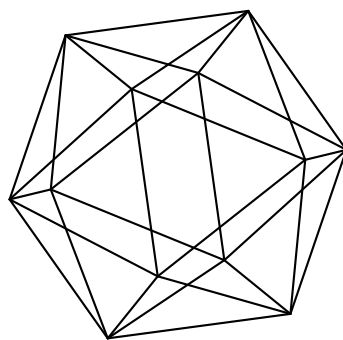


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Transfer factors for Jacquet-Mao's metaplectic
fundamental lemma

by

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Abstract

In an earlier paper we proved Jacquet-Mao's metaplectic fundamental lemma which is the identity between two orbital integrals (one is defined on the space of symmetric matrices and another one is defined on the 2-fold cover of the general linear group) corrected by a transfer factor. But we restricted to the case where the relevant representative is a diagonal matrix. Now, we show that we can extend this result for the more general relevant representative. Our proof is based on the concept of Shalika germs for certain Kloosterman integrals.

1 Introduction

This article extends the result of the author for Jacquet-Mao's metaplectic fundamental lemma in [2] (which is announced in [1]) and in [3]. To state the results, we introduce some notations.

Let F be a non-archimedean local field, \mathcal{O} its valuation ring, and k its residue field. Let p be the characteristic of k ($(p, 2) = 1$) and q be the cardinality of k . We choose once for all a uniformizing element $\varpi \in \mathcal{O}$ (i.e a generator of the maximal ideal of \mathcal{O}). We write v for the valuation of F and $|\cdot|$ for the norm, normalised such that $|x| = q^{-v(x)}$.

Let B_r be the standard Borel subgroup of GL_r (the subgroup of invertible upper triangular matrices) with unipotent radical N_r , and let T_r be a maximal split torus contained in B_r . Let S_r be the variety $\{g \in \mathrm{GL}_r \mid {}^t g = g\}$ and W_r be the Weyl group of T_r .

Let $\psi : k \rightarrow \mathbb{C}^*$ be a non-trivial additive character. We define an additive character on $\Psi : F \rightarrow \mathbb{C}^*$ of F : $\Psi(x) = \psi(\mathrm{res} x d\varpi)$, and a character $\theta : N_r(F) \rightarrow \mathbb{C}^*$ of $N_r(F)$: $\theta(n) = \Psi\left(\frac{1}{2} \sum_{i=2}^r n_{i-1,i}\right)$.

The local metaplectic cover $\widetilde{\mathrm{GL}}_r(F)$ of $\mathrm{GL}_r(F)$ is an extension of $\mathrm{GL}(F)$ by $\{\pm 1\}$ (cf. [8]). We can write the elements of $\widetilde{\mathrm{GL}}_r(F)$ in the form $\tilde{g} = (g, z)$, with $g \in \mathrm{GL}_r(F)$ and $z \in \{\pm 1\}$ and the group multiplication is defined by

$$(g, z)(g', z') = (gg', \chi(g, g')zz'),$$

where χ is a certain cocycle (cf. loc.cit. for the definition of χ). This cover splits (canonically) over $N_r(F)$ (the splitting σ over $N_r(F)$ is simply defined by $\sigma(n) = (n, 1)$); it splits also over $\mathrm{GL}_r(\mathcal{O})$. The splitting κ^* over $\mathrm{GL}_r(\mathcal{O})$ is defined by $\kappa^*(g) = (g, \kappa(g))$. We denote by $\mathrm{GL}_r^*(\mathcal{O})$ the image of $\mathrm{GL}_r(\mathcal{O})$ via the splitting κ^* .

We say a function f on $\widetilde{\mathrm{GL}}_r(F)$ is *genuine* if it satisfies $f(g, z) = f(g, 1) \cdot z$.

The group N_r acts on S_r by $n \cdot s = {}^t n s n$ and $N_r \times N_r$ acts on GL_r by $(n, n') \cdot g = n^{-1} g n$. We say an orbit $N_r \cdot s$ (resp. $(N_r \times N_r) \cdot g$) is *relevant* if the restriction of θ^2 (resp. $(n, n') \mapsto \theta(n^{-1} n')$) on the stabilizer $(N_r)_s$ (resp. $(N_r \times N_r)_g$) of s (resp. of g) is trivial. The relevant orbits $N_r \cdot s$ have the representative of the form $w \mathbf{t}$ (where w is the longest Weyl element of a standard parabolic subgroup in GL_r and \mathbf{t} lies in the center of the corresponding Levi subgroup [9, Theorem 1]). The relevant orbits $(N_r \times N_r) \cdot g$ have the representative of the form $w_{\mathrm{GL}_r} w \mathbf{t}$ with w, \mathbf{t} being as above and $w_{\mathrm{GL}_r} = \mathrm{antdiag}(1, \dots, 1)$ being the longest Weyl element of GL_r . So we have then a bijection between the sets of relevant orbits.

More precisely, the element $w \mathbf{t}$ can be described as follows. Let M be the standard Levi-subgroup of GL_r of type (r_1, \dots, r_m) , i.e M is the group of matrices of the form $\mathrm{diag}(g_i)$ with $g_i \in \mathrm{GL}_{r_i}$. Let $w_M = \mathrm{diag}(w_{\mathrm{GL}_{r_i}})$. Let T_M be the group of matrices of the form $\mathrm{diag}(a_i \mathrm{Id}_{r_i})$ with Id_{r_i} is the identity matrix and $a_i \in F^*$ - the center of M . Then any element of the form $w_M \mathbf{t}$ with $\mathbf{t} \in T_M$ is relevant. All relevant elements are obtained in this way. We denote by W_r^R the set of relevant elements in W_r . If $w \in W_r^R$ then the unique M such that $w = w_M$ is denoted by M_w . We also write T_w for T_M . For instance, if $w = \mathrm{Id}_r$ then $M_{\mathrm{Id}_r} = \mathrm{GL}_r$ and $T_{\mathrm{Id}_r} = T_r$. The stabilizer of $\mathbf{t} \in T_r(F)$ (resp. $w_{\mathrm{GL}_r} \mathbf{t}$) in N_r (resp. $N_r \times N_r$) is trivial. In this sense the diagonal matrices are representatives of the largest orbits.

