A Guide to the Calculation of Surgery Obstruction Groups

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We describe the main steps in the calculation of surgery obstruction groups for finite groups. Some new results are given and extensive tables are included in the appendix.

The surgery exact sequence of C. T. C. Wall [**61**] describes a method for classifying manifolds of dimension ≥ 5 within a given (simple) homotopy type, in terms of normal bundle information and a 4–periodic sequence of obstruction groups, depending only on the fundamental group and the orientation character. These obstruction groups $L^s_n(\mathbf{Z}G, w)$ are defined by considering stable isomorphism classes of quadratic forms on finitely generated free modules over $\mathbf{Z}G$ (n even), together with their unitary automorphisms (n odd).

Carrying out the surgery program in any particular case requires a calculation of the surgery obstruction groups, the normal invariants, and the maps in the surgery exact sequence. For fundamental group $G = 1$, the surgery groups were calculated by Kervaire–Milnor as part of their study of homotopy spheres:

$$L^s_n(\mathbf{Z}) = \ 8\mathbf{Z},\ 0,\ \mathbf{Z}/2,\ 0 \qquad \text{for } n = 0, 1, 2, 3 \pmod 4\ ,$$

where the non–zero groups are detected by the signature or Arf invariant, and the notation $8\mathbf{Z}$ means that the signature can take on any value $\equiv 0 \pmod 8$. The Hirzebruch signature theorem can be used to understand the signature invariant, and a complete analysis of the normal data was carried out by Milgram [**41**], Madsen–Milgram [**42**] and Morgan–Sullivan [**45**].

The theory of non–simply connected surgery has been used to investigate three important problems in topology:

(i) *the spherical space form problem, or the classification of free finite group actions on spheres*

(ii) *the Borel and Novikov conjectures, or the study of closed aspherical manifolds and assembly maps*

(iii) *transformation groups, or the study of Lie group actions on manifolds.*

In the first problem, surgery is applied to manifolds with finite fundamental group and the surgery obstruction groups can be investigated by methods closely related to

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number theory and the representation theory of finite groups. In the second problem, the fundamental groups are infinite and torsion–free, and the methods available for studying the surgery obstruction groups are largely geometrical. The case of the n–torus was particularly important for its applications to the theory of topological manifolds. The third problem includes both finite group actions and actions by connected Lie groups. The presence of fixed point sets introduces many interesting new features.

In this paper we consider only $L_*(\mathbf{Z}G)$ for finite groups G. The Novikov conjectures and other topics connected with infinite fundamental groups are outside the scope of this article.

Before giving some notation, definitions and a detailed statement of results, it may be useful to list some general properties of the surgery obstruction groups for finite groups.

(1) *The groups $L_*(\mathbf{Z}G)$ are finitely generated abelian groups, the odd–dimensional groups $L_{2k+1}(\mathbf{Z}G)$ are finite, and in every dimension the torsion subgroup of $L_*(\mathbf{Z}G)$ is 2–primary.*

There is a generalization of the ordinary simply–connected signature, called the multi–signature [**61**, 13A], [**38**].

(2) *The multi–signature is a homomorphism $\sigma_G\colon L_{2k}(\mathbf{Z}G) \to R_{\mathbf{C}}^{(-)^k}(G)$ where $R_{\mathbf{C}}(G)$ denotes the ring of complex characters of G. The multi–signature has finite 2–groups for its kernel and cokernel.*

Complex conjugation acts as an involution on $R_{\mathbf{C}}(G)$, decomposing it as a sum of $\mathbf{Z}$'s from the real–valued (type I) characters, and a sum of free $\mathbf{Z}[\mathbf{Z}/2]$ modules generated by irreducible type II characters $\chi \neq \bar{\chi}$. The $(-1)^k$–eigenspaces of the complex conjugation action are denoted $R_{\mathbf{C}}^{(-)^k}(G)$.

The theory of Dress induction [**22, 23**] greatly simplifies the calculation of L–groups. A group G is called *p–hyperelementary* if $G = C \rtimes P$ where P is a p–Sylow subgroup and C is a cyclic group of order prime to p. Then G is determined by C, P and the structure homomorphism $t\colon P \to Aut(C)$. Further, G is *p–elementary* if it is p–hyperelementary and t is trivial (equivalently $G = C \times P$).

(3) *$L_*(\mathbf{Z}G)$ can be calculated from knowledge of the L–groups of hyperelementary subgroups of G, together with the maps induced by subgroup inclusions.*

Moreover, one can calculate $L_*(\mathbf{Z}G) \otimes \mathbf{Z}_{(2)}$, $R_{\mathbf{C}}(G) \otimes \mathbf{Z}_{(2)}$ and $\sigma_G \otimes 1$ from the 2–hyperelementary subgroups and the maps between them. Since (1) and (2) imply that

$$\begin{array}{ccc} L_*(\mathbf{Z}G) & \xrightarrow{\sigma_G} & R_{\mathbf{C}}(G) \\ \downarrow & & \downarrow \\ L_*(\mathbf{Z}G) \otimes \mathbf{Z}_{(2)} & \xrightarrow{\sigma_G \otimes 1} & R_{\mathbf{C}}(G) \otimes \mathbf{Z}_{(2)} \end{array}$$

is a pull–back, Dress's work computes $L_*(\mathbf{Z}G)$ in terms of representation theory and the L–theory of 2–hyperelementary groups. For this reason, most of the calculational work has been devoted to the 2–hyperelementary case.

These general properties are fine until one needs more precise information for computing surgery obstructions. An early result of Bak and Wall (worked out as an example in Theorem 10.1) is that for G of *odd order*

$$L_n^s(\mathbf{Z}G) = \ \Sigma \oplus 8\mathbf{Z},\ 0,\ \Sigma \oplus \mathbf{Z}/2,\ 0 \qquad \text{for } n = 0, 1, 2, 3 \pmod 4\ .$$

The terms $\Sigma = \oplus\, 4(\chi \pm \bar{\chi})$ comes from the multisignatures at type II characters, and the term $\mathbf{Z}$ is the summand of $R_{\mathbf{C}}(G)$ generated by the trivial character. The term $\mathbf{Z}/2$ is detected by the ordinary Arf invariant.

Another nice case is $G = C \times P$, where C is a cyclic 2–group and P has odd order (this includes arbitrary cyclic groups as well as p–hyperelementary groups G for p odd). Assuming C is non–trivial, we have:

$$L_n^s(\mathbf{Z}G) = \ \Sigma \oplus 8\mathbf{Z} \oplus 8\mathbf{Z},\ 0,\ \Sigma \oplus \mathbf{Z}/2,\ \mathbf{Z}/2 \qquad \text{for } n = 0,1,2,3 \,(\bmod 4)\ .$$

The signature group again has two sources, the term $\Sigma = \oplus\, 4(\chi \pm \bar{\chi})$ from the type II characters and the two $\mathbf{Z}$'s coming from the type I characters (just the trivial character and the linear character which sends a generator to -1). The $\mathbf{Z}/2$ in dimension 2 is the ordinary Arf invariant and the $\mathbf{Z}/2$ in dimension 3 is a "codimension one" Arf invariant. The special case $G = \mathbf{Z}/2^r$ is worked out in Example 11.1.

Many geometric results have been obtained just from the vanishing of the odd–dimensional L–groups of odd order groups, but unfortunately $L_n^s(\mathbf{Z}G)$ is usually not zero, and the torsion subgroup can be complicated (for example, even when G is a group of odd order times an abelian 2–group).

Nor does it help to relax the Whitehead torsion requirements, and allow surgery up to homotopy equivalence. For example, the group $L_{2k}^h(\mathbf{Z}[\mathbf{Z}/2^r])$ has torsion subgroup $\big([2(2^{r-2}+2)/3] - [r/2] - \epsilon\big)\mathbf{Z}/2$, where $\epsilon = 1$ if k is even and 0 if k is odd [**12**, Thm.A]. The notation $[x]$ means the greatest integer in x. The source of this torsion is $D(\mathbf{Z}G) \subseteq \tilde{K}_0(\mathbf{Z}G)$, a part of the projective class group that is often amenable to calculation [**47**].

The torsion subgroup of $L_n(\mathbf{Z}G)$ can also involve the ideal class groups of the algebraic number fields in the centre of the rational group algebra $\mathbf{Q}G$, and the computation of ideal class groups is a well–known and difficult problem in number theory. Another major complication is that computing the surgery obstruction groups often requires information about the Whitehead groups $Wh(\mathbf{Z}G)$, the algebraic home for the theory of Whitehead torsion.

Here the problem is that the torsion subgroup $SK_1(\mathbf{Z}G)$ of $Wh(\mathbf{Z}G)$ is highly non–trivial [**46**]. In particular, both the first optimistic claims for the Whitehead groups of abelian groups (tentatively quoted by Milnor in [**44**]) and Wall's conjecture [**70**, p.64,5.1.3] about the Tate cohomology of Whitehead groups, turned out to be incorrect.

In spite of these complications, the L–groups can be effectively computed in many cases of interest. The approach presented here (following the procedure established by Wall in [**63**]–[**70**]) will be to try and reduce the compution of $L_*(\mathbf{Z}G)$ to specific and independent questions in number theory and representation theory. From the statement of results in Section 2, we hope that the reader can get an overview of present knowledge, and useful references for further investigation. In the rest of the paper, we describe the main steps in the calculation and work out some relatively easy examples.

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1. L–groups, decorations and geometric anti–structures

We begin with some algebraic definitions. An *antistructure* is a triple (R, α, u), where R is a ring with unity, u is a unit in R and $\alpha: R \to R$ is an anti–automorphism such that $\alpha(u) = u^{-1}$ and $\alpha^2(r) = uru^{-1}$ for all $r \in R$. Such rings have L–groups, denoted $L_n(R, \alpha, u)$, and in [**63**]–[**70**] Wall developed effective techniques for computing them, especially for the case when $R = \mathbf{Z}G$, and G a finite group. The main idea is to compare quadratic and hermitian forms over $\mathbf{Z}G$ to those over local and global fields using the "arithmetic" pull–back square

$$\begin{array}{ccc} \mathbf{Z}G & \longrightarrow & \mathbf{Q}G \\ \downarrow & & \downarrow \\ \widehat{\mathbf{Z}}G & \longrightarrow & \widehat{\mathbf{Q}}G \end{array}$$

of rings with antistructure to obtain Mayer–Vietoris sequences in L–theory.

There is a class of antistructures which suffice for applications of L–groups to the topology of manifolds, and which have other good properties. We say (α, u) is a *geometric antistructure* on a group ring $\mathbf{Z}G$ provide that α is given by $\alpha(g) = w(g)\theta(g^{-1})$, where θ is an automorphism of G, $w: G \to \{\pm 1\}$ is a homomorphism and $u = \pm b$ for some $b \in G$ [**31**, p.110]. A geometric antistructure is *standard* if θ is trivial and $b = e$ and *oriented* if w is trivial. Clearly α determines θ, w and b uniquely. Conversely, given any automorphism θ, any homomorphism $w: G \to \{\pm 1\}$ and any $b \in G$ the pairs $(\alpha, \pm b)$ are antistructures provided $\theta^2(g) = bgb^{-1}$ for all $g \in G$, $w \circ \theta = w$, $\theta(b) = b$ and $w(b) = 1$. In particular, a geometric antistructure induces an antistructure on the group ring AG for any ring with unity A so they fit well with the arithmetic square. In general $L_n(R, \alpha, u) = L_{n+2}(R, \alpha, -u)$, so we will usually only consider the case $u = b$. Another useful observation is that geometric antistructures induce involutions on $Wh(\mathbf{Z}G)$.

Traditionally the L–groups with standard antistructure are denoted $L_*(\mathbf{Z}G)$ if the antistructure is oriented and $L_*(\mathbf{Z}G, w)$ if it is not. One of the main theorems of surgery [**61**] states that these algebraically defined groups are naturally isomorphic to the geometrically defined surgery obstruction groups. Wall discovered the more general geometric antistructures while studying codimension one submanifolds (they give an algebraic description of the Browder-Livesay groups LN, see [**61**, 12C]).

