NON-STABILITY OF AK-INVARIANT FOR SOME $Q-$PLANES

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The following question is of great interest to us:

Is the AK-invariant of a surface stable under reasonable geometric constructions?

In our previous work the cylinder over a surface played the role of a "reasonable" geometric construction. Here we are replacing the cylinder by algebraic line bundles. As a result we define a family of threefolds having trivial $AK(ML)$ invariant and non-trivial topology. In [BML] we built an example of a $\mathbb{Q}$-plane S with non-trivial cyclic fundamental group of prime order such that $AK(S) = \mathbb{C}[x]$ and $AK(S \times \mathbb{C}) = \mathbb{C}$. It appears that this construction works for any $\mathbb{Q}$-plane S with non-trivial cyclic fundamental group of prime order if we permit ourselves to consider non-trivial line bundles over S instead of $S \times \mathbb{C}$.

It would be interesting to generalize the numerous facts we know about cylinders over surfaces to the case of non-trivial line bundles. One of such generalizations is the following Proposition.

Proposition 1. *Let X be a smooth affine variety admitting a $\mathbb{C}$-action Φ. Let (L, π, X) be an algebraic line bundle over X. Then L admits a $\mathbb{C}$-action Φ' such that the image $\pi(F')$ of a general orbit F' of the action Φ' is a general orbit of the action Φ.*

Proof. Since L is an algebraically locally trivial bundle, there is an open set $W \subset X$, such that $\pi^{-1}(V) \cong W \times \mathbb{C} \subset L$. It follows that L contains a cylinder-like subset (see, for example [Mi2], Chapt. 2, 2.1 for definition of cylinder-like subset). Since X is affine, this implies that there exists a $\mathbb{C}$-action ψ such that its general orbit is a fiber of π ([Mi1], Lemma 2.2).

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On the other hand the $\mathbb{C}$-action Φ provides the existence of an open subset $U \cong Y \times \mathbb{C} \subset X$ which is a cylinder-like subset of X. (That means that Y is affine and the fibers of projection $p : U \to Y$ are the orbits of this action).

We consider a set $V = \pi^{-1}(U)$ and a composition map $q = p \circ \pi : V \to Y$. For a point $y \in Y$ the fiber $P_y = q^{-1}(y)$ is an affine surface which is an algebraic line bundle over $p^{-1}(y) \cong \mathbb{C}$. Thus, $P_y \cong \mathbb{C}^2$ ([Mi2], Chapt. 3, Th. 2.2.1). By the Main Theorem in [KZ] there exists an open subset $W \subset Y$ such that $q^{-1}(W) \cong W \times \mathbb{C}^2$ and q is a projection on the first factor. By Lemma 2.2 of [Mi1], there are two locally nilpotent commutative derivations and corresponding $\mathbb{C}$-actions ψ_1, ψ_2 on L, such that the general orbit of the group, generated by ψ_1, ψ_2 coincides with a general fiber of q. At least one of ψ_1, ψ_2 is non-equivalent to ψ. (Two $\mathbb{C}^+$-actions are equivalent of they have the same general orbit.)

Let it be ψ_1. Let $C \subset P_y$ be an orbit of ψ_1. Then $\pi(C)$ is not a point, hence $\pi(C) = \pi(P_y) = p^{-1}(y)$ is a general orbit of Φ. We may take $\Phi' = \psi_1$. □

Corollary 1. *If $AK(X) = \mathbb{C}$ and (L, π, X) is an algebraic line bundle over X then $AK(L) = \mathbb{C}$.*

We want to use this fact in order to compute the AK invariant for algebraic line bundles (and, in particular, a cylinder) over other $\mathbb{Q}-$planes admitting a $\mathbb{C}$-action.

In [BML] the following Proposition was proved.