We denote by $\mathcal{C}_c^\infty(S_r(F))$ (resp. $\mathcal{C}_c^\infty(\widetilde{\mathrm{GL}}_r(F))$) the space of the smooth function of compact support on $S_r(F)$ (resp. on $\widetilde{\mathrm{GL}}_r(F)$). Let ϕ be a function in $\mathcal{C}_c^\infty(S_r(F))$ and f be a genuine function in $\mathcal{C}_c^\infty(\widetilde{\mathrm{GL}}_r(F))$. For each $w \mathbf{t}$ as above, we consider the orbital integrals of the form :

$$I(w \mathbf{t}, \phi) = \int_{N_r / (N_r)_{w \mathbf{t}}} \phi({}^t n w \mathbf{t} n) \theta^2(n) dn$$

and

$$J(w \mathbf{t}, f) = \int_{N_r \times N_r / (N_r \times N_r)_{w_{\mathrm{GL}_r} w \mathbf{t}}} f(\sigma(n)^{-1} (w_{\mathrm{GL}_r} w \mathbf{t}, 1) \sigma(n')) \theta(n^{-1} n') d n d n'.$$

Our main result is

Theorem 1.1. *Suppose that a function $\phi \in \mathcal{C}_c^\infty(S_r(F))$ and a genuine function $f \in \mathcal{C}_c^\infty(\widetilde{\mathrm{GL}}_r(F))$ satisfy the following identity*

$$J(\mathrm{Id}_r \mathbf{t}, f) = \Delta_r(\mathbf{t})I(\mathrm{Id}_r \mathbf{t}, \phi)$$

with $\Delta(\mathbf{t})$ is an explicit function for all $\mathbf{t} \in T_r(F)$. Then for all $w \in W_r^R$ and all $\mathbf{t} \in T_w(F)$ there exists $\Delta_w(\mathbf{t})$ (an explicit function calculated with the help of Δ_i , $1 \leq i \leq r$) such that $J(w\mathbf{t}, f) = \Delta_w(\mathbf{t})I(w\mathbf{t}, \phi)$.

If the first relation holds, we say that ϕ and f are matching. It is conjectured that there is a matching relation between a Hecke function of $S(F)$ and of $\widetilde{\mathrm{GL}}_r(F)$ (**fundamental lemma**). Our theorem asserts that the matching relation for smaller orbits, i.e $w \neq \mathrm{Id}_r$ is determined by the matching relation of the largest orbits. In the case when

$$f = f_0((g, z)) = \begin{cases} \kappa(g)z, & g \in \mathrm{GL}_r(\mathcal{O}), \\ 0, & g \notin \mathrm{GL}_r(\mathcal{O}), \end{cases} \quad \phi = \phi_0(g) = \begin{cases} 1, & g \in \mathrm{GL}_r(\mathcal{O}) \cap S_r(F), \\ 0, & g \notin \mathrm{GL}_r(\mathcal{O}) \cap S_r(F), \end{cases}$$

the fundamental lemma has been proven for $r = 2$ by Jacquet ([4, 5]) and $r = 3$ by Mao ([9]) and, in the case of positive characteristic, for any r by author ([2]) and then, in the general case (with the residual characteristic large enough) by author ([3]).

The integral J above is in fact a Kloosterman integral which is considered in [6, 7] (see. §2). So we have the following density theorem for Kloosterman integral:

Theorem 1.2 (cf. [7]). *If the diagonal orbital integral $J(\mathbf{t}, f)$ of a function $f \in \mathcal{C}_c^\infty(\widetilde{\mathrm{GL}}_r(F))$ vanishes for all $\mathbf{t} \in T_r(F)$, then all the orbital integrals $J(w\mathbf{t}, f)$ with $w \in W_r^R$ and $\mathbf{t} \in T_w(F)$ of f vanish.*

Combining the theorem 1.1 and the theorem 1.2, we obtain the density theorem for the orbital integral I (a Kloosterman integral for symmetric matrices):

Theorem 1.3. *If the diagonal orbital integrals $I(\mathbf{t}, \phi)$ of a function $\phi \in \mathcal{C}_c^\infty(S_r(F))$ vanishes for all $\mathbf{t} \in T_r(F)$, then all the orbital integrals $I(w\mathbf{t}, \phi)$ with $w \in W_r^R$ and $\mathbf{t} \in T_w(F)$ of ϕ vanish.*

We now state the organization of this manuscript. The main tool we use to prove the main theorem (Theorem 1.1) is Shalika germs which describe the asymptotic behavior of the orbital integrals. In Section 2 (resp. 3) we recall this asymptotic behavior of the integral J (resp. the integral I) and do some calculation for the germ functions. The main theorem is proved by induction in Section 4. In this section, we introduce some intermediate integrals which are designed to use inductive argument. The germ relations are used to handle the cases when we can not use these intermediate integrals.

The proof of the main theorem closely follows the guidelines of [7]. Most of this work is written during my visit at MPIM. I thank the MPIM for a very pleasant and productive visit.

2 Computation of the germ on J side

For a convenience, we shall rewrite the orbital integral J . Suppose that $w \in W_r^R$. Then let $P_w = M_w N_w$ be the standard parabolic subgroup which has Levi factor M_w . Let $V_w = N_r \cap M_w$. We have

$$(N_r \times N_r)_{w_{\mathrm{GL}_r} w \mathbf{t}} = N_r^w := \{(n_1, n_2) | w_{\mathrm{GL}_r} n_1^{-1} w_{\mathrm{GL}_r} w n_2 = w\}$$

for all $\mathbf{t} \in T_w$. Furthermore, if $(n_1, n_2) \in N_r^w$ then $n_2, {}^t n_1^{w_{\mathrm{GL}_r}} := {}^t(w_{\mathrm{GL}_r} n_1 w_{\mathrm{GL}_r}) \in V_w$ and $n_2 = w n_1^{w_{\mathrm{GL}_r}} w$. It implies that any point of the orbit of $w_{\mathrm{GL}_r} w \mathbf{t}$ under the action of $N_r \times N_r$ can be uniquely written in the following form

$$w_{\mathrm{GL}_r} {}^t u_1 w a v u_2$$

with $u_i \in N_w(F)$ and $v \in V_w(F)$. Thus

$$J(w \mathbf{t}, f) = \int_{N_w(F) \times N_w(F) \times V_w(F)} f((w_{\mathrm{GL}_r} {}^t u_1 w a v u_2, 1)) \theta(u_1 u_2 v) du_1 dv du_2.$$

Denote by f' the function $g \mapsto f((w_{\mathrm{GL}_r} g, 1))$, $\forall g \in \mathrm{GL}_r$, then $f' \in \mathcal{C}_c^\infty(\mathrm{GL}_r(F))$. The integral J is then the orbital Kloosterman integral $\mathrm{Kloos}(w \mathbf{t}; f')$ which is considered in [6] (in loc. cit., it is denoted by $I(w \mathbf{t}; f')$). This integral converges and defines a smooth function on $T_w(F)$.