The surgery obstruction groups come with K–theory *decorations* depending on the goal of the surgery process. For surgery up to homotopy equivalence (resp. simple homotopy equivalence) on compact manifolds, the relevant L–groups are $L^h_*(\mathbf{Z}G)$ (resp. $L^s_*(\mathbf{Z}G)$). For surgery on non–compact manifolds up to proper homotopy equivalence, the appropriate groups are $L^p_*(\mathbf{Z}G)$. Cappell [**11**] introduced "intermediate" L–groups for any involution invariant subgroup of $\tilde{K}_0(\mathbf{Z}G)$ or $Wh(\mathbf{Z}G)$ for use in his work on Mayer–Vietoris sequences for amalgamated free products and HNN extensions. Each of these has the form $L^{\tilde{X}}_n(\mathbf{Z}G)$, denoting an algebraic L–group ([**52**]) with decorations in an involution–invariant subgroup $\tilde{X} \subseteq \tilde{K}_1(\mathbf{Z}G)$ or $\tilde{X} \subseteq \tilde{K}_0(\mathbf{Z}G)$. In the first case, $\tilde{X}$ is always the inverse image of an involution–invariant subgroup in $Wh(\mathbf{Z}G)$, so we often refer instead to the decoration subgroup in $Wh(\mathbf{Z}G)$. For general antistructures (α, u) $L^{\tilde{X}}_*(\mathbf{Z}G, \alpha, u)$ is not defined unless $u \in \tilde{X}$, so again geometric antistructures provide the right setting.

Intermediate L–groups appear in the arithmetic Mayer–Vietoris sequence as well: in particular, $L'_n(\mathbf{Z}G)$ based on

$$\tilde{X} = SK_1(\mathbf{Z}G) \subseteq Wh(\mathbf{Z}G),$$

where $SK_1(\mathbf{Z}G) = \ker\big(K_1(\mathbf{Z}G) \to K_1(\mathbf{Q}G)\big)$, is especially important. It turns out that these L'–groups are more accessible to computation than either L^s or L^h, and so they have a central role in this subject.

The L–groups are related by many interlocking exact sequences [**56**], involving change of K–theory and change of rings. Such sequences often have both an algebraic and an geometric interpretation, making them useful for topological applications.

REMARK: The L–groups mentioned so far only involve the K groups K_0 and K_1, and it is natural to wonder about decorations in other K_i. In fact, there are geometrically interesting L–theories for both higher and lower K_i, and these are related to the ones studied here via change of K–theory sequences. Since there has not been a great deal of calculational work done on them, they are omitted from this survey.

The exact sequences describing the change of K–theory decoration are often called "Ranicki–Rothenberg" sequences [**57**], [**52**]. Some important examples are

$$\dots \to L^h_{n+1}(\mathbf{Z}G) \to H^{n+1}(Wh(\mathbf{Z}G)) \to L^s_n(\mathbf{Z}G) \to L^h_n(\mathbf{Z}G) \to H^n(Wh(\mathbf{Z}G)) \to \dots$$

and

$$\dots \to L^p_{n+1}(\mathbf{Z}G) \to H^n(\tilde{K}_0(\mathbf{Z}G)) \to L^h_n(\mathbf{Z}G) \to L^p_n(\mathbf{Z}G) \to H^{n-1}(\tilde{K}_0(\mathbf{Z}G)) \to \dots$$

although the step between L^s and L^h can also be usefully divided into $L^s \to L'$, with relative group $H^*(SK_1(\mathbf{Z}G))$, and $L' \to L^h$, with relative group $H^*(Wh'(\mathbf{Z}G))$. Here $Wh'(\mathbf{Z}G) = Wh(\mathbf{Z}G)/SK_1(\mathbf{Z}G)$, and we use the convention that $H^*(X)$ denotes the Tate cohomology $H^*(\mathbf{Z}/2; X)$ for any $\mathbf{Z}/2$–module X. There are versions of these sequences for any geometric antistructure.

The exact sequences involving change of rings are particularly important for computing L–groups. The most important example is

$$\dots \to L'_{n+1}(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G) \to L'_n(\mathbf{Z}G) \to L'_n(\widehat{\mathbf{Z}}_2 G) \to L'_n(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G) \to \dots$$

This sequence remains exact for any geometric antistructure.

2. Statement of Results

What does it mean to *compute* L–groups ? Given a finite group G and a geometric antistructure (α, u), the rational group algebra $\mathbf{Q}G$ becomes an algebra with involution. Under the Wedderburn decomposition, $\mathbf{Q}G$ splits canonically into simple algebras with involution and it is reasonable to assume that the classical invariants for such algebras (type, reduced norms, Schur indices, ideal class groups of centre fields, etc.) can be worked out for the given group G.

GOAL: *Find an algorithm to compute $L_*(\mathbf{Z}G, \alpha, u)$ for geometric antistructures, in terms of the character theory of G and the classical invariants of $\mathbf{Q}G$.*

Here is a brief summary of the state of progress towards this goal. Properties (1) and (2) in the introduction hold for L–groups of finite groups with geometric antistructures and any involution–invariant torsion decoration in $Wh(\mathbf{Z}G)$ or $\tilde{K}_0(\mathbf{Z}G)$ **[23]**, **[69]**, **[70]**, although the description of the multi–signature becomes more complicated. Dress's results (3) certainly hold for the standard geometric antistructures (oriented or not) with decorations p, s, $'$, h and many others, but the general case has not been worked out. In the case of the standard antistructures, we will apply Dress induction to obtain calculations for odd order groups in §10, and p–hyperelementary groups, p odd, in §12.

One obvious difficulty in extending Dress induction is that a given geometric antistructure on G may not restrict to a geometric antistructure on enough subgroups to simplify the calculation. In any case, not much work has been done on the maps induced by subgroup inclusion (even for the standard oriented antistructure), so we will consider only 2–hyperelementary groups from now on.

(4) *For 2–hyperelementary groups G and any geometric antistructure (α, u), the groups $L^p_*(\mathbf{Z}G, \alpha, u)$ can be computed in terms of character theory of G and the number theory associated to $\mathbf{Q}G$. The torsion subgroup has exponent 2 for the standard oriented antistructure, and exponent 4 in general.* **[28]**

Since the goal has been achieved for the L^p–groups, we turn to the groups $L'_*(\mathbf{Z}G, \alpha, u)$ with $Wh(\mathbf{Z}G)$ decoration lying in the subgroup $SK_1(\mathbf{Z}G)$. The approach is to study $L'_*(\mathbf{Z}G, \alpha, u)$ using the exact sequence comparing it with $L'_*(\widehat{\mathbf{Z}}_2G, \alpha, u)$. The relative term is under control:

(5) *For 2–hyperelementary groups G and any geometric antistructure, the relative groups $L'_*(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2G, \alpha, u)$ can be computed in terms of character theory of G and number theory associated to $\mathbf{Q}G$. The torsion subgroup has exponent* 2 (see **[70]** or Tables 14.12–14.15, and Theorem 7.1 for the exponent of the torsion subgroup).

The remaining obstacles are the groups $L'_*(\widehat{\mathbf{Z}}_2G, \alpha, u)$, and the maps

$$\psi_n\colon L'_*(\widehat{\mathbf{Z}}_2G, \alpha, u) \to L'_*(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2G, \alpha, u) \ .$$

The groups $L'_*(\widehat{\mathbf{Z}}_2G, \alpha, u)$ can be studied by comparing them to $L^h_*(\widehat{\mathbf{Z}}_2G, \alpha, u)$ using the change of K–theory sequence

$$\ldots \to H^{n+1}(Wh'(\widehat{\mathbf{Z}}_2G)) \to L'_n(\widehat{\mathbf{Z}}_2G, \alpha, u) \to L^h_n(\widehat{\mathbf{Z}}_2G, \alpha, u) \to H^n(Wh'(\widehat{\mathbf{Z}}_2G)) \to \ldots$$

Every third term is easy to compute:

(6) *The groups $L^h_*(\widehat{\mathbf{Z}}_2 G, \alpha, u)$ for geometric antistructures are determined by the centre of $\mathbf{F}_2 G$, and the kernel of the discriminant $L^h_n(\widehat{\mathbf{Z}}_2 G, \alpha, u) \to H^n(Wh'(\widehat{\mathbf{Z}}_2 G))$ is computable from the characters and types* ([**66**] and [**27**, 1.16], see also Remark 8.5).

For the Tate cohomology terms we have:

(7) *$Wh'(\widehat{\mathbf{Z}}_2 G)$ is computable by restriction to the 2–elementary subgroups of G, and there is an algorithm to calculate $Wh'(\widehat{\mathbf{Z}}_2 G)$ for 2–elementary groups* ([**46**, Thm. 6.7, 12.3]). *If w is trivial or if G is a 2–group, the involution induced by the geometric antistructure has been computed fairly explicitly* (see [**46**, p.163] and [**47**, p.61]).

It follows that $L'_*(\widehat{\mathbf{Z}}_2 G, \alpha, u)$ is algorithmically computable up to extensions in the oriented case or in the case that G is a finite 2–group.

Most difficult of all is to describe the ψ_* maps. Some examples can be worked out, especially for the standard antistructure.

Example: *If G is an abelian group then $L'_*(\mathbf{Z}G, w)$ is computable in terms of the characters of G* (see [**70**], and Example 11.1 for cyclic 2–groups done in detail).

Example: *If $G = G_1 \times G_2$ is a direct product where G_1 has odd order, then $L'_*(\mathbf{Z}G, w)$ is computable in terms of $L'_*(\mathbf{Z}G_2, w)$ and the character theory of $\mathbf{Q}G$* ([**37**] and Proposition 12.1).

Example: *Computations (modulo some extension problems) are available for $L'_*(\mathbf{Z}G, w)$ in certain families of 2–hyperelementary groups G, including*

(i) *groups of 2–power order,* (see [**70**, §5] for partial results and Example 9.2 for a reduction of the maps ψ_* to K–theory),

(ii) *groups with periodic cohomology,* [**40**], [**37**]

(iii) *groups $G = C \rtimes P$ where $\ker(P \to Aut(C))$ is abelian and P is a 2–group* (see [**70**], and Proposition 13.4 for G dihedral done as an example).

Finally, what can one say about the L–groups of primary geometric interest ? To study $L^s_*(\mathbf{Z}G)$ or $L^h_*(\mathbf{Z}G)$ via $L'_*(\mathbf{Z}G)$ we need the groups $H^n(SK_1(\mathbf{Z}G))$ or $H^n(Wh'(\mathbf{Z}G))$. Observe that we only need $Wh(\mathbf{Z}G) \otimes \mathbf{Z}_{(2)}$ for computing Tate cohomology, and this can result in significant simplification.

(8) *Extensive computations are available for the groups $SK_1(\mathbf{Z}G)$, but it is not easy to determine the action of the involution induced by the antistructure* (see [**46**] as a general reference, and [**43**] for a nice application to L^s–groups).

There is a short exact sequence

$$0 \to Cl_1(\mathbf{Z}G) \otimes \mathbf{Z}_{(2)} \to SK_1(\mathbf{Z}G) \otimes \mathbf{Z}_{(2)} \to SK_1(\widehat{\mathbf{Z}}_2 G) \to 0$$

and Bak [**3**] (or [**46**, 5.12]) shows the standard oriented involution is trivial on $Cl_1(\mathbf{Z}G) \otimes \mathbf{Z}_{(2)}$. Oliver [**46**, 8.6] shows that the standard oriented involution on $SK_1(\widehat{\mathbf{Z}}_2 G)$ is multiplication by -1, at least for 2–groups G.

(9) *The groups $Wh'(\mathbf{Z}G)$ are free abelian with rank depending on the characters of G. For the standard oriented antistructure, the induced involution on $Wh'(\mathbf{Z}G)$ is the identity, and $H^1(Wh'(\mathbf{Z}G)) = 0$* [**67**] (see [**47**, 4.8] for the answer when $w \not\equiv 1$).

