An infinite commutator-product is not automatically trivial in the homology

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Abstract

We investigate the fundamental group of Griffiths' space, and the first singular homology group of this space and of the Hawaiian Earring by using (countable) reduced tame words. We prove that two such words represent the same element in the corresponding group if and only if they can be carried to the same tame word by a finite number of word transformations from a given list. Our technique is based on transformations of associated 2-complexes. As a corollary we prove that the two homology groups contain uncountably many different elements that can be represented by infinite concatenations of countably many commutators of loops. As another application we give a short proof that these homology groups contain the direct sum of $2^{\aleph_0}$ copies of $\mathbb{Q}$.

1 Introduction

The first singular homology group of a topological space X, denoted $H_1(X)$, is known to be the abelianization of the fundamental group $\pi_1(X)$. Therefore if p is a continuous compact path in X of the form "the (infinite) commutator concatenation of closed paths"

$$\gamma_1^{-1}\gamma_2^{-1}\gamma_1\gamma_2\,\gamma_3^{-1}\gamma_4^{-1}\gamma_3\gamma_4\ \ldots, \tag{1.1}$$

one can naturally expect that it is zero-homological. Of course this is true if this concatenation is finite. This still remains true for infinite concatenations in case if X is a semi-locally simply connected space (defined in Section 3), e.g. a manifold or a triangulable space. Indeed, since p is continuous, almost all of the loops γ_i are "small" and hence contractible in such X. This implies that the infinite commutator concatenation is in fact homotopic to a finite one and hence is zero-homological.

However, there are not semi-locally simply connected spaces, and the small loops there can already be noncontractible. The simplest example of that type is the Hawaiian Earring Z, see Figure 2 and Definition 3.1.

In [14], G. Higman introduced the so called unrestricted free product of groups, which was later used by H.B. Griffiths [13] and J.W. Morgan and I. Morrison [18] to characterize $\pi_1(Z)$. Correcting an error in the arguments of Griffiths (see comments in [18, page 563]), Morgan and Morrison proved in [18, Theorem 4.1], that $\pi_1(Z)$ is naturally isomorphic to a certain subgroup of the unrestricted product of countably many infinite cyclic groups.

Taking in mind this isomorphism, one can easily see, that in [14, page 76] Higman proved, in fact, that $H_1(Z)$ contains a nonzero element, which can be represented as an infinite commutator product of type (1.1). His short and nice proof uses a representation of a free group as a group of formal power series in non-commuting variables, that was invented by W. Magnus in [17]. One can easily modify his example (just by changing indices) to construct uncountably many such elements in $H_1(Z)$.

Another proof giving uncountably many such elements can be extracted from the very complicated proof of K. Eda [6, Theorem 4.14]. Both proofs use the fact that some special elements in a free group cannot be expressed as products of a given number of commutators. And because of this specific, they do not work for Griffith's space, which is closely related to the Hawaiian Earring. This remarkable space was constructed by H.B. Griffiths in the fifties (see [11, page 185]) as a one-point union of two cones over Hawaiian Earrings (see Figure 3 and Definition 3.3), for showing that a one-point union of two spaces with trivial fundamental groups can have a non-trivial and even uncountable fundamental group.

In this paper we suggest an alternative proof, which works for Griffith's space Y as well as for the Hawaiian Earring Z (see Corollary 8.7). It is based on a Tietze-like solution of the word problem in each of the groups $\pi_1(Y)$, $H_1(Y)$, $H_1(Z)$ (see Theorems 8.1–8.3), that is important itself. Since all these groups are uncountable and their elements are represented by countable words over countable alphabets, this solution cannot be algorithmic. However, it helps to distinguish the elements given by regular forms. Using this, we also prove Theorem 9.2, which states that $H_1(Z)$ and $H_1(Y)$ contain subgroups isomorphic to the direct sum of $2^{\aleph_0}$ copies of $\mathbb{Q}$. The first part of this theorem was proved earlier by Eda [6, Theorem 4.14].

To be more precise, we explain the sense of the main Theorems 8.1–8.3. Let $P = \{p_1, p_2, \dots\}$ and $Q = \{q_1, q_2, \dots\}$ be two disjoint infinite countable alphabets. In Section 2 we define the group $\mathcal{W}(P)$, consisting of all reduced tame words over the alphabet P. This group is a subgroup of the unrestricted free product of countably many infinite cyclic groups introduced by Higman in [14]. As it was proved by Morgan and Morrison in [18, Theorem 4.1], $\pi_1(Z)$ is canonically isomorphic to $\mathcal{W}(P)$. Thus, for the Hawaiian Earring Z we have the canonical homomorphisms

$$\mathcal{W}(P) \overset{\cong}{\to} \pi_1(Z) \to H_1(Z).$$

For Griffith's space Z we have the canonical homomorphisms

$$\mathcal{W}(P \cup Q) \to \mathcal{W}(P \cup Q)/\langle\langle \mathcal{W}(P), \mathcal{W}(Q) \rangle\rangle \overset{\cong}{\to} \pi_1(Y) \to H_1(Y),$$

where $\langle\langle\rangle\rangle$ means the normal closure, see Section 3 for details. Theorem 8.2, for example, states that two countable words from $\mathcal{W}(P \cup Q)$ represent the same element of $H_1(Y)$ if and only if each of them can be transformed to the same countable word by applying a finite number of transformations of type (i), then of type (ii) and finally of type (iii):

(i) Deletion of a countable subword, containing only p- or q-letters;
(ii) Deletion of two distinct mutually inverse countable subwords: $AXBX^{-1}C \to ABC$;
(iii) Permuting two consecutive countable subwords.

The groups $\pi_1(Z)$, $H_1(Z)$ and $\pi_1(Y)$ has been studied by several authors. They used technical different, but equivalent ways of describing the elements of this group: inverse limits [11, 12, 13, 18, 19], word sequences [21, 22, 23], and special countable words [3, 6, 7, 8]. In this paper we use "(reduced) tame words" as an improvement of the last approach (see Section 2 for exact definitions). Our terminology is motivated by Lemma 2.13, which states that any tame word can be factorized into a finite concatenation of reduced tame words, so that after cancelling inverse factors as in a free group, we get the reduced form of the original tame word. Note, however, that $\pi_1(Z)$ is not a free group in the classical sense (see, for example, [3, Theorem 2.5 (4)] or [19]).

The word transformations (i)-(iii) come from transformations of some associated 2-complex, which we call arch-line-band systems; we define them in Section 4. These systems are closely related, but not coincide with the band systems introduced in the paper of M. Bestvina and M. Feighn [1] for studying the stable actions of groups on real trees; see also [4]. Note that our Theorem 8.3 can be deduced from [4, Theorem 6.1] and conversely.

In the forthcoming paper [2], we investigate Karimov's space K and show, that $H_1(K)$ is uncountable, and each element of $H_1(K)$ can be represented as an infinite commutator product of type (1.1).

2 Countable words

Definition 2.1. Let X be an alphabet, $X^{-1} = \{x^{-1} \,|\, x \in X\}$ and $X^{\pm} = X \cup X^{-1}$. A *countable word* over X is sequence $(x_i)_{i\in I}$, where I is an arbitrary linearly ordered countable set and $x_i \in X^{\pm}$. The order on I will be denoted by $\preccurlyeq$. E.g., the natural numbers $\mathbb{N}$ with the classical order and $\mathbb{N} \times \mathbb{N}$ with lexicographical order are different types of countable linearly ordered sets. There are uncountably many types of countable linear orders, e.g. all ordinal numbers that are smaller than the smallest uncountable ordinal. Two countable words $U = (x_i)_{i\in I}$ and $V = (x_j)_{j\in J}$ are called *equal* if there exists an order preserving bijection $\varphi : I \to J$, such that $x_{\varphi(i)} = x_i$ for all $i \in I$. In this case we write $U \equiv V$.

A subset J of I is called *connected* if for any two elements $j_1, j_2 \in J$ and any element $i \in I$, we have

$$(j_1 \preccurlyeq i \preccurlyeq j_2) \Rightarrow (i \in J).$$

Let $W = (x_i)_{i\in I}$ be a countable word over X. A *subword* of W is a sequence $(x_j)_{j\in J}$, where J is a connected subset of I with the induced order. Two subwords $U = (x_p)_{p\in P}$ and $V = (x_q)_{q\in Q}$ of w are called *consecutive* if $P \cup Q$ is a closed subset of I and $P \cap Q = \emptyset$.

If we have two countable words $W_1 = (x_i)_{i\in I_1}$ and $W_2 = (x_i)_{i\in I_2}$, we can *concatenate* them, i.e. to build the new countable word $W = (x_i)_{i\in I_1\cup I_2}$, where we assume that the elements of I_1 are smaller than the elements of I_2. In this case we write $W = W_1W_2$, not using the dot between W_1 and W_2.

In order to define (ir)reducible countable words we first need:

Definition 2.2. (a visualization of countable words)

To each countable word $W = (x_i)_{i\in I}$ over the alphabet X we associate

(i) a closed segment $[a, b]$ on the real axis of the Euclidean plane together with

(ii) a set of points in the open interval (a, b), each marked by a letter from $X^{\pm}$ so, that moving from a to b along this segment we read the word W.

The segment together with the set of its marked points will be called *W-segment.* For brevity we will call these points *letters.* Note, that the visualization is not unique. The following notion will serve to visualize a cancelation process in a countable word.