We let M be the standard Levi-subgroup of type $(r-1, 1)$. The corresponding element w_M is $\begin{pmatrix} w_{\mathrm{GL}_{r-1}} & 0 \\ 0 & 1 \end{pmatrix}$. We denote by $T_{w_M}^{w_{\mathrm{GL}_r}}$ the set of matrices $\mathbf{t} \in T_{w_M}(F)$ such that $\det(\mathbf{t}) = \det(w_M) \det(w_{\mathrm{GL}_r})$. There exists a smooth function $K_{w_M}^{w_{\mathrm{GL}_r}}$ on $T_{w_M}^{w_{\mathrm{GL}_r}}$ (cf. [6, 7]) with the following property: for any $f \in \mathcal{C}_c^\infty(\mathrm{GL}_r(F))$ there is a smooth function of compact support ω_f on T_M such that

$$J(w_M \mathbf{t}, f) = \omega_f(\mathbf{t}) + \sum_{\alpha \beta = \mathbf{t}} K_{w_M}^{w_{\mathrm{GL}_r}}(\alpha) J(w_{\mathrm{GL}_r} \beta, f). \quad (2.1)$$

Here, we still denote by f the genuine function in $\mathcal{C}_c^\infty(\widetilde{\mathrm{GL}}_r(F))$ defined by $(g, z) \mapsto z f(g)$ with $f \in \mathcal{C}_c^\infty(\mathrm{GL}_r(F))$. The sum is over all pairs in

$$\{(\alpha, \beta) \in (T_{w_M}^{w_{\mathrm{GL}_r}}, T_{\mathrm{GL}_r}(F)) \mid \alpha \beta = \mathbf{t}\}.$$

The function $K_{w_M}^{w_{\mathrm{GL}_r}}$ is the **germ** (for the side J) along the subset $T_{w_M}^{w_{\mathrm{GL}_r}}$. It is not unique.

Let

$$J(a, r) := \mathrm{vol}(\varpi^m \mathcal{O})^{-1} \int \Psi \left[\frac{x_1 + x_2 + \cdots + x_r}{2a} \right] \otimes dx_i$$

The integral is over the subset of F^r defined by:

$$x_i \equiv \quad \mathrm{mod} \varpi^m \mathcal{O}, \quad \prod_{i=1}^r x_i \equiv 1 \quad \mathrm{mod} a \varpi^m \mathcal{O}.$$

Due to Jacquet [7], we have the following formula for the germ $K_{w_M}^{w_{\mathrm{GL}_r}}$.

Proposition 2.1 ([7, Proposition 3.1]). *Set*

$$\alpha = \text{diag}(a, \dots, a, a^{1-r} \det(w_M w_{\text{GL}_r})).$$

Then, for $|a|$ sufficiently small,

$$K_{w_M}^{w_{\text{GL}_r}}(\alpha) = |a|^{-1 - \frac{n(n-1)}{2}} J(a, r).$$

3 Computation of the germ on I side

As the previous section, we let M be the standard Levi-subgroup of type $(r-1, 1)$. The discussion of [6, §2] applies to our situation where $\text{GL}_r(F)$ is replaced by $S_r(F)$ and the group $N_r \times N_r$ ($N_r \times N_r$ acts on GL_r by $g \mapsto {}^t n g n'$) by the group $N_r(F)$ acting on $S_r(F)$. So, as before, there exists a smooth function $L_{w_M}^{w_{\text{GL}_r}}$ on $T_{w_M}^{w_{\text{GL}_r}}$ with the following property: for any $\phi \in \mathcal{C}_c^\infty(S_r(F))$ there is a smooth function of compact support ω_ϕ such that

$$I(w_M \mathbf{t}, \phi) = \omega_\phi(\mathbf{t}) + \sum_{\alpha\beta=\mathbf{t}} L_{w_M}^{w_{\text{GL}_r}}(\alpha) I(w_{\text{GL}_r} \beta, \phi).$$

The sum is over all pairs in $\{(\alpha, \beta) \in (T_{w_M}^{w_{\text{GL}_r}}, T_{\text{GL}_r}(F)) \mid \alpha\beta = \mathbf{t}\}$. The function $L_{w_M}^{w_{\text{GL}_r}}$ is the **germ** (for the side I) along the subset $T_{w_M}^{w_{\text{GL}_r}}$. It is not unique.

Let $\mathbf{t} = \text{diag}(a, \dots, a, a^{1-r} \det(w_M w_{\text{GL}_r}))$. Since ω_ϕ is a smooth function of compact support, we can choose $|a|$ small enough such that $\omega_\phi(\mathbf{t}) = 0$. We consider the pair $(\alpha, \beta) \in (T_{w_M}^{w_{\text{GL}_r}}, T_{\text{GL}_r}(F))$ such that $\alpha\beta = \mathbf{t}$. Since $\det(\alpha) = \det(w_M w_{\text{GL}_r}) = \det(\mathbf{t})$ (by definition), we have $\det(\beta) = 1$. Moreover, $\beta = \text{diag}(z, z, \dots, z)$ with $z^r = 1$.

We denote by $[x]$ the integral part of a real number x . Let $K_m := \text{Id}_r + \varpi^m \mathfrak{gl}_r(\mathcal{O})$ be the principal congruence subgroup of $\text{GL}_r(F)$. We let ϕ be a characteristic function of $w_{\text{GL}_r} K_m \cap S_r(F)$ and the scalar

$$c_1(r) := \text{vol}(\varpi^m \mathcal{O})^{-\lceil \frac{r^2}{4} \rceil}.$$

We have then the following lemma

Lemma 3.1. *Let $\beta = \text{diag}(z, z, \dots, z)$ with $z^r = 1$ and ϕ as above. For m large enough, we have then*

$$I(w_{\text{GL}_r} \beta, \phi) = \begin{cases} 1, & \text{if } z = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Firstly, we calculate the integral $I(w_{\text{GL}_r}, \phi)$. This integral can be written as follows

$$I(w_{\text{GL}_r}, \phi) = \int \phi(x) \Psi \left(\frac{\sum_{i+j=r+2} x_{i,j}}{2} \right) \otimes dx_{i,j}$$

where x is a symmetric matrix such that

$$x_{i,j} = \begin{cases} 0, & \text{if } i + j < r + 1, \\ 1, & \text{if } i + j = r + 1. \end{cases}$$

The variables are the entries $x_{i,j} \in F$ with $i + j \geq r + 2$, $i < j$, the entries $x_{i,i} \in F$ with $2i \geq r + 2$.

The number of entries $x_{i,j}$ with $i + j \geq r + 2$ is $\frac{r(r-1)}{2}$. The number of entries $x_{i,i}$ with $2i \geq r + 2$ is $r - \left\lfloor \frac{r+3}{2} \right\rfloor + 1 = \left\lfloor \frac{r}{2} \right\rfloor$. So the number of variables of above integral is $\frac{r(r-1) - \left\lfloor \frac{r}{2} \right\rfloor}{2} + \left\lfloor \frac{r}{2} \right\rfloor = \left\lfloor \frac{r^2}{4} \right\rfloor$.

