Principal Gelfand pairs

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Let $X = G/K$ be a connected Riemannian homogeneous space of a real Lie group G. We assume that the action $G : X$ of G on X is locally effective, i.e., K contains no non-trivial connected central subgroups of G. Denote by $\mathcal{D}(X)^G$ the algebra of G-invariant differential operators on X and by $\mathcal{P}(T^*X)^G$ the algebra of G-invariant functions on T^*X polynomial on fibres. It is well known that $\mathcal{P}(T^*X)^G$ is a Poisson algebra, the Poisson bracket being induced by the commutator in $\mathcal{D}(X)^G$.

The homogeneous space X is called *commutative* or the pair (G, K) is called a *Gelfand pair* if the following three equivalent conditions are satisfied:

1) the algebra of K-invariant measures on X with compact support is commutative with respect to the convolution;

2) the algebra $\mathcal{D}(X)^G$ is commutative;

3) the algebra $\mathcal{P}(T^*X)^G$ is commutative with respect to the Poisson bracket.

The equivalence of the first two conditions was proved by Thomas [20] and Helgason [9], independently. Evidently, condition 3) is a consequence of 2). The inverse implication is recently proved by Rybnikov [19]. Commutative spaces can also be characterised by several other conditions. For instance, X is commutative if and only if the representation of G in $L^2(X)$ has a simple spectrum, see [5].

Symmetric Riemannian homogeneous spaces introduced by Élie Cartan are commutative. The theory of symmetric spaces is well developed. Works of Élie Cartan and Helgason describe their structure and also deal with harmonic analysis on such manifolds. One can hope that some day commutative spaces will be as thoroughly studied as symmetric spaces.

We denote Lie algebras of Lie groups by corresponding small gothic letters, for example, $\mathfrak{g} = \mathrm{Lie}\, G$ and $\mathfrak{k} = \mathrm{Lie}\, K$.

Definition 1. A real or complex linear Lie group with finitely many connected components is said to be *reductive* if it is completely reducible.

If X is a commutative homogeneous space of G, then, up to the local isomorphism, G has a factorisation $G = N \rightthreetimes L$, where N is a nilpotent radical of G, $K \subset L$, L and K have the same invariants in $\mathbb{R}[\mathfrak{n}]$ and $[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] = 0$, see [21]. Without loss of generality we may assume that the center of L is compact and the commutator group L' of L is a real form of some complex semisimple group. Hence, L is a reductive group. For a reductive Lie group F acting on a linear space V, we denote by $F_*(V)$ a generic stabiliser for this action. The stabiliser of a point $y \in V$ is denoted by F_y. Set $\mathfrak{m} := \mathfrak{l}/\mathfrak{k}$. In section 1 we prove the following criterion of commutativity.

Theorem 0.1. *$X = (N \rightthreetimes L)/K$ is commutative if and only if all of the following three conditions hold:*

i) $\mathbb{R}[\mathfrak{n}]^L = \mathbb{R}[\mathfrak{n}]^K$*;*
ii) for any point $\gamma \in \mathfrak{n}^*$ *the homogeneous space* L_γ/K_γ *is commutative;*
iii) for any point $\beta \in \mathfrak{m}^*$ *the homogeneous space* $(N \rtimes K_\beta)/K_\beta$ *is commutative.*

Let F be a complex reductive Lie group and $H \subset F$ a reductive subgroup.

Definition 2. An affine F-variety X is called *spherical* if a Borel subgroup $B(F) \subset F$ has an open orbit in X. If X is a linear space and a spherical F-variety then it is called a spherical representation of F. If a homogeneous space F/H is spherical, then the pair (F, H) and the subgroup H are also called spherical.

Let G be a real form of a complex reductive group $G(\mathbb{C})$. Suppose $K \subset G$ is a compact subgroup. We call the real homogeneous space G/K, the subgroup K and the pair (G, K) spherical if the complexification $X(\mathbb{C}) = G(\mathbb{C})/K(\mathbb{C})$ is a spherical $G(\mathbb{C})$-variety.

The commutative homogeneous spaces of reductive Lie groups are just the real forms of the spherical affine homogeneous spaces, see, for example, [21]. The theory of spherical homogeneous spaces is well developed, in particular, they are classified in [11], [6] and [15]. Note that [6] deals only with so-called *principal* homogeneous spaces (see Definition 5 in the third part of the present article). In [15] one type of non-principal spherical homogeneous spaces is described. The final classification is obtained in [23]. The real forms of homogeneous spherical spaces, i.e., commutative homogeneous spaces are explicitly described in [23].

Let $X = (N \rtimes L)/K$ be a commutative homogeneous space. Denote by P the ineffective kernel of the action $L : \mathfrak{n}$. Evidently, P is a normal subgroup of L and G. Due to (i) we have $L/P \subset \mathrm{O}(\mathfrak{n})$. Hence the group L_γ is reductive for any $\gamma \in \mathfrak{n}^*$, so in (ii) we can replace the word "commutative" by "spherical" and look up the given homogeneous space in the list.

Because the orbits of the compact group K in $\mathfrak{n}$ are separated by polynomial invariants, L and K have the same invariants in $\mathbb{R}[\mathfrak{n}]$ if and only if they have the same orbits. In other words, condition (i) means that there is a factorisation $L = L_*(\mathfrak{n})K$ or equivalently $L/P = (L/P)_*(\mathfrak{n})K$. All nontrivial factorisations of compact groups into products of two subgroups are classified by Onishchik [16]. The classification results are also nicely presented in [17, Chapter 4].

Commutativity is a local property, i.e., it depends only on the pair of algebras $(\mathfrak{g}, \mathfrak{k})$, see [21]. We assume that both G and K are connected, N is simply connected, $L = Z(L) \times L_1 \times ... \times L_m$, where $Z(L)$ stands for the connected centre of L and L_i are connected non-commutative simple groups. We also assume that L_i are real forms of simply connected complex simple groups and the action of a factor group $Z(L)/(Z(L) \cap P^0)$ on $\mathfrak{n}$ is effective. It can happen, that for a given pair $(\mathfrak{g}, \mathfrak{k})$, there is no effective pair (G, K) satisfying these assumptions, so we admit not only effective actions $G : (G/K)$ but locally effective as well. In tables we write U_n instead of $U_1 \times \mathrm{SU}_n$ and sometimes SO_n instead of Spin_n.

Suppose $X = (N \rtimes L)/K$ is commutative. Let $\mathfrak{z}_0 \subset [\mathfrak{n}, \mathfrak{n}]$ be an L-invariant subspace and $Z_0 \subset N$ a corresponding connected subgroup. Then the homogeneous space $X/Z_0 =$

$((N/Z_0) \rightthreetimes L)/K$ is also commutative, see [21]. The passage from X to X/Z_0 is called a *central reduction*. A commutative homogeneous space is said to be *maximal*, if it can not be obtained by a non-trivial central reduction from a larger one. We consider only maximal commutative homogeneous spaces. All commutative homogeneous spaces could be obtained as their central reductions.

Definition 3. A homogeneous space G/K is called *indecomposable* if it cannot be represented as a product $G_1/K_1 \times G_2/K_2$, where $G = G_1 \times G_2$, $K = K_1 \times K_2$ and $K_i \subset G_i$.

Obviously, $G_1/K_1 \times G_2/K_2$ is commutative if and only if both spaces G_1/K_1 and G_2/K_2 are commutative. Hence, for the classification of commutative homogeneous spaces it is enough to describe indecomposable ones.

Denote by H_n the $2n+1$ dimensional Heisenberg group, i.e., $\mathfrak{h}_n = \operatorname{Lie} H_n \cong \mathbb{C}^n \oplus \mathbb{R}$. Simply connected commutative groups are denoted by $\mathbb{R}^n$ or $\mathbb{C}^n$. The simplest and most important results are obtained in case of simple L.

Theorem 0.2. *Suppose $X = (N \rightthreetimes L)/K$ is an indecomposable commutative space, where L is simple, $L \neq K$ and $\mathfrak{n} \neq 0$. Then X is one of the following eight spaces up to the local isomorphism.*

$$\begin{array}{lll}
(H_{2n} \rightthreetimes \mathrm{SU}_{2n})/\mathrm{Sp}_n; & (\mathbb{R}^7 \rightthreetimes \mathrm{SO}_7)/G_2; & ((\mathbb{R}^8 \times \mathbb{R}^2) \rightthreetimes \mathrm{SO}_8)/\mathrm{Spin}_7; \\
(\mathbb{C}^{2n} \rightthreetimes \mathrm{SU}_{2n})/\mathrm{Sp}_n; & (\mathbb{R}^8 \rightthreetimes \mathrm{Spin}_7)/\mathrm{Spin}_6; & (\mathbb{R}^8 \rightthreetimes \mathrm{SO}_8)/\mathrm{Spin}_7; \\
(\mathbb{R}^{2n} \rightthreetimes \mathrm{SO}_{2n})/\mathrm{SU}_n; & & (\mathbb{R}^8 \rightthreetimes \mathrm{SO}_8)/(\mathrm{Sp}_2 \times \mathrm{SU}_2);
\end{array}$$

First we describe commutative homogeneous spaces satisfying condition

$(*)$ $L \neq K$ and the action $L : \mathfrak{n}$ is locally effective, i.e., P is finite.

Theorem 0.3. *Suppose a commutative homogeneous space $X = G/K$ satisfies condition $(*)$. Then any non-commutative normal subgroup of K different from SU_2 is contained in some simple direct factor of L.*

In the present work we impose on X two technical conditions of "principality" and "Sp_1-saturation", see sections 3 and 5 for precise definitions and explanations. The first condition concerns the embedding of the connected centre of K into L and the action of the connected centre of L on $\mathfrak{n}$. The second one is essentially the same, but we consider normal subgroups of K and L isomorphic to Sp_1.

Theorem 0.4. *Let $X = (N \rightthreetimes L)/K$ be a maximal principal indecomposable commutative homogeneous space satisfying condition $(*)$. Suppose there is a simple normal subgroup $L_1 \lhd L$, such that $L_1 \neq \mathrm{SU}_2$ and $L_1 \not\subset K$. Then either L is simple (and X is listed in Theorem 0.2) or X is locally isomorphic to one of the following two spaces: $(H_{2n} \rightthreetimes \mathrm{U}_{2n})/(\mathrm{Sp}_n \cdot \mathrm{U}_1)$, $((\mathbb{R}^8 \times \mathbb{R}^2) \rightthreetimes (\mathrm{SO}_8 \times \mathrm{SO}_2))/(\mathrm{Spin}_7 \times \mathrm{SO}_2)$.*

These additional spaces are essentially the same as the spaces from Theorem 0.2. All of them are listed in Table 2b (section 2).

A commutative homogeneous space $(N \rtimes L)/K$ is said to be of *Heisenberg type* if $L = K$. Recently commutative homogeneous spaces of Heisenberg type were intensively studied by several people, see, e.g., [3], [13], [21], [22].

The following theorem is the main result of this article.

Theorem 0.5. *Any indecomposable maximal principal* Sp_1*-saturated commutative homogeneous space belongs to the one of the following four classes:*
1) affine spherical homogeneous spaces of reductive real Lie groups;
2) homogeneous spaces listed in Table 2b;
3) homogeneous spaces $((\mathbb{R}^n \rtimes \mathrm{SO}_n) \times \mathrm{SO}_n)/\mathrm{SO}_n$, $((H_n \rtimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$, *where the normal subgroups* SO_n *and* SU_n *of* K *are diagonally embedded into* $\mathrm{SO}_n \times \mathrm{SO}_n$ *and* $\mathrm{SU}_n \times \mathrm{SU}_n$*, respectively;*
4) commutative homogeneous spaces of Heisenberg type.

Commutative homogeneous spaces of Heisenberg type are considered in the sixth section. In this case $\mathcal{D}(G/K)^G \cong U(\mathfrak{n})^K$, where $U(\mathfrak{n})$ is the universal enveloping algebra of $\mathfrak{n}$. If $\mathfrak{n}$ is commutative then obviously G/K is also commutative and it is called a commutative space of *Euclidian type*. We assume that $\mathfrak{n}$ is not commutative.

Consider a homogeneous space $(N \rtimes K)/K$. Suppose $\mathfrak{n}$ is at most two-step nilpotent and $[\mathfrak{n}, \mathfrak{n}]$ is a trivial K-module. We can decompose $\mathfrak{n}$ into an K-invariant sum $\mathfrak{n} = (\mathfrak{w} \oplus \mathfrak{z}) \oplus V$, where V is an abelian ideal and $[\mathfrak{w}, \mathfrak{w}] = \mathfrak{z}$. Any point $\alpha \in \mathfrak{z}^*$ determines a skew-symmetric form $\hat{\alpha}$ on $\mathfrak{w}$, namely $\hat{\alpha}(\xi, \eta) = \alpha([\xi, \eta])$ for $\xi, \eta \in \mathfrak{w}$. The form $\hat{\alpha}$ is non-degenerate for a generic α. The complexification $\mathfrak{w}(\mathbb{C})$ is an orthogonal and a symplectic representation of $K(\mathbb{C})$ at the same time. Hence it is reducible $\mathfrak{w}(\mathbb{C}) = W \oplus W^*$. According to [3] and [25], $(N \rtimes K)/K$ is commutative if and only if W is a spherical representation of the complexification $K_*(V)(\mathbb{C})$ of $K_*(V)$. In the simplest situation when $V = 0$ the statement means that W is a spherical representation of $K(\mathbb{C})$.

Spherical representations of reductive Lie groups are classified by Kac [10] (irreducible representations), Brion [7], finally by Benson and Ratcliff [4] and Leahy [14], independently. Historical comments and the classification result can be found in [12]. The list of commutative homogeneous spaces $(N \rtimes K)/K$, where N is a product of several Heisenberg groups, is given in [3]. That article also claims to classify all commutative homogeneous spaces $(N \rtimes K)/K$ with $\mathfrak{n} = (\mathfrak{w} \oplus \mathfrak{z}) \oplus V$, where V is an abelian ideal and $\mathfrak{w} \oplus \mathfrak{z}$ is a direct sum of several Heisenberg algebras. The authors of [3] erroneously assume that if for a subgroup H with $\mathrm{Lie}\, H = \mathfrak{w} \oplus \mathfrak{z}$ the homogeneous space $(H \rtimes K)/K$ is commutative, then $(N \rtimes K)/K$ is also commutative. The simplest counterexample is $((\mathbb{C}^2 \times H_2) \rtimes \mathrm{SU}_2)/\mathrm{SU}_2$. This space is not commutative according to [21], but $(H_2 \rtimes \mathrm{SU}_2)/\mathrm{SU}_2$ is commutative.

Commutative homogeneous spaces of Heisenberg type with an irreducible action $K : (\mathfrak{n}/\mathfrak{n}')$ are classified in [21] and [22]. Generally, $\mathfrak{n}$ is a sum of a commutative ideal and algebras listed in [21, Table 3] and [22]. But the sum cannot be arbitrary.

The classification of *saturated* commutative homogeneous spaces of Heisenberg type was announced in [25]. The condition of saturation is a little bit stronger than both conditions of "principality" and "Sp_1-saturation". We present the result of [25] in Table 4 and give a proof.

We will frequently use the following results of [21].

Proposition 1. [21, Corollaries to Proposition 10] *Let G/K be commutative. Then*

1) for any normal subgroup $N \subset G$ the homogeneous space $G/NK = (G/N)/(K/(N \cap K))$ is commutative;

2) for any compact subgroup $F \subset G$ containing K the homogeneous space G/F is commutative;

3) for any subgroup $F \subset G$ containing K the homogeneous space F/K is commutative.

We fix some additional notation.

G' is the commutator group of G;

$G(\mathbb{C})$ is the complexification of a real group G;

$B(G(\mathbb{C}))$ and $U(G(\mathbb{C})) \subset B(G(\mathbb{C}))$ are a Borel and a maximal unipotent subgroups of a reductive group $G(\mathbb{C})$;

$X/\!\!/G$ stands for the categorical quotient of an affine algebraic variety X by the action of a reductive group G.

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1. Commutativity criterion

Let $U(\mathfrak{g})$ stand for the universal enveloping algebra of $\mathfrak{g}$. There is a natural filtration on $U(\mathfrak{g})$:

$$U_0(\mathfrak{g}) \subset U_1(\mathfrak{g}) \subset \ldots \subset U_m(\mathfrak{g}) \subset \ldots ,$$

where $U_m(\mathfrak{g})$ consists of all elements of order at most m.

Recall that the Poisson bracket on the symmetric algebra $S(\mathfrak{g}) = \operatorname{gr} U(\mathfrak{g})$ is determined by the formula

$$\{a + U_{n-1}(\mathfrak{g}), b + U_{m-1}(\mathfrak{g})\} = [a, b] + U_{n+m-2}(\mathfrak{g}) \quad \forall a \in U_n(\mathfrak{g}), b \in U_m(\mathfrak{g}).$$

Let $X = G/K$ be a Riemannian homogeneous space. It is well known, see, for example, [21], that there is an isomorphism of the associated graded algebras:

$$\operatorname{gr} U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K = \operatorname{gr} \mathcal{D}(X)^G = \mathcal{P}(T^*X)^G = S(\mathfrak{g}/\mathfrak{k})^K.$$

The space $(U(\mathfrak{g})\mathfrak{k})^K$ is an ideal of $U(\mathfrak{g})^K$, also $(S(\mathfrak{g})\mathfrak{k})^K$ is a Poisson ideal of $S(\mathfrak{g})^K$. The well defined Poisson bracket on the factor $S(\mathfrak{g})^K/(S(\mathfrak{g})\mathfrak{k})^K \cong S(\mathfrak{g}/\mathfrak{k})^K$ coincides up to a sign with the Poisson bracket on $\mathcal{P}(T^*X)^G$. In particular, X is commutative if and only if the Poisson algebra $S(\mathfrak{g}/\mathfrak{k})^K$ is commutative.