Definition 2.3. An *arch-system based on the W-segment* is a subset of the plane, consisting of the W-segment and arches (i.e. half-circles) which are attached to this segment according to the following rules:

(1) all arches lie in the upper half-plane with respect to the real axes;

(2) each arch connects a letter with one of its inverses;

(3) different arches do not intersect;

(4) if two letters are connected by an arch, then each letter between them is connected by an arch with another letter between them.

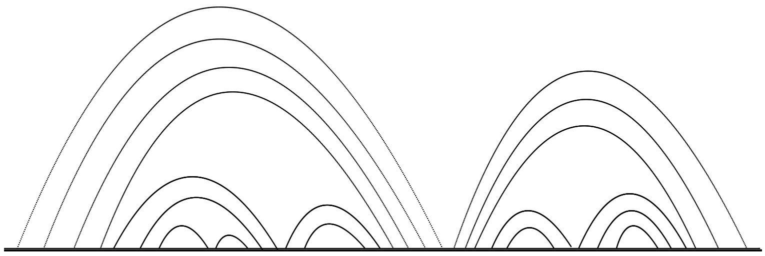

Fig. 1

An arch-system based on the W-segment is called *complete*, if each letter of W belongs to an arch of this system. The endpoints of an arc a are denoted by $\alpha(a)$ and $\omega(a)$; we assume that $\alpha(a) < \omega(a)$ and call them the *initial* and the *terminal* points of a, respectively.

Two arches a, b of a given arch-system are called *parallel* if after possible replacement of a by b and b by a the following holds:

1) $\alpha(a) < \alpha(b) < \omega(b) < \omega(a)$;

2) for any arch c of the arch-system, if $\alpha(a) < \alpha(c) < \alpha(b)$ then $\omega(b) < \omega(c) < \omega(a)$.

Definition 2.4. A countable word W is called *irreducible* (or *reduced*) if there does not exist an arch-system based on the W-segment with a nonempty set of arches. A countable word W is called *reducible* if it is not irreducible.

It is immediate from the definition that subwords of reduced words are reduced.

Let $U = (x_i)_{i \in I}$ be a countable word. A word $V = (x_j)_{j \in J}$ is called *inverse* to U and is denoted by U^{-1}, if there exists an order reversing bijection $\varphi : I \to J$ such that $x_{\varphi(i)} = x_i^{-1}$ for any $i \in I$. With $I = \varnothing$ the "*empty word*" is also defined.

Note, that reducible countable words not necessarily contain two consecutive inverse subwords, see an example in [3, page 227].

Now we discuss, how to define a reduced form of a countable word W, or how to reduce a countable word. One of the natural definitions could be the following: we withdraw from W the letters corresponding to the endpoints of arches of a maximal arch-system based on the W-segment and we will call the remaining countable word the reduced form of W. However different maximal arch-systems can give different reduced forms. The easiest example of this phenomenon is the following:

$$x\,\overset{\frown}{x^{-1}x}\,\overset{\frown}{x^{-1}x}\,\overset{\frown}{x^{-1}\ldots} \to x;$$

$$\overset{\frown}{x\,x^{-1}}\,\overset{\frown}{x\,x^{-1}}\,\overset{\frown}{x\,x^{-1}}\cdots \to \varnothing.$$

Moreover, the concatenation of two reduced countable words may have different reduced forms. Indeed, if $U = xxx\ldots$ is infinite, then U and U^{-1} are two reduced countable words, but UU^{-1} can be reduced to x^n for any integer n. To avoid such unlikable effects, we will work in a smaller class of countable words (that will be sufficient for our goals).

Definition 2.5. A countable word $W = (x_i)_{i\in I}$ is called *restricted* if every letter $x \in X^{\pm}$ occurs only finitely many times in W.

Definition 2.6. Let W be a countable word. A word U is called the *reduced form of* W if U can be obtained from W by deleting the letters corresponding to the endpoints of a maximal arch-system for W.

Clearly, for finite words we get the usual reduced form, which is, of course, unique. In Theorem 2.9, we will show that the reduced form of a restricted countable word is unique. This theorem is not new (see [3, Theorem 3.9] of J.W Cannon and G.R. Conner), however for the completeness we give a proof. In [3], it was proved, using topological properties of the plane, namely with the help of Jordan's theorem. Our proof is purely algebraic and it was inspired by the paper [6] of Eda. However, Eda defines reduced words and reduced forms of words differently. In Appendix we will show that our definitions and the definitions of Eda are equivalent.

For any subset $F \subseteq X$, let W_F denote the word obtained from W by deleting all letters of $(X \setminus F)^{\pm}$. Similarly, if $\mathcal{A}$ is an arch-system for W, we denote by $\mathcal{A}_F$ the system, obtained from $\mathcal{A}$ by deleting all arches with endpoints in $(X \setminus F)^{\pm}$. Clearly, $\mathcal{A}_F$ is an arch-system for W_F. For any finite word W we denote by $[W]$ its reduced form.

Lemma 2.7. *Let U, V be two reduced forms of a countable word W over X. Then for any finite $F \subseteq X$ holds $[U_F] \equiv [V_F]$.*

Proof. Let U is obtained from W by deleting the letter-endpoints of a maximal arch-system $\mathcal{A}$. Then, $[U_F]$ is obtained from W by the following consecutive operations:

1) Remove letter-endpoints of all arches of $\mathcal{A}$.
2) Remove all letters of $(X \setminus F)^{\pm}$.
3) Reduce the resulting word in the free group with base F.

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Similarly $[V_F]$ can be obtained from W by the last two operations. This completes the proof. $\square$

Note that a finite word is reducible to the empty word if and only if there is a complete arch-system for it.

Lemma 2.8. *Let W be a restricted countable word over X. If for any finite subset $F \subseteq X$, there is a complete arch-system for W_F, then there is a complete arch-system for W.*

Proof. We may assume that X is countable, say $X = \{x_1, x_2, \dots\}$. Denote $X_k = \{x_1, \dots, x_k\}$, $k = 1, 2 \dots$. For each $k \in \mathbb{N}$ we choose a complete arch-system associated with W_{X_k} and denote it by $\mathcal{A}(W_{X_k})$. The restrictions of $\mathcal{A}(W_{X_k})$ to the subword $W_{\{x_1\}}$, where $k = 1, 2, \dots$, are complete arch-systems for $W_{\{x_1\}}$. Since there is only a finite number of arch-systems for the finite word $W_{\{x_1\}}$, there exists an infinite subsequence $\mathcal{A}(W_{X_{k_1}})$, $\mathcal{A}(W_{X_{k_2}}), \dots$ with the same restrictions to $W_{\{x_1\}}$, denoted $\mathcal{A}_1$. From this sequence, we can choose an infinite subsequence with the same restrictions to $W_{\{x_2\}}$, denote it by $\mathcal{A}_2$, and so on. The systems $\mathcal{A}_1, \mathcal{A}_2, \dots$ do not conflict (i.e their arcs do not intersect each other), since any two of them are contained in a common finite arch-system. Therefore their union forms a complete arch-system for W. $\square$

Theorem 2.9. *Each restricted countable word W over X has a unique reduced form.*

Proof. The existence follows straightforward with the help of Zorn's lemma applied to the set of all arch-systems based on W, which is partially ordered by inclusion.

Let us prove the uniqueness. Let U, V be two reduced forms of W. By Lemma 2.7, for any finite $F \subseteq X$ holds $[U_F] \equiv [V_F]$. Let $\alpha \in X$. Consider U written in the form: $U \equiv U_0 A_1 U_1 A_2 U_2 \dots A_k U_k$, where each A_i is a positive or negative power of α and U_i does not contain α and α^{-1}, and U_i is nonempty for $i \neq 0, k$. Since U is reduced, all U_i's are also reduced and so, by Lemma 2.8, there exist finite subsets $F_i \subseteq X$ $(i = 1, \dots, k-1)$, such that $[(U_i)_{F_i}] \neq \varnothing$. Hence for any finite subset $F \subseteq X$ containing $\cup F_i \cup \{\alpha\}$, we have $[(U_i)_F] \neq \varnothing$ for all $i \neq 0, k$, and so $[U_F] \equiv [(U_0)_F] A_1 [(U_1)_F] A_2 \dots A_k [(U_k)_F]$. In particular, $U_{\{\alpha\}} \equiv [U_F]_{\{\alpha\}}$. Similarly, $V_{\{\alpha\}} \equiv [V_G]_{\{\alpha\}}$ for some finite subset $G \subseteq X$.

By Lemma 2.7, $[U_E] \equiv [V_E]$ for all finite subsets $E \subseteq X$. Hence, for a sufficiently large finite subset $E \subseteq X$ we have

$$U_{\{\alpha\}} \equiv [U_E]_{\{\alpha\}} \equiv [V_E]_{\{\alpha\}} \equiv V_{\{\alpha\}}.$$

In the same way one can prove that for any two letters $\alpha, \beta \in X$, there exists a finite subset $L \subseteq X$, such that

$$U_{\{\alpha,\beta\}} \equiv [U_L]_{\{\alpha,\beta\}} \equiv [V_L]_{\{\alpha,\beta\}} \equiv V_{\{\alpha,\beta\}}.$$

Using this, one can easily show that $U \equiv V$. $\square$

Definition 2.10. Let W_1, W_2 be two restricted reduced countable words over X. Their *product*, denoted $W_1 \cdot W_2$, is the reduced form of the concatenation $W_1 W_2$. By Theorem 2.9, the set of all reduced countable words over X with respect to this multiplication is a group. We denote this group by $\mathcal{W}(X)$.