Now we take ϕ be a product of the characteristic function of $w_{\text{GL}_r} K_m$ and the scalar $c_1(r)$, this integral is equal to

$$c_1(r) \int \Psi \left(\frac{\sum_{i+j=r+2} x_{i,j}}{2} \right) \otimes dx_{i,j}$$

integrated over the domain:

$$\begin{aligned} x_{i,j} &\equiv 1 \pmod{\varpi^m \mathcal{O}} \text{ for } i + j = r + 2, \\ x_{i,j} &\equiv 0 \pmod{\varpi^m \mathcal{O}} \text{ for } i + j > r + 2. \end{aligned}$$

Moreover, since Ψ is of order 0, we have $\Psi \left(\frac{\sum x_{i,j}}{2} \right) = 1$. It implies that

$$\int \Psi \left(\frac{\sum_{i+j=r+2} x_{i,j}}{2} \right) \otimes dx_{i,j} = \text{vol}(\varpi^m \mathcal{O})^{\left\lfloor \frac{r^2}{4} \right\rfloor} = c_1(r)^{-1}.$$

In consequence, the first assertion is proved.

Choosing m large enough such that $z \notin 1 + \varpi^m \mathcal{O}$ for all z which satisfy $z^r = 1$ and $z \neq 1$. We have then ${}^t n w_{\text{GL}_r} n \notin w_{\text{GL}_r} K_m, \forall n \in N_r(F)$. It follows the second assertion. \square

As a consequence, we obtain that

$$L_{w_M}^{w_{\text{GL}_r}}(\alpha) = I(w_M \alpha, \phi),$$

where

$$\alpha = \text{diag}(a, \dots, a, a^{1-r} \det(w_M w_{\text{GL}_r})),$$

and $|a|$ is small enough.

We can write the orbital integral $I(w_M\alpha, \phi)$ as follows

$$I(w_M\alpha, \phi) = \int \phi(g') \Psi \left(\frac{\sum_{i+j=r+1} x_{i,j}}{2} \right) \otimes dx_{i,j}, \quad (3.1)$$

where

$$g'_{i,j} = \begin{cases} 0, & \text{if } i + j < r, \\ a, & \text{if } i + j = r, \\ ax_{i,j}, & \text{if } i + j \geq r + 1, \end{cases}$$

and $x_{i,j} = x_{j,i}$, if $i + j \geq r + 1$. The variables are the entries $x_{i,j} \in F$ with $i + j \geq r + 1$, $i < j$, the entries $x_{i,i} \in F$ with $2i \geq r + 1$, except the entry $Z = x_{r,r}$ which is a dependent variable. The entry Z can be computed from the condition that $\det(g') = \det(w_{\text{GL}_r})$.

Denote by $c_2(r)$ the number of variables of above integral. We have then

$$c_2(r) = \left\lceil \frac{r^2 + 2r - 3}{4} \right\rceil.$$

After a change of variables, (3.1) can be written as

$$I(w_M\alpha, \phi) = |a|^{-c_2(r)} \int \phi(g) \Psi \left(\frac{\sum_{i=1}^r x_{i,r+1-i}}{2a} \right) \otimes dx_{i,j} \quad (3.2)$$

where

$$g_{i,j} = \begin{cases} 0, & \text{if } i + j < r, \\ a, & \text{if } i + j = r, \\ x_{i,j}, & \text{if } i + j \geq r + 1, \end{cases}$$

and $x_{i,j} = x_{j,i}$, if $i + j \geq r + 1$. The entry $x_{r,r}$ is defined by

$$a^{r-1} \det(w_M)x_{r,r} + \det \begin{pmatrix} 0 & \cdots & 0 & a & x_{1,r} \\ \vdots & \ddots & \cdot & \cdot & \vdots \\ 0 & \cdot & \cdot & x_{r-2,r-1} & x_{r-2,r} \\ a & \cdot & x_{r-1,r-2} & x_{r-1,r-1} & x_{r-1,r} \\ x_{r,1} & \cdots & x_{r,r-2} & x_{r,r-1} & 0 \end{pmatrix} = \det(w_{\text{GL}_r}).$$

Let \mathcal{I} be an subset of $\{(i,j) | 1 \leq i \leq j \leq r\}$. We denote by $g^{\mathcal{I}}$ the matrix obtained from g by replacing the entries $g_{i,j}$ and $g_{j,i}$ by 0 with $(i,j) \in \mathcal{I}$. For the instance, the above condition of $x_{r,r}$ can be written as follows

$$a^{r-1} \det(w_M)x_{r,r} + \det(g^{(r,r)}) = \det(w_{\text{GL}_r}).$$

Since ϕ is a product of the characteristic function of $w_{\text{GL}_r} K_m$ and the scalar $c_1(r)$, the above integral is equal to

$$|a|^{-c_2(r)} c_1(r) \int \Psi \left(\frac{\sum_{i=1}^r x_{i,r+1-i}}{2a} \right) \otimes dx_{i,j}$$

integrated over the set:

$$\begin{aligned} x_{i,j} &\equiv 1 \pmod{\varpi^m \mathcal{O}} \text{ for } i+j = r+1, \\ x_{i,j} &\equiv 0 \pmod{\varpi^m \mathcal{O}} \text{ for } i+j > r+1, j > i, (i,j) \neq (r,r), \\ x_{r,r} &\equiv 0 \pmod{\varpi^m \mathcal{O}}. \end{aligned}$$

The last condition can be written as follows

$$\det(g^{(r,r)}) \equiv \det(w_{\text{GL}_r}) \pmod{a^{r-1} \varpi^m \mathcal{O}}.$$

By single out the entries $x_{2,r} = x_{r,2}$ from the left hand side, we have:

$$2x_{2,r} a u + x_{2,r}^2 a^2 v + \det(g^{(r,r),(2,r)}) \equiv \det(w_{\text{GL}_r}) \pmod{a^{r-1} \varpi^m \mathcal{O}},$$

where $u, v \in 1 + \varpi^m \mathcal{O}$. Both u and v depend only on the variables $x_{i,j}$ with $(i,j) \neq (2,r)$.

We denote $T := \det(g^{(r,r),(2,r)}) - \det(w_{\text{GL}_r})$. Since $x_{2,r} \equiv 0 \pmod{\varpi^m \mathcal{O}}$, we have $T \equiv 0 \pmod{a \varpi^m \mathcal{O}}$. We have then $u - vT \in 1 + \varpi^m \mathcal{O}$. It implies that $u - vT$ has a square root in F for m large enough. We shall denote its square root by $\sqrt{u - vT}$. Moreover $\sqrt{u - vT} \in 1 + \varpi^m \mathcal{O}$. The last condition can be read

$$a^2 v \left(x_{2,r} - \frac{-u - \sqrt{u - vT}}{av} \right) \left(x_{2,r} - \frac{-u + \sqrt{u - vT}}{av} \right) \equiv 0 \pmod{a^{r-1} \varpi^m \mathcal{O}}.$$

For $|a|$ small enough (i.e the valuation of a large enough), from the definition of u, v and T , we have $\frac{-u - \sqrt{u - vT}}{av} \in a^{-1} \varpi^m \mathcal{O}$ and $\frac{-u + \sqrt{u - vT}}{av} \in -2a^{-1} + a^{-1} \varpi^m \mathcal{O}$. Thus above condition is equivalent to

$$x_{2,r} \equiv \frac{-u - \sqrt{u - vT}}{av} \pmod{a^{r-2} \varpi^m \mathcal{O}}.$$