Suppose $X = (N \rtimes L)/K$ is commutative. Then, as proved in [21], the following condition holds

i) $\mathbb{R}[\mathfrak{n}]^L = \mathbb{R}[\mathfrak{n}]^K$.

The orbits of a compact group are separated by polynomial invariants. Hence (i) is fulfilled if and only if L and K have the same orbits on $\mathfrak{n}$. There is a K-invariant positive-definite symmetric bilinear form on $\mathfrak{n}$ which is automatically L-invariant. In particular, vector spaces $\mathfrak{n}$ and $\mathfrak{n}^*$ are isomorphic as L-modules. Therefore, $\mathrm{ad}^*(\mathfrak{k})\gamma = \mathrm{ad}^*(\mathfrak{l})\gamma$ for each point $\gamma \in \mathfrak{n}^*$ and hence $\mathfrak{l} = \mathfrak{k} + \mathfrak{l}_\gamma$. Moreover, the natural restriction

$$\tau : \ \mathfrak{l}^* \longrightarrow \mathfrak{l}_\gamma^*$$

(which is also a homomorphism of L_γ-modules) determines an isomorphism of K_γ-modules $(\mathfrak{l}/\mathfrak{k})^*$ and $(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma)^*$.

Recall that $\mathfrak{g} = \mathfrak{l} + \mathfrak{n}$, where $\mathfrak{n}$ is a nilpotent ideal and $\mathfrak{l}$ is a reductive subalgebra. Let $\check{\mathfrak{n}}$ and $\check{\mathfrak{l}}$ be commutative Lie algebras of dimensions $\dim \mathfrak{n}$ and $\dim \mathfrak{l}$, respectively. We determine new Lie algebras $\check{\mathfrak{g}}_1 = \mathfrak{l} + \check{\mathfrak{n}}$ and $\check{\mathfrak{g}}_2 = \check{\mathfrak{l}} \oplus \mathfrak{n}$, where $\check{\mathfrak{l}}$, $\check{\mathfrak{n}}$ are commutative ideals and $\check{\mathfrak{n}} \cong \mathfrak{n}$ as an $\mathfrak{l}$-modules.

Denote by $\{\ ,\ \}_\mathfrak{l}$ and $\{\ ,\ \}_\mathfrak{n}$ the Poisson brackets on $S(\check{\mathfrak{g}}_1)$ and $S(\check{\mathfrak{g}}_2)$. There is a K-invariant bi-grading $S(\mathfrak{g}) = \bigoplus S^{n,l}(\mathfrak{g})$, where $S^{n,l}(\mathfrak{g}) = S^n(\mathfrak{n})S^l(\mathfrak{l})$. We identify elements of $S(\mathfrak{g})$ with the corresponding elements of $S(\check{\mathfrak{g}}_1)$ and $S(\check{\mathfrak{g}}_2)$.

Lemma 1. *We have*

$$\{\xi,\eta\} = \{\xi,\eta\}_\mathfrak{n} + \{\xi,\eta\}_\mathfrak{l}, \quad \textit{with } \{\xi,\eta\}_\mathfrak{n} \in S^{n+n'-1,l+l'}(\mathfrak{g}), \{\xi,\eta\}_\mathfrak{l} \in S^{n+n',l+l'-1}(\mathfrak{g}).$$

for any bi-homogeneous elements $\xi \in S^{n,l}(\mathfrak{g}), \eta \in S^{n',l'}(\mathfrak{g})$. In other words, the Poisson bracket on $S(\mathfrak{g})$ is a direct sum of the brackets $\{\ ,\ \}_\mathfrak{n}$ and $\{\ ,\ \}_\mathfrak{l}$.

Proof. Recall that the Poisson bracket of bi-homogeneous elements $\xi = \xi_1 ... \xi_n, \eta = \eta_1 ... \eta_m \in S(\mathfrak{g})$ is given by the formula

$$\{\xi,\eta\} = \sum_{i,j} [\xi_i,\eta_j]\xi_1 ... \widehat{\xi_i} ... \xi_n \eta_1 ... \widehat{\eta_j} ... \eta_m. \qquad (1)$$

This expression for $\{\xi,\eta\}$ contains summands of three different types, depending on whether ξ_i and η_j are elements of $\mathfrak{l}$ or $\mathfrak{n}$. Because $[\mathfrak{l},\mathfrak{n}] \subset \mathfrak{n}$ and $\mathfrak{l}$, $\mathfrak{n}$ are subalgebras, if $\xi_i, \eta_j \in \mathfrak{n}$, then $[\xi_i,\eta_j] \in S^{n+n'-1,l+l'}(\mathfrak{g})$, otherwise $[\xi_i,\eta_j] \in S^{n+n',l+l'-1}(\mathfrak{g})$. $\square$

We suppose that $\check{\mathfrak{k}}$ is embedded into $\check{\mathfrak{l}}$ as a commutative subalgebra of dimension $\dim \mathfrak{k}$. We also denote by $\{\ ,\ \}_{\mathfrak{n}}$ and $\{\ ,\ \}_{\mathfrak{l}}$ the corresponding Poisson brackets on the Poisson factors $S(\check{\mathfrak{g}}_2/\check{\mathfrak{k}})^K = S(\check{\mathfrak{g}}_2)^K/(S(\check{\mathfrak{g}}_2)\check{\mathfrak{k}})^K$ and $S(\check{\mathfrak{g}}_1/\mathfrak{k})^K = S(\check{\mathfrak{g}}_1)^K/(S(\check{\mathfrak{g}}_1)\mathfrak{k})^K$, where the actions $K : \check{\mathfrak{g}}_i$ are the same as $K : \mathfrak{g}$. We have $\{a,b\}_{\mathfrak{l}} \in S^{n+n',l+l'-1}(\mathfrak{g}/\mathfrak{k})$ and $\{a,b\}_{\mathfrak{n}} \in S^{n+n'-1,l+l'}(\mathfrak{g}/\mathfrak{k})$ for any $a \in S^{n,l}(\mathfrak{g}/\mathfrak{k}), b \in S^{n',l'}(\mathfrak{g}/\mathfrak{k})$ $(a, b \in S(\mathfrak{g}/\mathfrak{k})^K)$.

Lemma 2. *The Poisson bracket on $S(\mathfrak{g}/\mathfrak{k})^K$ is of the form $\{\ ,\ \} = \{\ ,\ \}_{\mathfrak{n}} + \{\ ,\ \}_{\mathfrak{l}}$.*

Proof. This is a straightforward consequence of Lemma 1. □

Corollary 1. *Let G/K be a commutative homogeneous space and $\check{N}$ a simply connected commutative Lie group with a Lie algebra $\check{\mathfrak{n}}$. Set $\check{G} := \check{N} \leftthreetimes L$. Then $\check{G}/K$ is also commutative.*

Theorem 1. *The homogeneous space $X = (N \leftthreetimes L)/K$ is commutative if and only if all of the following three conditions hold:*

i) $\mathbb{R}[\mathfrak{n}]^L = \mathbb{R}[\mathfrak{n}]^K$;

ii) for any point $\gamma \in \mathfrak{n}^*$ the homogeneous space L_γ/K_γ is commutative;

iii) for any point $\beta \in (\mathfrak{l}/\mathfrak{k})^*$ the homogeneous space $(N \leftthreetimes K_\beta)/K_\beta$ is commutative.

Remark 1. The statement of the theorem remains true if we replace arbitrary points by generic points in conditions (ii) and (iii).

Proof. As was already mentioned, Vinberg proved in [21] that the condition (i) holds for any commutative space. So let us assume that it is fulfilled.

Let γ be a point in $\mathfrak{n}^*$. Recall that the K_γ-modules $\mathfrak{l}/\mathfrak{k}$ and $\mathfrak{l}_\gamma/\mathfrak{k}_\gamma$ are isomorphic. Hence, $S(\mathfrak{l}/\mathfrak{k})$ is isomorphic to $S(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma)$ as a graded associative algebra and also as a K_γ-module.

Consider the homomorphism

$$\varphi_\gamma : \ S(\mathfrak{g}/\mathfrak{k}) \longrightarrow S(\mathfrak{g}/\mathfrak{k})/(\xi - \gamma(\xi) : \ \xi \in \mathfrak{n}) = S(\mathfrak{l}/\mathfrak{k}) = S(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma).$$

Evidently, $\varphi_\gamma(S(\mathfrak{g}/\mathfrak{k})^K) \subset S(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma)^{K_\gamma}$.

Let $\xi \in \mathfrak{l}_\gamma$, $\eta \in \mathfrak{n}$. Then $\gamma(\{\xi,\eta\}) = \gamma([\xi,\eta]) = -[\mathrm{ad}^*(\xi)\gamma](\eta) = 0 = \{\xi, \gamma(\eta)\}$.

It can easily be deduced from the above statement and from the formula (1), that for arbitrary bi-homogeneous elements $a, b \in S(\mathfrak{g}/\mathfrak{k})^K$, which can be regarded as elements of $S((\mathfrak{l}_\gamma \oplus \mathfrak{n})/\mathfrak{k}_\gamma)$, we have

$$\varphi_\gamma(\{a,b\}_{\mathfrak{l}}) = \{\varphi_\gamma(a), \varphi_\gamma(b)\},$$

where the second bracket is the Poisson bracket on $S(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma)^{K_\gamma}$. In other words, φ_γ is a homomorphism of the Poisson algebras $S(\check{\mathfrak{g}}_1/\mathfrak{k})^K$ and $S(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma)^{K_\gamma}$.

Recall that we denoted by $\mathfrak{m}$ the vector space $(\mathfrak{l}/\mathfrak{k})$. We repeat the procedure for the point $\beta \in \mathfrak{m}^*$. Consider the homomorphism

$$\varphi_\beta : \ S(\mathfrak{g}/\mathfrak{k}) \longrightarrow S(\mathfrak{g}/\mathfrak{k})/(\xi - \beta(\xi) : \ \xi \in \mathfrak{m}) = S(\mathfrak{n}).$$

Clearly, $\varphi_\beta(S(\mathfrak{g}/\mathfrak{k})^K) \subset S(\mathfrak{n})^{K_\beta}$. Note that φ_β is a homomorphism of Poisson algebras $S(\check{\mathfrak{g}}_2/\check{\mathfrak{k}})^K$ and $S(\mathfrak{n})^{K_\beta}$. For arbitrary bi-homogeneous elements $a, b \in S(\mathfrak{g}/\mathfrak{k})^K$ we have

$$\varphi_\beta(\{a,b\}_\mathfrak{n}) = \{\varphi_\beta(a), \varphi_\beta(b)\}.$$

Here the second bracket is a Poisson bracket on $S(\mathfrak{n})^{K_\beta}$.

Now we show that homomorphisms φ_γ and φ_β are surjective. We have $S(\mathfrak{g}) = \mathbb{R}[\mathfrak{g}^*]$, $S(\mathfrak{g}/\mathfrak{k})^K = \mathbb{R}[(\mathfrak{g}/\mathfrak{k})^*]^K = \mathbb{R}[(\mathfrak{g}/\mathfrak{k})^* /\!\!/ K]$ and $S(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma)^{K_\gamma} = \mathbb{R}[\mathfrak{m}^* /\!\!/ K_\gamma]$, $S(\mathfrak{n})^{K_\beta} = \mathbb{R}[\mathfrak{n}^* /\!\!/ K_\beta]$. Note that

$$\mathfrak{m}^* /\!\!/ K_\gamma \cong (K\gamma \times \mathfrak{m}^*) /\!\!/ K \subset (\mathfrak{g}/\mathfrak{k})^* /\!\!/ K;$$
$$\mathfrak{n}^* /\!\!/ K_\beta \cong (\mathfrak{n}^* \times K\beta) /\!\!/ K \subset (\mathfrak{g}/\mathfrak{k})^* /\!\!/ K.$$

Moreover, $K\gamma$ and $K\beta$ are closed in $\mathfrak{n}^*$ and $\mathfrak{m}^*$, respectively. Hence the subsets $(K\gamma \oplus \mathfrak{m}^*) /\!\!/ K$ and $(\mathfrak{n}^* \oplus K\beta) /\!\!/ K$ are closed in $(\mathfrak{g}/\mathfrak{k})^* /\!\!/ K$. Thus, the restrictions $\mathbb{R}[(\mathfrak{g}/\mathfrak{k})^*]^K \to \mathbb{R}[K\gamma \oplus \mathfrak{m}^*]^K$ and $\mathbb{R}[(\mathfrak{g}/\mathfrak{k})^*]^K \to \mathbb{R}[\mathfrak{n}^* \oplus K\beta]^K$ are surjective. It is therefore proved that φ_γ and φ_β are surjective.

($\Longleftarrow$) Suppose conditions (ii) and (iii) are satisfied. Clearly, X is commutative if and only if both Poisson brackets $\{\ ,\ \}_\mathfrak{n}$ and $\{\ ,\ \}_\mathfrak{l}$ equal zero on $S(\mathfrak{g}/\mathfrak{k})^K$. If $\{a,b\}_\mathfrak{l} \neq 0$ for some elements $a, b \in S(\mathfrak{g}/\mathfrak{k})^K$ then there is a (generic) point $\gamma \in \mathfrak{n}^*$ such that $\varphi_\gamma(\{a,b\}_\mathfrak{l}) \neq 0$. But $\varphi_\gamma(\{a,b\}_\mathfrak{l}) = \{\varphi_\gamma(a), \varphi_\gamma(b)\} = 0$. Analogously, if $\{a,b\}_\mathfrak{n} \neq 0$ for some elements $a, b \in S(\mathfrak{g}/\mathfrak{k})^K$ then there is a (generic) point $\beta \in \mathfrak{m}^*$ such that $\varphi_\beta(\{a,b\}_\mathfrak{l}) \neq 0$. But $\varphi_\beta(\{a,b\}_\mathfrak{l}) = \{\varphi_\beta(a), \varphi_\beta(b)\} = 0$.

($\Longrightarrow$) Suppose X is commutative. Then both Poisson brackets $\{\ ,\ \}_\mathfrak{n}$ and $\{\ ,\ \}_\mathfrak{l}$ vanish on $S(\mathfrak{g}/\mathfrak{k})^K$. Hence $\{\varphi_\gamma(a), \varphi_\gamma(b)\} = 0$, $\{\varphi_\beta(a), \varphi_\beta(b)\} = 0$ for any $a, b \in S(\mathfrak{g}/\mathfrak{k})^K$. The homomorphisms φ_γ and φ_β are surjective, so the Poisson algebras $S(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma)^{K_\gamma}$ and $S(\mathfrak{n})^{K_\beta}$ are commutative. $\square$

2. Properties of commutative spaces

Suppose $X = G/K = (N \leftthreetimes L)/K$ is a commutative homogeneous space. Denote by P the ineffective kernel of the action $L : \mathfrak{n}$. Note that P is a normal subgroup of L and G. Due to (i) we have $L/P \subset \mathrm{O}(\mathfrak{n})$. Hence, the stabiliser L_γ is reductive for any $\gamma \in \mathfrak{n}^*$ and the generic stabiliser $L_*(\mathfrak{n})$ is also reductive. Condition (i) holds if and only if $L = L_*(\mathfrak{n})K$ or equivalently L/P is a product of L_*/P and $K/(K \cap P)$. All nontrivial factorisations of compact groups as products of two subgroups are classified by Onishchik [16]. The group L_γ is reductive, hence the homogeneous space L_γ/K_γ considered in (ii), is commutative if and only if it is spherical.

Definition 4. Let M, F, G, K be Lie groups, with $F \subset M$ and $K \subset G$. The pair (M, F) is called an *extension* of (G, K) if

$$G \subsetneq M, \quad M = GF, \quad K = F \cap G.$$

The condition (i) means that (L,K) is an extension of $(L_*(\mathfrak{n}), K_*(\mathfrak{n}))$.

Evidently, $G/P = N \rtimes (L/P)$ and $(G/P)/[K/(K\cap P)]$ is a commutative homogeneous space of G/P.

In this section we are interested in commutative spaces satisfying condition

$(*)$ $L \neq K$ and the action $L : \mathfrak{n}$ is locally effective, i.e., P is finite.

In particular, this condition means that L is compact.