We stress, that $\mathcal{W}(X)$ is not generated by X, if X is an infinite set. The properties of this group will be discussed in Section 3. For technical reasons we will work with tame words, which we define next.

Definition 2.11. A restricted reduced countable word is called a *reduced* (or *irreducible*) *tame word.* A finite concatenation of such words is called a *tame word.*

Remark 2.12 The class of tame words over X is closed under taking subwords and finite concatenations.

In general, a tame word W can be presented as a concatenation of reduced tame words in different ways. The following lemma states, that at least one of such presentations can be transformed to the reduced form of W in the same way as in a free group.

Lemma 2.13. *Any tame word W can be presented as a concatenation of nonenempty reduced tame words $W \equiv v_1 v_2 \dots v_k$, $k \geqslant 0$, such that the following process yields the reduced form of W:*

(†) *if some adjacent factors v_i, v_{i+1} are mutually inverse, we delete them, and if this leaves us with mutually inverse adjacent factors, we also delete them. We repeat these steps until none of the remaining adjacent factors are mutually inverse.*

Proof. As W is a tame word, W is a finite concatenation of reduced tame words: $W \equiv u_1 u_2 \dots u_l$. We may assume that all u_i are nonempty and that $l \geqslant 2$, and we will use induction by l.

Let $l = 2$, that is $W \equiv u_1 u_2$. If W is reduced, we can set $v_1 = u_1$ and $v_2 = u_2$. If W is not reduced, there exists a nonempty maximal arch-system $\mathcal{A}$ for W; since u_1, u_2 are reduced, no arch of $\mathcal{A}$ has both endpoints in u_1 or in u_2. Hence the initial points of these arches lie in u_1 and the terminal one in u_2, and all the arches are parallel. Therefore there exist concatenations $u_1 \equiv v_1 v_2$, $u_2 \equiv v_3 v_4$, such that v_2, v_3 are mutually inverse and $v_1 v_4$ is reduced. Thus, we can set $W \equiv v_1 v_2 v_3 v_4 \equiv v_1 v_2 v_2^{-1} v_4$ and the above described process yields $v_1 v_4$ as the reduced form of W.

Now let $l \geqslant 3$. By induction, $W' \equiv u_1 u_2 \dots u_{l-1}$ can be presented as a concatenation $W' \equiv w_1 w_2 \dots w_k$, with the reduced form $W'_{\mathrm{red}} \equiv w_{i_1} w_{i_2} \dots w_{i_s}$, obtained from the concatenation by applying the process (†). Also $W'_{\mathrm{red}} u_l$ (as in case $l = 2$), can be presented as $v_1 v_2 v_3 v_4$, where $W'_{\mathrm{red}} \equiv v_1 v_2$, $u_l \equiv v_3 v_4$, and v_2, v_3 are mutually inverse, and $v_1 v_4$ is reduced.

Then there exists $1 \leqslant j \leqslant s$ and a subdivision $w_{i_j} \equiv pq$ such that $v_2 \equiv q w_{i_{j+1}} \dots w_{i_{s-1}} w_{i_s}$, and so $v_3 \equiv w_{i_s}^{-1} w_{i_{s-1}}^{-1} \dots w_{i_{j+1}}^{-1} q^{-1}$. The desired concatenation is

$$W \equiv w_1 w_2 \dots w_{i_j - 1} p q w_{i_j + 1} \dots w_{k-1} w_k w_{i_s}^{-1} w_{i_{s-1}}^{-1} \dots w_{i_{j+1}}^{-1} q^{-1} v_4.$$

□

3 Hawaiian Earring and Griffiths' space

Definition 3.1. The *Hawaiian Earring* is the topological space, which is the (countable) union of circles of radii $\frac{1}{n}$, $n = 1, 2, \dots$, embedded into the Euclidean plane in such a way that they have only one common point (called the *base-point*). The topology of the Hawaiian Earring is induced by the topology of the plane.

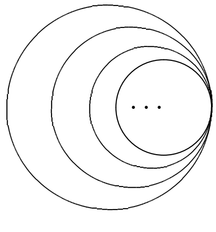

Fig. 2

Usually one assumes that these circles lie on same side of the common tangent (see Fig. 2). If we name the circle of radius $\frac{1}{n}$ by p_n, we will denote our Hawaiian Earring by $H(p_1, p_2, \dots)$ or shortly $H(P)$, where $P = \{p_1, p_2 \dots\}$. For convenience we will use the same name p_n for the path traversing p_n in counterclockwise direction, and also for the homotopy class of this path.

Note, that each homotopy class in $\pi_1(H(P))$ can be represented by a path without backtracking inside of each p_i. Because of continuity the path cannot pass through p_i infinitely many times. This yields the natural map $\pi_1(H(P)) \to \mathcal{W}(P)$. The following theorem is well known; see [18, Theorem 4.1].

Theorem 3.2. *The fundamental group of the Hawaiian Earring is isomorphic to the group $\mathcal{W}(P)$.*

The Hawaiian Earring has the following remarkable properties, some of them do not shared by tame spaces (e.g. manifolds or triangulable spaces).

• It is not semi-locally simply connected.
(A space X is called *semi-locally simply connected* if every point $x \in X$ has a neighborhood N such that the homomorphims $\pi_1(N, x) \mapsto \pi_1(X, x)$, induced by the inclusion map of N in X, is trivial. That is, every loop in N can be shrunk to a point in X.)

• It does not admit a simply connected covering space.
(For connected and locally path connected spaces this is equivalent to semi-locally simply connectedness, see Corollary 14 in Ch. 2, Sec. 5 of [20].)

• It is compact, but its fundamental group is uncountable.
(This is an easy exercise, see for example [3, Theorem 2.5 (1)].)

• It is one-dimensional, but its fundamental group is not free.
(This follows from [18, Theorem 4.1] and a remark in [14, page 80]. See also a short proof in [19].)

• Its fundamental group contains countable subgroups, which are not free.
(See [14, page 80] or [23, Theorem 3.9 (iii)].)

• Every finitely generated subgroup of its fundamental group is free.
(Since the fundamental group embeds in an inverse limit of free groups, see [18, Theorem 4.1].)

• Its first singular homology group is torsion free.
(See [8, Corollary 2.2]).

• Its first singular homology group is uncountable; it contains a divisible torsion-free subgroup of cardinality $2^{\aleph_0}$, i.e. $\bigoplus\limits_{2^{\aleph_0}} \mathbb{Q}$.
(See [8, Theorem 3.1] or Corollary 8.7 (a) and Theorem 9.2 of the present paper).

Short proofs of some of these statements are presented in [3, Theorem 2.5].

Definition 3.3. *Griffiths' space* can be constructed as follows: take two Hawaiian Earrings, build a cone over each of them and finally take their one-point union with respect to the base point of each of the Earrings.

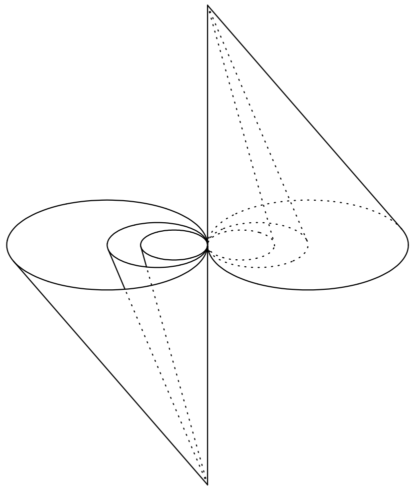

Fig. 3

We consider Griffiths' space as embedded into the 3-dimensional Euclidean space so, that both Hawaiian Earrings lie in the (x, y)-plane, have coinciding common tangents and that they are placed in different half-planes with respect to this tangent. Moreover, the cone points of their cones lie in the planes $z = 1$ and $z = -1$, respectively (see Fig. 3). Griffiths' space, if built from two Hawaiian Earrings $H(p_1, p_2, \dots)$ and $H(q_1, q_2, \dots)$, is denoted by $G(p_1, q_1, p_2, q_2, \dots)$, or shortly $G(P; Q)$. The following theorem is known; see [11, formula (3.55)]. For completeness we give here a straightforward proof.

Theorem 3.4. *The fundamental group of Griffiths' space $G(P; Q)$ is isomorphic to the factor group of $\mathcal{W}(P \cup Q)$ by the normal closure of $\langle \mathcal{W}(P), \mathcal{W}(Q) \rangle$.*

Proof. We consider Griffiths' space $Y = G(P; Q)$ with coordinates as in Definition 3.3. Let Y_{-1}, Y_0, Y_1 be the subsets of Y, consisting of all points with the z-coordinate from the intervals $[-1, -\frac{1}{3}), (-\frac{2}{3}, \frac{2}{3}), (\frac{1}{3}, 1]$ respectively. Note that $\pi_1(Y_{-1}) = \pi_1(Y_1) = 1$ and $\pi_1(Y_0) \cong \pi_1(H(P \cup Q))$. By the van Kampen Theorem, $\pi_1(Y_{-1} \cup Y_0)$ is the quotient of the group $\pi_1(H(P \cup Q))$ by the normal closure of its subgroup $\pi_1(H(P))$. Finally, $\pi_1(Y) = \pi_1((Y_{-1} \cup Y_0) \cup Y_1)$ is the factor group of the last quotient by the normal closure of the subgroup $\pi_1(H(Q))$. $\square$

Griffiths' space has the following remarkable properties.