Using this condition, we can integrate the variable $x_{2,r}$ away from the orbital integral I to obtain the scalar factor $|a|^{r-2} \text{vol}(\varpi^m \mathcal{O})$ multiple with a new integral. The new integral has the same form of the old one but the domain of integration is defined by:

$$\begin{aligned} x_{i,j} &\equiv 1 \pmod{\varpi^m \mathcal{O}} \text{ for } i+j = r+1, \\ x_{i,j} &\equiv 0 \pmod{\varpi^m \mathcal{O}} \text{ for } i+j > r+1, (i,j) \neq (r,r), (2,r) \\ T &\equiv 0 \pmod{a \varpi^m \mathcal{O}}. \end{aligned}$$

The determinant of $g^{(r,r),(2,r)}$ has the form

$$\prod_{i+j=r+1} x_{i,j} \det(w_G) + az$$

with $z \in \varpi^m \mathcal{O}$. Thus the condition on T can read

$$\prod_{i+j=r+1} x_{i,j} \equiv 1 \pmod{a\varpi^m \mathcal{O}}.$$

Now, we integrate over the variables $x_{i,j}$ with $i + j > r + 1$, $j > i$, $(i, j) \neq (r, r), (2, r)$ and we get

$$I(w_M \alpha, \phi) = |a|^{r-2-c_2(r)} c_1(r) \text{vol}(\varpi^m \mathcal{O})^{c_2(r) - [\frac{r+1}{2}] + 1} I(a, r),$$

where the function $I(a, r)$ is defined as follows.

If $r = 2\ell$, then

$$I(a, r) := \text{vol}(\varpi^m \mathcal{O})^{-1} \int \Psi \left(\frac{x_1 + x_2 + \cdots + x_\ell}{a} \right) \otimes dx_i. \quad (3.3)$$

The domain of integration is defined by

$$\begin{aligned} x_i &\equiv 1 \pmod{\varpi^m \mathcal{O}} \\ x_1^2 x_2^2 \cdots x_\ell^2 &\equiv 1 \pmod{a\varpi^m \mathcal{O}}. \end{aligned}$$

If $r = 2\ell + 1$, then

$$I(a, r) := \text{vol}(\varpi^m \mathcal{O})^{-1} \int \Psi \left(\frac{2x_1 + 2x_2 + \cdots + 2x_\ell + x_{\ell+1}}{2a} \right) \otimes dx_i. \quad (3.4)$$

The domain of integration is defined by

$$\begin{aligned} x_i &\equiv 1 \pmod{\varpi^m \mathcal{O}} \\ x_1^2 x_2^2 \cdots x_\ell^2 x_{\ell+1} &\equiv 1 \pmod{a\varpi^m \mathcal{O}}. \end{aligned}$$

We denote by $\gamma(a, \Psi)$ the Weil constant, which is defined by the formula

$$\int \Phi^\vee(x) \Psi \left(\frac{1}{2} ax^2 \right) dx = |a|^{-1/2} \gamma(a, \Psi) \int \Phi(x) \Psi \left(-\frac{1}{2} a^{-1} x^2 \right) dx,$$

where $\Psi : F \rightarrow \mathbb{C}^*$ is an additive character, Φ is a Schwartz function over F and Φ^\vee is its Fourier transform ($\Phi^\vee(x) = \int \Phi(y) \Psi(xy) dy$). We have:

Proposition 3.2. *Suppose that the residual characteristic of F is larger than $2r+1$. Then, if m is large enough:*

$$I(a, r) = |a|^{\lfloor \frac{r}{2} \rfloor} \gamma(a^{-1}, \Psi)^{-\lfloor \frac{r}{2} \rfloor} J(a, r),$$

for $|a|$ sufficiently small.

Proof. Firstly, we consider the case $r = 2\ell + 1$. We change variables

$$x_{\ell+1} = \left(\prod_{i=1}^{\ell} x_i^{-2} \right) t$$

with $t \in 1 + a\varpi^m\mathcal{O}$ and integrate over t . We obtain

$$I(a, 2\ell + 1) = |a| \int \Psi \left(\frac{\delta}{2a} \right) \otimes dx_i,$$

where the function

$$\delta = \sum_{i=1}^{\ell} 2x_i + \frac{1}{\prod_{i=1}^{\ell} x_i^2}.$$

We set $x_i = 1 + u_i$ with $u_i \in \varpi^m\mathcal{O}$. This function can be written as

$$\delta = 2\ell + \sum_{i=1}^{\ell} 2u_i + \prod_{i=1}^{\ell} \frac{1}{1 + 2u_i + u_i^2}.$$

Now we consider the Taylor expansion of this function at the origin. It has the form:

$$r + 3 \sum_{i=1}^{\ell} u_i^2 + 4 \sum_{1 \leq i < j \leq \ell} u_i u_j + \text{higher degree terms.}$$

Since the quadratic form

$$3 \sum_{i=1}^{\ell} X_i^2 + 4 \sum_{1 \leq i < j \leq \ell} X_i X_j$$

is equivalent to the quadratic form

$$\sum_{i=1}^{\ell} \frac{2i+1}{2i-1} Y_i^2$$

by a unipotent transformation (see. [7, Lemma 5.1]), after a unimodular change variables, this Taylor expansion may be written in the form:

$$\delta = r + \sum_{i=1}^{\ell} \frac{2i+1}{2i-1} x_i^2 + \text{higher degree terms.}$$

We can choose m large enough such that the origin is the only critical point in the domain of integration. Using the principle of stationary phase, there exists a neighbourhood $0 \in \Omega$ in F such that for $|a|$ small enough $I(a, r)$ is the product of the factors:

$$|a| \Psi \left(\frac{r}{2a} \right),$$

$$\int_{\Omega} \Psi \left(\frac{2i+1}{2(2i-1)a} x_i^2 \right) dx_i = \left| \frac{2i+1}{(2i-1)a} \right|^{-1/2} \gamma \left(\frac{2i+1}{(2i-1)a}, \Psi \right).$$

Using the property of Weil constant, we have:

$$\gamma \left(\frac{2i+1}{(2i-1)a}, \Psi \right) = \gamma \left(\frac{2i+1}{2i-1}, \Psi \right) \gamma(a^{-1}, \Psi) \left(\frac{2i+1}{2i-1}, a^{-1} \right).$$

Since the residual characteristic of F larger than $2r+1$, we have $\frac{2i+1}{2i-1} \in \mathcal{O}^*$. So above equation simplifies to

$$\gamma \left(\frac{2i+1}{(2i-1)a}, \Psi \right) = \gamma(a^{-1}, \Psi) \left(\frac{2i+1}{2i-1}, a^{-1} \right).$$

In consequence, we have:

$$I(a, 2\ell+1) = |a|^{\frac{\ell}{2}+1} \Psi \left(\frac{2\ell+1}{2a} \right) (2\ell+1, a^{-1}) \gamma(a^{-1}, \Psi)^{\ell}.$$

