{End} E)$ is well-defined and satisfies the Bianchi identity. Now the Chern-Weil theory of characteristic classes [2] can be played once we replace the ordinary cohomology groups by the twisted one. For instance

$$ch_D(E) = [Tr(exp(\frac{\sqrt{-1}}{2\pi} R^\nabla))] \in H^\bullet_D(M)$$

is independent of the choice of D-connection ∇ on E. Now we have

2.5. Theorem. *Let $D = d^F$ be an integrable derivation on M and let $F^\bullet : H^\bullet(M) \to H^\bullet_D(M)$ be the canonical map (1.8). Then for any complex vector bundle E on M,*

$$F^\bullet(ch(E)) = ch_D(E),$$

where $ch(E)$ is the ordinary Chern character.

PROOF: Let ∇ be any ordinary d-connection on E with the curvature R. Then for $D = d^F$, ∇^F is a D-connection with the curvature $F^\bullet(R) = \wedge^2 F^\vee(R)$. Now

$$F^\bullet ch(E) = [F^\bullet Tr(exp(\frac{\sqrt{-1}}{2\pi} R))] = [Tr(exp(\frac{\sqrt{-1}}{2\pi} F^\bullet R))] = ch_D(E). \quad \blacksquare$$

Remark. (1) More generally, the "Chern polynomial" can be replaced by any invariant polynomial $f : gl(r, \mathbb{C}) \to \mathbb{C}$, $r = \operatorname{rk} E$, and the theorem is

$$F^\bullet(f(E)) = f_D(E) \in H^\bullet_D(M)$$

for an integrable derivation $D = d^F$.

(2) When $D = \bar\partial$, then the existence of a holomorphic structure on a smooth vector bundle E over $(M, \bar\partial)$ is equivalent to the existence of a $\bar\partial$-connection on E with the vanishing "curvature" [1]. Thus if E admits a holomorphic structure, then $ch_{\bar\partial}(E) = 0$ or

$$f_{\bar\partial}(E) = 0 \in H^{o,\bullet}_{\bar\partial}(M) \subset H^\bullet_{\bar\partial}(M)$$

for every invariant polynomial f on $gl(r, \mathbb{C})$, as is well-known.

(3) In a functorial point of view, an object in our category is a pair (M, F) of a smooth manifold M and a smooth integrable endomorphism F of the complex tangent bundle TM, and the morphisms $\phi : (M_1, F_1) \to (M_2, F_2)$ are those that make the diagram

$$\begin{array}{ccc} A^0_{M_1} & \xrightarrow{F_1^\vee \circ d} & A^1_{M_1} \\ {\scriptstyle \phi^*}\uparrow & & \uparrow{\scriptstyle \phi^*} \\ A^0_{M_2} & \xrightarrow[F_2^\vee \circ d]{} & A^1_{M_2} \end{array}$$

commute. Then characteristic classes behave "naturally". Of course the ordinary category of smooth manifolds and the category of complex manifolds are full sub categories of ours.

REFERENCES

1. Atiyah, M. F., Hitchin, N. J., Singer, I. M., *Self-dualitiy in four-dimensional Riemannian geometry*, Proc. Roy. Soc. Lond. Ser. A **362** (1978), 425–461.
2. Chern, S.-s., *Differential geometry of fiber bundles*, Proc. Int. Con. Math. (1950), 397–411.
3. Kim, H.-J., *Curvatures and complex structures*, J. Korean Math. Soc. **24** (1987), 99–104.
4. Kobayashi, S., "Differential Geometry of Complex Vector Bundles," Publ. of the Math. Soc. of Japan **15**, Iwanami Shoten and Princeton Univ. Press, Tokyo, 1987.
5. Palais, R., *Natural operators on differential forms*, Trans. Amer. Math. Soc. **92** (1959), 125–141.
6. Simpson, C. T., *Higgs bundles and local systems*, preprint, Priceton University, 1989.
7. Witten, E., *Supersymmetry and Morse theory*, J. Differential Geom. (1982), 661–692.

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