- It is a one-point union of two contractible spaces, but itself it is not contractible. Moreover its fundamental group is uncountable [11, Claim 3.4].
- It is not semi-locally simply connected.
- It does not admit a simply connected covering space.

(See Corollary 14 in Ch. 2, Sec. 5 of [20] and [20, Example 2.5.18].)

- It is a space where the natural homomorphism from the fundamental group to the shape group is not an injection [9, Example 3].
- Its first singular homology group is torsion free.

(This follows from [7, Theorem 1.2] and [8, Corollary 2.2].)

• Its first singular homology group is uncountable; it contains a divisible torsion-free subgroup of cardinality $2^{\aleph_0}$, i.e. $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$.
(See Corollary 8.7 (b) and Theorem 9.2 of the present paper.)

Recall that a space X is called *homotopically Hausdorff*, if for every point $x \in X$ and any non-trivial homotopy class $c \in \pi_1(X, x)$, there is a neighborhood of x which contains no representative for c. In other words, a space X is homotopically Hausdorff, if for every point $x \in X$ the space of homotopy classes of closed curves in X emanating from x, sometimes denoted $\Omega(X, x)$, is Hausdorff. Note that any semi-locally simply connected space is homotopically Hausdorff. In particular, topological manifolds and CW-complexes enjoy this property. Every one-dimensional space and every planar set is homotopically Hausdorff, see [4, Corollary 5.4 (2)] and [5, Theorem 3.4], respectively. In particular, the Hawaiian Earring, is homotopically Hausdorff. However

• Griffiths' space is not homotopically Hausdorff, since it contains the loop $p_1q_1p_2q_2\dots$, which is homotopic into any neighborhood of the basepoint, but yet is not nullhomotopic. This was noticed in [4, Example 2.0.2], but this also follows from our Theorem 8.1.

4 Description of the strategy of the proof

The *commutator* of two elements $u, v \in \mathcal{W}(X)$ is the element $u^{-1} \cdot v^{-1} \cdot u \cdot v \in \mathcal{W}(X)$, and it is denoted by $[u, v]$.

We describe an idea of the proof of Theorem 8.2 (Theorems 8.1 and 8.3 are proved similarly). Thus, we consider the the Griffiths space $Y = G(P; Q)$. Let w_1, w_2 be two reduced tame words over the alphabet $P \cup Q$. Suppose that they represent the same element in $H_1(Y) = [\pi_1(Y), \pi_1(Y)]$. Then, by Theorem 3.4 we have

$$w_2 = w_1 \cdot \prod_{j=1}^{m} \theta_j^{-1} \cdot \eta_j \cdot \theta_j \cdot \prod_{i=1}^{n} [\sigma_i, \tau_i] \tag{4.1}$$

for some reduced tame words $\theta_j, \sigma_i, \tau_i, \eta_j \in \mathcal{W}(P \cup Q)$, where $\eta_j \in \mathcal{W}(P)$ or $\eta_j \in \mathcal{W}(Q)$. Therefore w_2 is the reduced form of the concatenation

$$w \equiv w_1 \prod_{j=1}^{m} \theta_j^{-1} \eta_j \theta_j \prod_{i=1}^{n} [\sigma_i, \tau_i] \tag{4.2}$$

By Lemma 2.13, w can be written as

$$w \equiv t_1 u_1 t_2 u_2 \dots t_k u_k t_{k+1}, \tag{4.3}$$

where the reduced form of each u_q is the empty word and

$$w_2 \equiv t_1 t_2 \dots t_{k+1}. \tag{4.4}$$

With the concatenations (4.2)-(4.3) we will associate a geometric object called arch-line-band system (see Section 5). In this system we define leafs and parallelity classes of leafs. By Proposition 6.6, the number of these classes is finite and we will call it *complexity* of concatenations (4.2)-(4.3).

Using transformations (i) and (ii) from Theorem 8.2, we will simplify w, w_1 and w_2 in several steps, so that

1) at each step the new tame words w_1' and w_2' represent the same elements of $H_1(Y)$;

2) the new tame words w', w_1' and w_2' are related by equations, which are analogous to (4.2)-(4.4), have the same parameters n, m, k, but smaller complexity:

$$w' \equiv w_1' \prod_{j=1}^{m} \theta_j'^{-1} \eta_j' \theta_j' \prod_{i=1}^{n} [\sigma_i', \tau_i'], \tag{4.2$'$}$$

$$w' \equiv t_1' u_1' t_2' u_2' \dots t_k' u_k' t_{k+1}', \tag{4.3$'$}$$

and

$$w_2' \equiv t_1' t_2' \dots t_{k+1}', \tag{4.4$'$}$$

where $t_s', u_l', w_1', \theta_j', \eta_j', \sigma_i', \tau_i'$ are obtained from $t_s, u_l, w_1, \theta_j, \eta_j, \sigma_i, \tau_i$ by removing a finite number of subwords, and where the reduced form of each u_q' is the empty word;

3) the final word w_2' can be obtained from the final word w_1' by applying a finite number of transformations (iii).

5 A visualization of cancellation processes. Arch-line-band systems

In this section we define geometric objects – arch-line-band systems, which are associated with concatenations (4.2) and (4.3) of tame words. They are designed to control cancellations in these concatenations. In Section 7 we introduce transformations of these systems, that will be used in Section 8 to prove our main Theorems 8.1, 8.2 and 8.3.

First we define components of an arch-line-band system, and after that we will combine all together to define this system at whole. The following definitions are based on the visualization of countable words as described in Definition 2.2.

(a) Bands. Here we define two types of bands, line-bands and arch-bands.

Line-bands. Let t be a countable word and T be a t-segment. We glue the rectangle $[0,1] \times [0,1]$ to the t-segment by a homeomorphism which identifies the side $[0,1] \times \{0\}$ with this segment. The opposite side $[0,1] \times \{1\}$ is called *free*. Inside of this rectangle we draw parallel *vertical lines*, starting from the letters of this segment and ending at its free side. The system, consisting of the t-segment, this rectangle and all these lines is called the *line-band* associated with T. We will say that this line-band is *glued* to T.

Arch-bands. Now let T be a t-segment and T_1 be a t^{-1}-segment. We glue the rectangle $[0,1] \times [0,1]$ to T and T_1 along the sides $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ with the help of homeomorphisms which preserve and reverse an orientation, respectively. Then we connect the corresponding inverse letters of T and T_1 by parallel *arcs* inside of this rectangle. The system, consisting of the segments T, T_1, this rectangle and all these arches is called the *arch-band* associated with T and T_1. We will say that this arch-band *connects* T and T_1.

(b) Arch-band system. Let u be a tame word, and suppose that u is reducible to the empty word. According to Lemma 2.13, there exists a cancelation pattern for u, that is a subdivision $u \equiv z_1 z_2 \dots z_{2k}$, where some neighboring factors $z_{i(1)}, z_{j(1)}$ are mutually inverse, and after removing them some other neighboring factors $z_{i(2)}, z_{j(2)}$ will be mutually inverse, and so on until we get the empty word. The factors $z_{i(s)}, z_{j(s)}$ are called *associated factors*.

This cancelation procedure may be not unique, but for any fixed one we define the corresponding arch-band system in the following way. First we visualize the words z_i on the real line so that the z_{i+1}-segment is situated to the right of the z_i-segment in the adjacent position, $i = 1, \dots, 2k-1$. Clearly, the union of these z_i-segments is a u-segment.

We connect the associated $z_{i(s)}$- and $z_{j(s)}$-segments by an arch-band. The system, consisting of the u-segment and all these arch-bands is called the *arch-band system* associated with u.

Now, let w be a tame word, which is represented as a concatenation in two ways, see (4.2) and (4.3), where u_i's are tame words, which are reducible to the empty word. We visualize w by an w-segment.

(c) Upper arch-line-band system. Consider $2k+1$ consecutive subsegments of the w-segment, which correspond to the factors in the right side of (4.3). For every t_i-segment we draw the associated line-band and for every u_i-segment we draw the associated arch-band system. The system, consisting of the w-segment, the line-bands and the arch-bands is called the *upper arch-line-band system* associated with the concatenation (4.3). The corresponding bands are called *upper*.

(d) Lower arch-line-band system. Consider $1 + 3m + 4n$ consecutive subsegments of the w-segment, which correspond to the factors in the right side of (4.2). Then for each i, j we

(i) connect θ_j^{-1}- and θ_j-segments by an arch-band,

(ii) connect σ_i^{-1}- and σ_i-segments and also connect τ_i^{-1}- and τ_i-segments by arch-bands,

(iii) glue line-bands to the w_1- and to the η_j-segments.

The system, consisting of the w-segment, all these line-bands and arch-bands is called the *lower arch-line-band system* associated with the concatenation (4.2). The corresponding bands are called *lower*.