For the case $r = 2\ell$. We set

$$x_{\ell} = t \left(\prod_{i=1}^{\ell-1} x_i \right)^{-1}.$$

Since $\prod_{i=1}^{\ell} x_i^2 \equiv 1 \pmod{a\varpi^m\mathcal{O}}$, we have $t^2 \equiv 1 \pmod{a\varpi^m\mathcal{O}}$. For m large enough and $|a|$ small enough, this condition is equivalent to $t \equiv \pm 1 \pmod{a\varpi^m\mathcal{O}}$. Moreover $t = \prod_{i=1}^{\ell} x_i \equiv 1 \pmod{\varpi^m\mathcal{O}}$, so $t \equiv 1 \pmod{\varpi^m\mathcal{O}}$. We now integrate over t to get

$$I(a, 2\ell) = |a| \int \Psi \left(\frac{\delta}{a} \right) \otimes dx_i,$$

where the function

$$\delta = \sum_{i=1}^{\ell-1} x_i + \frac{1}{\prod_{i=1}^{\ell-1} x_i}.$$

Using the proof of [7, Proposition 2.1], we have:

$$I(a, 2\ell) = |a|^{\frac{\ell+1}{2}} \Psi \left(\frac{\ell}{a} \right) (\ell, a^{-1}) \gamma(a^{-1}, \Psi)^{\ell-1}.$$

In summary, we get

$$I(a, r) = |a|^{\left[\frac{r-1}{2}\right]+1} \Psi \left(\frac{r}{2a} \right) (r, a^{-1}) (1/2, a^{-1})^{r-1} \gamma(a^{-1}, \Psi)^{\left[\frac{r-1}{2}\right]}. \quad (3.5)$$

In the other hand, reusing the proof of [7, Proposition 2.1], we have:

$$J(a, r) = |a|^{\frac{r+1}{2}} \Psi \left(\frac{r}{2a} \right) \left(\frac{r}{2r-1}, a^{-1} \right) \gamma(a^{-1}, \Psi)^{r-1}. \quad (3.6)$$

Comparing (3.5) and (3.6) we obtain the proposition. \square

Resuming the argument above, we obtain the following proposition:

Proposition 3.3. *For*

$$\alpha = \text{diag}(a, a, \dots, a^{1-r} \det(w_M) \det(w_{\text{GL}_r}))$$

and $|a|$ is small enough,

$$L_{w_M}^{w_{\text{GL}_r}}(\alpha) = |a|^{r-2 - \left[\frac{r^2+2r-3}{4}\right] + \frac{1}{2}\left[\frac{r}{2}\right]} \gamma(a^{-1}, \Psi)^{-\left[\frac{r}{2}\right]} J(a, r).$$

In particular,

$$L_{w_M}^{w_{\text{GL}_r}}(\alpha) = |a|^{\left[\frac{r^2}{4}\right] + \frac{1}{2}\left[\frac{r}{2}\right]} \gamma(a^{-1}, \Psi)^{-\left[\frac{r}{2}\right]} K_{w_M}^{w_{\text{GL}_r}}(\alpha).$$

Note that to simplify the formula, we used the following identities:

$$\left[\frac{r^2 + 2r - 3}{4} \right] - \left[\frac{r+1}{2} \right] - \left[\frac{r^2}{4} \right] = -1$$

and

$$\frac{r(r-1)}{2} + 1 + (r-2) - \left[\frac{r^2 + 2r - 3}{4} \right] = \left[\frac{r^2}{4} \right].$$

The second relation follows from the first and the Proposition 2.1.

4 Proof of the main theorem

We shall prove the Theorem 1.1 by induction on r . It is trivial when $r = 1$. We suppose that it is true for $1 \leq r' < r$.

Firstly we consider $W_r^R \ni w \neq w_{\text{GL}_r}$. We have then that the relevant element $w\mathbf{t}$ has the following form

$$w\mathbf{t} = \begin{pmatrix} w_1\mathbf{t}_1 & 0 \\ 0 & w_2\mathbf{t}_2 \end{pmatrix}$$

with $w_i\mathbf{t}_i$ is the relevant element of GL_{r_i} . For a convenience, we shall introduce some intermediate orbital integrals.

On the side J , for a function $f \in \mathcal{C}_c^\infty(\text{GL}_{r_1+r_2}(F))$ we define the intermediate integral

$$\begin{aligned} & J_{r_2}^{r_1} \left[\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}, f \right] \\ & := \int f \left[w_{\text{GL}_{r_1+r_2}} \begin{pmatrix} \text{Id}_{r_1} & \\ & \text{Id}_{r_2} \end{pmatrix} \begin{pmatrix} w_{\text{GL}_{r_1}}g_1 & \\ & w_{\text{GL}_{r_2}}g_2 \end{pmatrix} \begin{pmatrix} \text{Id}_{r_1} & Y \\ & \text{Id}_{r_2} \end{pmatrix} \right] \times \\ & \quad \times \theta \left[\begin{pmatrix} \text{Id}_{r_1} & X+Y \\ & \text{Id}_{r_2} \end{pmatrix} \right] dXdY \\ & = \int f \left[w_{\text{GL}_{r_1+r_2}} \begin{pmatrix} A_{r_1} & A_{r_1}Y \\ & tXA_{r_1}Y + B_{r_2} \end{pmatrix} \right] \theta \left[\begin{pmatrix} \text{Id}_{r_1} & X+Y \\ & \text{Id}_{r_2} \end{pmatrix} \right] dXdY \end{aligned}$$

where $A_{r_1} := w_{\text{GL}_{r_1}}g_1 \in \text{GL}_{r_1}(F)$, $B_{r_2} := w_{\text{GL}_{r_2}}g_2 \in \text{GL}_{r_2}(F)$ and the domain of integration is $M_{r_1 \times r_2}(F)$ - the set of matrices of size $r_1 \times r_2$.

Fixing $f \in \mathcal{C}_c^\infty(\text{GL}_{r_1+r_2}(F))$, the function $\mathfrak{J}_{r_2}^{r_1}(g_1, g_2) := J_{r_2}^{r_1} \left[\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}, f \right]$ is a smooth function of support compact on $\text{GL}_{r_1}(F) \times \text{GL}_{r_2}(F)$.