Lemma 3. *Let a symmetric pair $(M = F\times F, F)$ with a simple compact group F be an extension of a spherical pair (G,H). Then G contains either $F\times\{e\}$ or $\{e\}\times F$.*

Proof. Let G_1 and G_2 be the images of the projections of G onto the first and the second factors respectively. The group $G_1\times G_2$ acts spherically on $F \cong M/F \cong G/H$. If neither G_1 nor G_2 equals F, then due to [2, Theorem 4] we have $\dim B(G_i(\mathbb{C})) \le \dim U(F(\mathbb{C}))$. Hence, $\dim B((G_1\times G_2)(\mathbb{C})) \le 2\dim U(F(\mathbb{C})) < \dim F(\mathbb{C})$ and the action $(G_1\times G_2) : F$ cannot be spherical. Assume that $G_1 = F$ but $F\times\{e\}$ is not contained in G. Then $G\cong F$ and $H = \{e\}$. But the pair $(F, \{e\})$ cannot be spherical. $\square$

Lemma 4. *Suppose a compact group $F \subset \mathrm{Sp}_n$ acts irreducibly on $\mathbb{H}^n$ and $F|_{\xi\mathbb{H}} = \mathrm{Sp}_1$ for every $\xi\in\mathbb{H}^n$, $\xi\neq 0$. Then $F = \mathrm{Sp}_n$.*

Proof. Let $F(\mathbb{C}) \subset \mathrm{Sp}_{2n}(\mathbb{C})$ be the complexification of F. Then $F(\mathbb{C})$ acts on a generic subspace $\mathbb{C}^2\subset\mathbb{C}^{2n}$ as $\mathrm{SL}_2(\mathbb{C})$. Hence it acts on $\mathbb{C}^{2n}$ locally transitively. It was proved by Panyushev [18] in a classification-free way, that $F(\mathbb{C}) = \mathrm{Sp}_{2n}(\mathbb{C})$. $\square$

Lemma 5. *Suppose $\mathfrak{l}\subset\mathfrak{so}(V)$ is a Lie algebra. Let $\mathfrak{l}_1$ be a non-abelian simple ideal of $\mathfrak{l}$. Denote by π the projection onto $\mathfrak{l}_1$. If $\pi(\mathfrak{l}_*(V)) = \mathfrak{l}_1$ and W_1 is a non-trivial irreducible $\mathfrak{l}$-submodule of V that is also non-trivial as an $\mathfrak{l}_1$-module, then $\mathfrak{l}_1 = \mathfrak{su}_2$ and W_1 is of the form $\mathbb{H}^1\otimes_{\mathbb{H}}\mathbb{H}^n$, where $\mathfrak{l}$ acts on $\mathbb{H}^n$ as $\mathfrak{sp}_n$.*

Proof. Set $\mathfrak{l} = \mathfrak{l}_1\oplus\mathfrak{l}_2$. We may assume that $V = W_1$. The vector space V can be decomposed into a tensor product $V = V_{1,1}\otimes_{\mathbb{D}} V_1^1$ of $\mathfrak{l}_1$ and $\mathfrak{l}_2$-modules, where $\mathbb{D}$ is one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Here $\mathfrak{l}_1$ acts trivially on V_1^1 and $\mathfrak{l}_2$ acts trivially on $V_{1,1}$. Both actions $\mathfrak{l}_1 : V_{1,1}$ and $\mathfrak{l}_2 : V_1^1$ are irreducible.

Let $x = x_{1,1}\otimes x_1^1 \in V$ be a non-zero decomposable vector. Because $V_{1,1}$ is a non-trivial irreducible $\mathfrak{l}_1$-module, $(\mathfrak{l}_1)_x \neq \mathfrak{l}_1$. We have $\mathfrak{l}_*\subset\mathfrak{l}_x$ up to conjugation. Evidently, $\mathfrak{l}_x\subset\mathfrak{n}_1(x)\oplus\mathfrak{n}_2(x)$, where $\mathfrak{n}_i(x) = \{\xi\in\mathfrak{l}_i : \xi x\in\mathbb{D}x\}$. Since $\mathfrak{l}_1 = \pi(\mathfrak{l}_*)\subset\mathfrak{n}_1(x)$, we have $\mathfrak{n}_1(x) = \mathfrak{l}_1$. Hence, $\mathbb{D}x_{1,1}$ is an $\mathfrak{l}_1$-invariant subspace of $V_{1,1}$. Thus $V_{1,1} = \mathbb{D}x_{1,1}$ and $\mathfrak{l}_1\subset\mathfrak{gl}_1(\mathbb{D})$. If $\mathbb{D}$ equals $\mathbb{R}$ or $\mathbb{C}$, $\mathfrak{gl}_1(\mathbb{D})$ is commutative. If $\mathbb{D} = \mathbb{H}$ then $\mathfrak{l}_1(x)\subset\mathfrak{sp}_1$. Thus we have shown that $\mathfrak{l}_1 = \mathfrak{sp}_1 = \mathfrak{su}_2$ and $W_1 = \mathbb{H}^1\otimes_{\mathbb{H}}\mathbb{H}^n$. Moreover, $\mathfrak{l}_2|_{\mathbb{H}x} = \mathfrak{sp}_1$. To conclude, notice that $\mathfrak{l}$ has to act on $\mathbb{H}^n$ as $\mathfrak{sp}_n$ by Lemma 4. $\square$

Set $L_* := L_*(\mathfrak{n})$, $K_* := K_*(\mathfrak{n})$. Recall that there is a factorisation $L = Z(L)\times L_1\times\ldots\times L_m$. Denote by π_i the projection onto L_i.

Theorem 2. *Suppose a commutative homogeneous space $X = G/K$ satisfies condition $(*)$. Then any non-commutative normal subgroup of K distinct from SU_2 is contained in some simple factor of L.*

Proof. Assume K_1 is a normal subgroup of K having non-trivial projections onto, say, L_1 and L_2. Consider the subgroup $M = Z(L) \times \pi_1(K) \times \pi_2(K) \times L_3 \times ... \times L_m$. Evidently, $K \subset M$. Without loss of generality we can replace L by M or better assume from the beginning that $L_i = \pi_i(K) = \pi_i(K_1) \cong K_1$ $(i = 1, 2)$. Denote by $\pi_{1,2}$ the projection onto $L_1 \times L_2$. According to condition (i), $L_1 \times L_2 = K_1\pi_{1,2}(L_*)$. Recall that due to condition (ii), L_*/K_* is commutative, hence (L_*, K_*) is a spherical pair. The pair $(\pi_{1,2}(L_*), \pi_{1,2}(K_*))$ is also spherical as a factor of a spherical pair. Clearly, $\pi_{1,2}(K_*) \subset \pi_{1,2}(K) \cap \pi_{1,2}(L_*)$. Thus the symmetric pair $(L_1 \times L_2, K_1)$ is an extension of the spherical pair $(\pi_{1,2}(L_*), \pi_{1,2}(K) \cap \pi_{1,2}(L_*))$. By Lemma 3 the group $\pi_{1,2}(L_*)$ contains L_1 or L_2 (we can assume that it contains L_1). Then $\pi_1(L_*) = L_1$ and by Lemma 5 we have $L_1 = \mathrm{SU}_2$. $\square$

In Table 1 we present the list of all factorisations of compact simple Lie algebras obtained in [16]. Here $\mathfrak{g}^1$, $\mathfrak{g}^2$ are subalgebras of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{g}^1 + \mathfrak{g}^2$, $\mathfrak{u} = \mathfrak{g}^1 \cap \mathfrak{g}^2$. In all cases $n > 1$, φ^1 and φ^2 are the restrictions of the defining representation of the complexification $\mathfrak{g}(\mathbb{C})$ to $\mathfrak{g}^1(\mathbb{C})$ and $\mathfrak{g}^2(\mathbb{C})$ (whose highest weights are indicated), ϖ_m are the fundamental weights, $\mathbf{I}$ is the trivial representation.

Table 1.

$\mathfrak{g}$	$\mathfrak{g}^1$	φ^1	$\mathfrak{g}^2$	φ^2	$\mathfrak{u}$
$\mathfrak{su}_{2n}$	$\mathfrak{sp}_n$	ϖ_1	$\mathfrak{su}_{2n-1}$	$\varpi_1 + \mathbf{I}$	$\mathfrak{sp}_{n-1}$
			$\mathfrak{su}_{2n-1} \oplus \mathbb{R}$		$\mathfrak{sp}_{n-1} \oplus \mathbb{R}$
$\mathfrak{so}_{2n+4}$	$\mathfrak{so}_{2n+3}$	$\varpi_1 + \mathbf{I}$	$\mathfrak{su}_{n+2}$	$\varpi_1 + \varpi_{n+1}$	$\mathfrak{su}_{n+1}$
			$\mathfrak{su}_{n+2} \oplus \mathbb{R}$		$\mathfrak{su}_{n+1} \oplus \mathbb{R}$
$\mathfrak{so}_{4n}$	$\mathfrak{so}_{4n-1}$	$\varpi_1 + \mathbf{I}$	$\mathfrak{sp}_n$	$\varpi_1 + \varpi_1$	$\mathfrak{sp}_{n-1}$
			$\mathfrak{sp}_n \oplus \mathbb{R}$		$\mathfrak{sp}_{n-1} \oplus \mathbb{R}$
			$\mathfrak{sp}_n \oplus \mathfrak{su}_2$		$\mathfrak{sp}_{n-1} \oplus \mathfrak{su}_2$
$\mathfrak{so}_{16}$	$\mathfrak{so}_{15}$	$\varpi_1 + \mathbf{I}$	$\mathfrak{so}_9$	ϖ_4	$\mathfrak{so}_7$
$\mathfrak{so}_8$	$\mathfrak{so}_7$	ϖ_3	$\mathfrak{so}_7$	$\varpi_1 + \mathbf{I}$	G_2
$\mathfrak{so}_7$	G_2	ϖ_1	$\mathfrak{so}_5$	$\varpi_1 + \mathbf{I} + \mathbf{I}$	$\mathfrak{su}_2$
			$\mathfrak{so}_5 \oplus \mathbb{R}$		$\mathfrak{su}_2 \oplus \mathbb{R}$
			$\mathfrak{so}_6$	$\varpi_1 + \mathbf{I}$	$\mathfrak{su}_3$

Note that in Table 1 all algebras $\mathfrak{g}^1$ are simple; if $\mathfrak{g}^2$ is not simple, then $\mathfrak{g}^2 = \mathfrak{g}^3 \oplus \mathfrak{a}$, where $\mathfrak{g}^3$ is simple, $\mathfrak{a} = \mathbb{R}$ or $\mathfrak{a} = \mathfrak{su}_2$ and $\mathfrak{g} = \mathfrak{g}^1 + \mathfrak{g}^3$.

Suppose $X = (N \rtimes L)/K$ is commutative and $K \neq L$. Let $L_1 \neq \mathrm{SU}_2$ be a simple non-commutative normal subgroup of L such that $\pi_1(K) \neq L_1$ and $L_1 \not\subset P$. Due to Lemma 5 and

Theorem 1 there is a non-trivial factorisation $L_1 = \pi_1(K)\pi_1(L_*)$. Because L_1 is compact, this equality holds if and only if $\mathfrak{l}_1 = \pi_1(\mathfrak{k}) + \pi_1(\mathfrak{l}_*)$. Note that if X satisfies condition $(*)$, then no simple factor L_i of L is contained in P. Besides, if X satisfies condition $(*)$ and $L_i \neq \mathrm{SU}_2$, then, by Theorem 2, $\pi_i(K) \neq L_i$ if and only if $L_i \not\subset K$.

Let $X = (N \rtimes L)/K$ be commutative and L_1 be a simple normal subgroup of L such that $\pi_1(K) \neq L_1$ and $L_1 \not\subset P$. Let $\hat{V}$ stand for the sum of those irreducible L-invariant subspaces of $\mathfrak{n}$, on which L_1 acts non-trivially. Denote by $\hat{P}$ the identity component of the ineffective kernel of $L : \hat{V}$. Set $\hat{L} = L/\hat{P}$. The group $\hat{L}$ can be considered as the maximal connected subgroup of L, which acts on $\hat{V}$ locally effectively. Denote by $\hat{K}$ the image of the projection of K onto $\hat{L}$, i.e., $\hat{K} \cong K/(K \cap \hat{P})$. Clearly, $\hat{V}$ need not be a subalgebra of $\mathfrak{n}$, but it can be considered as a factor algebra. Let $\hat{\mathfrak{a}}$ stand for the maximal L-invariant subspace of $\mathfrak{n}$ on which L_1 acts trivially. Evidently, $\hat{\mathfrak{a}}$ is a subalgebra. Moreover, because different L-invariant summands of $\mathfrak{n}$ commute, see [21, Proposition 15], it is an ideal. We identify $\hat{V}$ with $\mathfrak{n}/\hat{\mathfrak{a}}$.

Proposition 2. *All triples $(\hat{L}, \hat{K}, \hat{V})$ which can be obtained from a commutative space X as a result of the procedure described above are contained in Table 2b.*

Proof. The homogeneous space $(\hat{V} \rtimes L)/K$ is commutative. Let $L = L_1 \times L^1 \times Z(L)$, where L^1 is the product of all simple factors of L except L_1. The space $\hat{V}$ can be represented as a sum

$$\hat{V} = (V_{1,1} \otimes_{\mathbb{D}_1} V_1^1) \oplus \ldots \oplus (V_{1,p} \otimes_{\mathbb{D}_p} V_p^1),$$

where $V_{1,i}$ are irreducible L_1-modules, V_i^1 are L^1-modules, L_1 acts trivially on each V_i^1 and L^1 acts trivially on each $V_{1,i}$. In each summand the tensor product is taken over the skew-field $\mathbb{D}_i$, which equals $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ depending on $V_{1,i}$ and V_i^1.

Set $V_1 := V_{1,1} \oplus \ldots \oplus V_{1,p}$. Let $x_1 = x_{1,1} + \ldots + x_{1,p}$ be a generic vector of V_1. Then there is a sum of decomposable vectors $x := x_{1,1} \otimes x_1^1 + \ldots + x_{1,p} \otimes x_p^1 \in V$, such that x_i^1 are linearly independent.

Note that $(\pi_1(L_x), \pi_1(K_x))$ is a spherical pair as a factor of the spherical pair (L_x, K_x). Clearly, $\pi_1(K_x) \subset \pi_1(L_x) \cap \pi_1(K)$. According to condition (i), $L_1 = \pi_1(L_x)\pi_1(K)$. Hence, the pair $(L_1, \pi_1(K))$ is an extension of the spherical pair $(\pi_1(L_x), \pi_1(L_x) \cap \pi_1(K))$.

The generic stabiliser $(L_1)_*(V_1)$ is defined up to conjugation. We may assume that $(L_1)_*(V_1) = (L_1)_{x_1} \subset \pi_1(L_x)$. Obviously, $\pi_1(L_x)x_1 \subset \mathbb{D}_1 x_{1,1} + \ldots + \mathbb{D}_p x_{1,p}$. Hence, $(L_1)_*(V_1)$ is a normal subgroup of $\pi_1(L_x)$ and $\pi_1(L_x)/(L_1)_*(V_1)$ is locally isomorphic to a direct product of $(\mathrm{SU}_2)^k$ and $(\mathrm{U}_1)^n$. Recall that there is a non-trivial factorisation $L_1 = \pi_1(L_x)\pi_1(K)$. Looking in Table 1 one can see that $(L_1)_*(V_1)^0 \neq \{E\}$, $L_1 = \pi_1(L_x)(L_1)_*(V_1)$, and $\pi_1(L_x)/(L_1)_*(V_1)$ is locally isomorphic to SU_2 or U_1, or trivial. To conclude the proof, we need the following lemma.

Lemma 6. *All triples $(L_1, \pi_1(K), V_1)$ which can be obtained as a result of the above procedure are contained in Table 2a.*

Proof. As we already know, $(L_1, \pi_1(K))$ is an extension of a spherical pair $(\pi_1(L_x), \pi_1(L_x) \cap \pi_1(K))$; in particular, it is spherical. The Lie algebra $\mathfrak{l}_1$ is simple, hence this extension is contained in Table 1. Suppose $\mathfrak{l}_1 = \mathfrak{g}$ for some $\mathfrak{g}$ from Table 1 and $(\mathfrak{g}, \mathfrak{g}_i)$ is an extension of a spherical pair $(\mathfrak{g}_j, \mathfrak{u})$. Then $(\mathfrak{l}_1)_*(V_1)$ equals to one of the following three algebras: $\mathfrak{g}_j$, $\mathfrak{g}_j'$, $\mathfrak{g}_j/\mathfrak{sp}_1$. The last case is only possible if $\mathfrak{g}_j = \mathfrak{sp}_n \oplus \mathfrak{sp}_1$. We have to check if any of these three algebras is a generic stabiliser for some representation of $\mathfrak{l}_1$. The representations of complex simple algebras with non-trivial generic stabiliser are classified by Élashvili [8]. We are interested in the real forms of the orthogonal representations with non-trivial generic stabiliser.

The algebra $\mathfrak{l}_1$ is either $\mathfrak{su}_{2n}$ or $\mathfrak{so}_m$. If $\mathfrak{l}_1 = \mathfrak{su}_{2n}$, $(\mathfrak{l}_1)_*(V_1)$ have to be one of the following three algebras $\mathfrak{sp}_n$, $\mathfrak{su}_{2n-1}$, $\mathfrak{u}_{2n-1}$. According to [8], V_1 is either $\mathbb{C}^{2n}$ or $\mathbb{R}^6$ for $\mathfrak{l}_1 = \mathfrak{su}_4$.

Suppose $\mathfrak{l}_1 = \mathfrak{so}_m$. According to [8], if $m > 15$, then V_1 is the sum of k copies of $\mathbb{R}^m$ and $(\mathfrak{l}_1)_*(V_1) = \mathfrak{so}_{m-k}$. Table 1 tell and Krämer's classification [11] tell us that $k = 1$, so $\mathfrak{g}_j = \mathfrak{so}_{m-1}$ and $(\mathfrak{g}_j, \mathfrak{u})$ is spherical only in one case, namely $(\mathfrak{so}_{2n+3}, \mathfrak{su}_{n+1} \oplus \mathbb{R})$. For smaller m one has to check several cases by direct computations. The result is given in rows 2a, 2b, 4a and 4b of Table 2a.