Proposition 2. *Let S be a $\mathbb{Q}$-plane admitting a $\mathbb{C}$-action. Let $\pi_1(S) = \mathbb{Z}/m\mathbb{Z}$, where $m > 1$ is prime. Then $S \cong V/\mathcal{G}$, where*

- *V is a hypersurface in $\mathbb{C}^3$ with coordinates $(u, y, v) : V = \{u^k y = v^m - v_1^m + uq(u, v)\}$, where $v_1 \in \mathbb{C}^*$ and $q(u, v)$ is a polynomial of degree less than m relative to v ;*
- *$\mathcal{G}$ is the group of transformations generated by $g(u, y, v) = (u\varepsilon, y\varepsilon^{-k},\ v\varepsilon^{-\alpha})$, where $\alpha, k \in \mathbb{N},\ (\alpha, m) = 1$ and $\varepsilon = e^{\frac{2i\pi}{m}}$.*

We want to use this explicit representation for proving the following

Theorem 1. *Let S be as in Proposition 2. Then there is an algebraic line bundle (L, π, S) such that $AK(L) = \mathbb{C}$.*

Proof of Theorem 1. We may assume that $v_1 = 1$. Surface V admits a fixed point-free $\mathbb{C}^+-$ action, defined by the locally nilpotent derivation

$$\partial u = 0, \partial v = u^k, \partial y = mv^{m-1} + uq_v(u, v). \tag{1}$$

All the components of the fiber $u = 0$ are reduced (see Proposition 2 in [BML]).

Consider a standard Danielewsky surface

$$V_0 = \{ut = z^m - 1\} \subset \mathbb{C}^3.$$

It admits a locally nilpotent derivation ∂_0, such that: $\partial_0 u = 0,\ \partial_0 z = u,\ \partial_0 t = mz^{m-1}$. Then the Danielewsky-Fieseler factors ([D], [F])of these two actions on V and V_0 respectively coincide, and $\tilde{T} = V \times \mathbb{C}$ is isomorphic to $V_0 \times \mathbb{C}$ (see [D], [F]). We will need now the following three Lemmas.

Lemma 1. *The affine variety $\tilde{T}$ is isomorphic to the set T defined in $\mathbb{C}^6$ with coordinates (u, y, v, t, z, h) by the following equations:*

$$T = \begin{cases} u^k y = v^m - 1 + uq(u, v), \\ ut = z^m - 1, \\ hu = z - v, \\ t = u^{k-1} y - q(u, v) + mhv^{m-1} + \sum_2^m \binom{m}{r} u^{r-1} h^r v^{m-r}. \end{cases}$$

(cf. Lemma 6 in [BML]*).*

Proof of Lemma 1. Let X stand for the Danielewsky-Fieseler factor of V and V_0 relative to ∂ and ∂_0 respectively. The construction of T provides that the projection $\pi_0 : T \to V_0$, $\pi_0(u, y, v, t, z, h) = (u, t, z)$ and $\pi : T \to V,\ \ \pi(u, y, v, t, z, h) = (u, y, v)$, and projections τ_0 and τ from V and V_0 onto X respectively may be included into the following commutative diagram.

Diagram 1.

$$\begin{tikzcd} & T \arrow[dl, "\pi_0"'] \arrow[dr, "\pi"] & \\ V_0 \arrow[dr, "\tau_0"'] & & V \arrow[dl, "\tau"] \\ & X & \end{tikzcd} .$$

The inverse image $\pi^{-1}(\tilde{v})$ for any point $\tilde{v} \in V$ is a line; thus T is connected. Since it is a smooth variety, it has to be irreducible.

We want to show that $A = \mathcal{O}(T) = \mathcal{O}(V)[h]$.

Indeed, by the definition of T (see above) $z = v + hu$ and $t = u^{k-1}y - q(u, v) + hmv^{m-1} + \sum_2^m \binom{m}{r} u^{r-1} h^r v^{m-r}$.

So $T = V \times \mathbb{C}$ and the projection π is a projection onto the first factor as well.

Since $T = V \times_X V_0$, it follows (see ([D], [F]) that $T = V_0 \times \mathbb{C}$ and π_0 is the projection of T onto V_0. $\square$

Let ∂_1 be a locally nilpotent derivation corresponding to such an action in T, that its general orbit is a fiber of morphism π_0 (see formula (2) below). It follows that there is a function $\varphi \in \mathcal{O}(T)$ such that for each $c \in \mathbb{C}$ the set $T \cap \{\varphi = c\}$ is isomorphic to V_0 and $\partial_1 \varphi = const$. It is not defined uniquely: any function $\varphi' = c\varphi + h(u,t,z)$ has the same property for any function $h \in \mathcal{O}(V_0)$.