The study of $L^h_*(\mathbf{Z}G)$ via $L^p_*(\mathbf{Z}G)$ looks promising, but we need knowledge of $H^n(\tilde{K}_0(\mathbf{Z}G))$. In general this is not easy to compute, however:

Example: *The groups $L^h_*(\mathbf{Z}G, w)$ can be computed (up to extensions) in terms of the character theory of G, for G a finite 2–group* (See [**29**], [**47**], [**12**] for pieces of the solution, but there is no complete account in the literature).

REMARK: In order to keep this paper reasonably short, we have omitted any discussion of hermitian K–theory and form parameters. This approach is fully developed in [**9**], [**10**], and [**1**]–[**6**].

3. Round decorations

There are two detours to be made along the way towards systematic computations of the surgery obstruction groups. The first is to use the "round" algebraic L–theory $L^X_n(R, \alpha, u)$ for a ring R with antistructure, based on involution–invariant subgroups X of $K_i(R)$ (see [**30**, §2]), instead of the geometrically useful groups. These round groups have several algebraic advantages. They respect products, and are invariant under quadratic Morita equivalence of rings [**31**]. These are related to the usual L–groups by exact sequences which can be analyzed and largely determined in the cases considered here. The definitions also make sense for higher and lower L–theory, but all of our actual computations are for the L–groups based on subgroups of K_0 or K_1. To our knowledge, no calculational work has been done on higher L–theory for finite groups and very little has been done for lower L–theory.

A particular example of quadratic Morita equivalence is the notion of *scaling*. If (α, u) is an antistructure on R and $v \in R$ is a unit, (α_v, u_v) is also an antistructure where $\alpha_v(x) = v\alpha(x)v^{-1}$ and $u_v = v\alpha(v^{-1})u$: furthermore $L^X_n(R, \alpha, u) \cong L^X_n(R, \alpha_v, u_v)$ [**31**, p.74] for any subgroup $X \subset K_i(R)$ invariant under the involution induced by α.

The second step is to choose the "right" K–theory decoration. For any ring R, let $O_i(R) = 0 \subseteq K_i(R)$ and let $X_i(R) = \ker\big(K_i(R) \to K_i(R \otimes \mathbf{Q})\big)$. Note that $L^{O_i}_n(R) = L^{K_{i+1}}_n(R)$, so there is a Ranicki–Rothenberg sequence relating $L^{O_i} \to L^{O_{i-1}}$ with relative term $H^*(K_i(R))$. For group rings AG and $i = 1$, we define

$$Y_1(AG) = X_1(AG) \oplus \{\pm G^{ab}\}$$

where G^{ab} denotes the set of elements in the abelianization of G. If $i \le 0$ then we define $Y_i(AG) = X_i(AG)$. For higher L–theory ($i \ge 2$) it seems that the right definition of $Y_i(AG)$ would be the image of the assembly map in K–theory.

Fortunately, the passage between round and geometric L–theory is very uniform so the round results suffice. Given a geometric antistructure and any invariant subgroup $\widetilde{U} \subset \widetilde{K}_i(\mathbf{Z}G)$, let $U \subset K_i(\mathbf{Z}G)$ denote the inverse image. There is a natural map $\tau_U \colon L^U_*(\mathbf{Z}G, \alpha, u) \to L^{\widetilde{U}}_*(\mathbf{Z}G, \alpha, u)$ from the round to the geometric theory. For subgroups of $\widetilde{K}_0$, τ is an isomorphism, and for $U = Y_1$, the following sequence is exact (see [**30**, 3.2]).

$$0 \to L^{Y_1}_{2k}(\mathbf{Z}G, \alpha, u) \xrightarrow{\tau_{Y_1}} L'_{2k}(\mathbf{Z}G, \alpha, u) \to \mathbf{Z}/2 \to L^{Y_1}_{2k-1}(\mathbf{Z}G, \alpha, u) \xrightarrow{\tau_{Y_1}} L'_{2k-1}(\mathbf{Z}G, \alpha, u) \to 0.$$

The map into $\mathbf{Z}/2$ is given by the rank (mod 2) of the underlying free module.

THEOREM 3.1: *For any geometric antistructure, $L^{Y_1}_{2k}(\mathbf{Z}G,\alpha,u) \cong L'_{2k}(\mathbf{Z}G,\alpha,u)$ and $L'_{2k-1}(\mathbf{Z}G,\alpha,u)$ is obtained from $L^{Y_1}_{2k-1}(\mathbf{Z}G,\alpha,u)$ by dividing out a single $\mathbf{Z}/2$ summand.*

The intermediate projective L–groups we can compute are the $L^{Y_0}_n(\mathbf{Z}G) = L^{X_0}_n(\mathbf{Z}G)$ (denoted $L^{X_0}_n(\mathbf{Z}G)$ in [**25**, §3]) based on the subgroup $X_0(\mathbf{Z}G)$. These are related to the usual projective L–groups, $L^p = L^{K_0}$, by the exact sequence [**30**, 3.2], [**25**, 3.8])

$$0 \to L^{X_0}_{2k}(\mathbf{Z}G) \to L^p_{2k}(\mathbf{Z}G) \to \mathbf{Z}/2 \to L^{X_0}_{2k-1}(\mathbf{Z}G) \to L^p_{2k-1}(\mathbf{Z}G) \to 0\ .$$

The map into $\mathbf{Z}/2$ is given by the rank (mod 2) of the underlying projective module.

THEOREM 3.2: *For any geometric antistructure, $L^{X_0}_{2k}(\mathbf{Z}G,\alpha,u) \cong L^p_{2k}(\mathbf{Z}G,\alpha,u)$ and $L^p_{2k-1}(\mathbf{Z}G,\alpha,u)$ is obtained from $L^{X_0}_{2k-1}(\mathbf{Z}G,\alpha,u)$ by dividing out a single $\mathbf{Z}/2$ summand.*

We now follow the procedure outlined in the first two sections to compute $L^{Y_i}_n$, $i = 0$, 1 and then use Theorems 3.2 and 3.1 to compute L^p_* or L'_*.

4. The main exact sequence

Exact sequences for computing L–groups come from the arithmetic square [**69**], [**56**, §6], where the basic form is

$$\ldots \to L^{X_i}_n(R) \to L^{X_i}_n(\hat{R}) \oplus L^{O_i}_n(S) \to L^{O_i}_n(\hat{S}) \to L^{X_i}_{n-1}(R) \to \ldots$$

where R is a ring with antistructure, $S = R \otimes \mathbf{Q}$, $\hat{R} = R \otimes \hat{\mathbf{Z}}$ and $\hat{S} = \hat{R} \otimes \mathbf{Q}$. This assumes that excision holds in algebraic L–theory, which is known for $i \leq 1$.

Most of the difficulties involved in computing $L^{Y_i}_n(\mathbf{Z}G,\alpha,u)$ for a geometric antistructure concern the group $L^{Y_i}_n(\widehat{\mathbf{Z}}_2G,\alpha,u)$. We therefore reorganize the calculation by considering the exact sequence

$$\ldots \to L^{Y_i}_{n+1}(\mathbf{Z}G\to\widehat{\mathbf{Z}}_2G) \longrightarrow L^{Y_i}_n(\mathbf{Z}G) \longrightarrow L^{Y_i}_n(\widehat{\mathbf{Z}}_2G) \xrightarrow{\psi_n} L^{Y_i}_n(\mathbf{Z}G\to\widehat{\mathbf{Z}}_2G) \to \ldots \tag{4.1}$$

valid for any antistructure. On the other hand, we have isomorphisms of relative groups

$$L^{Y_i}_n(\mathbf{Z}G\to\widehat{\mathbf{Z}}_2G) \cong L^{X_i}_n(\mathbf{Z}G\to\widehat{\mathbf{Z}}_2G)$$

so we are free to use the L^{X_i} relative groups for computation. By excision

$$L^{X_i}_n(\mathbf{Z}G\to\widehat{\mathbf{Z}}_2G) \cong L^{O_i}_n(\widehat{R}_{odd} \oplus S \to \hat{S}), \tag{4.2}$$

where $\widehat{R}_{odd}$ is the product of the ℓ–adic completions of R at all odd primes ℓ.

We now introduce the groups

$$CL^{O_i}_n(S) = L^{O_i}_n(S \to S_A) \tag{4.3}$$

where $S_A = \hat{S} \oplus (S \otimes \mathbf{R})$ is the adelic completion of S. Let $T = S \otimes \mathbf{R}$. Then by the arithmetic sequence and (4.2) we have a long exact sequence

$$\ldots CL^{O_i}_{n+1}(S) \to L^{X_i}_{n+1}(\mathbf{Z}G\to\widehat{\mathbf{Z}}_2G) \to L^{O_i}_n(\widehat{R}_{odd} \oplus T) \xrightarrow{\gamma_n} CL^{O_i}_n(S) \to \ldots \tag{4.4}$$

valid for any geometric antistructure. This is the main exact sequence for computing the relative groups, and then (4.1) is used to compute the absolute groups. It is a major ingredient in Wall's program that the groups $CL^{O_i}_n(S)$ are actually computable [**68**], although not finitely generated. In fact, they are elementary abelian 2–groups depending only on the idèle class groups of the centre of S (see Tables 14.9–14.11). This is a form of the Hasse principle for quadratic forms, and follows from the work of Kneser on Galois cohomology.

5. Dress induction and idempotents

The calculation of L–groups of finite groups can be reduced to calculating a limit of L–groups for hyperelementary subgroups. This process is known as Dress induction. More generally, Dress assumes that some Green ring, say $\mathcal{G}$, acts on a Mackey functor $\mathcal{M}$. Write

$$\delta_{\mathcal{G}}^{\mathcal{H}}\colon \bigoplus_{H\in\mathcal{H}} \mathcal{G}(H) \to \mathcal{G}(G)$$

for the sum of the induction maps.

THEOREM 5.1: *If there exists $y \in \bigoplus_{H\in\mathcal{H}} \mathcal{G}(H)$ such that $\delta_{\mathcal{G}}^{\mathcal{H}}(y) = 1 \in \mathcal{G}(G)$, then both Amitsur complexes for $\mathcal{M}$ are contractible.*

REMARK: One writes the conclusion as $\mathcal{M}(G) = \varprojlim_{\mathcal{H}} \mathcal{M}(H) = \varinjlim_{\mathcal{H}} \mathcal{M}(H)$ where the first limit made up of restrictions and the second of inductions. The result above follows from [**22**, Prop.1.2, p.305] and the remark just above [**23**, Prop.1.3, p.190].

Dress also proves a local result which says the following about $\mathcal{M}(G)$. Fix a prime p, let $\mathcal{H}_p$ denote the family of p–hyperelementary subgroups and let $\mathcal{M}(G)_{(p)}$ denote the p– localization of $\mathcal{M}(G)$. Then

$$\mathcal{M}(G)_{(p)} = \varprojlim_{\mathcal{H}_p} \mathcal{M}(H)_{(p)} = \varinjlim_{\mathcal{H}_p} \mathcal{M}(H)_{(p)}\ . \tag{5.2}$$

By [**69**] the 2–localization map is an injection on L–groups of finite groups. To apply these results to computation of L–groups, Dress defined a suitable Green ring (see also [**70**]). For any commutative ring R, let $y(G,\theta,R)$ be the Grothendieck group of finitely–generated, projective left R modules with an R bilinear form $\lambda\colon M \times M \to R$ which is *equivariant* in that $\lambda(m_1, gm_2) = \lambda(\theta(g^{-1})m_1, m_2)$, *symmetric* in that $\lambda(m_2, m_1) = \lambda(b^{-1}m_1, m_2)$, and *non–singular* in that the adjoint of λ is an isomorphism. Define $GU(G,\theta,R)$ and $GW(G,\theta,R)$ by equating two forms with isomorphic Lagrangians or moding out forms with a Lagrangian. Thomas [**60**] produces the necessary formulae to check that $GW(G,\theta,R)$ acts on $L_n^P(RG,\alpha,b)$ where α is the antistructure associated to any geometric antistructure θ, b, w where θ and b are fixed but w is allowed to vary subject only to $w \circ \theta = w$. Dress proves generation results for the case $\theta = 1_G$ which yield

THEOREM 5.3: *For the standard geometric antistructures, the L^p–groups of finite groups are computable from the family of 2–hyperelementary and p–elementary subgroups, p odd. The torsion in the L^p–groups and the L^p–groups localized at 2 are computable from the family of 2–hyperelementary subgroups.*

REMARK 5.4: One can also show that the round groups L^{O_i}, L^{X_i} and L^{Y_i} localized at 2 are 2–hyperelementary computable. This can be done either by refining the groups GW or by studying the Ranicki–Rothenberg sequences.