Note that, we assume that a triple $(L_1, \pi_1(K), W)$ is contained in Table 2a, if W is an L_1-invariant subspace of some V_1 and $(L_1, \pi_1(K), V_1)$ is a triple precisely listed in Table 2a. $\square$

Proceed to the proof of Proposition 2. As can be easily seen, Table 2a does not contain symplectic representations, i.e., those that could be a factor in a tensor product over $\mathbb{H}$. It contains only one Hermitian representation (in the first line).

Set $W_1 := V_{1,1} \otimes V_1^1$. Our next goal is to describe all possible W_1. The pair $(L_1, \pi_1(K), V_{1,1})$ is contained in Table 2a, i.e., $V_{1,1}$ is an L_1-invariant subspace of V_1. In particular, the tensor product in W_1 is taken over $\mathbb{R}$ or $\mathbb{C}$.

Set $n_1 = \dim V_{1,1}$, $n_2 = \dim V_1^1$ and $s = min(n_1, n_2)$ (here we consider the dimensions over the same field as the tensor product in W_1.) Let us prove that up to conjugation

$$[\pi_1(L_*(W_1))]' = [(L_1)_{\xi_1} \cap ... \cap (L_1)_{\xi_s}]',$$

where $\xi_i \in V_{1,1}$ are linear independent generic vectors. In particular, this equality means that if $\dim V_{1,1} \leq \dim V_1^1$, then $[\pi_1(L_*(W_1))]' = \{E\}$. Note that the right hand side is evidently a subset of the left hand side. So, it is enough to proof the inclusion "$\subset$".

The group L_1 is compact, so we can assume that $L_1 \subset \mathrm{O}_{n_1}$ if $\mathbb{D}_1 = \mathbb{R}$ or $L_1 \subset \mathrm{U}_{n_1}$ if $\mathbb{D}_1 = \mathbb{C}$. Without loss of generality it can be also assumed that the considered action is $L_1 \cdot \mathrm{O}_{n_2} : V_{1,1} \otimes_{\mathbb{R}} V_1^1$ or $L_1 \cdot \mathrm{U}_{n_2} : V_{1,1} \otimes_{\mathbb{C}} V_1^1$. The left hand side of the proving equality could become only larger after such replacement. The required inclusion can be deduced from a well known fact: if $n \leq m$, then the generic stabilisers for $\mathrm{O}_n \times \mathrm{O}_m : \mathbb{R}^n \otimes \mathbb{R}^m$ and $\mathrm{U}_n \times \mathrm{U}_m : \mathbb{C}^n \otimes \mathbb{C}^m$ are $\mathrm{O}_{m-n} \times (\mathbb{Z}/2\mathbb{Z})^n$ and $\mathrm{U}_{m-n} \times \mathrm{U}_1^n$, respectively.

Recall that there should be factorisations $L_1 = \pi_1(\hat{K})\pi_1(\hat{L}_*)$ and $L_1 = \pi_1(\hat{K})[\pi_1(\hat{L}_*)]'$. This imposes rather strong restrictions on s. Thus $s = 1$ in cases 1a, 1b, 2b, 3 and 4b; $s < 3$ in case 2a; $s < 4$ in case 4a. If V_1 is irreducible L_1-module, then $\hat{V} = W_1$. Hence, $\hat{V} = V_1$ in cases 1a, 1b, 2b, 3 and 4b. Note that in general

$$\pi_1(\hat{L}_*(\hat{V})) \subset \pi_1(\hat{L}_*(V_{1,1} \otimes V_1^1) \cap \ldots \cap \hat{L}_*(V_{1,p} \otimes V_p^1)).$$

So there are only a few possibilities for $\hat{V}$ and hence for $(\hat{L}, \hat{K})$. Namely, $p \leq 2$, $(\dim V_1^1 + \dim V_2^1) \leq 2$ in case 2a; and $p \leq 3$, $(\dim V_1^1 + \dim V_2^1 + \dim V_3^1) \leq 3$ in case 4a. For each representation one can easily verify whether conditions (i) and (ii) hold. The result is shown in Table 2b. $\square$

Remark 2. In the first row of Table 2b $\hat{V}$ is written as $\mathbb{C}^{2n}(\oplus\mathbb{R})$. Of course, $\mathbb{R}$ is a trivial L-module. So, formally it cannot be put into the table. But it is done here to indicate that in this case $\hat{V}$ can be a non-commutative subspace of $\mathfrak{n}$.

Table 2a.

	L_1	$\pi_1(K)$	V_1
1a	SU_{2n}	Sp_n	$\mathbb{C}^{2n}$
1b	SU_4	U_3	$\mathbb{R}^6$
2a	SO_7	G_2	$\mathbb{R}^7 \oplus \mathbb{R}^7$
2b	Spin_7	Spin_6 $\mathrm{Spin}_5 \cdot \mathrm{U}_1$	$\mathbb{R}^8$
3	SO_{2n}	SU_n U_n	$\mathbb{R}^{2n}$
4a	SO_8	Spin_7	$\mathbb{R}^8 \oplus \mathbb{R}^8 \oplus \mathbb{R}^8$
4b	SO_8	$\mathrm{Sp}_2 \times \mathrm{SU}_2$	$\mathbb{R}^8$

Table 2b.

	$\hat{L}$	$\hat{K}$	$\hat{V}$
1a	$\mathrm{(S)U}_{2n}$	$\mathrm{Sp}_n(\cdot \mathrm{U}_1)$	$\mathbb{C}^{2n}(\oplus\mathbb{R})$
1b	SU_4	U_3	$\mathbb{R}^6$
2a	SO_7	G_2	$\mathbb{R}^7$
2b	Spin_7	Spin_6	$\mathbb{R}^8$
3	SO_{2n}	U_n	$\mathbb{R}^{2n}$
4a	$\mathrm{SO}_8 \times \mathrm{SO}_2$	$\mathrm{Spin}_7 \times \mathrm{SO}_2$	$\mathbb{R}^8 \otimes \mathbb{R}^2$
4b	SO_8	Spin_7	$\mathbb{R}^8 \otimes \mathbb{R}^2$
4c	SO_8	Spin_7	$\mathbb{R}^8$
4d	SO_8	$\mathrm{Sp}_2 \times \mathrm{SU}_2$	$\mathbb{R}^8$

So far nothing was said about the Lie algebra structure on $\hat{V}$. Denote by $\hat{\mathfrak{n}}$ the subalgebra of $\mathfrak{n}$ generated by $\hat{V}$, i.e., $\hat{\mathfrak{n}} = \hat{V} + [\hat{V}, \hat{V}]$.

Suppose $[\hat{V}, \hat{V}] = 0$. Then we have a commutative homogeneous space $(\hat{N} \leftthreetimes \hat{L})/\hat{K}$, where $\hat{N}$ is a simply connected commutative Lie group.

Suppose $X = (N \leftthreetimes L)/K$ is commutative, $L_1 \neq \mathrm{SU}_2$ is a simple normal subgroup of L, such that $\pi_1(K) \neq L_1$, $L_1 \not\subset P$ and a triple $(\hat{L}, \hat{K}, \hat{V})$ corresponds to L_1 in the aforementioned way. By Proposition 2, $(\hat{L}, \hat{K}, \hat{V})$ is contained in Table 2b.

Lemma 7. *If $(\hat{L}, \hat{K}, \hat{V})$ is a triple from Table 2b but not from the first row of it, then $\hat{V}$ has to be a commutative subspace of $\mathfrak{n}$.*

Proof. Assume that $[\hat{V},\hat{V}]\neq 0$. There in an L-invariant surjection $\Lambda^2\hat{V}\mapsto[\hat{V},\hat{V}]$. Because representation of L in $\Lambda^2\hat{V}$ is completely reducible, $[\hat{V},\hat{V}]$ can be regarded as an L-invariant subspace of $\Lambda^2\hat{V}$. If $[\hat{V},\hat{V}]$ is a non-trivial L_1-module, then it is an $\hat{L}$-invariant subspace of $\hat{V}$ (by definition of $\hat{V}$). Recall that $\mathfrak{n}$ is nilpotent. Hence, $[\hat{V},\hat{V}]\neq\hat{V}$. In particular, if $\hat{V}$ is an irreducible $\hat{L}$-module, then L_1 has to act on $[\hat{V},\hat{V}]$ trivially. In all rows of Table 2b except 4b the representation $\hat{L}:\hat{V}$ is irreducible. The space $\Lambda^2\hat{V}$ contains non-trivial $\hat{L}$-invariants only in cases 1a, 4a and 4b.

Consider case 4a. Here $\hat{K}_*(\mathfrak{m})=\mathrm{SU}_4\times\mathrm{SO}_2$ and $\mathbb{R}^8\otimes\mathbb{R}^2=\mathbb{C}^4\otimes_{\mathbb{R}}\mathbb{C}\cong\mathbb{C}^4\oplus\mathbb{C}^4$ is a sum of two isomorphic $K_*(\mathfrak{m})$-modules. According to [21, Proposition 15], these two submodules are commutative and commute with each other. Hence, $[\hat{V},\hat{V}]=0$. In case 4b $\hat{V}=\mathbb{R}^8\oplus\mathbb{R}^8$ is a sum of two isomorpich $\hat{L}$-modules. Hence, it is commutative. □

The first case is different. It corresponds to six commutative spaces, namely $\hat{L}$ can be SU_{2n} or U_{2n}. In the second case there are also two possibilities $\hat{K}=\mathrm{Sp}_n$ or $\hat{K}=\mathrm{Sp}_n\times\mathrm{U}_1$. Independently $\hat{\mathfrak{n}}$ can be either $\mathbb{C}^{2n}$ or $\mathbb{C}^{2n}\oplus\mathbb{R}$, with $\hat{N}$ being commutative or the Heisenberg group H_{2n}.

So, the first row gives rise to six commutative spaces and each of the other to only one. In each case the commutativity can easily be proved by means of Theorem 1. Two cases are considered in detail.

Example 1. Let us prove that $(H_n\leftthreetimes\mathrm{U}_{2n})/\mathrm{Sp}_n$ is commutative. Since U_{2n} and Sp_n are transitive on the sphere in $\mathbb{C}^{2n}$, $\mathbb{R}[\mathbb{C}^{2n}]^{\mathrm{U}_{2n}}=\mathbb{R}[q]=\mathbb{R}[\mathbb{C}^{2n}]^{\mathrm{Sp}_n}$, where q is the invariant of degree 2.

The generic stabiliser for $\mathrm{Sp}_n:\mathbb{C}^{2n}$ is equal to Sp_{n-1}. The space $\mathrm{U}_{2n-1}/\mathrm{Sp}_{n-1}$ is a compact real form of the complex spherical space $\mathrm{GL}_{2n-1}(\mathbb{C})/\mathrm{Sp}_{2n-2}(\mathbb{C})$, and, hence, is commutative.

So, only (iii) is left. Here we have $\mathfrak{m}=\mathfrak{u}_{2n}/\mathfrak{sp}_n=\bigwedge^2\mathbb{C}^{2n}$. It is a classical result that $K_*(\bigwedge\mathbb{C}^{2n})=\underbrace{\mathrm{SU}_2\times\ldots\times\mathrm{SU}_2}_{n}$. As a $K_*(\mathfrak{m})$-module $\mathfrak{n}=\mathfrak{v}_1\oplus\ldots\oplus\mathfrak{v}_n\oplus\mathbb{R}$, where $\mathfrak{v}_i=\mathbb{C}^2$ for every i. Each $\mathfrak{v}_i$ is acted upon by its own SU_2. Note that $[\mathfrak{v}_i,\mathfrak{v}_j]=0$ for $i\neq j$. For $K_*(\mathfrak{m})$-invariants we have $S(\mathfrak{n})^{K_*(\mathfrak{m})}=\mathbb{R}[t_1,\ldots t_n,\xi]$, where t_i is the quadratic SU_2 invariant in $S^2(\mathfrak{v}_i)$ and $\xi\in\mathfrak{h}_n'$. Evidently, t_i and t_j commute as elements of the Poisson algebra $S(\mathfrak{n})$, and ξ lies in the centre of $S(\mathfrak{n})$.

Example 2. The homogeneous space $(\mathbb{R}_{2n}\leftthreetimes\mathrm{SO}_{2n})/\mathrm{U}_n$ is also commutative. Here $\mathfrak{n}$ is commutative, so we do not need to check condition (iii). For (i) we have $\mathbb{R}[\mathbb{R}^{2n}]^{\mathrm{SO}_{2n}}=\mathbb{R}[q]=\mathbb{R}[\mathbb{R}^{2n}]^{\mathrm{U}_n}$. It can be easily seen that $L_*=\mathrm{SO}_{2n-1}$ and $K_*=\mathrm{U}_{n-1}$. The corresponding homogeneous space $\mathrm{SO}_{2n-1}/\mathrm{U}_{n-1}$ is spherical by the Krämer's classification [11].

Note that the triple $(\mathrm{SU}_4,\mathrm{U}_3,\mathbb{R}^6)$ is locally isomorphic to a triple from the row 3 of Table 2b with n=3.

Proposition 3. *Suppose $X = (N \rightthreetimes L)/K$ is an indecomposable commutative space, $\mathfrak{n} \neq 0$, L is simple and $L \neq K$. Then X is locally isomorphic to a homogeneous space from Table 2b.*

Proof. The action $L : \mathfrak{n}$ is non-trivial, otherwise $X = N \times (L/K)$. Consider the triple $(L, K, \hat{V})$ corresponding to $L_1 = L$. By Proposition 2, it is contained in Table 2b. Assume that $\hat{\mathfrak{n}} \neq \mathfrak{n}$ and let $\mathfrak{a}$ be an L-invariant complement of $\hat{\mathfrak{n}}$ in $\mathfrak{n}$. By definition of $\hat{V}$, L acts on $\mathfrak{a}$ trivially. Thus $\mathfrak{a}$ is an abelian ideal and X is decomposable $X = ((\hat{N} \rightthreetimes L)/K) \times A$, where $A \subset N$ and $\mathrm{Lie}\, A = \mathfrak{a}$. $\square$

Lemma 8. *Let $(N \rightthreetimes L)/K$ be a commutative homogeneous space satisfying condition $(*)$ and $K_1 \cong \mathrm{SU}_2$ a normal subgroup of K. Then either $K_1 \subset \mathrm{SO}_8$, where SO_8 is a simple direct factor of L contained in the row 4d of Table 2b, or K_1 is the diagonal of a product of at most three simple direct factors of L isomorphic to SU_2.*

Proof. Suppose $\pi_i(K_1) \neq \{E\}$ and $L_i \neq \mathrm{SU}_2$. Then $\pi_i(K) \neq L_i$. Consider a triple $(\hat{L}, \hat{K}, \hat{V})$ corresponding to L_i. By Proposition 2, it is contained in Table 2b. Note that K_1 is a normal subgroup of $\hat{K}$. Thus $\hat{K} = \mathrm{Sp}_2 \times \mathrm{SU}_2$, $\hat{L} = \mathrm{SO}_8$. Assume that K_1 has a non-trivial projections onto some other simple direct factor of L. Then the pair $(\mathrm{SO}_8 \times \mathrm{SU}_2, \mathrm{Sp}_2 \times \mathrm{SU}_2)$ with SU_2 embedded into SO_8 as a centraliser of Sp_2 and into SU_2 isomorphically, should be spherical, but it is not. To conclude, note that the pair $(\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2, \mathrm{SU}_2)$ is not spherical either. $\square$

3. Principal commutative spaces

Let $G/K = (N \rightthreetimes L)/K$ be a commutative homogeneous space. Recall that by our assumptions $L = Z(L) \times L_1 \times \ldots L_m$, where $Z(L)$ is the connected centre of L. The ineffective kernel of $L : \mathfrak{n}$ is denoted by P. Denote by $Z(K)$ the connected centre of K. Decompose $\mathfrak{n}/\mathfrak{n}'$ into a sum of irreducible L-invariant subspaces $\mathfrak{n}/\mathfrak{n}' = \mathfrak{w}_1 \oplus \ldots \oplus \mathfrak{w}_p$.

Definition 5. Let us call a commutative homogeneous space G/K *principal* if P is semisimple, $Z(K) = Z = Z(L) \times (L_1 \cap Z) \times \ldots \times (L_m \cap Z)$ and $Z(L) = C_1 \times \ldots \times C_p$, where $C_i \subset \mathrm{GL}(\mathfrak{w}_i)$.

The classification of commutative homogeneous spaces can be divided in two parts: the classification of principal commutative spaces and description of the possible centres of L and K in the general case. Note that Table 2b contained only two non-principal homogeneous spaces, namely $(H_{2n} \rightthreetimes \mathrm{U}_{2n})/\mathrm{Sp}_n$ and $(\mathbb{C}^{2n} \rightthreetimes \mathrm{U}_{2n})/\mathrm{Sp}_n$.

Example 3. We have proved that the homogeneous spaces $X = (H_{2n} \rightthreetimes \mathrm{SU}_{2n})/\mathrm{Sp}_n$ and $Y = (H_{2n} \rightthreetimes \mathrm{U}_{2n})/\mathrm{Sp}_n$, where $N = H_{2n}$ is the Heisenberg group with $\mathrm{Lie}\, H_{2n} = \mathfrak{h}_{2n} = \mathbb{C}^{2n} \oplus \mathbb{R}$, are commutative. Denote by $X^m = (H_{2n} \rightthreetimes \mathrm{SU}_{2n})^m/(\mathrm{Sp}_n)^m$ the product of m copies of

X. Let L and K be the products of m copies of SU_{2n} and Sp_n respectively. Suppose $F_1 \subset F_2 \subset (\mathrm{U}_1)^m$. Then, evidently, $((H_{2n})^m \leftthreetimes (F_2 \times L))/(F_1 \times K)$ is a commutative homogeneous space, in general indecomposable but non-principal.