Lemma 2. *Let $\mathcal{G}'$ be the group generated by the action*

$$g'(u,y,v,h,t,z) = (u\varepsilon, y\varepsilon^{-k},\ v\varepsilon^{-\alpha}, h\varepsilon^{-(\alpha+1)}, t\varepsilon^{-1}, z\varepsilon^{-\alpha}).$$

Then there is such a function $\varphi' = c\varphi + h(u,t,z), c \in \mathbb{C}$ that

$$\varphi'(g'(a)) = \varepsilon^r \varphi'(a)$$

for some $r \in \mathbb{N}, c \in \mathbb{C}$ and any point $a \in T$.

Proof of Lemma 2. Let $\mathcal{G}''$ be the group acting on V_0 and generated by the map $g''((u,t,z)) = (u\varepsilon, t\varepsilon^{-1}, z\varepsilon^{-\alpha})$. Since g' and g'' act on the (u,t,z) in the same way, a fiber $\pi^{-1}(s) \subset T$ over a point $s \in V_0$ is mapped by g' to the fiber over the point $g''(s)$.

That means that the point $a = (s, \varphi) \in T$ goes to the point $(g''(s), \varphi_1) \in T$, and $\varphi_1 = \alpha(s)\varphi + \beta(s)$. Moreover, $\alpha(s)$ never vanishes on V_0, hence should be constant. Since $g'^m = id$, $\alpha = \varepsilon^r$ for some $r \in \mathbb{N}$.

We can represent the function β by

$$\beta(s) = \sum_0^m b_i(s),$$

where $b_i(g''(s)) = \varepsilon^i b_i(g''(s))$. Let

$$\gamma(s) = \sum_{i \neq r} \frac{b_i(s)}{\varepsilon^r - \varepsilon^i}$$

and

$$\varphi'(a) = \varphi(a) + \gamma(s).$$

Then $\gamma(g''(s)) = \sum_{i\neq r} \frac{\varepsilon^i b_i(s)}{\varepsilon^r - \varepsilon^i}$ and

$$\varphi'(g'(a)) = \varepsilon^r \varphi(a) + \sum_0^m b_i(s) + \sum_{i \neq r} \frac{\varepsilon^i b_i(s)}{\varepsilon^r - \varepsilon^i} =$$

$$\varepsilon^r(\varphi(a) + \gamma(s)) + b_r(s) = \varepsilon^r \varphi'(a) + b_r(s).$$

So

$$\varphi'((g')^2(a)) = \varepsilon^r\varphi'(g'(a))+b_r(g''(s)) = (\varepsilon^r)((\varepsilon^r)\varphi'(a)+b_r(s))+(\varepsilon^r)b_r(s) = (\varepsilon^r)^2\varphi'(a)+2(\varepsilon^r)b_r(s).$$

Repeating this computation, we get:

$$\varphi'((g')^m(a)) = \varphi'(a) + m(\varepsilon^r)^{m-1}b_r(s).$$

On the other hand, $\varphi'((g')^m(a)) = \varphi'(a)$; thus, $b_r(s) = 0$ and φ' is precisely the needed function. $\square$

Let $Z = T/\mathcal{G}'$ and $S_0 = V_0/\mathcal{G}''$. Denote by μ the natural projection $T \to T/\mathcal{G}' = Z$, by ν_0 the natural projection $V_0 \to V_0/\mathcal{G}'' = S_0$, and by ν the natural projection $V \to V/\mathcal{G} = S$.

Since g' and g act on (u, v, y) in the same way, and g' and g'' act on (u, t, z) in the same way, there are morphisms $\sigma : T/\mathcal{G}' \to V/\mathcal{G}$ and $\sigma_0 : T/\mathcal{G}' \to V_0/\mathcal{G}''$ such that the following diagrams are commutative.

Diagram 2.

$$\begin{array}{ccc} T & \xrightarrow{\mu} & T/\mathcal{G}' = Z \\ \pi_0 \downarrow & & \downarrow \sigma_0 \\ V_0 & \xrightarrow{\nu_0} & V_0/\mathcal{G}'' = S_0. \end{array}$$

Diagram 3.

$$\begin{array}{ccc} T & \xrightarrow{\mu} & T/\mathcal{G}' = Z \\ \pi \downarrow & & \downarrow \sigma \\ V & \xrightarrow{\nu} & V/\mathcal{G} = S. \end{array}$$

Lemma 3. *(Z, σ, S) and (Z, σ_0, S_0) are algebraic line bundles.*

Proof of Lemma 3. Both surfaces S and S_0 are smooth affine surfaces (the surface S_0 is described in [MiMa1], [MiMa2]).