Associated with the action of $(N_{r_1} \times N_{r_1}) \times (N_{r_2} \times N_{r_2})$ on $\text{GL}_{r_1} \times \text{GL}_{r_2}$ we consider the double orbital integral

$$\begin{aligned} & J(w_1\mathbf{t}_1, w_2\mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1}) \\ & := \int \mathfrak{J}_{r_2}^{r_1}(n_1^{-1}w_{\text{GL}_{r_1}}w_1\mathbf{t}_1n'_1, n_2^{-1}w_{\text{GL}_{r_2}}w_2\mathbf{t}_2n'_2)\theta(n_1n'_1)dn_1dn'_1\theta(n_2n'_2)dn_2dn'_2, \end{aligned}$$

where the (n_i, n'_i) is integrated over $N_{r_i} \times N_{r_i}$ divided by the stabilizer of $w_{\text{GL}_{r_i}}w_i\mathbf{t}_i$. We have then

$$J \left(\begin{pmatrix} w_1\mathbf{t}_1 & \\ & w_2\mathbf{t}_2 \end{pmatrix}, f \right) = J(w_1\mathbf{t}_1, w_2\mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1}) \quad (4.1)$$

We can also define the partial orbital integrals $J_1(w_1\mathbf{t}_1, g_2; \mathfrak{J}_{r_2}^{r_1})$ and $J_2(g_1, w_2\mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1})$. For example

$$J_1(w_1\mathbf{t}_1, g_2; \mathfrak{J}_{r_2}^{r_1}) := \int \mathfrak{J}_{r_2}^{r_1}(n_1^{-1}w_{\text{GL}_{r_1}}w_1\mathbf{t}_1n'_1, g_2)\theta(n_1n'_1)dn_1dn'_1,$$

where the (n_1, n'_1) is integrated over $N_{r_1} \times N_{r_1}$ divided by the stabilizer of $w_{\text{GL}_{r_1}} w_1 \mathbf{t}_1$. If we fix $w_1 \mathbf{t}_1$, this integral defines a smooth function of compact support on $\text{GL}_{r_2}(F)$. Moreover, we have:

$$J(w_1 \mathbf{t}_1, w_2 \mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1}) = J(w_2 \mathbf{t}_2, J_1(w_1 \mathbf{t}_1, \bullet; \mathfrak{J}_{r_2}^{r_1}))$$

and

$$J(w_1 \mathbf{t}_1, w_2 \mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1}) = J(w_1 \mathbf{t}_1, J_2(\bullet, w_2 \mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1})).$$

On the side I , for a function $\phi \in \mathcal{C}_c^\infty(\text{GL}_{r_1+r_2}(F))$ we define the intermediate integral

$$\begin{aligned} I_{r_2}^{r_1} \left[\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}, f \right] \\ &:= \int \phi \left[\begin{pmatrix} \text{Id}_{r_1} & & \\ & \text{Id}_{r_2} & \\ & & X \end{pmatrix} \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} \begin{pmatrix} \text{Id}_{r_1} & X \\ & \text{Id}_{r_2} \end{pmatrix} \right] \theta \left[\begin{pmatrix} \text{Id}_{r_1} & 2X \\ & \text{Id}_{r_2} \end{pmatrix} \right] dX dY \\ &= \int f \left[\begin{pmatrix} g_1 & & \\ & g_1 X & \\ X g_1 & & X g_1 X + g_2 \end{pmatrix} \right] \theta \left[\begin{pmatrix} \text{Id}_{r_1} & 2X \\ & \text{Id}_{r_2} \end{pmatrix} \right] dX dY \end{aligned}$$

where $g_i \in \text{GL}_{r_i}(F)$ and the domain of integration is $M_{r_1 \times r_2}(F)$ - the set of matrices of size $r_1 \times r_2$.

Fixing $f \in \mathcal{C}_c^\infty(\text{GL}_{r_1+r_2}(F))$, the function $\mathfrak{J}_{r_2}^{r_1}(g_1, g_2) := I_{r_2}^{r_1} \left[\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}, f \right]$ is a smooth function of support compact on $\text{GL}_{r_1}(F) \times \text{GL}_{r_2}(F)$.

Associated with the action of $N_{r_1} \times N_{r_2}$ on $\text{GL}_{r_1} \times \text{GL}_{r_2}$ we consider the double orbital integral

$$I(w_1 \mathbf{t}_1, w_2 \mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1}) := \int \mathfrak{J}_{r_2}^{r_1}({}^t n_1 w_1 \mathbf{t}_1 n_1, {}^t n_2 w_2 \mathbf{t}_2 n_2) \theta_{r_1}(n_1) dn_1 \theta_{r_2}(n_2) dn_2,$$

where the n_i is integrated over N_{r_i} divided by the stabilizer of $w_i \mathbf{t}_i$. We have then

$$I \left(\begin{pmatrix} w_1 \mathbf{t}_1 & \\ & w_2 \mathbf{t}_2 \end{pmatrix}, \phi \right) = I(w_1 \mathbf{t}_1, w_2 \mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1}) \quad (4.2)$$

We can also define the partial orbital integrals $I_1(w_1 \mathbf{t}_1, g_2; \mathfrak{J}_{r_2}^{r_1})$ and $I_2(g_1, w_2 \mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1})$. For example

$$I_1(w_1 \mathbf{t}_1, g_2; \mathfrak{J}_{r_2}^{r_1}) := \int \mathfrak{J}_{r_2}^{r_1}({}^t n_1 w_1 \mathbf{t}_1 n_1, g_2) \theta_{r_1}(n_1) dn_1,$$

where the n_1 is integrated over N_{r_1} divided by the stabilizer of $w_1 \mathbf{t}_1$. If we fix $w_1 \mathbf{t}_1$, this integral defines a smooth function of compact support on $\text{GL}_{r_2}(F)$. Moreover, we have:

$$I(w_1 \mathbf{t}_1, w_2 \mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1}) = I(w_2 \mathbf{t}_2, I_1(w_1 \mathbf{t}_1, \bullet; \mathfrak{J}_{r_2}^{r_1}))$$

and

$$I(w_1 \mathbf{t}_1, w_2 \mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1}) = I(w_1 \mathbf{t}_1, I_2(\bullet, w_2 \mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1})).$$

Now we can continue with the induction argument. If $w_i = \text{Id}_{r_i}$ then $w = \text{Id}_r$. From the hypothesis of the theorem and the identities 4.1 and 4.2 we have:

$$J(\mathbf{t}_1, \mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1}) = \Delta_r(\text{diag}(\mathbf{t}_1, \mathbf{t}_2))I(\mathbf{t}_1, \mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1}).$$

Using the relation between the double integrals and the partial integral, it implies the following

$$J(\mathbf{t}_1, J_2(\bullet, \mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1})) = \Delta_r(\text{diag}(\mathbf{t}_1, \mathbf{t}_2))I(\mathbf{t}_1, I_2(\bullet, \mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1})). \quad (4.3)$$

Since the identity (4.3) is true for all $\mathbf{t}_i \in T_{r_i}$, so when we fix \mathbf{t}_2 we obtain a matching relation over GL_{r_1} . Using the inductive hypothesis, there exists Δ_{w_1} such that

$$J(w_1 \mathbf{t}_1, J_2(\bullet, \mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1})) = \Delta_{w_1}(\text{diag}(\mathbf{t}_1, \mathbf{t}_2))I(w_1 \mathbf{t}_1, I_2(\bullet, \mathbf{t}_2; \mathfrak{J}_{r_2}^{r_1})).$$