Dress proves these results by studying the Burnside ring and its p–localizations. He also constructed idempotents in the p–local Burnside ring and in [**25**, §6] and [**46**, §11] these idempotents are combined with induction theory to do calculations. We discuss the p–local case on a finite group G. Dress constructs one idempotent $e_E(G)$ in the

p–local Burnside ring for each conjugacy class of cyclic subgroups of order prime to p in G and shows that they are orthogonal. One can then split any p–local Mackey functor, F, using these idempotents into pieces $e_E(G) \cdot F(G)$ plus a piece left over since $1_G \neq \sum_E e_E(G)$ in the p–local Burnside ring. If $F(G)$ is generated by the images under induction from the p–hyperelementary subgroups, then the leftover piece vanishes. The additional result we want is Oliver's identification of the pieces [**46**, 11.5, p.256]. Let F be a p–local Mackey functor on G which is p–hyperelementary generated. In general Oliver describes $e_E(G) \cdot F(G)$ as a limit over subgroups H of the form $E \lhd H \twoheadrightarrow P$ where P is a p–group. He then makes the observation that the limit takes place inside $N_G(E)$ so

$$e_E(G) \cdot F(G) = e_E\big(N_G(E)\big) \cdot F\big(N_G(E)\big) .$$

If one could compute conjugations, induction and restriction maps for index p–inclusions of p–hyperelementary, then one could work out the limit in general. This makes 2–hyperelementary groups especially important for L–theory.

The L–theory case has an additional feature: for general geometric antistructures, the L–groups are not Mackey functors. The theory of "twisting diagrams", [**31**] or [**55**], can be used to overcome this difficulty.

Given an extension

$$G \lhd \widehat{G} \xrightarrow{\phi} \mathbf{Z}/2,$$

one can use ϕ to pull–back a non–trivial line bundle over a surgery problem with fundamental group $\widehat{G}$. This gives rise to a transfer map $tr: L_n(\mathbf{Z}\widehat{G}, w\phi) \to L_{n+1}(\mathbf{Z}G \to \mathbf{Z}\widehat{G}, w)$. Selecting a generator $t \in \widehat{G} - G$ gives rise to an automorphism θ of G given by conjugation by t. We assume that w extends over $\widehat{G}$ with $w(t) = 1$. Setting $b = t^2 \in G$ gives a geometric antistructure on G and there is a long exact sequence

$$\cdots \to L_{n+2}(\mathbf{Z}G \to \mathbf{Z}\widehat{G}, w) \to L_n(\mathbf{Z}G, \alpha, b) \to L_n(\mathbf{Z}\widehat{G}, w\phi) \xrightarrow{tr} L_{n+1}(\mathbf{Z}G \to \mathbf{Z}\widehat{G}, w) \to \cdots$$

where α is the antistructure associated to θ, b and $s\phi$. Conversely, any geometric antistructure on G arises from such a procedure. The group $L_n(\mathbf{Z}\widehat{G}, \lambda w)$ is a Mackey functor and the relative group $L_{n+1}(\mathbf{Z}G \to \mathbf{Z}\widehat{G}, w)$ sits in a long exact sequence where the other two terms are Mackey functors. This allows one to produce decompositions of $L_n(\mathbf{Z}G, \alpha, b)$ mimicking the Mackey functor case even though $L_n(\mathbf{Z}G, \alpha, b)$ has no obvious Mackey functor structure.

The most important application of these decomposition techniques applies to the 2–hyperelementary case $G = C \rtimes P$, because here there is a further identification of the $e_C(G) \cdot F(G)$ with a twisted group ring of P. In principle, this reduces the calculation to 2–groups where numerous simplifications occur. Wall [**70**, §4] was the first to explore this decomposition. He produced the idempotent decomposition by hand but had to restrict to groups with abelian 2–Sylow group. Hambleton–Madsen do the general case using the Burnside ring idempotents, [**25**, §6].

If C_m is cyclic of odd–order m, let ζ_m denote a primitive mth root of unity. Any $d|m$ determines a unique cyclic subgroup of odd order of G and we denote the corresponding summand of our functor by $F(G)(d)$. The map $t: P \to Aut(C_m)$ can be regarded as a map $t: P \to Gal\big(\mathbf{Q}(\zeta_m)/\mathbf{Q}\big)$ and we let $\mathbf{Z}[\zeta_m]^t P$ denote the corresponding *twisted group ring*. Any geometric antistructure induces an antistructure on $\mathbf{Z}[\zeta_m]^t P$ which we continue to denote by (α, u). The main exact sequence can be applied again.

THEOREM 5.5:([**25**, 6.13,7.2]) *For $i = 0, 1$ and $G = C_m \rtimes P$ with any geometric antistructure, there is a natural splitting*

$$L_n^{Y_i}(\mathbf{Z}G, \alpha, u)_{(2)} = \sum\nolimits^{\oplus} \big\{L_n^{Y_i}(\mathbf{Z}G, \alpha, u)(d) : d \mid m\big\}$$

(i) $L_n^{Y_i}(\mathbf{Z}G, \alpha, u)(1) \cong L_n^{X_i}(\mathbf{Z}P, \alpha, u)_{(2)}$ via the restriction map,
(ii) for $d \neq 1$, $L_n^{Y_i}(\mathbf{Z}G, \alpha, u)(d) \cong L_n^{X_i}(\mathbf{Z}G, \alpha, u)(d)$,
(iii) $L_n^{Y_i}(\mathbf{Z}G, \alpha, u)(d)$ maps isomorphically under restriction to $L_n^{Y_i}(\mathbf{Z}[C_d \rtimes P], \alpha, u)(d)$,
(iv) for each $d \mid m$ there is a long exact sequence

$$\to CL_{n+1}^{O_i}(S(d)) \to L_n^{X_i}(\mathbf{Z}G, \alpha, u)(d) \to \prod_{\ell \nmid d} L_n^{X_i}(\widehat{R}_\ell(d)) \oplus L_n^{O_i}(T(d)) \to CL_n^{O_i}(S(d)) \to$$

where $R(d)$ is the twisted group ring $\mathbf{Z}[\zeta_d]^t P$ with antistructure (α, u), and $S(d) = R(d) \otimes \mathbf{Q}$, $\widehat{R}_\ell(d) = R(d) \otimes \widehat{\mathbf{Z}}_\ell$, $T(d) = R(d) \otimes \mathbf{R}$.

There are similar splittings and calculations for the relative L–groups and the 2–adic L–groups.

For certain purposes, it is useful to be able to detect surgery obstructions. Dress's results for the standard antistructures say that we can detect by transfer to the collection of 2–hyperelementary subgroups. In some cases we can also reduce from hyperelementary groups to a smaller collection, the *basic* groups. More explicitly [**32**, 3.A.6], a 2–hyperelementary group $G = C \rtimes P$ is basic provided

$$P_1 = \ker\big(t\colon P \to Aut(C)\big)$$

is cyclic, dihedral, semi-dihedral or quaternion.

In [**32**, 1.A.4] we introduced the category RG–Morita, for any commutative ring R. The category $\mathbf{Q}G$–Morita is defined as follows. The objects are subgroups, $H \subset G$, and the morphisms from H_1 to H_2 are generated by the H_2–H_1 bisets X, modulo some relations spelled out in [**32**, p.249-250]. From [**32**, 1.A.12(i), p.251], $R_{\mathbf{Q}}(G)$ is a functor on $\mathbf{Q}G$–Morita defined by sending a rational representation $V$of H_1 to $\mathbf{Q}[X] \otimes_{\mathbf{Q}H_1} V$. Note if V is a permutation module on the H_1–set Y, then

$$\mathbf{Q}[X] \otimes_{\mathbf{Q}H_1} \mathbf{Q}[Y] = \mathbf{Q}[X \times_{H_1} Y]\ .$$

We proved in [**32**, 1.A.9, p.251] that the morphisms in $\mathbf{Q}G$–Morita are generated by generalized inductions and restrictions corresponding to homomorphisms $f\colon G_1 \to G_2$ which are either injections (subgroups) or surjections (quotient groups). Let $\mathcal{M}$ be a functor on $\mathbf{Q}G$–Morita.

THEOREM 5.6:([**32**], 1.A.11, p.251) *The sum of the generalized restriction maps,*

$$\mathcal{M}(G) \to \bigoplus_{B \in \mathcal{B}_G} \mathcal{M}(B)$$

is a split injection where $\mathcal{B}_G$ denotes the set of basic subquotients of G. The sum of the generalized induction maps is a split surjection.

This result has an analogue for quadratic Morita theory, and it applies to detect the relative groups $L_*^{X_i}(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G)(d)$ for $i = 0, 1$ since these are functors out of $(\mathbf{Q}G, -)$–Morita. To detect $L_*^p(\mathbf{Z}G, w)$ we need the w–basic subquotients of G, defined in [**32**]. Combining [**32,1.B.7**] with [**32,6.2**] gives:

THEOREM 5.7:([**28**, Thm.A]) *Let G be a 2-hyperelementary group. Then the sum of all the (generalized) restriction maps*

$$L_n^p(\mathbf{Z}G, w) \longrightarrow L_n^p(\mathbf{Z}[\overline{G}], w) \oplus \sum \{L_n^p(\mathbf{Z}[H/N], w) : H/N \ \ \text{a } w\text{–basic subquotient of } G\}$$

is a natural (split) injection, where $\overline{G} = G/[P_1, P_1]$.

We remark that [**28**, Thm.B] lists specific invariants which detect all oriented L^p surgery obstructions, and [**28**, 5.21] shows that the torsion subgroup of the L^p–groups has exponent 4 in general (exponent 2 in the standard oriented case).

6. Central simple algebras with involution

We collect here some terminology and results about the building blocks for our calculations. These are the L–groups $L_n^{O_i}(R, \alpha, u)$ where (R, α, u) denotes an antistructure over an algebra. When the cases $i = 0, 1$ are being considered separately, we will use Wall's notation $L^S = L^{O_1}$ and $L^K = L^{O_0}$.

First we summarize some of the standard facts about quadratic forms on simple algebras with centre field continuous, local (of characteristic 0), and finite. For our purposes, the main references are [**64**] and [**68**]. Since we are only interested in the applications to surgery theory, we will restrict ourselves to the simple algebras which arise from the rational group rings of finite groups. This assumption will simplify the arguments at various points. More precisely, if D denotes such a skew field with centre E, and $A \subseteq E$ the ring of integers, then E is an abelian extension of $\mathbf{Q}$. We fix an odd integer d such that $\hat{D}_\ell$ is *split*, and E_ℓ is an *unramified* extension of $\hat{\mathbf{Q}}_\ell$ for all finite primes ℓ with $\ell \nmid 2d$. We also assume that D has "uniformly distributed invariants": the Schur indices of D at all primes $\ell \in E$ over a fixed rational prime are equal, and the Hasse invariants are Galois conjugate. This holds for the algebras arising from group rings by the Benard-Schacher Theorem [**72**, Th. 6.1].

In addition to listing the values of the groups, we mention explicit invariants (such as signature and discriminant) used to detect them. From these facts we can compute the $CL_n^{O_i}$ and prepare for the computation of the maps γ_i. The L^S to L^K Rothenberg sequences are tabulated in Tables 14.1–14.8.