The above example is rather simple, because there are no conditions on F_1 and F_2. In other cases the situation is more difficult. For commutative homogeneous spaces of reductive Lie groups the description of possible centres of L and K is given in [23]. The same problem for commutative homogeneous spaces $(H_n \leftthreetimes K)/K$, where $K \subset \mathrm{U}_n$ is solved in [14] and [4]. In the present article we concentrate on principal commutative spaces.

Theorem 3. *Let $X = (N \leftthreetimes L)/K$ be a maximal indecomposable principal commutative homogeneous space satisfying condition $(*)$. Then either X is contained in Table 2b (and L' is simple); or (L, K) is isomorphic to a product of pairs* $(\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2, \mathrm{SU}_2)$, $(\mathrm{SU}_2 \times \mathrm{SU}_2, \mathrm{SU}_2)$ *or* $(\mathrm{SU}_2, \mathrm{U}_1)$ *and a pair (K^1, K^1), where K^1 is a compact Lie group.*

Proof. Suppose there is a simple normal subgroup $L_i \neq \mathrm{SU}_2$ of L, which is not contained in K. Then by Theorem 2 $\pi_i(K) \neq L_i$. Consider the corresponding triple $(\hat{L}, \hat{K}, \hat{\mathfrak{n}})$. It is a homogeneous space from Table 2b. Recall that by definition $\hat{L}$ is the maximal connected subgroup of L acting on $\hat{V}$ locally effectively, $\hat{P}$ is the identity component of the ineffective kernel of $L : \hat{V}$ and $L = \hat{L} \cdot \hat{P}$. Because X is principal, $Z(L) = \hat{C} \times C^1$, where $\hat{C} = \mathrm{GL}(\hat{V}) \cap C$, hence $\hat{C} \subset \hat{L}$, and $C^1 \subset \hat{P}$. In particular, $L = \hat{L} \times \hat{P}$. Similarly, the connected centre $Z = Z(K)$ is a product $Z = Z(L) \times \hat{Z} \times Z^1$, where $\hat{Z} \subset \hat{L}'$ and $Z^1 \subset \hat{P}'$. According to Theorem 2 each normal subgroup $K_i \ncong \mathrm{SU}_2$ of K is contained in some simple direct factor of L, hence either in $\hat{L}$ or in $\hat{P}$. Suppose a normal subgroup $K_j \cong \mathrm{SU}_2$ of K is not contained in any simple direct factor of L. Then, by Lemma 8, it is diagonal in a product of at most three direct factors of L isomorphic to SU_2. The group $\hat{L}$ has no normal subgroups isomorphic to SU_2. Hence, $K_j \subset \hat{P}$. Thus $\hat{K} = \hat{L} \cap K$. Moreover, $K = \hat{K} \times F$, where $F \subset \hat{P}$. Recall that $\mathfrak{n} = \hat{V} \oplus \hat{\mathfrak{a}}$, where $\hat{\mathfrak{a}}$ is an ideal and L_i acts on $\hat{\mathfrak{a}}$ trivially. Let $\hat{A} \subset N$ be a corresponding connected subgroup. Note that either $\hat{L} = L_i$ or $\hat{L} = \mathrm{U}_1 \times L_i$. Anyway $\hat{L}$ acts on $\hat{\mathfrak{a}}$ trivially. Thus in case $[\hat{V}, \hat{V}] = 0$, i.e., $\hat{\mathfrak{n}} = \hat{V}$, we have obtained a decomposition $X = ((\hat{N} \leftthreetimes \hat{L})/\hat{K}) \times ((\hat{A} \leftthreetimes \hat{P})/F)$. But X in indecomposable, hence $X = (\hat{N} \leftthreetimes \hat{L})/\hat{K}$ and it is contained in table 2b.

There is only one possibility for non-commutative $\hat{\mathfrak{n}}$, namely $L_i = \mathrm{SU}_{2n}$, $\mathfrak{n} = \hat{V} \oplus \mathfrak{z}$, where $\mathfrak{z} \in \hat{\mathfrak{a}}$ and $\mathfrak{z} \cong \mathbb{R}$ is a trivial L-module. Let $\mathfrak{a}$ be an L-invariant complement of $\mathfrak{z}$ in $\hat{\mathfrak{a}}$. If $\mathfrak{a}$ is a subalgebra (then it is an ideal), we again have a decomposition of X. Assume that $\mathfrak{z} \subset [\mathfrak{a}, \mathfrak{a}]$. We have an L-invariant decomposition $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{z} \oplus \hat{V}$, where $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a} \oplus \mathfrak{z}$, $[\hat{V}, \hat{V}] = \mathfrak{z}$ and $[\mathfrak{z}, \mathfrak{n}] = 0$. Thus X is a central reduction of $((\hat{N} \leftthreetimes \hat{L})/\hat{K}) \times ((\hat{A} \leftthreetimes \hat{P})/F)$ by a one dimensional subgroup embedded diagonally into $\hat{N}' \times \hat{A}'$. Hence X is not maximal.

We have proved that if there is a simple normal subgroup $L_i \neq \mathrm{SU}_2$ of L, which is not contained in K, then X is a homogeneous space from Table 2. If this not the case the spherical pair (L, K) is a product of the "SU_2-pairs" and (K^1, K^1), where K^1 contains the connected center of L and all its simple normal subgroups different from SU_2. $\square$

Note that the homogeneous space $(\hat{N} \rtimes \hat{L})/\hat{K}$ corresponding to the first row of Table 2b is maximal if and only if $\mathfrak{n} = \mathfrak{h}_{2n}$, and it is principal if and only if $\hat{L} = \mathrm{SU}_{2n}$, $\hat{K} = \mathrm{Sp}_n$ or $\hat{L} = \mathrm{U}_{2n}$, $\hat{K} = \mathrm{U}_1 \cdot \mathrm{Sp}_n$. Homogeneous spaces corresponding to other rows of Table 2b are maximal and principal.

Let X be a commutative homogeneous space. Denote by $(L^\triangle, K^\triangle)$ a spherical subpair of (L, K) of the type $(\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2, \mathrm{SU}_2)$, $(\mathrm{SU}_2 \times \mathrm{SU}_2, \mathrm{SU}_2)$ or $(\mathrm{SU}_2, \mathrm{U}_1)$, and by $\pi^\triangle$ a projection onto $L^\triangle$.

Lemma 9. *If $(L^\triangle, K^\triangle) = (\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2, \mathrm{SU}_2)$ or $(L^\triangle, K^\triangle) = (\mathrm{SU}_2, \mathrm{U}_1)$ then $\pi^\triangle(L_*) = L^\triangle$, if $(L^\triangle, K^\triangle) = (\mathrm{SU}_2 \times \mathrm{SU}_2, \mathrm{SU}_2)$ then $\pi^\triangle(L_*)$ equals $L^\triangle$, or $\mathrm{SU}_2 \times \mathrm{U}_1$.*

Proof. The group SU_2 has only trivial factorisations, besides, $(\pi^\triangle(L_*), \pi^\triangle(L_*) \cap K^\triangle)$ is a spherical pair. In particular, $\pi^\triangle(L_*) \cap K^\triangle$ is not empty. This reasoning explains the first and the second cases. It remains to observe that in the third case the group $\pi^\triangle(L_*)$ can not be $\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{U}_1$, because the pair $(\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{U}_1, \mathrm{U}_1)$ is not spherical. $\square$

We complete our classification modulo a description of possible actions of normal subgroups of L isomorphic to SU_2 on $\mathfrak{n}$. This description is a very intricate problem, which may be a subject of another article.

4. The ineffective kernel

Suppose $X = (N \rtimes L)/K$ is a commutative homogeneous Riemannian space. Let P be the ineffective kernel of the action $L : \mathfrak{n}$. Then L can be decomposed as $L = P \cdot L^\diamond$, where $L^\diamond$ is the maximal connected normal subgroup of L acting on $\mathfrak{n}$ locally effectively. We assume that G is not reductive, hence $P \neq L$. From the classification of spherical pairs we know that any normal subgroup K_1 of K not locally isomorphic to SU_2 can have non-trivial projections only on two different simple factors of L.

Lemma 10. *Let X be commutative. Suppose a normal subgroup $K_1 \neq \mathrm{SU}_2$ of K is not contained in either P or $L^\diamond$. Then there are simple factors P_1, $L^\diamond_1$ of P, $L^\diamond$ such that $K_1 \subset P_1 \times L^\diamond_1$, $P_1 \cong L^\diamond_1 \cong K_1$. Moreover, either $K_1 = \mathrm{SO}_{n+1}$, where $n \geq 4$; or $K_1 = \mathrm{SU}_{n+1}$, where $n \geq 2$.*

Proof. It can be seen from the classification of spherical pairs that $K_1 \subset L_i \times L_j$. We can assume that $K_1 \subset P_1 \times L^\diamond_1$. The action $K_1 : \mathfrak{n}$ is non-trivial, otherwise K_1 would be a subgroup of P. Denote by π^K_1 the projection onto K_1 in K and by $\pi_{1,1}$ the projection onto $P_1 \times L^\diamond_1$ in L. By Lemma 5, $\pi^K_1(K_*) \neq K_1$. Recall that (L_*, K_*) is spherical. Hence, the pair

$(\pi_{1,1}(L_*), \pi_{1,1}(K_*))$ is also spherical. Note that $L_* = P \cdot L_*^\diamond(\mathfrak{n})$. Hence, $\pi_{1,1}(L_*) = P_1 \times \pi_1^\diamond(L_*)$, where $\pi_1^\diamond$ is a projection onto $L_1^\diamond$ in L.

We claim that $(K_1 \times \pi_1^K(K_*), \pi_1^K(K_*))$ is spherical. Without loss of generality, we can assume that $P_1 \cong L_1^\diamond \cong K_1$. If it is not the case, we replace L by a smaller subgroup containing K, namely each of P_1 and $L_1^\diamond$ is replaced by a projection of K onto it. We illustrate the embedding $\pi_{1,1}(K_*) \subset \pi_{1,1}(L_*)$ by the following diagram.

$$\begin{tikzcd}[column sep=small]
\pi_{1,1}(L_*) & \cong & K_1 \arrow[dr, dash] & \times & \pi_1^\diamond(L_*) \arrow[dl, dash] \\
\pi_{1,1}(K_*) & = & & \pi_1^K(K_*) &
\end{tikzcd}$$

Because the pair $(\pi_{1,1}(L_*), \pi_{1,1}(K_*))$ is spherical, $(K_1 \times \pi_1^K(K_*), \pi_1^K(K_*))$ is also spherical. According to the classification of spherical pairs, there are only two possibilities: either $K_1 = \mathrm{SO}_{n+1}$, $\pi_1^K(K_*) = \mathrm{SO}_n$; or $K_1 = \mathrm{SU}_{n+1}$, $\pi_1^K(K_*) = \mathrm{U}_n$. Assume that $P_1 \cong K_1$. If $L_1^\diamond$ is larger than K_1, then $(P_1 \times L_1^\diamond, \pi_{1,1}(K))$ is one of the following three pairs: $(\mathrm{SO}_{n+1} \times \mathrm{SO}_{n+2}, \mathrm{SO}_{n+1})$; $(\mathrm{Sp}_2 \times \mathrm{Sp}_{m+2}, \mathrm{Sp}_2 \times \mathrm{Sp}_m)$; $(\mathrm{SU}_{n+1} \times \mathrm{SU}_{n+2}, \mathrm{U}_{n+1})$. Recall that $L_1^\diamond = \pi_1^\diamond(L_*)\pi_1^\diamond(K)$. The group Sp_{m+2} has no non-trivial factorisation, hence the second case is not possible. Also, we know that $\pi_1^\diamond(L_*) \cap \pi_1^\diamond(K)$ contains SO_n or U_n, depending on K_1. Thus, only one possibility is left $(K_1 \times L_1^\diamond, \pi_{1,1}(K)) = (\mathrm{SU}_3 \times \mathrm{SU}_4, \mathrm{U}_3)$. According to Table 1, $\pi_1^\diamond(L_*) = \mathrm{Sp}_2$, $\mathrm{Sp}_2 \cap \mathrm{U}_3 = \mathrm{Sp}_1 \times \mathrm{U}_1$. We have $\pi_{1,1}(L_*) = \mathrm{SU}_3 \times \mathrm{Sp}_2$. The subgroup $\pi_{1,1}(K_*)$ is contained in $\mathrm{Sp}_1 \times \mathrm{U}_1$, which is not spherical in $\mathrm{SU}_3 \times \mathrm{Sp}_2$. Hence, in this case the pair $(\pi_{1,1}(L_*), \pi_{1,1}(K_*))$ is not spherical. We illustrate this case by the following diagram.

$$\begin{tikzcd}[column sep=small]
\pi_{1,1}(L_*) & = & \mathrm{SU}_3 \arrow[dr, dash] & \times & \mathrm{Sp}_2 \arrow[dl, dash] \\
\pi_{1,1}(K_*) & \subset & & \mathrm{Sp}_1 \times \mathrm{U}_1 &
\end{tikzcd}$$

Thus, we have proved that $L_1^\diamond \cong K_1$.

To conclude, we show that $P_1 \cong K_1$. Denote by π_1^P the projection onto P_1. We can decompose $\pi_1^P(K)$ into a locally direct product $\pi_1^P(K) \cong F \cdot K_1$. The subgroups $F \cdot K_1$ and $F \cdot \pi_1^K(K_*)$ are spherical in P_1. Moreover, the pairs $(P_1 \times K_1, F \times K_1)$ and $(P_1 \times \pi_1^K(K_*), F \times \pi_1^K(K_*))$ are also spherical. There are the same three possibilities for $(P_1 \times K_1, F \times K_1)$, in which $P_1 \neq K_1$, namely: $(\mathrm{SU}_{n+2} \times \mathrm{SU}_{n+1}, \mathrm{U}_{n+1})$, $(\mathrm{Sp}_{m+2} \times \mathrm{Sp}_2, \mathrm{Sp}_m \times \mathrm{Sp}_2)$ and $(\mathrm{SO}_{n+2} \times \mathrm{SO}_{n+1}, \mathrm{SO}_{n+1})$. But even the pair $(P_1, F \cdot \pi_1^K(K_*))$ is not spherical in any of these cases. $\square$

Example 4. We show that the homogeneous spaces $((\mathbb{R}^n \leftthreetimes \mathrm{SO}_n) \times \mathrm{SO}_n)/\mathrm{SO}_n$ and $((H_n \leftthreetimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$ are commutative. We have $L_* = \mathrm{SO}_n \times \mathrm{SO}_{n-1}$ for the first space and $L_* = \mathrm{SU}_n \times \mathrm{U}_{n-1}$ for the second one, K_* is either SO_{n-1} or U_{n-1}. The stabiliser L_* contains the first direct factor and K is the diagonal multiplied by the connected centre of L. Hence, $L = L_* K$. According to [6] and [15], L_*/K_* is spherical. For the second space we have to check condition (iii) of Theorem 1. We have $K_*(\mathfrak{m}) = (\mathrm{U}_1)^n$. As a $K_*(\mathfrak{m})$-module $\mathfrak{n} = \mathfrak{v}_1 \oplus ... \oplus \mathfrak{v}_n \oplus \mathbb{R}$, where $\mathfrak{v}_i = \mathbb{R}^2$ for every $1 \le i \le n$. Each $\mathfrak{v}_i$ is acted upon by its own U_1. Note that $[\mathfrak{v}_i, \mathfrak{v}_j] = 0$ for $i \ne j$. For $K_*(\mathfrak{m})$-invariants we have $S(\mathfrak{n})^{K_*(\mathfrak{m})} = \mathbb{R}[t_1, ...t_n, \xi]$, where t_i is the quadratic U_1 invariant in $S^2(\mathfrak{v}_i)$ and $\xi \in \mathfrak{n}'$. Evidently, t_i and t_j commute as elements of the Poisson algebra $S(\mathfrak{n})$, and ξ lies in the centre of $S(\mathfrak{n})$.

Note that the homogeneous space $((\mathbb{C}^n \rtimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$ is a central reduction of $((H_n \rtimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$ and it is not maximal.

Theorem 4. *Suppose $X = (N \rtimes L)/K$ is a maximal principal indecomposable commutative homogeneous space. Then either X is one of the spaces $((\mathbb{R}^n \rtimes \mathrm{SO}_n) \times \mathrm{SO}_n)/\mathrm{SO}_n$, $((H_n \rtimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$ or each non-commutative simple normal subgroup $K_1 \ne \mathrm{SU}_2$ of K is contained in either P or $L^\diamond$.*

Proof. Essentially, this is a corollary of Lemma 10. Let $K_1 \ne \mathrm{SU}_2$ be a non-commutative simple normal subgroup of K that is not contained in either $L^\diamond$ or P. Then either $K_1 = \mathrm{SO}_n$ or $K_1 = \mathrm{SU}_n$ and there are $P_1 \cong L_1^\diamond \cong K_1$ such that $K_1 \subset P_1 \times L_1^\diamond$. Besides, $\pi_1^K(K_*) = \mathrm{SO}_{n-1}$ or $\pi_1^K(K_*) = \mathrm{U}_{n-1}$, depending on K_1. According to [8], $L_1^\diamond$ can act non-trivially only on its simplest module V ($V = \mathbb{R}^n$ or $V = \mathbb{C}^n$) and $V \subset \mathfrak{n}$ is an L-invariant subspace. Set $C_V = Z(L) \cap \mathrm{GL}(V)$. Because X is principal, $C_V \subset K$ and if C_V is trivial, then $\pi_1^K(K_*) = (K_1)_*(V)$. For $K_1^\diamond = \mathrm{SU}_n$ we have $(\mathrm{SU}_n)_*(\mathbb{C}^n) = \mathrm{SU}_{n-1}$, so $C_V = \mathrm{U}_1$, $(\mathrm{U}_n)_*(\mathbb{C}^n) \cong \pi_1^K(K_*) = \mathrm{U}_{n-1}$. In case $K_1 = \mathrm{SO}_n$ the group C_V is trivial.