(e) The arch-line-band system associated with the equations (4.2) and (4.3) is the union of the corresponding upper and lower arch-line-band systems based on the same w-segment (see Fig. 4). It will be denoted by

$$\frac{t_1 u_1 t_2 u_2 \dots t_k u_k t_{k+1}}{w_1 \prod_{j=1}^{m} \theta_j^{-1} \eta_j \theta_j \prod_{i=1}^{n} [\sigma_i, \tau_i]}. \tag{5.1}$$

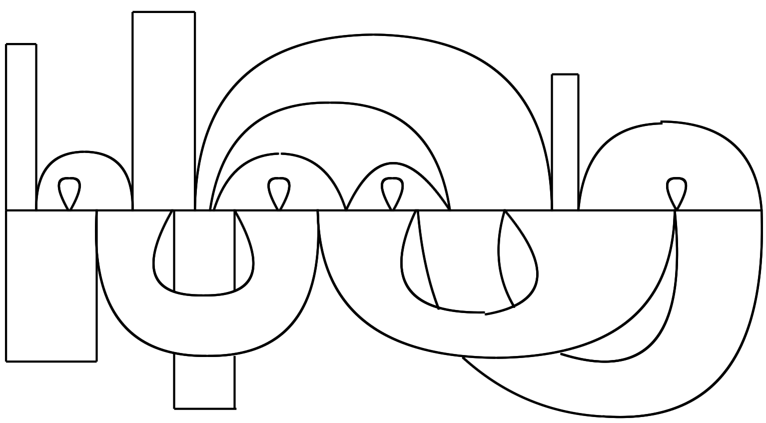

Fig. 4

Definition. Let $(\dots, \sigma_{i-1}, \sigma_i, \sigma_{i+1}, \dots)$ be a finite or infinite sequence of arches or vertical lines of our system, which lie alternately in upper or lower bands such that the terminal point of each σ_j coincides with the initial point of its successor σ_{j+1}. The union of these vertical lines and arches is called a *curve.*

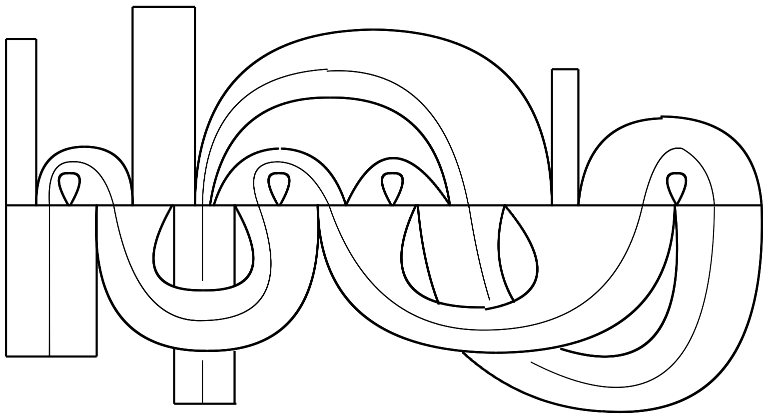

Fig. 5

A *leaf* in the arch-line-band system is a maximal curve (see Fig. 5). Clearly, vertical lines can appear only at the beginning or at the end of the leaf. Note that a leaf cannot intersect itself and it never splits.

6 Properties of leafs of an arch-line-band system

Proposition 6.1. *Any leaf of an arch-line-band system consists of a finite number of arcs and vertical lines.*

Proof. We will use notations of Section 5. Any leaf intersects the w-segment of the system at points, which are marked by letters of $\{x, x^{-1}\}$ for some $x \in X$. Since w is a tame word, each letter of $X^{\pm}$ occurs only finitely many times in w. Thus, any lief intersects the w-segment in a finite set of points. $\square$

Therefore each leaf σ can be written in the form $\sigma = \sigma_1\sigma_2 \dots \sigma_p$, where p is finite and all σ_i are arches, except for σ_1 and σ_p, which also may be vertical lines.

Definition 6.2. a) Two arches (lines) of an arch-line-band system are called *parallel* if they lie in the same arch-band (line-band).

b) Two nonclosed leafs $\sigma_1\sigma_2 \dots \sigma_p$ and $\tau_1\tau_2 \dots \tau_q$ are called *parallel* if $p = q$ and σ_i, τ_i are parallel for any $i = 1, \dots, p$.

Two closed leafs $\sigma_1\sigma_2 \dots \sigma_p$ and $\tau_1\tau_2 \dots \tau_q$ are called *parallel* if $p = q$ and there is a natural k, such that the arches σ_i, τ_{i+k} are parallel for any $i = 1, \dots, p$ (addition modulo p).

Remark 6.3. Let $\sigma_1\sigma_2 \dots \sigma_p$ and $\tau_1\tau_2 \dots \tau_q$ be two nonclosed leafs such that $p \leqslant q$ and σ_i, τ_i are parallel for any $i = 1, \dots, p$. Then σ_p ends on the free side of some line-band and so does τ_p. Therefore $p = q$.

Definition 6.4. Let S be a topological space with a distinguished subset D. Two non-closed curves l_1, l_2 in S with endpoints in D will be called *homotopic relatively* D if there exists a homotopy carrying l_1 to l_2, which moves the endpoints of l_1 and l_2 inside of D. If we apply this term to two closed curves, we mean that they are freely homotopic.

Sometimes we will consider an arch-line-band system as a topological space, which is the union of the corresponding rectangles glued to a segment of the real line.

Lemma 6.5. *Let A be an arch-line-band system and $D = b_1 \cup \cdots \cup b_t$ be the union of the free sides of its line-bands. Two leafs of A are parallel if and only if they are homotopic relatively D.*

Proof. If two leafs of A are parallel, then, clearly, they are homotopic relatively D. We prove the converse statement.

Let R be a rose with one vertex v, petals corresponding to the arch-bands and with thorns corresponding to the line-bands of A (see Figure 6).

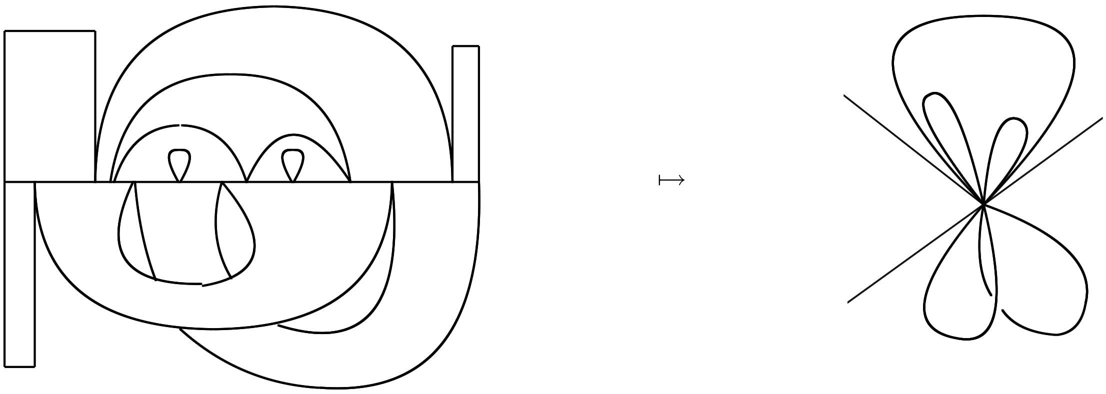

Fig. 6

There is a natural homotopy equivalence $\pi : A \to R$, which sends the w-segment to v, arch-bands to petals and line-bands to thorns. Free sides of line-bands go to the vertices of thorns different from v. The set of these vertices is $\pi(D)$.

Let l_1, l_2 be two nonparallel leafs in A, both closed or both nonclosed. If they are closed, their images in R are different cyclic cyclically reduced paths. If l_1, l_2 are nonclosed, their images in R are different reduced paths with endpoints in $\pi(D)$. In both cases these paths are not homotopic in R relatively $\pi(D)$, and so the leafs are not homotopic in A relatively D. $\Box$

Proposition 6.6. *There is only a finite number of parallelity classes of leafs in the arch-line-band system* (5.1).

Proof. Let A be the topological space associated with the system (5.1). We embed this space into a surface, so that this embedding will be a homotopy equivalence. To construct the surface we will glue to A a finite number of triangles and rectangles.

More exactly, any two adjacent upper (lower) bands in A are either disjoint or meet at a point of the w-segment; in the first case we glue a rectangle, in the second case we glue a triangle as it shown in Figure 7. Denote the resulting space by T and note that it is a finite genus orientable surface.

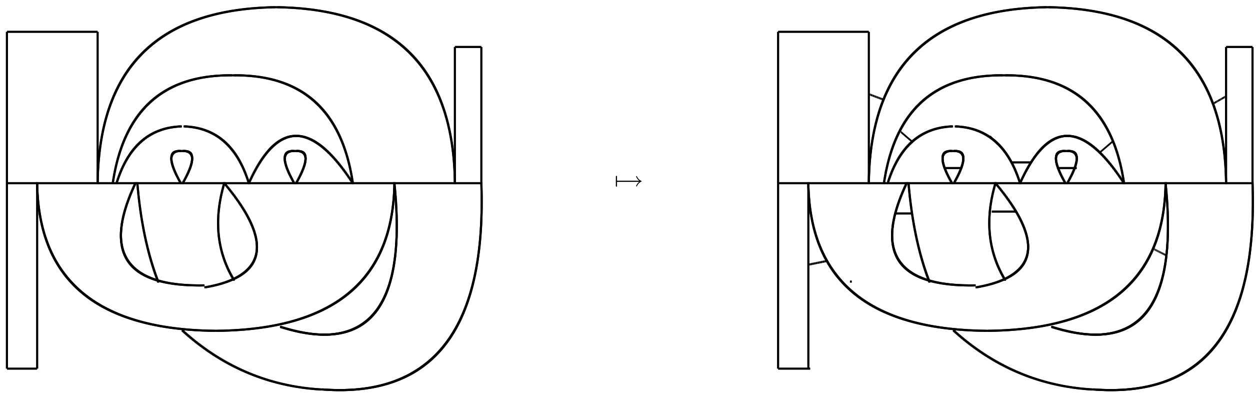

Fig. 7

The boundary of T contains parts of the lateral sides of all bands, some sides of the triangles and rectangles, and the free sides of all line-bands. Within ∂T we consider these free sides $b_1, \dots, b_t$ as *distinguished boundary segments.*

Since the space A is contained in T, we may assume that all leafs of A are contained in T. These leafs are of two types: closed curves in the interior of T and curves with ends on distinguished boundary segments.