If (D, α, u) denotes an antistructure on a division algebra with centre E (and $A \subseteq E$ the ring of integers), then we distinguish as usual types U , Sp and O (see [**70**, §1.2]). We further subdivide into types OK if $D = E$, type OD if $D \neq E$ and similarly for type Sp. If an involution-invariant factor is the product of two simple rings interchanged by the involution, this is type GL. Such factors make no contribution to L-theory. When the anti-structure is understood, we will say "D has type ..." for short. Recall that $L_n^K(D, \alpha, -u) = L_{n+2}^K(D, \alpha, u)$ and types O and Sp are interchanged, so we usually just list type O.

(6.1) Continuous Fields: For continuous fields ($E = \mathbf{R}$ or $\mathbf{C}$) the signature gives an explicit isomorphism of $L_0^K(\mathbf{C}, c, 1)$, $L_2^K(\mathbf{C}, c, 1)$, $L_0^K(\mathbf{R}, 1, 1)$ and $L_0^K(\mathbf{H}, c, 1)$ onto $2\mathbf{Z}$

(the types are U, U, O and Sp); in all these cases except for $(\mathbf{H}, c, 1)$ the discriminant map $L_0^K(E) \to H^0(E^\times)$ is onto. Indeed the groups $H^n(K_1(\mathbf{H})) = 0$ so $L_n^S(\mathbf{H}, c, 1) = L_n^K(\mathbf{H}, c, 1)$. The discriminant also gives an isomorphism for $L_1^K(\mathbf{R}, 1, 1) = \mathbf{Z}/2$ and $L_1^K(\mathbf{C}, 1, 1) = \mathbf{Z}/2$. The other L^K–groups are zero. In the final calculation we wish to keep track of the divisibility of the signatures. The notation $2\mathbf{Z}$ stands for an infinite cyclic group of signatures taking on any even value.

(6.2) Local Fields: Over local fields (of characteristic 0), in type U: $L_{2n}^K(D) \cong H^0(E^\times) = \mathbf{Z}/2$ via the discriminant and $L_{2n+1}^K(D) = 0$. In type OD, $L_0^K(D) \cong H^0(E^\times)$ and the others are zero. In type OK, $L_1^K(E) \cong H^1(E^\times) = \mathbf{Z}/2$ by the discriminant and $L_0^K(E)$ is an extension of $H^0(E^\times)$ by $\mathbf{Z}/2$, while $L_2^K(E) = L_3^K(E) = 0$. The natural map $L_1^S(E) \to L_1^K(E)$ is zero.

(6.3) Finite Fields: For finite fields in type U, $L_n^S = L_n^K = 0$, and in type O characteristic 2, $L_n^S = L_n^K = \mathbf{Z}/2$ for each n . For type O odd characteristic, the discriminant gives isomorphisms $L_0^K \cong \mathbf{Z}/2$, $L_1^K = \mathbf{Z}/2$ and $L_2^K = L_3^K = 0$. The groups $L_n^S = 0$ for $n = 0, 3$ and $L_1^S = L_2^S = \mathbf{Z}/2$. The map $L_1^S \to L_1^K$ is zero.

7. Computation of the relative group, $L_n^{X_i}(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G)$

We can now compute the map

$$\gamma_n^{O_i}(d) \colon L_n^{O_i}(\widehat{R}_{odd}(d) \oplus T(d)) \longrightarrow CL_n^{O_i}(S(d))$$

from (4.4) for each involution-invariant factor of $S(d)$. The main result about the relative groups is:

THEOREM 7.1: *For 2–hyperelementary groups G and any geometric antistructure, there is an isomorphism*

$$L_n^{X_i}(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G)(d) \cong \operatorname{cok} \gamma_n^{O_i}(d) \oplus \ker \gamma_{n-1}^{O_i}(d)$$

and each of the summands decomposes according to the types of the simple components of $\mathbf{Q}G$.

Each simple component of $\mathbf{Q}G$ is a matrix algebra over a skew field, and by Morita equivalence it suffices to study γ_n for an antistructure (D, α, u) on a skew field D. Its centre E is an abelian extension of $\mathbf{Q}$ with ring of integers $A \subseteq E$. We fix an odd integer d such that $\hat{D}_\ell$ is *split*, and E_ℓ is an *unramified* extension of $\hat{\mathbf{Q}}_\ell$ for all finite primes ℓ with $\ell \nmid 2d$. Tables 14.12-14.14 (for $i = 1$) and Tables 14.16-14.22 (for $i = 0$) list the domain, range, kernel and cokernel of each summand of $\gamma_n^{O_i}(d)$.

In order to use the tables, it is necessary to determine the types (O, Sp, U or GL) and centre fields for all the rational representations of $G = \mathbf{Z}/d \rtimes P$, following the method given in [**27**, p.148], or [**31**, Appendix I]. Recall that for a simple factor of type Sp, the groups $\operatorname{cok} \gamma_n$ and $\ker \gamma_n$ are equal to $\operatorname{cok} \gamma_{n+2}$ and $\ker \gamma_{n+2}$ respectively. For the d–component we need to consider only those representations which are faithful on $\mathbf{Z}/d$. Here is a list of the possible types, subdivided according to the behaviour at infinite primes.

Type O:

$OK(\mathbf{R})$ if $D = E$ and E has a real embedding,

$OK(\mathbf{C})$	if $D = E$ and E has no real embedding.
$OD(\mathbf{H})$	if $D \neq E$ and D is nonsplit at infinite primes,
$OD(\mathbf{R})$	if $D \neq E$ is split at infinite primes and E has a real embedding,
$OD(\mathbf{C})$	if $D \neq E$ is split at infinite primes and E has no real embedding.

Type U:

$U(\mathbf{C})$	if D_∞ has type U,
$U(\mathbf{GL})$	if D_∞ has type $\mathbf{GL}$.

We remark that in type $U(\mathbf{C})$ the centre field of D_∞ at each infinite place is the complex numbers with complex conjugation as the induced involution. Type $U(\mathbf{GL})$ algebras are isomorphic to matrix rings over $\mathbf{C} \times \mathbf{C}$ or $\mathbf{R} \times \mathbf{R}$, at each infinite place, with the induced involution interchanging the two factors of $\mathbf{C}$ or $\mathbf{R}$.

In the remainder of this section we give a brief discussion of the computation in the case $i = 0$, including the definition of the maps Φ, Φ', the group $\Gamma(E)$, and related notation (see **[28]** or **[37]** for more details).

First we consider type U where $H^0(C(E)) = \mathbf{Z}/2$ lies in the sequence

$$0 \to H^0(E^\times) \to H^0(E_A^\times) \to H^0(C(E)) \to 0.$$

At finite primes $L_n^K(\hat{A}_\ell) = L_n^K(\hat{A}_\ell/Rad) = 0$, since the right-hand side is the sum of L^K-groups of finite fields. At the infinite places we have the signature group $L_{2n}^K(D_\infty)$. This is non-zero when D_∞ remains type U (a change to type $\mathbf{GL}$ is possible) and the fixed field $E_0 \subseteq E$ of the involution is real. In type $U(\mathbf{C})$, each factor $2\mathbf{Z}$ maps surjectively to $H^0(C(E)) = \mathbf{Z}/2$ so $\operatorname{cok}\gamma_{2n} = 0$ and $\ker\gamma_{2n} = \Sigma$, where Σ is a subgroup of index 2 in a direct sum of factors $2\mathbf{Z}$, one for each complex place.

Next we consider type O. It is convenient to introduce the "discriminant part" $\tilde{\gamma}_n$ of γ_n for a factor $(D, \alpha, u) = (E, 1, 1)$ of type OK to fit into the following commutative diagram:

$$\text{(7.2)}\qquad \begin{array}{ccc} \prod\limits_{\ell \nmid 2d} L_n^K(\widehat{A}_\ell) \times L_n^K(E_\infty) & \xrightarrow{\gamma_n} & CL_n^K(E) \\ \downarrow & & \downarrow \\ H^n(\hat{A}_{2d'}^\times) \times H^n(E_\infty^\times) & \xrightarrow{\tilde{\gamma}_n} & H^n(C(E)) \end{array}$$

where $\hat{A}_{2d'}^\times = \prod\limits_{\ell \nmid 2d} \hat{A}_\ell^\times$. Below we will also use the notation $\hat{A}_{2d}^\times = \prod\limits_{\ell | 2d} \hat{A}_\ell^\times$. Since $\tilde{\gamma}_n$ has the same kernel and cokernel as the map

$$H^n(\hat{A}_{2d'}^\times) \times H^n(E_\infty^\times) \times H^n(E^\times) \longrightarrow H^n(E_A^\times)\ ,$$

we are led to consider the following commutative diagram (for $n = 0$):

$$\begin{array}{ccccccccccc} 0 & \to & \ker\tilde{\gamma}_0 & \to & H^0(\hat{A}_{2d'}^\times) \times H^0(E_\infty^\times) \times H^0(E^\times) & \to & H^0(\hat{E}_A^\times) & \to & \operatorname{cok}\tilde{\gamma}_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \downarrow & & \\ 0 & \to & E^{(2)}/E_2^\times & \to & H^0(\hat{A}^\times) \times H^0(E_\infty^\times) \times H^0(E^\times) & \to & H^0(\hat{E}_A^\times) & \to & H^0(\Gamma(E)) & \to & 0 \\ & & \downarrow & & \downarrow & & & & & & \\ \text{(7.3)} & & H^0(\hat{A}_{2d}^\times) & = & H^0(\hat{A}_{2d}^\times) & & & & & & \end{array}$$

Let $E^{(2)}$ denotes the elements of E with even valuation at all finite primes and $\Gamma(E)$ is the ideal class groups defined by

$$1 \to E^\times/A^\times \to \hat{E}^\times/\hat{A}^\times \to \Gamma(E) \to 1\ .$$

To obtain the middle sequence, add $H^0(\hat{A}^\times_{2d})$ to the domain of $\tilde{\gamma}_0$, then the map to $H^0(E^\times_{\mathbb{A}})$ has the same kernel and cokernel as $H^0(\hat{E}^\times) \to H^0(I(E))$ where $I(E) = \hat{E}^\times/\hat{A}^\times$ is the ideal group of E .

From (7.3) we obtain the following exact sequence

$$(7.4) \qquad 0 \to \ker\tilde{\gamma}_0 \to E^{(2)}/E^{\times 2} \xrightarrow{\Phi/} H^0(\hat{A}^\times_{2d}) \to \operatorname{cok}\tilde{\gamma}_0 \to H^0(\Gamma(E)) \to 0$$

for the computation of $\tilde{\gamma}_0$ in type OK. In type $OD(\mathbf{H})$ when (D,α,u) is non-split at all infinite primes, the term $H^0(E^\times_\infty)$ is missing from the top row of 7.3. This produces instead:

$$(7.5) \qquad 0 \to \ker\tilde{\gamma}_0 \to E^{(2)}/E^{\times 2} \xrightarrow{\Phi'} H^0(\hat{A}^\times_{2d}) \oplus H^0(E^\times_\infty) \to \operatorname{cok}\tilde{\gamma}_0 \to H^0(\Gamma(E)) \to 0$$

For the map $\tilde{\gamma}_1$ in type OK a similar but easier analysis gives $\ker\tilde{\gamma}_1 = 0$ and an exact sequence

$$(7.6) \qquad 0 \to H^1(A^\times) \to H^1(\hat{A}^\times_{2d}) \to \operatorname{cok}\tilde{\gamma}_1 \to 0.$$

In type OD, nonsplit at infinite primes, $H^1(E^\times_\infty)$ is added to the middle term.

The maps Φ and Φ' occur in number theory, and the 2–ranks of their kernel and cokernels are determined by classical invariants of E (see [**70**, 4.6]). A similar discussion can be carried out for the maps $\gamma^{\mathcal{O}_1}(d)$, and it turns out that the same maps Φ, Φ' appear in the calculation.