Denote by $\mathfrak{n}_1 := V + [V, V]$ the Lie subalgebra generated by V, by $N_1 \subset N$ the corresponding connected subgroup. If $K_1 = \mathrm{SO}_n$, then $(K_1)_*(\mathfrak{m}) = (\mathrm{U}_1)^{[n/2]}$. According to condition (iii) of Theorem 1, $\mathfrak{n}_1$ has to be commutative, i.e, $\mathfrak{n}_1 = V$. If $K_1 = \mathrm{SU}_n$, then $\mathfrak{n}_1$ can be either a Heisenberg or a commutative algebra.

Assume that $X \ne ((N_1 \rtimes (L_1 \times C_V)) \times P_1)/(K_1 \times C_V)$. Then $L = (L_1 \times C_V \times P_1) \times F$, $K = (K_1 \times C_V) \times H$, where $H \subset F$. Let $\mathfrak{a}$ be an L invariant complement of $\mathfrak{n}_1$ in $\mathfrak{n}$. Recall that the actions $(L_1 \times C_V \times P_1) : \mathfrak{a}$ and $F : \mathfrak{n}_1$ are trivial.

To conclude repeat the proof of Theorem 3. If $\mathfrak{a}$ is a subalgebra (ideal), then X is decomposable. Assume that $[\mathfrak{a}, \mathfrak{a}] \not\subset \mathfrak{a}$. The action $L_1 : \mathfrak{a}$ is trivial, hence $[\mathfrak{a}, \mathfrak{a}] \cap V = 0$. Thus $[V, V] \subset [\mathfrak{a}, \mathfrak{a}]$ and X is not maximal. $\square$

Let $K_1 = \mathrm{SU}_2$ be a normal subgroup of K. Suppose it has a non-trivial projections onto P_1 and $L_1^\diamond$. If $L_1^\diamond \ne \mathrm{SU}_2$, then, as we can see from Table 2a, $L_1^\diamond \cong \mathrm{Spin}_8$. But as was already mentioned, the pair $(\mathrm{SU}_2 \times \mathrm{Spin}_8, \mathrm{SU}_2 \times \mathrm{Sp}_2)$ is not spherical. So $L_1^\diamond = \mathrm{SU}_2$.

If $\pi_1^K(K_*) \ne K_1$, i.e., $\pi_1^K(K_*)^0 = \mathrm{U}_1$, then $K_1 \subset P_1 \times L_1^\diamond$ and $P_1 = \mathrm{SU}_2$. But if $\pi_1^K(K_*) = K_1$ (and this can be the case), then P_1 can be larger and K_1 can have a non-trivial projection onto some other simple factor P_2 or $L_2^\diamond = \mathrm{SU}_2$.

Example 5. Set $P := \mathrm{Sp}_m \times \mathrm{Sp}_l$, $L^\diamond := \mathrm{Sp}_1 \times \mathrm{Sp}_n$, $K := \mathrm{Sp}_{m-1} \times \mathrm{Sp}_{l-1} \times \mathrm{Sp}_1 \times \mathrm{Sp}_n$ and take for N a commutative group $\mathbb{H}^n$. The inclusions and actions are illustrated by the following diagram.

$$
\begin{tikzcd}[column sep=small]
\mathrm{Sp}_m \arrow[d, dash] \arrow[drr, dash] & \mathrm{Sp}_l \arrow[d, dash] \arrow[dr, dash] & \mathrm{Sp}_1 \arrow[d, dash] \arrow[drr] & \mathrm{Sp}_n \arrow[d, dash] \arrow[dr] & \\
Sp_{m-1} & \mathrm{Sp}_{l-1} & \mathrm{Sp}_1 & \mathrm{Sp}_n & \mathbb{H}^n
\end{tikzcd}
$$

The homogeneous space $(N \leftthreetimes (P \times L^{\diamond}))/K$ is commutative. Here $L_* = \mathrm{Sp}_m \times \mathrm{Sp}_l \times \mathrm{Sp}_1 \times \mathrm{Sp}_{n-1}$ and $K_* = \mathrm{Sp}_{m-1} \times \mathrm{Sp}_{l-1} \times \mathrm{Sp}_1 \times \mathrm{Sp}_{n-1}$.

To avoid complicated technical details concerning actions and inclusions of normal subgroup isomorphic to Sp_1 we impose on X a condition of Sp_1-saturation.

5. Sp_1-saturated spaces

Let $X = (N \leftthreetimes L)/K$ be a commutative homogeneous space. Let L_i be a simple direct factor of L. By our assumptions L is a product $L = Z(L) \times L_i \times L^i$, where L^i contains all direct factor L_j with $j \neq i$.

Definition 6. A commutative homogeneous space X is called Sp_1-*saturated*, if

(1) any normal subgroup $K_1 \cong \mathrm{SU}_2$ of K is contained in either P or $L^{\diamond}$;

(2) if a simple direct factor L_i is not contained in P and $\pi_i(L_*) = L_i$, then $L_i \subset K$;

(3) if there is an L-invariant subspace $\mathfrak{w}_j \subset (\mathfrak{n}/\mathfrak{n}')$ such that for some L_i the action $L_i : \mathfrak{w}_j$ is non-trivial and the action $Z(L) \times L^i : \mathfrak{w}_j$ is irreducible, then L_i acts on $(\mathfrak{n}/\mathfrak{n}')/\mathfrak{w}_j$ trivially.

Example 6. Set $X = ((N \leftthreetimes (\mathrm{Sp}_n \times \mathrm{Sp}_1)) \times \mathrm{Sp}_1)/(\mathrm{Sp}_n \times \mathrm{Sp}_1)$, where $\mathfrak{n} = \mathbb{H}^n \oplus \mathbb{H}_0$ is a two-step nilpotent non-commutative Lie algebra with $[\mathbb{H}^n, \mathbb{H}^n] = \mathbb{H}_0$, $\mathbb{H}_0$ is the space of purely imaginary quaternions, the normal subgroup Sp_1 of K is the diagonal of the product $\mathrm{Sp}_1 \times \mathrm{Sp}_1$. Here $\mathbb{H}^n = \mathbb{H}^n \otimes_{\mathbb{H}} \mathbb{H}$, where Sp_n acts on $\mathbb{H}^n$ and Sp_1 acts on $\mathbb{H}^1$; $\mathbb{H}_0 \cong \mathfrak{sp}_1$ as an L-module, i.e., Sp_n acts on it trivially and Sp_1 via adjoint representation.

Evidently, $X = (N \leftthreetimes L)/K$ is not Sp_1-saturated. We show that it is commutative. First we compute the generic stabiliser L_*. Recall that $(\mathrm{Sp}_n \times \mathrm{Sp}_1)_*(\mathbb{H}^n) = \mathrm{Sp}_{n-1} \times \mathrm{Sp}_1$. Clearly $L_*(\mathbb{H}_0) = \mathrm{Sp}_n \times \mathrm{U}_1 \times \mathrm{Sp}_1$, $(\mathrm{Sp}_n \times \mathrm{U}_1)_*(\mathbb{H}^n) = \mathrm{Sp}_{n-1} \times \mathrm{U}_1$. We have $L_* = \mathrm{Sp}_{n-1} \times \mathrm{U}_1 \times \mathrm{Sp}_1$, $K_* = K \cap L_* = \mathrm{Sp}_{n-1} \times \mathrm{U}_1$, $K_*(\mathfrak{m}) = \mathrm{Sp}_n \times ((\mathrm{Sp}_1)_*(\mathfrak{sp}_1)) = \mathrm{Sp}_n \times \mathrm{U}_1$. One can easily verify conditions (i) and (ii) of Theorem 1. Tables of [21] and [22] shows that (iii) is also satisfied.

The description of non Sp_1-saturated spaces seems to be an interesting problem. We are going to consider this problem in a forthcoming paper.

Let X be a non Sp_1-saturated commutative homogeneous space. It can be made Sp_1-saturated by enlarging K, L and possibly N too. For instance, if a simple factor Sp_1 of K has non-trivial projections onto P and $L^{\diamond}$, then we replace P by $P \times \mathrm{Sp}_1$ or $P \times \mathrm{Sp}_1 \times \mathrm{Sp}_1$ (the second replacement is needed if Sp_1 has non-trivial projections onto two simple factors of P). The group K is replaced by $K \times \mathrm{Sp}_1$. Starting with the commutative spaces from Example 6 we construct an Sp_1-saturated commutative homogeneous space $(\mathrm{Sp}_1 \times \mathrm{Sp}_1/\mathrm{Sp}_1) \times (N \leftthreetimes K/K)$, where $K = \mathrm{Sp}_1 \times \mathrm{Sp}_n$ and N is the same as before.

Example 7. Set $L = K = \mathrm{Sp}_n \times \mathrm{Sp}_1 \times \mathrm{Sp}_m$, $\mathfrak{n} = \mathbb{H}^n \oplus \mathbb{H}^m \oplus \mathbb{H}_0$, where both algebras $\mathbb{H}^n$ and $\mathbb{H}^m$ are not commutative and $[\mathbb{H}^n, \mathbb{H}^n] = [\mathbb{H}^m, \mathbb{H}^m] = \mathbb{H}_0$. We have $S(\mathfrak{g}/\mathfrak{k})^K = S(\mathfrak{n})^K = \mathbb{R}[\xi_1, \xi_2, \eta]$, where $\xi_1 \in S^2(\mathbb{H}^n)^{\mathrm{Sp}_n}$, $\xi_2 \in S^2(\mathbb{H}^m)^{\mathrm{Sp}_m}$, $\eta \in S^2(\mathbb{H}_0)^{\mathrm{Sp}_1}$, so the corresponding homogeneous space $(N \rtimes K)/K$ is commutative. The third condition of Definition 6 is not fulfilled. If we want to enlarge L, we also need to enlarge N. As a Sp_1-saturation we have a product of two commutative spaces $(N_i \rtimes K_i)/K_i$, where $\mathfrak{n}_1 = \mathbb{H}^n \oplus \mathbb{H}_0$, $K_1 = \mathrm{Sp}_n \times \mathrm{Sp}_1$; $\mathfrak{n}_2 = \mathbb{H}^m \oplus \mathbb{H}_0$, $K_2 = \mathrm{Sp}_m \times \mathrm{Sp}_1$.

The procedure that is inverse to Sp_1-saturation can have steps of three different types. First, one simple factor Sp_1 of K is replaced by U_1; second, two of three simple factors Sp_1 of K are replaced by the diagonal of their product; third, several simple factors Sp_1 of L are replaced by the diagonal of their product, K is replaced by the intersection of the former K and new L and probably N is decreased.

According to Lemma 9, if (L, K) contains any of the "SU_2" pairs condition (2) of Definition 6 is not satisfied. So an Sp_1-saturated maximal principal commutative homogeneous space is a product of the spaces from Table 2b, spaces of the type $((\mathbb{R}^n \rtimes \mathrm{SO}_n) \times \mathrm{SO}_n)/\mathrm{SO}_n$, $((\mathbb{C}^n \rtimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$, $((H_n \rtimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$, spherical homogeneous space of a reductive Lie group and a space of Heisenberg type. Below we prove a more precise statement.

6. Commutative homogeneous spaces of Heisenberg type

In this section we consider homogeneous spaces of the type $(K \ltimes N)/K$. In this case $S(\mathfrak{g}/\mathfrak{k})^K = S(\mathfrak{n})^K$. We assume that $\mathfrak{n}$ is not commutative.

Let us decompose $\mathfrak{n}/\mathfrak{n}'$ into a sum of irreducible K-invariant subspaces, namely $\mathfrak{n}/\mathfrak{n}' = \mathfrak{w}_1 \oplus ... \oplus \mathfrak{w}_p$. As was proved in [21], if X is commutative then $[\mathfrak{w}_i, \mathfrak{w}_j] = 0$ for $i \neq j$, also $[\mathfrak{w}_i, \mathfrak{w}_i] = 0$ if there is $j \neq i$ such that $\mathfrak{w}_i \cong \mathfrak{w}_j$ as a K-module. Denote by $\mathfrak{n}_i := \mathfrak{w}_i \oplus [\mathfrak{w}_i, \mathfrak{w}_i]$ the subalgebra generated by $\mathfrak{w}_i$. Let $\mathfrak{v}^i$ stand for a K-invariant complement of $\mathfrak{n}_i$ in $\mathfrak{n}$ and set $K^i := K_*(\mathfrak{v}^i)$.

Theorem 5. *([25, Theorem 1]) In the above notation, G/K is commutative if and only if each Poisson algebra $S(\mathfrak{n}_i)^{K^i}$ is commutative.*

Note that the irreducibility of $\mathfrak{w}_i$ is not important here. The statement of the theorem remains true for any K-invariant subspace $\mathfrak{w} \subset \mathfrak{n}/\mathfrak{n}'$.

For convenience of the reader we present here the classification results of [21] and [22]. All maximal commutative homogeneous spaces of Heisenberg type with $\mathfrak{n}/\mathfrak{n}'$ being an irreducible K-module and $\dim \mathfrak{n}' > 1$ are listed in Table 3. The following notation is used:

$\mathfrak{n} = \mathfrak{w} \oplus \mathfrak{z}$, where $\mathfrak{z} = \mathfrak{n}'$ is the centre of $\mathfrak{n}$;

$\mathbb{H}_0$ is the space of purely imaginary quaternions;

$\mathbb{C}^m \otimes \mathbb{H}^n$ is the tensor product over $\mathbb{C}$;

$\mathbb{H}^m \otimes \mathbb{H}^n$ is the tensor product over $\mathbb{H}$;

$H\Lambda^2\mathbb{D}^n$, where $\mathbb{D} = \mathbb{C}$ or $\mathbb{H}$, is the skew-Hermitian square of $\mathbb{D}$;

$HS_0^2\mathbb{H}^n$ is the space of Hermitian quaternion matrices of order n with zero trace.

For all cases in Table 3 the commutation operation $\mathfrak{w} \times \mathfrak{w} \mapsto \mathfrak{z}$ is uniquely determined by the condition of K-equivariance. Notation $(\mathrm{U}_1\cdot)F$ means that K can be either F or $\mathrm{U}_1 \cdot F$. Cases in which U_1 is necessary are indicated in the column "U_1". Some spaces are not always maximal. This is indicated in the column "max".

Table 3.

	K	$\mathfrak{w}$	$\mathfrak{z}$	U_1	max
1	SO_n	$\mathbb{R}^n$	$\Lambda^2\mathbb{R}^n = \mathfrak{so}_n$		
2	Spin_7	$\mathbb{R}^8$	$\mathbb{R}^7$		
3	G_2	$\mathbb{R}^7$	$\mathbb{R}^7$		
4	$(\mathrm{U}_1\cdot)\mathrm{SU}_n$ (n even)	$\mathbb{C}^n$	$\Lambda^2\mathbb{C}^n \oplus \mathbb{R}$		
5	$(\mathrm{U}_1\cdot)\mathrm{SU}_n$ (n odd)	$\mathbb{C}^n$	$\Lambda^2\mathbb{C}^n$		
6	U_n	$\mathbb{C}^n$	$H\Lambda^2\mathbb{C}^n = \mathfrak{u}_n$		
7	$(\mathrm{U}_1\cdot)\mathrm{Sp}_n$	$\mathbb{H}^n$	$HS_0^2\mathbb{H}^n \oplus \mathbb{H}_0$		
8	$\mathrm{U}_1 \cdot \mathrm{Spin}_7$	$\mathbb{C}^8$	$\mathbb{R}^7 \oplus \mathbb{R}$		
9	$\mathrm{Sp}_1 \times \mathrm{Sp}_n$	$\mathbb{H}^n$	$\mathbb{H}_0 = \mathfrak{sp}_1$		$n \geqslant 2$
10	$\mathrm{Sp}_2 \times \mathrm{Sp}_n$	$\mathbb{H}^2 \otimes \mathbb{H}^n$	$H\Lambda^2\mathbb{H}^2 = \mathfrak{sp}_2$		
11	$(\mathrm{U}_1\cdot)\mathrm{SU}_2 \times \mathrm{SU}_n$	$\mathbb{C}^2 \otimes \mathbb{C}^n$	$H\Lambda^2\mathbb{C}^2 = \mathfrak{u}_2$	$n = 2$	
12	$\mathrm{U}_2 \times \mathrm{Sp}_n$	$\mathbb{C}^2 \otimes \mathbb{H}^n$	$H\Lambda^2\mathbb{C}^2 = \mathfrak{u}_2$		

In general case, the classification of maximal principal Sp_1-saturated commutative spaces of Heisenberg type is being done in the following way. If $(N \rtimes K)/K$ is a commutative homogeneous space of non-Euclidian type, then there is a non-commutative subspace $\mathfrak{w}_1 \subset (\mathfrak{n}/\mathfrak{n}')$. Denote by K_e the maximal connected subgroup of K acting on $\mathfrak{w}_1$ effectively and by π_e the projection onto K_e in K. Recall that $\mathfrak{n}_1 := \mathfrak{w}_1 \oplus [\mathfrak{w}_1, \mathfrak{w}_1]$.