Taking into account that A and T are homotopy equivalent relatively $D = b_1 \cup \dots \cup b_t$, we conclude from Lemma 6.5 that two leafs of A are parallel if and only if they are homotopic in T relatively D. Now the statement follows from Lemma 6.7. $\square$

For any compact surface S we denote by $g(S)$ its genus and by $b(S)$ the number of its boundary components.

Lemma 6.7. *Let S be a compact surface with a finite set of distinguished segments $b_1, \dots, b_t$ on its boundary. Let L be a set of disjoint simple curves in S, such that each curve of L is either*

(1) *a closed curve in the interior of S, or*

(2) *a nonclosed curve, which has both endpoints in $\{b_1 \cup \dots \cup b_t\}$ and lies, apart from them, in the interior of S.*

Then the number of relatively-$\{b_1 \cup \dots \cup b_t\}$ homotopy classes of these curves does not exceed a constant depending only on $g(S)$, $b(S)$ and t.

Proof. We will proceed by induction on $(g(S), b(S))$ trying to decrease the pair in the lexicographical order $\preccurlyeq$ on $\mathbb{N} \times \mathbb{N}$ by cutting S along a curve $l \in L$. There are 2 cases.

Case 1. l is a nonseparating curve. Then, after cutting along l, we get a new surface S_1. In this case there are 3 subcases.

a) l is a closed curve. Then $g(S_1) = g(S) - 1$ and $b(S_1) = b(S) + 2$.

b) l is a nonclosed curve with endpoints at different boundary components of S. Then $g(S_1) = g(S)$ and $b(S_1) = b(S) - 1$.

c) l is a nonclosed curve with endpoints at the same boundary component of S. Then $g(S_1) = g(S) - 1$ and $b(S_1) = b(S) + 1$.

Case 2. l is a separating curve. Then, if we cut along l, we get two surfaces S_1, S_2. In this case there are 2 subcases.

a) l is a closed curve. Then $g(S_1) + g(S_2) = g(S)$ and $b(S_1) + b(S_2) = b(S) + 2$.

b) l is a nonclosed curve with endpoints at the boundary of S. Since l is separating, its endpoints lie at the same boundary component of S. Then $g(S_1)+g(S_2) = g(S)$ and $b(S_1)+b(S_2) = b(S) + 1$, and $b(S_1) > 0$, $b(S_2) > 0$.

We see, that in Case 1 the pair $(g(\cdot), b(\cdot))$ decreases in the lexicographical order $\preccurlyeq$. In Case 2 we have also $(g(S_j), b(S_j)) \prec (g(S), b(S))$ for $j = 1, 2$ except of three subcases, where for some j holds

(i) l is closed and $(g(S_j), b(S_j))$ equals to $(0, 1)$,
(ii) l is closed and $(g(S_j), b(S_j))$ equals to $(0, 2)$,
(iii) l is nonclosed and $(g(S_j), b(S_j))$ equals to $(0, 1)$.

Note, that the curve l lies on ∂S_j.

In Subcase (i), S_j is a disk and $l = \partial S_j$. In particular l is null-homotopic in S.

In Subcase (ii), S_j is an annulus and l is one of its boundary components. In particular l is parallel to another boundary component of S_j, which is a boundary component of S.

In Subcase (iii), S_j is a disc and l, having endpoints on distinguished segments of ∂S, is parallel to ∂S.

Thus, if L contains more than $b(S)+2t^2+1$ relatively-$\{b_1 \cup \dots \cup b_t\}$ homotopy classes of curves, then one of them does not fall into the exceptional subcases (i), (ii) or (iii). Cutting S along this curve, we get one or two surfaces with smaller $(g(\cdot), b(\cdot))$. Note that for each of the resulting surfaces $g(\cdot)$ cannot increase, $b(\cdot)$ can increase by at most 2 and t can increase by at most 2. Thus for each of the new surfaces we can apply the induction.

Note, that the number of the relatively-$\{b_1 \cup \dots \cup b_t\}$ homotopy classes of curves in L does not exceed the sum of the corresponding numbers after cutting of S along a curve $l \in L$. This completes the proof. $\square$

7 Transformations of arch-line-band systems

Now we consider an important transformation of an arch-line-band system:

removing the parallelity class of a leaf. Let σ be a leaf in the arch-line-band system (5.1) and $[\sigma]$ be the parallelity class of σ. We then "remove $[\sigma]$", i.e. we remove from the factors of (5.1) all letters that lie on the intersection of the w-segment and leafs from $[\sigma]$, and we also remove these leafs from our picture. Due to Proposition 6.6, only finitely many subwords will be removed from each of the factors $t_s, u_l, w_1, \theta_j, \eta_j, \sigma_i, \tau_i$ of (5.1). The remaining parts of these factors form new factors $t'_s, u'_l, w'_1, \theta'_j, \eta'_j, \sigma'_i, \tau'_i$ and we obtain a new system:

$$\frac{t'_1 u'_1 t'_2 u'_2 \dots t'_k u'_k t'_{k+1}}{w'_1 \prod_{j=1}^{m} \theta'^{-1}_j \eta'_j \theta'_j \prod_{i=1}^{n} [\sigma'_i, \tau'_i]}. \tag{7.1}$$

Note, that the remaining letters are still connected by the same upper or lower arches as before. Therefore we conclude:

Remark 7.1. 1) The words u'_s are reducible to the empty word.

2) We have $\theta_j^{-1\prime} \equiv \theta'^{-1}_j$, $\sigma_i^{-1\prime} \equiv \sigma'^{-1}_i$, $\tau_i^{-1\prime} \equiv \tau'^{-1}_i$, i.e. letters were deleted from pairs of inverse factors in such a way, that the new factors remained inverse to each other.

3) By Remark 2.12, all the words in the numerator and the denominator of (7.1) are tame, since they are finite concatenations of subwords of tame words.

4) Summing up 1)–3) we obtain: the system (7.1) indeed satisfies all conditions of an arch-line-band system, and has complete analogous structure of factors as (5.1) (although some factors might have become empty). In particular, the numerator and the denominator in (7.1) are equal.

We are especially interested in the exact description of the structure of $t'_1, \dots, t'_k$ and w'_1. This structure depends on the type of the parallelity class of σ.

(1) If σ is closed it cannot run through w_1 and t_i. In this case we have $w_1 \equiv w'_1$ and $t_i \equiv t'_i$ for every i.

(2) If σ starts at an upper t_i-band and ends at another upper t_j-band, then there exists a tame word W, such that t_i contains W, t_j contains W^{-1} and we have the following transformation:

$$\begin{array}{lll} t_i \equiv A_i W B_i & \mapsto & t'_i \equiv A_i B_i, \\ t_j \equiv A_j W^{-1} B_j & \mapsto & t'_j \equiv A_j B_j, \\ t_l & \mapsto & t'_l \equiv t_l \qquad \text{for} \quad l \neq i, j, \\ w_1 & \mapsto & w'_1 \equiv w_1. \end{array}$$

(2') If σ starts and ends at the same upper t_i-band, then t_i has the form $t_i \equiv A_i W B_i W^{-1} C_i$ and we have the following transformation:

$$\begin{array}{lll} t_i \equiv A_i W B_i W^{-1} C_i & \mapsto & t'_i \equiv A_i B_i C_i, \\ t_l & \mapsto & t'_l \equiv t_l \qquad \text{for} \quad l \neq i, \\ w_1 & \mapsto & w'_1 \equiv w_1. \end{array}$$

(3) If σ starts at a lower η_i-band and ends at an upper t_j-band, then there is a common subword W of η_i and t_j, such that we have the following transformation:

$$\begin{array}{lll} t_j \equiv A_j W B_j & \mapsto & t'_j \equiv A_j B_j, \\ t_l & \mapsto & t'_l \equiv t_l \qquad \text{for} \quad l \neq j, \\ w_1 & \mapsto & w'_1 \equiv w_1. \end{array}$$

(4) If σ starts at the lower w_1-band and ends at an upper t_j-band, then there is a common subword W of w_1 and t_j, such that we have the following transformation:

$$\begin{array}{lll} t_j \equiv A_j W B_j & \mapsto & t'_j \equiv A_j B_j, \\ t_l & \mapsto & t'_l \equiv t_l \qquad \text{for} \quad l \neq j, \\ w_1 \equiv AWB & \mapsto & w'_1 \equiv AB. \end{array}$$

(5) If σ starts at a lower η_i-band and ends at the lower w_1-band, then there exists a tame word W, such that η_i contains W, w_1 contains W^{-1} and we have the following transformation:

$$\begin{array}{lll} t_l & \mapsto & t'_l \equiv t_l \qquad \text{for} \quad l = 1, \dots, k+1, \\ w_1 \equiv A W^{-1} B & \mapsto & w'_1 \equiv AB. \end{array}$$

(6) If σ starts at a lower η_i-band and ends at a lower η_j-band, then $w'_1 \equiv w_1$ and $t'_l \equiv t_l$ for all l.