PROPOSITION 7.7:([**28**, 2.18])

(i) The 2-rank of $\ker\Phi_E$ *(resp.* $\ker\Phi'_E$*) is* $\gamma^*(E,2d)$ *(resp.* $\gamma(E,2d)$ *).*

(ii) The 2-rank of $\operatorname{cok}\Phi_E$ *(resp* $\operatorname{cok}\Phi'_E$*) is* $g_{2d}(E)+r_2+\gamma^*(E,2d))-\gamma(E)$ *(resp.* $g_{2d}(E)+r_1+\gamma(E,2d))-\gamma(E)$*).*

Here $\gamma(E,m)$ $(\gamma^*(E,m))$ *denotes the 2–rank of the (strict) class group of* $A[\frac{1}{m}]$, $g_m(E)$ *is the number of primes in* E *which divide* m *and* r_1 (r_2) *is the number of real (complex) places of* E.

EXAMPLE 7.8: If $\Gamma(E)$ has odd order then the exact sequence

$$0 \to H^0(A^\times) \to E^{(2)}/E^{\times 2} \to H^1(\Gamma(E)) \to 0$$

gives an isomorphism $H^0(A^\times) \cong E^{(2)}/E^{\times 2}$. Then the map Φ is just the reduction map $H^0(A^\times) \to H^0(\hat{A}^\times_{2d})$. For example, if $E \subseteq \mathbf{Q}(\zeta_{2^k})$ and $d = 1$, then $\ker\Phi = 0$ and $\operatorname{cok}\Phi = (\mathbf{Z}/2)^{r_2+1}$. ∎

8. The 2–adic calculation, $L_n^{Y_i}(\widehat{\mathbf{Z}}_2 G)$

We want to state the main result of [**27**, 1.16] which which computes the map

$$\Psi_n : L_n^K(\widehat{\mathbf{Z}}_2 G, \alpha, u) \to L_n^K(\widehat{\mathbf{Q}}_2 G, \alpha, u) \tag{8.1}$$

for $G = C \rtimes P$ a 2–hyperelementary group with a geometric antistructure (α, u) with $K = O_0$. This map appears in the calculation of the ψ maps in the next section as well as in the Ranicki–Rothenberg sequence for computing $L_n^{Y_i}(\widehat{\mathbf{Z}}_2 G, \alpha, u)$. Since Ψ_n splits as in (5.5), it is enough to give the answer for the d-component for each $d \mid |C|$ which we know is determined by the top component for the various $G = C_d \rtimes P$. If $T \in C$ denotes a generator, then θ induces an automorphism of C given by $\theta(T) = T^{\vartheta}$ (there is a misprint in the formula in [**27**, p.148,line-9]). The domain of Ψ_n is easy to compute:

THEOREM 8.2:
$$L_n^{Y_0}(\widehat{\mathbf{Z}}_2 G)(d) = L_n^K(\widehat{\mathbf{Z}}_2 G)(d) = g_2(d) \cdot (\mathbf{Z}/2)$$
where $g_2(d) = g_2\big(\mathbf{Q}(\zeta_d)^P\big)$ denotes the number of primes ℓ dividing 2 in the field $\mathbf{Q}(\zeta_d)^P$, where P acts as Galois automorphisms via the action map t.

Recall that if $P_1 = \ker(t \colon P \to (C_d)^\times)$, then any irreducible complex character of G which is faithful on C_d is induced up from $\chi \otimes \xi$ on $\mathbf{Z}/d \times P_1$ where χ is a linear character of $\mathbf{Z}/d$ and ξ is an irreducible character of P_1. These are the representations in the semi–simple algebra $S(d)$. They are divided as usual into the types O, Sp and U, and we say that the *order* of a linear character ξ is the order of its image $\xi(P_1)$. Let $S(d, \xi)$ denote the simple factor of $S(d)$ associated to an involution-invariant character $(\chi \otimes \xi)^*$, induced up from $\chi \otimes \xi$.

THEOREM 8.3: ([**27**, 1.16]) *Let (α, u) be a geometric antistructure. If $d > 1$ and there is no element $g_0 \in P$ satisfying $t(g_0) = -\vartheta^{-1}$, then $L_n^K(\widehat{R}_2(d), \alpha, u) = 0$. Otherwise if $d > 1$ pick such a g_0 (or when $d = 1$ set $g_0 = e$), and let $m = n$ if $w(g_0) = 1$ (resp. $m = n+2$ if $w(g_0) = -1$). For each irreducible complex character ξ of P_1 the composite*

$$L_n^K(\widehat{R}_2(d), \alpha, u) \xrightarrow{\Psi_n(d)} L_n^K(\widehat{S}_2(d), \alpha, u) \xrightarrow{proj.} L_n^K(\widehat{S}_2(d, \xi), \alpha, u)$$

is injective or zero and detected by the discriminant. It is injective if and only if the character ξ is
(a) linear type O (and $m \equiv 0$ or $1 \pmod 4$)
(b) linear type Sp (and $m \equiv 2$ or $3 \pmod 4$)
(c) linear type U (and m even), order 2^ℓ and $\xi(b_0^{2^{\ell-1}}) = -1$.
Here the types refer to the antistructure $(\widehat{\mathbf{Q}}_2[P_1], \alpha_0, b_0)$, with $\alpha_0(a) = g_0 \alpha(a) g_0^{-1}$ and $b_0 = g_0 \alpha(g_0^{-1}) b w(g_0) \in \pm P_1$.

REMARK: A type I linear character ξ has type O (resp. Sp) if $\xi(b_0) = 1$ (resp. $\xi(b_0) = -1$). If P_1 has a linear character ξ of type 8.3(c), then (by projecting onto the $\mathbf{Z}/2$ quotient of $\xi(P_1)$) it also has linear characters of type O and Sp.

COROLLARY 8.4: *For $X = O_1(\widehat{R}_2)$ or $X = X_1(\widehat{R}_2)$, the discriminant map*

$$d_n^{K/X} \colon L_n^K(\widehat{R}_2(d), \alpha, b) \to H^n(K_1(\widehat{R}_2(d))/X)$$

is injective or zero. For $X = X_1(\widehat{R}_2)$*, it is injective if* $m \equiv 0,1 \pmod 4$*, or if* $m \equiv 2,3 \pmod 4$ *and there exists a linear character* ξ *with type Sp.*

REMARK 8.5: The first statement in Corollary 8.4 holds for any decoration subgroup $X(\widehat{R}_2)$ that decomposes completely over the primes $\ell \mid d$ in $\mathbf{Q}[\zeta_d]^P$, but O_1 and X_1 are the usual examples. If $m \equiv 2,3 \pmod 4$ and under certain assumptions, this condition is also known to be necessary for the discriminant maps d_n^K to be non–zero (see [**37**, 3.14]). In either case ($X = O_1$ or X_1), when $d_n = 0$ the kernel $\ker d_n \cong g_2 \cdot \mathbf{Z}/2$ can be lifted to a direct summand of $L_n^S(\widehat{\mathbf{Z}}_2 G)$ or $L_n^{X_1}(\widehat{\mathbf{Z}}_2 G)$ isomorphic to $(\mathbf{Z}/4)^\kappa$ if n is odd, and to $(\mathbf{Z}/2)^\kappa$ if n is even. A basis for these summands is represented by flips or rank 2 quadratic forms with Arf invariant one at primes $\ell \mid 2$ in $\mathbf{Q}[\zeta_d]^P$.

It is not hard to check that these flips and Arf invariant one planes in $L_n^{X_1}(\widehat{\mathbf{Z}}_2 G)$ map to zero under

$$\Psi_n^{X_1}(d) \colon L_n^{X_1}(\widehat{\mathbf{Z}}_2 G) \to L_n^S(\widehat{\mathbf{Q}}_2 G) \ ,$$

so the computation of $\Psi_n^{X_1}(d)$ is reduced to the map

$$H^{n+1}(K_1(\widehat{R}_2(d))/X_1) \to H^{n+1}(K_1(\widehat{S}_2(d))) \ ,$$

which only involves K–theory. ∎

When the discriminant map into $H^n(K_1(\widehat{S}_2(d,\xi)))$ is injective, its image in $H^n(\hat{A}_\ell^\times)$ for $\ell \mid 2$ is either $\langle 1-4\beta \rangle$ if $n = 0,2$, or $\langle -1 \rangle$ if $n = 1,3$. The element $\beta \in \hat{A}_\ell^\times$ is a unit whose residue class has non–zero trace in $\mathbf{F}_2$. This description allows us to identify the image of the discriminant in $H^n(K_1(\widehat{R}_2(d)))$ once the Tate cohomology group has been calculated, and thus compute $L_*^{X_1}(\widehat{\mathbf{Z}}_2 G)$.

EXAMPLE 8.6: Consider the simplest case, where $G = 1$. Then $K_1(\widehat{\mathbf{Z}}_2) = \widehat{\mathbf{Z}}_2^\times$ is generated by the units $\langle 5, -1 \rangle$. Therefore $H^n(K_1(\widehat{\mathbf{Z}}_2)) = \mathbf{Z}/2 \oplus \mathbf{Z}/2$ (n even) or $\mathbf{Z}/2$ (n odd), and $L_n^K(\widehat{\mathbf{Z}}_2) = \mathbf{Z}/2$ in each dimension. In particular, the element $1 - 4\beta = 5$ is the discriminant of the generator in $L_0^K(\widehat{\mathbf{Z}}_2)$.

By the results above, we get

$$L_n^S(\widehat{\mathbf{Z}}_2) = 0,\ \mathbf{Z}/2 \oplus \mathbf{Z}/2,\ \mathbf{Z}/2 \oplus \mathbf{Z}/2,\ \mathbf{Z}/4 \qquad \text{for } n \equiv 0,1,2,3 \pmod 4 \ .$$

The $\mathbf{Z}/4$ in L_3 is generated by the flip automorphism $\tau(e) = f$, $\tau(f) = -e$ of the hyperbolic plane over $(\widehat{\mathbf{Z}}_2, 1, -1)$. Since $X_1(\widehat{\mathbf{Z}}_2) = 0$, we also have computed $L_*^{X_1}(\widehat{\mathbf{Z}}_2) = L_*^S(\widehat{\mathbf{Z}}_2)$. ∎

9. The maps $\psi_n^{Y_i} \colon L_n^{Y_i}(\widehat{\mathbf{Z}}_2 G) \to L_n^{X_i}(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G)$

The final step in computing the main exact sequence is to determine the maps ψ_n for 2–hyperelementary groups. First we consider the case $i = 0$ needed for computing the L^p–groups:

$$\psi_n(d) : L_n^K(\widehat{\mathbf{Z}}_2 G)(d) \longrightarrow L_n^{X_0}(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G)(d) \ .$$

Here ψ_n factors through

$$\bar{\psi}_n \colon L_n^K(\widehat{\mathbf{Z}}_2 G) \xrightarrow{\Psi_n} L_n^K(\widehat{\mathbf{Q}}_2 G) \to CL_n^K(\mathbf{Q}G) \to \operatorname{cok} \gamma_n^K$$

and after taking the d–component these can be studied one simple component of $\mathbf{Q}G$ at a time. The maps $\Psi_n(d)$ are given in Theorem 8.3, and the other maps in the composite are contained in Tables 14.16–14.22.

Computing $\bar\psi_n(d)$ also computes the kernel and cokernel of $\psi_n(d)$ since $\ker\psi_n(d) = \ker\bar\psi_n(d)$, and $\operatorname{cok}\psi_n(d) \cong \operatorname{cok}\bar\psi_n(d) \oplus \ker\gamma_{n-1}(d)$.