If $\mathfrak{n}_1$ is not a Heisenberg algebra (i.e., $\dim \mathfrak{n}_1' > 1$), then the pair $(K_e, \mathfrak{n}_1)$ is a central reduction of some pair from Table 3. If $\mathfrak{n}_1$ is a Heisenberg algebra, then $(K_e, \mathfrak{n}_1)$ corresponds to an irreducible spherical representation $K_e(\mathbb{C}) : W$, in a sense that $\mathfrak{w}_1$ is a K_e-invariant real form of $W \oplus W^*$. According to Kac's list [10], there are 11 such cases.

For any commutative homogeneous space $(N_1 \rtimes K_e)/K_e$, where $\mathfrak{n}_1/\mathfrak{n}_1' = \mathfrak{w}_1$, we have to find out if it arise in the aforementioned way from some larger commutative homogeneous space $(\tilde{N} \rtimes K)/K$, and if so, list all of them. Note that K^1 acts on $\mathfrak{n}_1$ as $\pi_e(K^1)$. Due to lemma 5 and the third condition of Definition 6, $\pi_e(K^1)$ is a proper subgroup of K_e.

As was proved in [3], homogeneous space $(N \rtimes F)/F$, where $\mathfrak{n} = \mathbb{R}^n \oplus \Lambda^2\mathbb{R}^n$, is not commutative for any proper subgroup $F \subset \mathrm{SO}_n$. So if $\mathfrak{n}_1$ corresponds to the first row of Table 3, then, by Lemma 5, $\mathfrak{n} = \mathfrak{n}_1$. Below we consider the remaining 22 cases.

We say that the action $K : \mathfrak{n}$ is *commutative* if the corresponding homogeneous space

$(N \leftthreetimes K)/K$ is commutative.

Lemma 11. *Suppose that K_e' is simple and $K_e = K_e' \times \mathrm{U}_1$, $[\mathfrak{w}_1, \mathfrak{w}_1] \neq 0$ and $\mathfrak{w}_1 = W \otimes \mathbb{R}^2$, where K_e' acts on W and U_1 on $\mathbb{R}^2$. Let F be a proper subgroup of K_e'. If the action $F : W$ is reducible then $(N_1 \leftthreetimes (F \times \mathrm{U}_1))/(F \times \mathrm{U}_1)$ is not commutative.*

Proof. We show that the action $H = (\mathrm{SO}_n \times \mathrm{SO}_m) \times \mathrm{SO}_2 : \mathfrak{n}_1 \cong (\mathbb{R}^n \oplus \mathbb{R}^m) \otimes \mathbb{R}^2 \oplus [\mathfrak{w}_1, \mathfrak{w}_1]$ cannot be commutative. Assume that it is commutative and apply Theorem 5. We have $H_*(\mathbb{R}^n \otimes \mathbb{R}^2)^0 = \mathrm{SO}_{n-1} \times \mathrm{SO}_m$. The subspace $\mathbb{R}^m \otimes \mathbb{R}^2$ is a sum of two isomorphic $\mathrm{SO}_{n-1} \times \mathrm{SO}_m$-modules. Hence, $\mathbb{R}^m \otimes \mathbb{R}^2$ is a commutative subalgebra of $\mathfrak{n}_1$. This can happen only if $[\mathfrak{w}_1, \mathfrak{w}_1] = 0$. □

Lemma 12. *Let $(N \leftthreetimes K)/K$ be a commutative homogeneous space from Table 3, but not from the second, third or ninth row. Suppose a subgroup $F \subset K$ acts on $\mathfrak{n}/\mathfrak{n}'$ reducibly. Then $(N \leftthreetimes F)/F$ is not commutative.*

Proof. Assume that $(N \leftthreetimes F)/F$ is commutative. Then due to Proposition 15 of [21] there are at list two subspaces $V_1, V_2 \subset \mathfrak{n}/\mathfrak{n}'$, such that $V_1 \oplus V_2 = \mathfrak{n}/\mathfrak{n}'$ and $[V_1, V_2] = 0$. Evidently, this is not true in cases 1, 4, 5, 6. For the same reason, in cases 10, 11 and 12 F contains the first simple factor of K, either Sp_2 or SU_2.

In the seventh case F has to be a subgroup of $Sp_m \times \mathrm{Sp}_{n-m}$. Subspaces $\mathbb{H}^m$ and $\mathbb{H}^{n-m}$ do not commute with each other. For the eighth case we apply Lemma 11.

In case 10, we have $F \subset \mathrm{Sp}_2 \times \mathrm{Sp}_m \times \mathrm{Sp}_{n-m}$, $F_*(\mathbb{H}^2 \otimes \mathbb{H}^m) \subset \mathrm{Sp}_1 \times \mathrm{Sp}_1 \times \mathrm{Sp}_n$. The subspace $\mathbb{H}^2 \otimes \mathbb{H}^{n-m}$ is a sum of two isomorphic $F_*(\mathbb{H}^2 \otimes \mathbb{H}^m)$-modules. Hence, $\mathbb{H}^2 \otimes \mathbb{H}^{n-m}$ is a commutative subalgebra of $\mathfrak{n}$. But this is not the case.

In case 11, F is a subgroup of either $\mathrm{SU}_2 \times \mathrm{U}_m \times \mathrm{U}_{n-m}$ or $\mathrm{SU}_2 \times \mathrm{Sp}_{m/2}$ for even m. The first case is just the same as the previous one. For the second case note that as a $\mathrm{SU}_2 \times \mathrm{Sp}_{m/2}$-module $\mathbb{C}^2 \otimes \mathbb{C}^n \cong \mathbb{H}^{m/2} \oplus \mathbb{H}^{m/2}$. The 12-th case is exactly the same. □

Corollary (of the proof). Let $(N \leftthreetimes K)/K$ be a non-Euclidian central reduction of a homogeneous space from row 11 or 12 of Table 3. Then $(N \leftthreetimes F)/F$ is not commutative for any proper subgroup of $F \subset K$ acting on $\mathfrak{n}/\mathfrak{n}'$ reducibly.

Lemma 13. *Suppose $\pi_e(K^1) = (\mathrm{U}_1)^n$ and a homogeneous space $(N_1 \leftthreetimes K^1)/K^1$ is commutative. Then $\mathfrak{n}_1 = \mathbb{R}^{2n} \oplus \mathbb{R}$, $K_e = \mathrm{U}_n$.*

Proof. An irreducible representation of U_1 on a real vector space is either trivial ($\mathbb{R}$) or $\mathbb{R}^2$. If $\mathfrak{w}_1$ is the sum of more than n K^1-invariant summands, then two of them are isomorphic and there is a non-zero $\eta \in \mathfrak{w}_1$ such that $[\eta, \mathfrak{w}_1] = 0$. But $K_e\eta = \mathfrak{w}_1 \subset \mathfrak{z}(\mathfrak{n}_1)$. By the same reason $\mathfrak{w}_1^{(\mathrm{U}_1)^n} = 0$. Because the action $(U_1)^n : \mathfrak{w}_1$ is locally effective, $\mathfrak{w}_1 = \mathbb{R}^{2n}$. We have $\Lambda^2\mathbb{R}^2 = \mathbb{R}$, hence K^1 acts on $\mathfrak{n}_1'$ trivially. Each element of K_e is contained in some maximal torus, that is up to conjugation in $\pi_e(K^1)$. Hence K_e acts on $\mathfrak{n}_1'$ trivially and $K_e \subset \mathrm{U}_n$.

The group U_n has no proper subgroups of rank n acting on $\mathbb{R}^{2n}$ irreducibly. Thus we have $K_e = \mathrm{U}_n$, $\mathfrak{n}'_1 = (\Lambda^2\mathbb{R}^n)^{\mathrm{U}_n} \cong \mathbb{R}$. □

From now on, let $(N \rtimes K)/K$ be an indecomposable maximal Sp_1-saturated principal commutative space with $\mathfrak{n}_1 \neq \mathfrak{n}$. In particular, the connected centre $Z(K_e)$ of K_e acts on $\mathfrak{v}^1$ trivially. Let $\mathfrak{a} \subset \mathfrak{n}$ be a K-invariant subalgebra. Clearly, if the action $K : \mathfrak{n}$ is commutative, then $K : \mathfrak{a}$ is also commutative. We assume that $K : \mathfrak{n}$ is not a "subaction" of some large commutative action.

Assume for the time being that K'_e is simple and denote it by K_1. Decompose $\mathfrak{n}$ into a sum of K-invariant subspaces $\mathfrak{n} = \mathfrak{n}_1 \oplus V_2 \otimes_{\mathbb{D}_2} V^2 \oplus ... \oplus V_q \otimes_{\mathbb{D}_q} V^q \oplus V_{\mathrm{tr}}$, where V_i are pairwise non-isomorphic irreducible non-trivial K_1-modules, V_{tr} and V^i are trivial K_1-modules and all the other simple normal subgroups of K act on V_i trivially. First we have to describe possible V_i, then dimensions of V^i, afterwards the actions $K : \bigoplus_{i=2}^{q} V_i \otimes V^i$ and $K : V_{\mathrm{tr}}$. Once again we use Élashvili's classification [8]. Lemma 13 tells us that the adjoint representation can be among V_i only for $(K_e, \mathfrak{n}_1) = (\mathrm{U}_n, \mathbb{C}^n \oplus \mathbb{R})$. Note that if V_{tr} is not contained in $\mathfrak{n}'$, then there is a simple factor of K acting non-trivially both on $\bigoplus_{i=2}^{q} V_i \otimes V^i$ and V_{tr}.

Recall that $\pi_e(K^1)$ is contained in the product $Z(K_e) \cdot N(\xi)$, where $N(\xi) := \{k \in K_1 | k\xi \in \mathbb{D}_i\xi\}$ is the "normaliser" of a generic vector $\xi \in V_i$. Suppose $(K_1)_*(V_i)$ is trivial or finite. Then $\pi_e(K^1)$ is finite for $\mathbb{D}_i = \mathbb{R}$ and commutative for $\mathbb{D}_i = \mathbb{C}$. In the following lemma we prove that for $\mathbb{D}_i = \mathbb{H}$ the projection $\pi_e(K^1)$ is also commutative.

Lemma 14. *Let $F \subset \mathrm{Sp}_n$, $n \geq 2$ and $(F_*(\mathbb{H}^n))^0 = E$. Then the image of the generic stabiliser $(F \times \mathrm{Sp}_m)_*(\mathbb{H}^n \otimes \mathbb{H}^m)$ under the projection on F is commutative.*

Proof. Assume that this is not the case, i.e., the image contains Sp_1. Then this is true not only for generic points, but for all of them, in particular, for decomposable vectors. In other words, the restriction $F|_{\xi\mathbb{H}}$ for $\xi \neq 0, \xi \in \mathbb{H}^n$ contains Sp_1. Due to Lemma 5, we have $F = \mathrm{Sp}_n$. But then $F_*(\mathbb{H}^n) = \mathrm{Sp}_{n-1}$. □

Example 8. Let $(K_e, \mathfrak{n}_1)$ be a pair from the second row of Table 3. We show that $\mathfrak{n} \subset \mathfrak{n}_1 \oplus \mathbb{R}^7 \otimes \mathbb{R}^2$. All representations of Spin_7 are orthogonal, so here all $\mathbb{D}_i$ equal $\mathbb{R}$. The group Spin_7 has only three irreducible representations with infinite generic stabiliser, namely $\mathfrak{so}_7$, $\mathbb{R}^7$ and $\mathbb{R}^8$. If $V_i = \mathbb{R}^8$ for some i, then K^1 has a non-zero invariant in $\mathfrak{w}_1$, which commute with $\mathfrak{w}_1$. This is a contradiction. Thus $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathbb{R}^7 \otimes V^2 \oplus V_{\mathrm{tr}}$. If $\dim V^2 \geq 3$, then $\pi_e(K^1) \subset \mathrm{SU}_2 \times \mathrm{SU}_2$. But the action $\mathrm{SU}_2 \times \mathrm{SU}_2 : (\mathbb{C}^2 \oplus \mathbb{C}^2) \oplus \mathbb{R}^7$ is not commutative. In case $\dim V^2 = 2$ we have $\pi_e(K^1) = \mathrm{Spin}_5 = \mathrm{Sp}_2$. The pair $(\mathrm{Sp}_2, \mathbb{H}^2 \oplus \mathbb{R}^7)$ is a central reduction of the pair from the 7-th row of Table 3 with $n = 2$ ($K = \mathrm{Sp}_2$) by a subgroup corresponding to $\mathbb{H}_0$ (here $HS_0^2\mathbb{H}^2 \cong \mathbb{R}^7$ as a K-module).

We have seen that $\dim V^2 \leq 2$, so no simple normal subgroup of K except Spin_7 acts non-trivially on $V_2 \otimes V^2$. Assume that there is a non-commutative subspace $\mathfrak{w}_2 \subset \mathfrak{n}$. It is either $\mathbb{R}^7$ or $\mathbb{R}^7 \otimes \mathbb{R}^2$. The first case is not possible, because $\Lambda^2\mathbb{R}^7 \cong \mathfrak{so}_7$ as a Spin_7-module. In the second case we apply Theorem 5 to $\mathfrak{w}_2 = \mathbb{R}^7 \otimes \mathbb{R}^2$. Recall that, $K = \mathrm{Spin}_7$ or $K = \mathrm{Spin}_7 \times \mathrm{SO}_2$, $(\mathrm{Spin}_7)_*(\mathbb{R}^8 \oplus \mathbb{R}^7) \subset \mathrm{Spin}_6$. By Lemma 11, the action $\mathrm{SO}_6 \times \mathrm{SO}_2 : \mathfrak{w}_2 \oplus [\mathfrak{w}_2, \mathfrak{w}_2]$ is commutative only if $[\mathfrak{w}_2, \mathfrak{w}_2] = 0$. The corresponding commutative space is indicated in the 13-th row of Table 4.

Other cases are done in almost the same way. There are 7 cases in which K_e has two simple factors, 4 last rows in Table 3 and 3 irreducible spherical representations: $(\mathbb{C}^*\cdot)\mathrm{SL}_m \times \mathrm{SL}_n : \mathbb{C}^m \otimes \mathbb{C}^n$, $(\mathbb{C}^*\cdot)\mathrm{SL}_n \times \mathrm{Sp}_4 : \mathbb{C}^n \otimes \mathbb{C}^4$, $\mathrm{GL}_3 \times \mathrm{Sp}_n : \mathbb{C}^3 \otimes \mathbb{C}^{2n}$. We have to carry the same procedure for both simple normal subgroups of K. Note that, if $(K_e, \mathfrak{n}_1)$ is the pair from the 9-th row of Table 3, then due to the third condition of Definition 6, the normal subgroup Sp_1 of K_e acts on $\mathfrak{v}^1$ trivially.

For convenience of the reader, we list here all irreducible representations of $\mathfrak{su}_n$ with non-trivial generic stabiliser. They are described by the highest weights of the complexifications $V(\mathbb{C})$ with respect to $\mathfrak{sl}_n$.

Table A_{n-1}.

n	representation	$(\mathrm{SU}_n)_*(V)^0$
	$R(\varpi_1) \oplus R(\varpi_1)^*$	SU_{n-1}
	$R(\varpi_2) \oplus R(\varpi_2)^*$	$(\mathrm{SU}_2)^{[n/2]}$
	$R(2\varpi_1) \oplus R(2\varpi_1)^*$	$(\mathrm{U}_1)^{[n/2]}$
	$R(\varpi_1 + \varpi_1^*)$	$(\mathrm{U}_1)^{n-1}$
4	$R(\varpi_2)$	Sp_2
6	$2R(\varpi_3)$	$(\mathrm{U}_1)^2$

Note that the action $(\mathrm{SU}_n)_*(V) : \mathbb{C}^n$ is irreducible only in one case, namely $n = 4$, $V = R(\varphi_2) \cong \mathbb{R}^6$.

For almost all pairs from Table 3, the action $K^1 : \mathfrak{w}_1$ have to be irreducible. This leaves only a few possibilities for V_i. More precisely, there are seven spaces $(N \rtimes K)/K$ with $K_e \neq K$ and non-trivial action $K : \mathfrak{n}/\mathfrak{n}'$ (rows 2, 4, 5, 10, 11, 12 and 13 of Table 4).

If $\mathfrak{n}'$ is a trivial K-module, the calculations are even simpler. However, we have more such cases. Assume that $\mathfrak{n}_1$ is not commutative. Recall that $\mathfrak{w}_1(\mathbb{C}) = W_1 \oplus W_1^*$ as a K-module. We have to check whether the action $\pi_e(K^1) : W_1$ is spherical or not. We will consider one example in full details.