(7) If σ starts and ends at the lower w_1-band, then w_1 has the form $w_1 \equiv AWBW^{-1}C$ and we have the following transformation:

$$\begin{aligned} t_l \qquad &\mapsto \quad t'_l \equiv t_l \qquad \text{for} \quad l = 1, \dots, k+1, \\ w_1 \equiv AWBW^{-1}C \quad &\mapsto \quad w'_1 \equiv ABC. \end{aligned}$$

8 Main theorems

Theorem 8.1. *Let Y be the Griffiths space $G(p_1, q_1, p_2, q_2, \dots)$. Two reduced tame words w_1, w_2 over the alphabet $\{p_1, q_1, p_2, q_2, \dots\}$ represent the same element in $\pi_1(Y)$ if and only if each of them can be transformed to the same reduced tame word by applying a finite number of transformations of type* (i) *and finally the transformation of type* (ii)*:*

(i) *Deletion of a countable subword, containing only p- or q-letters;*
(ii) *Reducing the resulting countable words.*

Proof. The part "if" follows from Theorem 3.4: by this theorem any restricted countable word containing only p- or q-letters represents the identity element of $\pi_1(Y)$. It remains to prove the part "only if".

Let w_1, w_2 be two reduced tame words over the alphabet $\{p_1, q_1, p_2, q_2, \dots\}$ representing the same element in $\pi_1(Y)$. By Theorem 3.4, the word w_2 is the reduced form of a finite concatenation

$$w_1 \prod_{j=1}^{m} \theta_j^{-1} \eta_j \theta_j, \tag{8.1}$$

where $\theta_j \in \mathcal{W}(p_1, q_1, p_2, q_2, \dots)$ and $\eta_j \in \mathcal{W}(p_1, p_2, \dots)$ or $\eta_j \in \mathcal{W}(q_1, q_2, \dots)$. By Lemma 2.13, there exists a decomposition of (8.1),

$$w_1 \prod_{j=1}^{m} \theta_j^{-1} \eta_j^{-1} \theta_j \equiv t_1 u_1 t_2 u_2 \dots t_k u_k t_{k+1}, \tag{8.2}$$

where all the subwords u_i are reducible to the empty word, all t_i are irreducible and

$$w_2 \equiv t_1 t_2 \dots t_{k+1}.$$

Now we consider the arch-line-band system

$$\frac{t_1 u_1 t_2 u_2 \dots t_k u_k t_{k+1}}{w_1 \prod_{j=1}^{m} \theta_j^{-1} \eta_j \theta_j}. \tag{8.3}$$

Remove one by one from it all (a finite number by Proposition 6.6) parallelity classes of leafs which start at the lower η_j-bands. Then the words η_j completely disappear and we get a new arch-line-band system

$$\frac{t'_1 u'_1 t'_2 u'_2 \dots t'_k u'_k t'_{k+1}}{w'_1 \prod_{j=1}^{m} \theta'^{-1}_j \theta'_j}. \tag{8.4}$$

By analogy with $w_2 \equiv t_1 t_2 \dots t_{k+1}$ we denote $w'_2 \equiv t'_1 t'_2 \dots t'_{k+1}$. According to Cases (3), (5), (6) from Section 7, the countable words w'_2 and w'_1 are obtained from w_2 and w_1 by applying a finite number of transformations (i).

Since the numerator and the denominator in (8.4) are equal as countable words and still tame (see statements 4) and 3) of Remark 7.1), they have the same reduced form by Theorem 2.9. Hence w_2' and w_1' have the same reduced form. In other words, after applying to w_2' and w_1' the transformation (ii), we get the same reduced tame word. □

Theorem 8.2. *Let Y be the Griffiths space $G(p_1, q_1, p_2, q_2, \dots)$. Two reduced tame words w_1, w_2 over the alphabet $\{p_1, q_1, p_2, q_2, \dots\}$ represent the same element in $H_1(Y)$ if and only if each of them can be transformed to the same tame word by applying a finite number of transformations of type* (i)*, then of type* (ii) *and finally of type* (iii)*:*

(i) *Deletion of a countable subword, containing only p- or q-letters;*
(ii) *Deletion of two distinct mutually inverse countable subwords: $AXBX^{-1}C \to ABC$;*
(iii) *Permuting two consecutive countable subwords.*

Proof. We will prove the nontrivial "only if" part. Let w_1, w_2 be two reduced tame words over the alphabet $\{p_1, q_1, p_2, q_2, \dots\}$ representing the same element in $H_1(Y)$. By Proposition 3.4, the word w_2 is the reduced form of a finite concatenation

$$w \equiv w_1 \prod_{j=1}^{m} \theta_j^{-1}\eta_j\theta_j \prod_{i=1}^{n}[\sigma_i, \tau_i], \tag{8.5}$$

where each η_j contains only p- or q-letters. This means, that there exists a decomposition of (8.5),

$$w_1 \prod_{j=1}^{m} \theta_j^{-1}\eta_j\theta_j \prod_{i=1}^{n}[\sigma_i, \tau_i] \equiv t_1u_1t_2u_2 \dots t_ku_kt_{k+1}, \tag{8.6}$$

where all the subwords u_i are reducible to the empty word and

$$w_2 \equiv t_1t_2 \dots t_{k+1}.$$

Now we consider the arch-line-band system

$$\frac{t_1u_1t_2u_2 \dots t_ku_kt_{k+1}}{w_1 \prod_{j=1}^{m} \theta_j^{-1}\eta_j\theta_j \prod_{i=1}^{n}[\sigma_i, \tau_i]}. \tag{8.7}$$

We will simplify this system by removing certain parallelity classes of leafs. First we eliminate all η_j exactly in the same way as in the proof of Theorem 8.1 (so, we use transformations (i) several times). Thus, without loss of generality, we may assume that we have the arch-line-band system

$$\frac{t_1u_1t_2u_2 \dots t_ku_kt_{k+1}}{w_1 \prod_{j=1}^{m} \theta_j^{-1}\theta_j \prod_{i=1}^{n}[\sigma_i, \tau_i]}. \tag{8.8}$$

The only line-bands of this system are the upper t_s-bands, $s = 1, \dots, k+1$, and the lower w_1-band. Note that $t_1t_2 \dots t_{k+1}$ is a tame word, but may be already not reduced.

Now suppose that the system contains a leaf σ starting at some t_p-band and ending at the same or another t_q-band. By Cases (2) and (2') from Section 7, the tame word $w_2 \equiv t_1t_2 \dots t_{k+1}$ has the form $AXBX^{-1}C$. Removing the parallelity class of σ, we do not change the form of (8.8) and do not change w_1, but we replace w_2 by ABC (this is the transformation (ii)).

We remove all parallelity classes starting and ending at the union of the upper t-bands. Analogously we remove all parallelity classes starting and ending at the lower w_1-band.

Thus we may assume, that every parallelity class starting at an upper t-band ends at the lower w_1-band. Conversely, every parallelity class starting at the lower w_1-band ends at some upper t-band. Moreover, every letter of $w_2 \equiv t_1 t_2 \dots t_{k+1}$ and every letter of w_1 lies on some leaf. Therefore there is a partition $w_2 \equiv v_1 v_2 \dots v_r$ and a permutation $\sigma \in S_r$, such that $w_1 \equiv v_{1\sigma} v_{2\sigma} \dots v_{r\sigma}$. It remains to apply a finite number of transformations (iii). □

Theorem 8.3. *Let Z be the Hawaiian Earring $H(p_1, p_2, \dots)$. Two reduced tame words over the alphabet $\{p_1, p_2, \dots\}$ represent the same element in the first homology group $H_1(Z)$ if and only if each of them can be transformed to the same tame word by applying a finite number of transformations of type* (ii) *and finally of type* (iii)*:*

(ii) *Deletion of two distinct mutually inverse countable subwords: $AXBX^{-1}C \to ABC$;*
(iii) *Permuting two consecutive countable subwords.*

Proof. We will prove the nontrivial "only if" part. The word w_2 is the reduced form of a finite concatenation

$$w_1 \prod_{i=1}^{n} [\sigma_i, \tau_i] \tag{8.9}$$

(compare with (8.5)). The further proof is the same as the proof of Theorem 8.2, starting from formula (8.8) with empty θ_j. □

We fix two disjoint infinite countable alphabets $P = \{p_1, p_2 \dots\}$ and $Q = \{q_1, q_2, \dots\}$ for the rest of this section. Below we introduce legal and (P,Q)-legal words, which, in view of Lemma 8.6, will be used later to construct uncountably many words with certain nice properties.

Definition 8.4.
1) A countable word U over the alphabet P is called *legal* if it is restricted and for any presentation $U \equiv AXBX^{-1}C$ the subword X is finite.
2) A countable word U over the alphabet $P \cup Q$ is called (P,Q)-*legal* if it is restricted, every p- or q-subword of U is finite, and for any presentation $U \equiv AXBX^{-1}C$ the subword X is finite.

The proof of the following useful lemma is straightforward.

Lemma 8.5. *The class of legal words over the alphabet P and the class of (P,Q)-legal words over the alphabet $P \cup Q$ both are closed under the transformations* (i) *and* (ii) *from* Theorem 8.2.