First we are going to show that both (Z, σ, V) and (Z, σ_0, V_0) are analytically locally-trivial $\mathbb{C}-$fibrations. That follows from the following observations:

- the groups $\mathcal{G}', \mathcal{G}, \mathcal{G}''$ have no fixed points, and so the morphisms μ, ν, ν_0 are non-ramified (étale) coverings;
- (T, π, V), (T, π_0, V_0) are trivial line bundles.

For (Z, σ, S) we choose an open (in the analytic topology) set $U \subset S$ such that ν is an isomorphism of $W = \nu^{-1}(U)$ onto U, i.e. $g_i(w) \notin W$ for any $g_i \neq 1$ in $\mathcal{G}$ and any point $w \in W$. Since diagram (3) is commutative there are no points in $\pi^{-1}(W)$ conjugate by the action of $\mathcal{G}'$, i.e. $\mu\big|_{\pi^{-1}(W)}$ is an isomorphism as well. On the other hand, $\pi^{-1}(W) \cong W \times \mathbb{C} \subset T$. Hence,

$$\sigma^{-1}(U) = \mu(\pi^{-1}(W)) \cong \pi^{-1}(W) \cong W \times \mathbb{C} \cong U \times \mathbb{C}.$$

The case of (Z, π_0, V_0) is dealt with in a similar way.

Now the Lemma follows from the Theorem of J. Kollár which we found in [Ka]. Let us cite it.

Theorem K. *Let $\pi : X \to S$ be a morphism of smooth algebraic varieties which is also a locally trivial analytic $\mathbb{C}-$fibration. Then X is a complement to a section of an algebraic $\mathbb{P}^1-$bundle over S. Furthermore, if S is affine then X is the total space of an algebraic line bundle, and if, in addition, $\pi : X \to S$ is a quotient morphism of a free $\mathbb{C}^+$-action, X is isomorphic to $S \times \mathbb{C}$ over S.*

□

Remark 1. We could use this Theorem to prove Lemma 1. It is easy to show that (T, π_0, V_0) is analytically locally trivial $\mathbb{C}$-fibration and the fibers of the projection π_0 are the general orbits of a free action corresponding to the locally nilpotent derivation ∂_1:

$$\partial_1 u = \partial_1 t = \partial_1 z = 0, \; \partial_1 v = \partial_1 h = u^k, \; \partial_1 y = mv^{m-1} + uq_v(u, v). \tag{2}$$

To complete the proof of Theorem 1 we have to note only, that $AK(S_0) = \mathbb{C}$ ([MiMa1], [MiMa2]). Due to Corollary 1,

$$AK(Z) = \mathbb{C}$$

as well. Thus, (Z, π, S) is the line bundle we were looking for.

□

Remark 2. If $\alpha = -1$, then $AK(S \times \mathbb{C}) = \mathbb{C}$.

Proof. In this case the action

$$g'(u, y, v, h, t, z) = (u\varepsilon, y\varepsilon^{-k}, \; v\varepsilon, h, t\varepsilon^{-1}, z\varepsilon)$$

leaves h invariant.

Hence

$$Z = T/\mathcal{G}' = (V \times \mathbb{C})/\mathcal{G}' = (V/\mathcal{G}) \times \mathbb{C} = S \times \mathbb{C}.$$

□

We also can define explicitly three locally nilpotent derivations acting on $\mathcal{O}(Z)$ which have only constants as common elements of their kernels. These are the derivations, defined on $\mathcal{O}(T)$ and invariant under the action of group $\mathcal{G}'$. Namely, in notations of Lemma 1, Lemma 2:

1. ∂, defined by

$$\partial u = 0,\ \partial v = u^{mk-\alpha},\ \partial y = (mv^{m-1} + uq_v(u,v))u^{mk-k-\alpha},\ \partial h = 0. \tag{3}$$

2. δ, defined by

$$\delta h = u^{m-(\alpha+1)},\ \delta u = \delta y = \delta v = 0,\ \delta z = u^{m-\alpha},\ \delta t = mz^{m-1}u^{m-(\alpha+1)}, \tag{4}$$

3. $\tilde{\partial}$, defined by

$$\tilde{\partial} t = 0,\ \tilde{\partial} z = t^{m+\alpha},\ \tilde{\partial} u = mz^{m-1}t^{m+\alpha-1},\ \tilde{\partial}\varphi = 0. \tag{5}$$

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