Lemma 15. *Let $(N \rtimes K)/K$, where $K = \mathrm{SU}_2 \times F$ be a principal Sp_1-saturated commutative space. Suppose that the image of the projection of $K_*(\mathfrak{n})$ on SU_2 contains U_1. Then there are three possibilities: $K = \mathrm{SU}_2$, $\mathfrak{n} = \mathbb{R}^3$; $F = (\mathrm{S})U_s$ or $F = (\mathrm{U}_1\cdot)\mathrm{Sp}_{s/2}$, $\mathfrak{n} = \mathbb{C}^2 \otimes \mathbb{C}^s \oplus \mathbb{R}$; $F = (\mathrm{S})U_4$, $\mathfrak{n} = (\mathbb{C}^2 \otimes \mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6$.*

Proof. Note that the intersection of two distinct general subgroups $\mathrm{U}_1 \subset \mathrm{SU}_2$ is finite. Hence, $\mathfrak{n}$ contains only one irreducible K-invariant subspace on which SU_2 acts non-trivially. The Lie algebra $\mathfrak{su}_2$ has only two non-trivial representations with a non-trivial generic normaliser, namely $\mathbb{R}^3$ and $\mathbb{C}^2$. The first one is orthogonal and can not be contained in $\mathfrak{n}$ in the form $\mathbb{R}^3 \otimes \mathbb{R}^s$ for $s > 1$. Assume that $\mathbb{C}^2 \otimes \mathbb{C}^s \subset \mathfrak{n}$. The group F acts on a generic subspace $\mathbb{C}^2 \subset \mathbb{C}^s$ as U_1 or as SU_2. It can be easily verified that the first case is not possible. In the second one F and U_m have the same orbits in $\mathbb{C}^m$, i.e., $\mathrm{U}_m = F\mathrm{U}_{m-1}$. According to the classification [16] , there are only 4 listed in the lemma possibilities for F. Suppose F acts (non-trivially) on some other K-invariant subspace $V \subset \mathfrak{n}$. Then $\mathbb{C}^s$ is an irreducible $F_*(V)$-module. This can only happen for $s = 4$, $F = (\mathrm{S})U_4$ and $V = \mathbb{R}^6$. $\square$

Example 9. We describe all possible principal Sp_1-saturated maximal commutative pairs $(K, \mathfrak{n})$ for $(K_e, \mathfrak{n}_1) = ((\mathrm{U}_1\cdot)\mathrm{SU}_n, \mathbb{C}^n \oplus \mathbb{R})$.

First assume that $n = 2$. The action $\mathrm{U}_1 \cdot \mathrm{U}_1 : \mathbb{C}^2 \oplus \mathbb{R}$ is commutative. Hence we can "replace" SU_2 by U_1. This yields the three possibilities from Lemma 15.

Suppose $n > 2$. From Table A_{n-1} we know that $\mathfrak{su}_n$ has the following irreducible representations with non-trivial generic stabiliser: $\mathfrak{su}_n$, $\mathbb{C}^n$, $\Lambda^2\mathbb{C}^n$ for $n > 4$ and $\mathbb{R}^6$ for $n = 4$. All of them can occur in $\mathfrak{v}^1$ as V_i. The first one is orthogonal, so for $s > 1$ the space $\mathfrak{su}_n \otimes \mathbb{R}^s$ could not be a subspace of $\mathfrak{v}^1$. Moreover, if $V_2 = \mathfrak{su}_n$, then $\mathfrak{v}^1 = \mathfrak{su}_n$, because a rank of any proper subalgebra $\mathfrak{f} \subset \mathfrak{su}_n$ is smaller then $n - 1$. The action $\mathrm{U}_n : \mathfrak{n}_1 \oplus \mathfrak{su}_\mathfrak{n}$ is commutative. In the following we assume that $\mathfrak{su}_n$ is not contained in $\mathfrak{v}^1$.

Consider the case $n = 4$. We have $\mathfrak{v}^1 = \mathbb{R}^6 \otimes \mathbb{R}^s \oplus \mathbb{C}^4 \otimes V^3 \oplus V_{\mathrm{tr}}$. Note that $(\mathrm{SU}_4)_*(\mathbb{R}^6 \otimes \mathbb{R}^3)$ is finite, so $s \le 2$. Also $(\mathrm{U}_1 \cdot \mathrm{U}_4)_*(\mathbb{R}^6 \otimes \mathbb{R}^2 \oplus \mathbb{C}^4) = (\mathrm{U}_1)^3$ and $\mathbb{C}^4$ is not a spherical representation of $(\mathbb{C}^*)^3$. Hence, for $s = 2$ we have $(K, \mathfrak{n}) = ((\mathrm{S})\mathrm{U}_4(\cdot\mathrm{SO}_2), \mathbb{C}^4 \oplus \mathbb{R} \oplus \mathbb{R}^6 \otimes \mathbb{R}^2)$. Another possible pair is $(\mathrm{U}_4 \cdot \mathrm{U}_1, (\mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6 \oplus (\mathbb{C}^4 \oplus \mathbb{R}))$. The rest of this case is the same for general n and is dealt upon below.

Note that $(\mathrm{SU}_n)_*(\Lambda^2\mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n) = \mathrm{U}_1$ and $(\mathrm{SU}_n)_*(\Lambda^2\mathbb{C}^n \oplus \mathbb{C}^n) = \{E\}$. Thus we have either $\mathfrak{n} = \mathfrak{n}_1 \oplus (\Lambda^2\mathbb{C}^n \oplus \mathbb{R})$ or $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathbb{C}^n \otimes_{\mathbb{D}_2} V^2 \oplus V_{\mathrm{tr}}$. Here $\mathbb{D}_2$ equals $\mathbb{C}$ or $\mathbb{R}$. If $\mathbb{D}_2 = \mathbb{R}$ and $\dim V^2 > 1$, then $\pi_e(K^1)$ is contained in $\mathrm{U}_1 \cdot \mathrm{U}_{n-2} \subset \mathrm{U}_2 \cdot \mathrm{U}_{n-2}$. Evidently, the $\mathrm{U}_1 \cdot \mathrm{U}_{n-2}$-module $\mathbb{C}^n$ is not spherical. Hence, $\mathbb{D}_2 = \mathbb{C}$.

Similar to the proof of Lemma 15 we obtain that $K_*(V_{\mathrm{tr}})/\mathrm{SU}_n$ acts on $V^2 = \mathbb{C}^s$ as $(\mathrm{S})\mathrm{U}_s$ or $\mathrm{Sp}_{s/2}$ for even s. In the second case (for $s \ge 4$) we have $\pi_e(K^1) \subset (\mathrm{U}_1)^2 \cdot \mathrm{SU}_{n-3}$, but the action $(\mathbb{C}^*)^2 : \mathbb{C}^3$ is not spherical. Thus $V_{\mathrm{tr}} = \mathbb{R}$ is a trivial K-module and $\mathfrak{n} \subset \mathfrak{n}_1 \oplus \mathbb{C}^2 \otimes \mathbb{C}^s \oplus \mathbb{R}$.

In Table 4 the algebra $\mathfrak{n}$ is given in the following way. Each subspace in parentheses represent a subalgebra $\mathfrak{w}_i \oplus [\mathfrak{w}_i, \mathfrak{w}_i]$. The spaces given outside parentheses are commutative. The action of K is uniquely determined by irreducibility and by Table 3.

Notation $(\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1 \cdot \mathrm{Sp}_{n/2})$ means that this normal subgroup of K can be equal either of these three groups. Symbol $\mathrm{Sp}_{n/2}$ has sense only for even n. For n odd, the group $\mathrm{Sp}_{n/2}$ does not exist, so it cannot be a subgroup of K.

Table 4.

	K	$\mathfrak{n}_{\max}$
1	U_n	$(\mathbb{C}^n \oplus \mathbb{R}) \oplus \mathfrak{su}_n$
2	U_4	$(\mathbb{C}^4 \oplus \Lambda^2\mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6$
3	$\mathrm{U}_1 \cdot \mathrm{U}_n$	$(\mathbb{C}^n \oplus \mathbb{R}) \oplus (\Lambda^2\mathbb{C}^n \oplus \mathbb{R})$
4	SU_4	$(\mathbb{C}^4 \oplus HS_0^2\mathbb{H}^2 \oplus \mathbb{R}) \oplus \mathbb{R}^6$
5	$\mathrm{U}_2 \cdot \mathrm{U}_4$	$(\mathbb{C}^2 \otimes \mathbb{C}^4 \oplus H\Lambda^2\mathbb{C}^2) \oplus \mathbb{R}^6$
6	$\mathrm{SU}_4 \cdot \mathrm{U}_m$	$(\mathbb{C}^4 \otimes \mathbb{C}^m \oplus \mathbb{R}) \oplus \mathbb{R}^6$
7	$\mathrm{U}_m \cdot \mathrm{U}_n$	$(\mathbb{C}^m \otimes \mathbb{C}^n \oplus \mathbb{R}) \oplus (\mathbb{C}^m \oplus \mathbb{R})$
8	$\mathrm{U}_m \cdot \mathrm{SU}_2 \cdot \mathrm{U}_p$	$(\mathbb{C}^m \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^p \oplus \mathbb{R})$
9	$\mathrm{U}_1 \cdot \mathrm{U}_1 \cdot \mathrm{Sp}_n$	$(\mathbb{H}^n \oplus \mathbb{R}) \oplus (\mathbb{H}^n \oplus \mathbb{R})$
10	$\mathrm{Sp}_n \cdot \mathrm{Sp}_1 \cdot (\mathrm{Sp}_1, \mathrm{U}_1, \{E\})$	$(\mathbb{H}^n \oplus \mathbb{H}_0) \oplus \mathbb{H}^1 \otimes \mathbb{H}^n$
11	$\mathrm{Sp}_n \cdot (\mathrm{Sp}_1, \mathrm{U}_1) \cdot \mathrm{Sp}_m$	$(\mathbb{H}^n \oplus \mathbb{H}_0) \oplus \mathbb{H}^n \otimes \mathbb{H}^m$
12	$\mathrm{Sp}_n \cdot (\mathrm{Sp}_1, \mathrm{U}_1, \{E\})$	$(\mathbb{H}^n \oplus \mathbb{H}_0) \oplus HS_0^2\mathbb{H}^n$
13	$\mathrm{Spin}_7 \cdot (\mathrm{SO}_2, \{E\})$	$(\mathbb{R}^8 \oplus \mathbb{R}^7) \oplus \mathbb{R}^7 \otimes \mathbb{R}^2$
14	$\mathrm{U}_1 \cdot \mathrm{Spin}_7$	$(\mathbb{C}^7 \oplus \mathbb{R}) \oplus \mathbb{R}^8$
15	$\mathrm{U}_1 \cdot \mathrm{U}_1 \cdot \mathrm{Spin}_8$	$(\mathbb{C}^8_+ \oplus \mathbb{R}) \oplus (\mathbb{C}^8_- \oplus \mathbb{R})$
16	$\mathrm{U}_1 \cdot \mathrm{Spin}_{10}$	$(\mathbb{C}^{16} \oplus \mathbb{R}) \oplus \mathbb{R}^{10}$
17	$(\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1 \cdot \mathrm{Sp}_{n/2}) \cdot \mathrm{SU}_2$	$(\mathbb{C}^n \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus \mathfrak{su}_2$
18	$(\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1 \cdot \mathrm{Sp}_{n/2}) \cdot \mathrm{U}_2$	$(\mathbb{C}^n \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus (\mathbb{C}^2 \oplus \mathbb{R})$
19	$(\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1 \cdot \mathrm{Sp}_{n/2}) \cdot \mathrm{SU}_2 \cdot (\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1 \cdot \mathrm{Sp}_{n/2})$	$(\mathbb{C}^n \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^n \oplus \mathbb{R})$
20	$(\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1 \cdot \mathrm{Sp}_{n/2}) \cdot \mathrm{SU}_2 \cdot \mathrm{U}_4$	$(\mathbb{C}^n \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6$
21	$\mathrm{U}_4 \cdot \mathrm{U}_2$	$\mathbb{R}^6 \oplus (\mathbb{C}^4 \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus \mathfrak{su}_2$
22	$\mathrm{U}_4 \cdot \mathrm{U}_2 \cdot \mathrm{U}_4$	$\mathbb{R}^6 \oplus (\mathbb{C}^4 \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6$
23	$\mathrm{U}_1 \cdot \mathrm{U}_1 \cdot \mathrm{SU}_4$	$(\mathbb{C}^4 \oplus \mathbb{R}) \oplus (\mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6$
24	$(\mathrm{U}_1 \cdot)\mathrm{SU}_4(\cdot \mathrm{SO}_2)$	$(\mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6 \otimes \mathbb{R}^2$

Theorem 6. *All indecomposable* Sp_1*-saturated maximal principal commutative homogeneous spaces* $(N \rtimes K)/K$ *with non-commutative* $\mathfrak{n}$ *and reducible* $\mathfrak{n}/\mathfrak{n}'$ *are given in Table 4 in a sense that* $\mathfrak{n}$ *is a* K*-invariant subalgebra of* $\mathfrak{n}_{max}$ *.*

Proof. We have seen above how one can prove that all such commutative spaces are contained in Table 4. Now we prove that all listed spaces are commutative. We do it using Theorem 5 and the list of the spherical representations given in [12]. It is proved in [3], [4] and [14] that spaces listed in rows $3, 7, 8, 9, 15, 18$ and 19 are commutative.

Suppose $\mathfrak{n}$ contains a commutative K-invariant ideal $\mathbb{R}^6$. According to Theorem 5, $K : \mathfrak{n}$ is commutative if and only if $K_*(\mathbb{R}^6) : \mathfrak{n}/\mathbb{R}^6$ is commutative. For second, 4-th and 5-th rows of Table 4 pairs $K_*(\mathbb{R}^6) : \mathfrak{n}/\mathbb{R}^6$ are contained in Table 3, hence, commutative. For 6-th, 20-th and 23-d rows of Table 4 pairs $K_*(\mathbb{R}^6) : \mathfrak{n}/\mathbb{R}^6$ correspond to spherical representations listed in [4], [14]. Analogously, for the 22-d row pair $K_*(\mathbb{R}^6\oplus\mathbb{R}^6):\mathfrak{n}/(\mathbb{R}^6\oplus\mathbb{R}^6)$ corresponds to a spherical representation.

Let $(N\leftthreetimes L)/K$ be a commutative homogeneous space. Then the action $K:\mathfrak{n}\oplus(\mathfrak{l}/\mathfrak{k})$, where $\mathfrak{l}/\mathfrak{k}$ is a commutative ideal, is commutative. The pairs from the first and 12-th rows of Table 4 are obtained in such a way from commutative spaces $((H_n\leftthreetimes \mathrm{U}_n)\times \mathrm{SU}_n)/\mathrm{U}_n$ and $(H_{2n}\leftthreetimes \mathrm{U}_{2n})/\mathrm{Sp}_n$. In case $[\mathfrak{n},\mathfrak{n}]=0$, we obtain a commutative homogeneous space of Euclidian type. Thus essentially these are the only non-trivial examples given by this construction.

In the remaining eight cases we use Theorem 5. For instance, take the 11-th row with $K=\mathrm{Sp}_n\times\mathrm{Sp}_m$. Here $\mathfrak{n}$ contains only one non-commutative subspace $\mathfrak{w}_1\cong\mathbb{H}^n$. Set $d=|n-m|$ and $s=\min(n,m)$. Then $K^1=K_*(\mathbb{H}^n\times\mathbb{H}^m)=(\mathrm{Sp}_1)^d\times\mathrm{Sp}_s$. Anyway $K^1/(K^1\cap P_1)$ contains $(\mathrm{Sp}_1)^n$. To conclude note that the action $Sp_1:(\mathbb{H}\oplus\mathbb{H}_0)$ is commutative according to Table 3. □

7. Conclusion

Theorem 7. *Any maximal indecomposable principal* Sp_1*-saturated commutative homogeneous space belongs to the one of the following four classes:*

1) affine spherical homogeneous spaces of reductive real Lie groups;

2) spaces corresponding to the rows of Table 2b;

3) homogeneous space $((\mathbb{R}^n\leftthreetimes\mathrm{SO}_n)\times\mathrm{SO}_n)/\mathrm{SO}_n$, $((H_n\leftthreetimes\mathrm{U}_n)\times\mathrm{SU}_n)/\mathrm{U}_n$, *where the normal subgroups* SO_n *and* SU_n *of* K *are diagonally embedded into* $\mathrm{SO}_n\times\mathrm{SO}_n$ *and* $\mathrm{SU}_n\times\mathrm{SU}_n$, *respectively;*

4) commutative homogeneous spaces of Heisenberg type.

Proof. Let $X=G/K$ be a commutative homogeneous space. If G is reductive, X belongs to the first class. If $L=K$ then it is a space of Heisenberg type.

Assume that G is not reductive and $L\neq K$. Suppose a simple factor K_1 of K has non-trivial projections onto both P and $L^\diamond$. Then due to condition (1) of the definition of Sp_1-saturated commutative spaces, $K_1\neq\mathrm{SU}_2$. By theorem 4, X belongs to the 3-d class. If all simple factors of K are contained in either P or $L^\diamond$, then, because X is principal, $P^0/(P^0\cap K)$ is a factor of X. But X is indecomposable and G is not reductive, so P^0 is trivial. Thus, X satisfies condition $(*)$. If there is a simple factor L_1 of L such that $\pi_1(L_*)\neq K$ and $L_1\subsetneq K$, then, according to Theorem 3, X is contained in the second class.

If there is not any such factor, then also by Theorem 3, (L,K) is a product of pairs of the type $(\mathrm{SU}_2\times\mathrm{SU}_2\times\mathrm{SU}_2,\mathrm{SU}_2)$, $(\mathrm{SU}_2\times\mathrm{SU}_2,\mathrm{SU}_2)$ or $(\mathrm{SU}_2,\mathrm{U}_1)$ and a pair (K^1,K^1), where K^1 is a compact Lie group. But these pairs (except (K^1,K^1)) are not allowed in Sp_1-saturated commutative space. The second condition of the definition of Sp_1-saturated commutative space contradicts the conditions of Lemma 8. Thus, L would be equal K, but this is not the case. □

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