Lemma 8.6. 1) *A reduced legal word U over the alphabet P determines a nonzero element in $H_1(Z)$ if and only if it is infinite.*
2) *A reduced (P,Q)-legal word U over the alphabet $P \cup Q$ determines a nonzero element in $H_1(Y)$ if and only if it is infinite.*

Proof. We will prove only the more difficult statement 2). By Theorem 8.2, U determines the zero element in $H_1(Y)$ if and only if it can be carried to the empty word by applying a finite number of transformations (i), (ii). By Lemma 8.5, each transformation leaves words in the class of (P,Q)-legal words. Therefore at each step at most two finite subwords can be deleted. Thus, if U is infinite, we cannot get the empty word in a finite number of steps. If U is finite, we can delete all its letters with the help of several transformations (i).
The proof of the statement 1) is analogous and uses Theorem 8.3. □

Recall, that $\aleph_0$ denotes the cardinality of natural numbers, and so $2^{\aleph_0}$ is the cardinality of continuum.

Corollary 8.7. *For the Hawaiian Earring $Z = H(p_1, p_2, \dots)$ and for Griffiths' space $Y = G(p_1, q_1, p_2, q_2, \dots)$, there are $2^{\aleph_0}$ functions $k : \mathbb{N} \to \mathbb{N}$, such that*

(a) *all the elements $[p_{k(1)}, p_{k(2)}][p_{k(3)}, p_{k(4)}] \dots$ are different in $H_1(Z)$;*

(b) *all the elements $[p_{k(1)}, q_{k(1)}][p_{k(2)}, q_{k(2)}] \dots$ are different in $H_1(Y)$.*

Proof. Consider the set of all strictly monotone functions $k : \mathbb{N} \to \mathbb{N}$. It has cardinality $2^{\aleph_0}$. Two such functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ will be called *equivalent*, if there exist natural n, m such that for every $l \geqslant 0$ holds $f(n + l) = g(m + l)$. Roughly speaking the tails of equivalent functions are the same. Since each equivalence class of these functions is countable, the cardinality of the set of equivalence classes is $2^{\aleph_0}$. We choose a system $\mathcal{K}$ of representatives of these classes to define $2^{\aleph_0}$ infinite products according to (a) and (b), respectively.

It is easy to see that for any two such products U, V, the concatenation UV^{-1} is reduced legal in Case (a) or reduced (P, Q)-legal in Case (b). The proof completes by applying Lemma 8.6. □

9 Divisible subgroups of $H_1(Y)$ and $H_1(Z)$

Definition 9.1. An element g of a group G is called $\mathbb{N}$-*divisible* if for any $n \in \mathbb{N}$, there exists an element $x \in G$ such that $g = x^n$. An abelian group G is called *divisible* if every element of $g \in G$ is $\mathbb{N}$-divisible.

Theorem 9.2. *The homology groups $H_1(Z)$ and $H_1(Y)$ of the Hawaiian Earring $Z = H(p_1, p_2, \dots)$ and of Griffiths' space $Y = G(p_1, q_1, p_2, q_2, \dots)$ contain subgroups isomorphic to $\bigoplus\limits_{2^{\aleph_0}} \mathbb{Q}$.*

Proof. a) We construct a nontrivial $\mathbb{N}$-divisible element in $H_1(Y)$ using limits of word sequences. Let $a_1, a_2, \dots$ be an infinite sequence of countable words. First we define a countable word w as a limit word[1] resulting from the following recursive process: set $w_1 = a_1$ and construct w_{i+1} from w_i by insertion the a_{i+1}^{i+1} after each occurrence of a_i:

$$\begin{array}{llllll} w_1 = & a_1 & & & & \\ w_2 = & a_1 & a_2 & & a_2 & \\ w_3 = & a_1 & a_2 & a_3a_3a_3 & a_2 & a_3a_3a_3 \\ \vdots & & & & & \end{array}$$

Specifically for $a_i = [p_i, q_i]$, $i = 1, 2, \dots$, we denote the limit word w by W. Clearly, W is restricted and reduced, and so W is a nontrivial element of $G = \mathcal{W}(P \cup Q)$. By Theorem 3.4, there is a natural homomorphism $G \to \pi_1(Y)$. We compose it with the homomorphism $\pi_1(Y) \to H_1(Y)$ and denote the image of W in $H_1(Y)$ by $\overline{W}$.

Note, that if we delete in W a finite number of a_i's, the resulting word will not change modulo $[G, G]$. Thus, by deleting all occurrences of a_1 in W, we see that W is a square modulo $[G, G]$. Similarly, for any natural n, if we delete all occurrences of $a_1, a_2, \dots, a_n$ in W, we see that W is the $(n + 1)$-th power in G modulo $[G, G]$. Therefore $\overline{W}$ is $\mathbb{N}$-divisible in $H_1(Y)$. Moreover, $\overline{W}$ is nontrivial in $H_1(Y)$ by Lemma 8.6.2).

b) Now we construct $2^{\aleph_0}$ $\mathbb{N}$-divisible elements in $H_1(Y)$. For that we will use functions $k : \mathbb{N} \to \mathbb{N}$ from the class $\mathcal{K}$, which was defined in the proof of Corollary 8.7. For each function $k \in \mathcal{K}$, we

[1] This limit word is a modification of those described in [3, pages 235-236] and [23, page 14].

define the limit word W_k just by putting $a_i = [p_{k(i)}, q_{k(i)}]$ in the above construction. And as above, one can show, that these words determine $\mathbb{N}$-divisible elements $\overline{W_k}$ in $H_1(Y)$.

It remains to prove, that the words W_k, $k \in \mathcal{K}$, determine different elements in $H_1(Y)$. This follows from Lemma 8.6.2), if we prove that for different $k, l \in \mathcal{K}$ the reduced form $R_{k,l}$ of the word $W_k^{-1} W_l$ is infinite and (P, Q)-legal. The word $R_{k,l}$ is infinite, since

$$W_k^{-1} W_l \equiv (\dots [p_{k(3)}, q_{k(3)}]^{-1} [p_{k(2)}, q_{k(2)}]^{-1} [p_{k(1)}, q_{k(1)}]^{-1} [p_{l(1)}, q_{l(1)}][p_{l(2)}, q_{l(2)}][p_{l(3)}, q_{l(3)}] \dots),$$

and so the reduction can meet only a finite number of the middle commutators.

The word $R_{k,l}$ is (P, Q)-legal, since

(1) p- and q-subwords of R_{kl} are finite;

(2) if $R_{k,l} \equiv AXBX^{-1}C$, then X is finite.

The first assertion is clear, since p- and q-subwords of W_k and W_l contain only one letter. For the second one we need the following claim, which can be proved straightforward.

Claim. Every infinite subword of the limit word w contains an infinite subword of the form $a_n a_{n+1} \dots$, which is ordered as $\mathbb{N}$. Moreover, each subword of w, which is ordered as $\mathbb{N}$ has such form.

If (2) were not true, then W_k and W_l would have a common infinite subword. Then, by Claim, we would have $[p_{k(n)}, q_{k(n)}][p_{k(n+1)}, q_{k(n+1)}] \cdots \equiv [p_{l(m)}, q_{l(m)}][p_{l(m+1)}, q_{l(m+1)}] \dots$ for some natural n, m. But this is impossible since k, l are different in $\mathcal{K}$.

c) Note that the group $H_1(Y)$ is torsion-free by [7, Theorem 1.2] and [8, Corollary 2.2]. Therefore its minimal divisible subgroup containing $\{\overline{W_k} \mid k \in \mathcal{K}\}$ is isomorphic to the direct sum $\oplus_{i \in I} \mathbb{Q}$ for some index set I. Since the set $\{\overline{W_k} \mid k \in \mathcal{K}\}$ has cardinality $2^{\aleph_0}$, the set I has cardinality $2^{\aleph_0}$ too.

d) The statement of Theorem 9.2 concerning $H_1(Z)$ can be proven analogously, if in a) we take $a_i = [p_{2i-1}, p_{2i}]$ and in b) we take $a_i = [p_{k(2i-1)}, p_{k(2i)}]$. $\square$

10 Appendix

We compare our and Eda's [6] definitions of reduced words and of reduced forms of countable words. Note that Eda works in a more general context: he considers words of arbitrary cardinality, furthermore these words consist of elements of a given set of groups $G_i (i \in I)$ (and not of letters of an alphabet as in the present paper). As by us, his words contain only finitely many elements of each group. Adapting his definitions to our notations from Section 2, we get

Definition 10.1 (compare with [6, Definition 1.3]). A countable restricted word $U = (x_i)_{i \in I}$ is called *reduced*, if whenever $U \equiv ABC$ for a nonempty subword B, there exists a finite subset $F \subseteq I$ such that $[B_F] \neq 1$.

Definition 10.2. (compare with [6, Theorem 1.4]). Let $W = (x_i)_{i \in I}$ be a countable restricted word. A word U is called the *reduced form of* W if it is reduced and for any finite subset $F \subseteq I$ holds $[W_F] = [U_F]$.

In [6, Theorem 1.4], Eda proves in particular, that any restricted countable word has a unique reduced form in the sense of Definition 10.2.

Finally we note, that Definition 10.1 is equivalent to our Definition 2.4 due to Lemma 2.8, and Definition 10.2 is equivalent to our Definition 2.6 due to the proof of Theorem 2.9.

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