EXAMPLE 9.1: We continue with the example $G = 1$ from the last section. The group $\operatorname{cok}\gamma_0 = \mathbf{Z}/2$ generated by the class $\langle 5\rangle \in H^0(K_1(\widehat{\mathbf{Z}}_2))$ (see Example 7.8) and otherwise is zero. It follows that the map $\bar\psi_0$ is an isomorphism, and we get the values

$$L_n^{X_0}(\mathbf{Z},1,1) = 8\mathbf{Z},\ \mathbf{Z}/2,\ \mathbf{Z}/2,\ \mathbf{Z}/2 \qquad \text{for } n \equiv 0,1,2,3 \,(\operatorname{mod} 4)\ .$$

To obtain the geometric L^p–groups, we cancel the terms $\mathbf{Z}/2$ in odd dimensions. Note that since $\tilde K_0(\mathbf{Z}) = 0 = Wh(\mathbf{Z})$, the other geometric L–groups are isomorphic to $L_*^p(\mathbf{Z})$. ∎

The calculation of the maps $\psi_n\colon L_n^{Y_1}(\widehat{\mathbf{Z}}_2G) \to L_n^{Y_1}(\mathbf{Z}G\to\widehat{\mathbf{Z}}_2G)$ needed to determine the L'–groups is more involved. Notice, however, that by Theorem 5.6 it is enough in principle to do the calculation for basic subquotients of G and then compute some generalized restriction maps. Because of the difficulties involved computing restriction maps on $L_*^{Y_1}(\widehat{\mathbf{Z}}_2G)$ this approach remains more a theoretical simplification than a practical one.

We can again define $\bar\psi_n$ as the composite

$$\bar\psi_n\colon L_n^{X_1}(\widehat{\mathbf{Z}}_2G)\xrightarrow{\Psi_n}L_n^{S}(\widehat{\mathbf{Q}}_2G) \to CL_n^{S}(\mathbf{Q}G) \to \operatorname{cok}\gamma_n^{S}$$

but this time we only have a commutative diagram

$$\begin{array}{ccccc}
H^{n+1}(K_1(\widehat{\mathbf{Z}}_2G)/X_1) & \to & L_n^{X_1}(\widehat{\mathbf{Z}}_2G) & \xrightarrow{\bar\psi_n} & \operatorname{cok}\gamma_n^{S} \\
\downarrow & & \downarrow & & \downarrow \\
H^{n+1}(K_1(\widehat{\mathbf{Z}}_2G)/Y_1) & \to & L_n^{Y_1}(\widehat{\mathbf{Z}}_2G) & \xrightarrow{\psi_n} & L_n^{X_i}(\mathbf{Z}G\to\widehat{\mathbf{Z}}_2G)
\end{array}$$

where the right–hand vertical arrow is a (split) injection. Note that the quotient group $K_1(\widehat{\mathbf{Z}}_2G)/Y_1 = Wh'(\widehat{\mathbf{Z}}_2G)$, which has been studied intensively in [**46**], [**47**]. Applying the idempotent splitting partly solves the problem: on the d–component for $d > 1$ the groups $L_n^{Y_1}(\widehat{\mathbf{Z}}_2G)(d) \cong L_n^{X_1}(\widehat{\mathbf{Z}}_2G)(d)$, and in Remark 8.5 we pointed out that the calculation of $\bar\psi_n(d)$ is now reduced to K–theory.

For $d = 1$ we may assume that G is a finite 2–group. Then $L_n^{K}(\widehat{\mathbf{Z}}_2G) = \mathbf{Z}/2$ and the map ψ_n also factors through $\operatorname{cok}\gamma_n^{S} \cong \operatorname{cok}\gamma_n^{Y_1}$. (both assertions follow because there is just one prime $\ell \mid 2$ in the centre fields of $\mathbf{Q}G$). We may assume that $\bar\psi_n$ is known from Theorem 8.3.

To proceed, we first compute $L_n^{Y_1}(\widehat{\mathbf{Z}}_2G)$ via the discriminant map $d_n^{K/Y_1}\colon L_n^{Y_1}(\widehat{\mathbf{Z}}_2G) \to H^n(Wh'(\widehat{\mathbf{Z}}_2G))$, either directly (starting with the known map d_n^{K/X_1}), or using the long exact sequence

$$\ldots \to H^{n+1}(\{\pm 1\}\oplus G^{ab}) \to L_n^{X_1}(\widehat{\mathbf{Z}}_2G) \to L_n^{Y_1}(\widehat{\mathbf{Z}}_2G) \to H^n(\{\pm 1\}\oplus G^{ab}) \to \ldots$$

It is quite likely that the "twisting diagram" method introduced in [**24**] and [**31**] would be useful here.

Next, we must compute ψ_n. One remark that may be helpful is that the the image of $H^{n+1}(\{\pm 1\}\oplus G^{ab})$ in $L_n^{X_1}(\widehat{\mathbf{Z}}_2 G)$ is mapped by Ψ_n into integral units, hence mapped to zero under $\bar{\psi}_n$. Hence, if $L_n^{X_1}(\widehat{\mathbf{Z}}_2 G) \to L_n^{Y_1}(\widehat{\mathbf{Z}}_2 G)$ happens to be surjective, we are done.

An alternate approach is to apply Remark 8.5 (valid for $Y_1(\widehat{\mathbf{Z}}_2 G)$ if G is a finite 2–group) to the L^{Y_1} to L^K Rothenberg sequence for $\widehat{\mathbf{Z}}_2 G$. As before, this reduces the computation of ψ_n to K–theory calculation. We can compute the composite

$$H^{n+1}(K_1(\widehat{\mathbf{Z}}_2 G)/Y_1) \to H^{n+1}(K_1(\widehat{\mathbf{Q}}_2 G)/Y_1) \to CL_n^S(\mathbf{Q}G)$$

using the algorithm from **[46]**, **[47]** for computing $Wh'(\widehat{\mathbf{Z}}_2 G)$. For a general geometric antistructure, this can involve a lot of book–keeping. For the standard oriented antistructure, things are not so difficult.

EXAMPLE 9.2: For the standard oriented antistructure, we can always split off $L_n(\mathbf{Z})$ from $L_n(\mathbf{Z}G)$ or $L_n(\widehat{\mathbf{Z}}_2 G)$ by using the inclusion and projection $1 \to G \to 1$.

If G is a finite 2–group, then

$$L_n^K(\widehat{\mathbf{Z}}_2 G) \cong L_n^K(\widehat{\mathbf{Z}}_2 G/Rad) \cong L_n^K(\mathbf{F}_2)$$

so the image of $L_n^{Y_1}(\widehat{\mathbf{Z}}_2 G) \to CL_n^S(\mathbf{Q}G)$ is just the image of the composite

$$H^{n+1}(K_1(\widehat{\mathbf{Z}}_2 G)/Y_1) \to H^{n+1}(K_1(\widehat{\mathbf{Q}}_2 G)/Y_1) \to L_n^{Y_1}(\widehat{\mathbf{Q}}_2 G) \to CL_n^S(\mathbf{Q}G)\ .$$

This directly reduces the calculation of ψ_n to a K_1–calculation. ∎

10. Groups of odd order

We prove a well–known vanishing result, as an example of the techniques developed so far.

THEOREM 10.1: *Let G be a finite group of odd order. Then in the standard oriented antistructure, $L_{2k+1}^{?}(\mathbf{Z}G) = 0$ for $? = s, \prime, h$ and p.*

Proof: For groups of odd order, the 2–hyperelementary subgroups are cyclic, so it is enough to let $G = C_m$ denote a cyclic group of odd order m. We have a decomposition into components $L_*^{Y_i}(\mathbf{Z}G)(d)$ indexed by the divisors $d \mid m$, and there are two distinct cases according as $d = 1$ or $d \neq 1$.

Let's start with $i = 0$ or L^p–groups. By Theorem 5.5, when $d = 1$ we are computing $L_*^p(\mathbf{Z})$ which was done in Example 9.1. For $d > 1$, all the summands in $S(d)$ have type $U(\mathbf{C})$, so by Table 14.21 we have

$$L_n^{X_0}(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G) = 0,\ \Sigma,\ 0,\ \Sigma \quad \text{for } n \equiv 0, 1, 2, 3 \pmod 4\ .$$

Next $L_n^K(\widehat{R}_2(d)) = 0$, for all n, since $\widehat{R}_2(d) = \widehat{\mathbf{Z}}_2 \otimes \mathbf{Z}[\zeta_d]$ reduces modulo the radical to a product of finite fields with type U antistructure. Therefore

$$L_n^p(\mathbf{Z}G)(d) = \Sigma,\ 0,\ \Sigma,\ 0 \quad \text{for } n \equiv 0, 1, 2, 3 \pmod 4$$

and in particular the L^p–groups vanish in odd dimensions.

Next we consider the $d > 1$ components in the main exact sequence for $L_*^{X_1}(\mathbf{Z}G)(d)$. Since type U factors of $\widehat{R}_{odd}$ or $\mathbf{Q}G$ make no contribution to the relative L'–groups, we have

$$L_n^{X_1}(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G) = 0,\ \Sigma,\ 0,\ \Sigma \quad \text{for } n \equiv 0,1,2,3 \,(\mathrm{mod}\, 4)\ .$$

as before. Now consider the 2–adic contribution. Here

$$H^{n+1}(K_1(\widehat{R}_2(d))/X_1) \cong L_n^{X_1}(\widehat{R}_2(d))$$

and $X_1(\widehat{R}_2(d)) = 0$ since the ring is abelian, so K_1 is just the group of units. Let $A = \widehat{\mathbf{Z}}_2[\zeta_d]$ and consider the sequences

$$1 \to (1 + 2A)^\times \to A^\times \to (A/2A)^\times \to 1$$

and

$$1 \to (1 + 2^{r+1}A)^\times \to (1 + 2^r A)^\times \xrightarrow{\varphi} A/2A \to 1$$

for $r \geq 1$, where $\varphi(1 + 2^r a) = a \,(\mathrm{mod}\, 2)$. Since $(A/2A)^\times$ has odd order and $A/2A = \mathbf{F}_2[\zeta_d]$ has non-trivial involution, both are cohomologically trivial as $\mathbf{Z}/2$–modules. Therefore $H^*(A^\times) = 0$ and

$$L'_n(\mathbf{Z}G)(d) = \Sigma,\ 0,\ \Sigma,\ 0 \quad \text{for } n \equiv 0,1,2,3 \,(\mathrm{mod}\, 4)$$

so once again the L–group vanish in odd dimensions. For G cyclic, $SK_1(\mathbf{Z}G) = 0$ and so $L' = L^s$. Also, $H^1(Wh'(\mathbf{Z}G)) = 0$ in the standard oriented antistucture, so $L'_{2k+1}(\mathbf{Z}G)$ surjects onto $L^h_{2k+1}(\mathbf{Z}G)$. ∎

REMARK 10.2: We don't want to leave the impression that all the L–groups of odd order groups G are easy to compute. For G cyclic of odd order, the groups $L^h_{2k}(\mathbf{Z}G)$ have torsion subgroup $H^0(\tilde{K}_0(\mathbf{Z}G))$ and this can be highly non–trivial. ∎

11. Groups of 2–power order

In [**31**] the groups L^p-groups for $\mathbf{Z}G$ were completely determined, for G a finite 2-group with any geometric anti-structure. For $L'_*(\mathbf{Z}G)$ with the standard oriented antistructure, there is in principle an algorithm for carrying out the computation. We have already discussed the steps in computing the main exact sequence (see Example 9.2) and mentioned that results of Oliver give an algorithm for computing $K'_1(\widehat{\mathbf{Z}}_2 G) = K_1(\widehat{\mathbf{Z}}_2 G)/X_1$, together with the action of the antistructure, by using the integral logarithm [**46**, Thm. 6.6]. Thus we can regard the L'–groups for 2–groups as computable up to extensions, although the method can be difficult to carry out in practice.

EXAMPLE 11.1: Let's compute $L'_*(\mathbf{Z}G)$ for G a cyclic 2–group of order $2^k \geq 2$ in the standard oriented antistructure (done in [**70**, 3.3]). Since $SK_1(\mathbf{Z}G) = 0$, this also gives us $L^s_*(\mathbf{Z}G)$. Note that $L^p_*(\mathbf{Z}G)$ is tabulated in [**31**], and $L^h_*(\mathbf{Z}G)$ was reduced to the computation of $H^0(D(\mathbf{Z}G))$ in [**29**] or [**2**]. The final step, the computation of $H^0(D(\mathbf{Z}G))$ was carried out independently in [**47**] and [**12**].