Reusing the relation between the double integrals and the partial integrals, we obtain the matching relation over GL_{r_2}

$$J(\mathbf{t}_2, J_1(w_1 \mathbf{t}_1, \bullet; \mathfrak{J}_{r_2}^{r_1})) = \Delta_{w_1}(\text{diag}(\mathbf{t}_1, \mathbf{t}_2))I(\mathbf{t}_2, I_1(w_1 \mathbf{t}_1, \bullet; \mathfrak{J}_{r_2}^{r_1})).$$

We use the inductive hypothesis on r_2 , to get $\Delta_{w_1, w_2}(\text{diag}(\mathbf{t}_1, \mathbf{t}_2))$ such that

$$J(w_2 \mathbf{t}_2, J_1(w_1 \mathbf{t}_1, \bullet; \mathfrak{J}_{r_2}^{r_1})) = \Delta_{w_1, w_2}(\text{diag}(\mathbf{t}_1, \mathbf{t}_2))I(w_2 \mathbf{t}_2, I_1(w_1 \mathbf{t}_1, \bullet; \mathfrak{J}_{r_2}^{r_1})).$$

So we obtain the transfer factor $\Delta_w(t)$ for the relevant element $w\mathbf{t}$ with w of the form $\text{diag}(w_1, w_2)$. In particular, we have Δ_{w_M} (recall that M is the standard parabolic subgroup of type $(r-1, 1)$ of GL_r).

The only transfer factor which can not be obtained from above processing is $\Delta_{w_{\text{GL}_r}}$. To find it, we shall use the germ relation which we mentioned in the section 2 and 3.

Recall that we have:

$$J(w_M \mathbf{t}, f) = \omega_f(\mathbf{t}) + \sum_{\alpha\beta=\mathbf{t}} K_{w_M}^{w_{\text{GL}_r}}(\alpha)J(w_{\text{GL}_r} \beta, f)$$

and

$$I(w_M \mathbf{t}, \phi) = \omega_\phi(\mathbf{t}) + \sum_{\alpha\beta=\mathbf{t}} L_{w_M}^{w_{\text{GL}_r}}(\alpha)I(w_{\text{GL}_r} \beta, \phi),$$

where w_f, w_ϕ are smooth function of compact support, $\mathbf{t} \in T_{w_M}$ and the sums are over all pairs in $\mathfrak{S} := \{(\alpha, \beta) \in (T_{w_M}^{w_{\text{GL}_r}}, T_{w_{\text{GL}_r}}) | \alpha\beta = \mathbf{t}\}$. Given $(\alpha, \beta) \in (T_{w_M}^{w_{\text{GL}_r}}, T_{w_{\text{GL}_r}})$ then all the pair $(\alpha', \beta') \in (T_{w_M}^{w_{\text{GL}_r}}, T_{w_{\text{GL}_r}})$ satisfied $\alpha'\beta' = \alpha\beta$ has a form $(z\alpha, z^{-1}\beta)$ with z is a r -th root of unity.

Given $\beta \in T_{w_{\text{GL}_r}}$, we choose $\alpha = \text{diag}(a, \dots, a, a^{1-r} \det(w_M w_{\text{GL}_r}))$ with $|a|$ so small that $\omega_f(\alpha\beta) = \omega_\phi(\alpha\beta) = 0$. We get then:

$$J(w_M \alpha \beta, f) = \sum_{z|z^r=1} K_{w_M}^{w_{\text{GL}_r}}(z\alpha) J(w_{\text{GL}_r} z^{-1} \beta, f)$$

and

$$I(w_M \alpha \beta, \phi) = \sum_{z|z^r=1} L_{w_M}^{w_{\text{GL}_r}}(z\alpha) I(w_{\text{GL}_r} z^{-1} \beta, \phi).$$

Moreover, with $|a|$ small enough, using the proposition 3.3, we have:

$$K_{w_M}^{w_{\text{GL}_r}}(z\alpha) = c.L_{w_M}^{w_{\text{GL}_r}}(z\alpha)$$

with $c(a, z) := |az|^{-\left[\frac{r^2}{4}\right] - \frac{1}{2}\left[\frac{r}{2}\right]} \gamma((az)^{-1}, \Psi)^{\left[\frac{r}{2}\right]}$. Combining them with the matching relation of ϕ and f on the orbit of $w_M \alpha \beta$ (which is had been proved), we obtain

$$\sum_{z|z^r=1} K_{w_M}^{w_{\text{GL}_r}}(z\alpha) (J(w_{\text{GL}_r} z^{-1} \beta, f) - c.\Delta_{w_M}(\alpha\beta) I(w_{\text{GL}_r} z^{-1} \beta, \phi)) = 0.$$

As follows from the proof of the [7, Lemma 4.1], this condition implies that

$$J(w_{\text{GL}_r} z^{-1} \beta, f) - c(a, z).\Delta_{w_M}(\alpha\beta) I(w_{\text{GL}_r} z^{-1} \beta, \phi) = 0$$

for all z . In particular, we have:

$$J(w_{\text{GL}_r} \beta, f) = c(a, 1).\Delta_{w_M}(\alpha\beta) I(w_{\text{GL}_r} \beta, \phi).$$

Since the above equations don't depend on the choice of a , we can denote $c(a, 1).\Delta_{w_M}(\alpha\beta)$ by $\Delta_{w_{\text{GL}_r}}(\beta)$. This is a transfer factor which we want to find for the matching relation on the orbit of $w_{\text{GL}_r} \beta$.

References

- [1] V. C. Do, *Le lemme fondamental métalectique de Jacquet et Mao en caractéristique positive*, C. R. Acad. Sci. Paris 349 (2011), 1077-1081.
- [2] V. C. Do, *Le lemme fondamental métalectique de Jacquet et Mao en égales caractéristiques*, Bull. Soc. math. France, 143 (1), 2015, 125-196.
- [3] V. C. Do, *Transfer to characteristic zero of Jacquet-Mao's metaplectic fundamental lemma*, preprint.
- [4] H. Jacquet, *On the non vanishing of some L-functions*, Proc. Indian Acad. Sci. (Math. Set.) 97 (1987), 117-155.

- [5] H. Jacquet, *Représentations distinguées pour le groupe orthogonal*, C. R. Acad. Sci. Paris Sér. I Math. No. 13 312 (1991), 957-961.
- [6] H. Jacquet, Y. Ye, *Distinguished representations and quadratic base change for $GL(3)$* , Proceedings of the American mathematical society, Vol. 348, No. 3 (1996), 913-939.
- [7] H. Jacquet *A theorem of density for Kloosterman integrals. Mikio Sato: a great Japanese mathematician of the twentieth century* Asian J. Math. 2 (1998), no. 4, 759-778.
- [8] D. Kazhdan, S. Patterson, *Metaplectic forms*, Inst. Hautes Études Sci. Publ. Math. No. 59 (1984), 35-142.
- [9] Z. Mao, *A fundamental lemma for the metaplectic correspondence*, J. Reine Angew. Math. 496 (1998), 107-129.