We begin as usual with the relative groups, this time from Table 14.23 and Table 14.15. The types are $U(\mathbf{C})$ and $OK(\mathbf{R})$, where the latter are the two quotient representations arising from the projection $G \to \mathbf{Z}/2$. We get

$$L_n^{X_1}(\mathbf{Z}G\to\widehat{\mathbf{Z}}_2G) = 0,\ \Sigma\oplus(8\mathbf{Z})^2\oplus(\mathbf{Z}/2)^2,\ 0,\ \Sigma \quad \text{for } n=0,1,2,3\,(\text{mod}\,4)\ .$$

Here $\Sigma = \oplus 4\mathbf{Z}$ is the is the signature group from the type $U(\mathbf{C})$ representations.

Next we compute $L_n^{Y_1}(\widehat{\mathbf{Z}}_2G)$ by comparing it to $L_n^{K}(\widehat{\mathbf{Z}}_2G)$. Since the antistructure is oriented, we can split off $L_n^{Y_1}(\widehat{\mathbf{Z}}_2) = L_n^{X_1}(\widehat{\mathbf{Z}}_2)$ computed in Example 8.6, and obtain

$$L_n^{Y_1}(\widehat{\mathbf{Z}}_2G) = L_n^{X_1}(\widehat{\mathbf{Z}}_2)\oplus H^{n+1}\big((1+I)^\times/G\big)$$

where $I = I(\widehat{\mathbf{Z}}_2G)$ is the augmentation ideal of $\widehat{\mathbf{Z}}_2G$. It is not hard to see that

$$H^n\big((1+I)^\times/G\big) = \mathbf{Z}/2,\ 0 \quad \text{for } n=0,1\,(\text{mod}\,2),$$

and a generator for the non–trivial element in H^0 is given by $\langle 3-g-g^{-1}\rangle$ where $g\in G$ is a generator. Since this element has projection $\langle 5\rangle$ at the non–trivial type $OK(\mathbf{R})$ representation (where $g\mapsto -1$), the map $\bar\psi_1$ has image $\mathbf{Z}/2$ in this summand of the relative group. This is an example of the "book–keeping" process mentioned in Example 9.2. Putting the summand from the trivial group back in, we get the well–known answer

$$L'_n(\mathbf{Z}G) = \Sigma\oplus 8\mathbf{Z}\oplus 8\mathbf{Z},\ 0,\ \Sigma\oplus\mathbf{Z}/2,\ \mathbf{Z}/2 \quad \text{for } n=0,1,2,3\,(\text{mod}\,4)\ .$$

∎

12. Products with odd order groups

Here we correct an error in the statement of [**28**, 5.1] where $G=\sigma\times\rho$ with σ an abelian 2–group and ρ odd order. More generally, for $G = G_1\times G_2$ where G_1 has odd order, we can reduce the calculation of $L_*(\mathbf{Z}G,w)$ to knowledge of $L_*(\mathbf{Z}G_2,w)$ and the character theory of G.

PROPOSITION 12.1: *Let $G=G_1\times G_2$, where G_1 has odd order. Then for $i=0,1$*

$$L_n^{X_i}(\mathbf{Z}G,w) = L_n^{X_i}(\mathbf{Z}G_2,w)\oplus L_n^{X_i}(\mathbf{Z}G_2\to\mathbf{Z}G,w)$$

where $w\colon G\to\{\pm1\}$ is an orientation character. For $n=2k$, the second summand is free abelian and detected by signatures at the type $U(\mathbf{C})$ representations of G which are non–trivial on G_1. For $i=0$ and $n=2k+1$, the second summand is a direct sum of $\mathbf{Z}/2$'s, one for each type $U(\mathbf{GL})$ representation of G which is non–trivial on ρ.

REMARK 12.2: In the important special case when G_2 is an abelian 2–group, note that type $U(\mathbf{C})$ representations of G exist only when $w\equiv 1$, and type $U(\mathbf{GL})$ representations of G exist only when $w\not\equiv 1$. In both cases, the second summand is computed by transfer to cyclic subquotients of order $2^r q$, $q>1$ odd, with $r\geq 2$. If G_2 is a cyclic 2–group, then $L^s = L'$ by [**46**, Ex.3,p.14].

Proof: The given direct sum decomposition follows from the existence of a retraction of the inclusion $G_2 \to G$ compatible with w. It also follows that

$$L^{X_i}_{n+1}(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G, w) \cong L^{X_i}_{n+1}(\mathbf{Z}G_2 \to \widehat{\mathbf{Z}}_2 G_2, w) \oplus L^{X_i}_{n}(\mathbf{Z}G_2 \to \mathbf{Z}G, w)$$

since the map induced by inclusion $L^{X_i}_n(\widehat{\mathbf{Z}}_2 G_2, w) \to L^{X_i}_n(\widehat{\mathbf{Z}}_2 G, w)$ is an isomorphism ([**37**, 3.4] for $i = 1$).

The computation of the relative groups for $\mathbf{Z} \to \widehat{\mathbf{Z}}_2$ can be read off from Table 14.22: for each centre field E of a type $U(\mathbf{GL})$ representation, the contribution is $H^0(C(E)) \cong \mathbf{Z}/2$ if $i \equiv 1 \pmod 2$.

The detection of $L^p_n(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G, w)$ by cyclic subquotients is proved in [**32**, 1.B.7, 3.A.6, 3.B.2]. ∎

COROLLARY 12.3: *Let $G = C_{2^r q}$, for q odd and $r \geq 2$. If $q = 1$ assume that $r \geq 3$. Then the group*

$$L^p_{2k-1}(\mathbf{Z}G, w)(q) = \bigoplus_{i=2}^{r} CL^K_{2k}(E_i) \cong (\mathbf{Z}/2)^{r-1}$$

when $w \not\equiv 1$, where the summand $CL^K_{2k}(E_i) = H^0(C(E_i))$, $2 \leq i \leq r$, corresponds to the rational representation with centre field $E_i = \mathbf{Q}(\zeta_{2^i q})$.

13. Dihedral groups

Wall [**70**], Laitinen and Madsen [**40**], [**37**] did extensive computations for the L'–groups of the groups G with periodic cohomology, because of the importance of these computations for the spherical space form problem.

As a final, and much easier example, we will consider the dihedral groups $G = C_d \rtimes \mathbf{Z}/2$. These are the simplest kind of 2–hyperelementary groups which are not 2–elementary. Here the action map is injective, as a generator of the $\mathbf{Z}/2$ quotient group acts by inversion on C_d. We take the standard oriented antistructure. Note that $L' = L^s$ for dihedral groups [**46**, p.15].

It is enough to do the $d > 1$ component, and we see that $S(d)$ contains a single type $OK(\mathbf{R})$ representation with centre field $E = \mathbf{Q}(\zeta_d + \zeta_d^{-1})$ and ring of integers A. Let g_p denote the number of primes in this field lying over the rational prime p. For any integer d, let $g_d = \sum\{g_p : p \mid m\}$.

For the L^p calculation, we have relative groups

$$L^{X_0}_n(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G)(d) = \operatorname{cok} \gamma^K_n \oplus \ker \gamma^K_{n-1}$$

which can be read off from Table 14.16. The groups $L^K_n(\widehat{\mathbf{Z}}_2 G) = g_2 \cdot (\mathbf{Z}/2)$ in each dimension and $\Psi_n(d)$ is injective for $n \equiv 0, 1 \pmod 4$ but zero for $n \equiv 2, 3 \pmod 4$.

Since the image of $\Psi_1(d)$ hits the classes $\langle -1 \rangle$ at primes lying over 2, it follows that $\bar{\psi}_1(d)$ is injective with cokernel $H^1(\hat{A}^\times_d)/H^1(A^\times)$, an elementary abelian 2–group of rank $g_d - 1$.

Similarly, the image of $\Psi_0(d)$ hits the classes $g_2\langle 1 - 4\beta \rangle$ in $H^0(\hat{A}^\times_2)$, so we must compute the kernel and cokernel of the map

$$\bar{\Phi}\colon E^{(2)}/E^{\times 2} \to H^0(\hat{A}^\times_{2d})/g_2(1 - 4\beta) \ . \tag{13.1}$$

It is not hard to see that $\ker\bar\Phi = \ker\Phi \oplus \ker\bar\psi_0(d)$ and $\operatorname{cok}\bar\Phi = \operatorname{cok}\bar\psi_0(d)$ (see [**28**, p.566]). In the short exact sequences

$$0 \to \operatorname{cok}\bar\psi_{n+1}(d) \oplus \ker\gamma_n(d) \to L_n^{X_0}(\mathbf{Z}G)(d) \to \ker\bar\psi_n(d) \to 0,$$

the only potential extension problem occurs for $n = 0$. Let $\lambda_E = g_d + \gamma^*(E, d)$. Then a similar argument to that in [**28**, p. 551], together with [**28**, 5.19], shows that

PROPOSITION 13.2: *Let $G = C_d \rtimes \mathbf{Z}/2$ be a dihedral group, with $d > 1$ odd. Then*

$L_0^p(\mathbf{Z}G)(d) = \Sigma \oplus (\mathbf{Z}/2)^{\lambda_E - 1}$
$L_1^p(\mathbf{Z}G)(d) = 0$
$L_2^p(\mathbf{Z}G)(d) = g_2 \cdot \mathbf{Z}/2$
$L_3^p(\mathbf{Z}G)(d) = g_2 \cdot \mathbf{Z}/2 \oplus (\mathbf{Z}/2)^{\lambda_E}$

REMARK 13.3: The signature divisibility is given by

$$\Sigma = 8\mathbf{Z} \oplus (4\mathbf{Z})^{m - \bar r_E - 1} \oplus (2\mathbf{Z})^{\bar r_E}$$

where $\bar r_E$ is the 2–rank of the image of $(\Theta|\ker\bar\Phi)$ as in [**28**, p.550] and $m = \phi(d)/2$ is the number of real places in E. The formula in [**28**, 5.17(ii)] is incorrect. It should read

$$(8\mathbf{Z})^{r(S)} \oplus (4\mathbf{Z})^{r_1(S) - r_O(S) - r(S)} \oplus (2\mathbf{Z})^{r_O(S)}$$

where $r(S)$ denotes the number of type $OK(\mathbf{R})$ factors in $S(d)$, and $r_1(S)$, $r_O(S)$ are as defined in [**28**] ∎

For the L' calculation, we have relative groups

$$L_n^{X_1}(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G)(d) = \operatorname{cok}\gamma_n^S \oplus \ker\gamma_{n-1}^S$$

which can be read off from Table 14.12. The groups $L_n^{X_1}(\widehat{\mathbf{Z}}_2 G)(d)$ are computed from the Rothenberg sequence using the same method as in Section 9. We have

$$L_0^S(\widehat{\mathbf{Z}}_2 G)(d) = \begin{cases} 0 & n \equiv 0 \pmod 4, \\ H^0(\hat A_2^\times) & n \equiv 1 \pmod 4, \\ H^1(\hat A_2^\times) \oplus g_2 \cdot \mathbf{Z}/2 & n \equiv 2 \pmod 4, \\ \frac{H^0(\hat A_2^\times)}{g_2\langle 1-4\beta, -1\rangle} \oplus g_2 \cdot \mathbf{Z}/4 & n \equiv 3 \pmod 4, \end{cases}$$

where $H^0(\hat A_2^\times)$ has 2–rank $m + g_2$. The analogous number theoretic map to (13.1) is

$$\tilde\Phi_E \colon E^{(2)}/E^{\times 2} \to H^0(\hat A_d^\times)\ .$$

and the 2–ranks of its kernel and cokernel can again be given in terms of classical invariants (see [**70**, p.56]). We then have the torsion subgroup of $L_1^{X_1}(\mathbf{Z}G)$ isomorphic to

$$\ker\bar\psi_1 \oplus \operatorname{cok}\Phi_E \oplus (g_d - 1)\cdot \mathbf{Z}/2 \cong \ker\tilde\Phi_E \oplus (g_d - 1)\cdot\mathbf{Z}/2\ .$$

However, the exact sequence

$$0 \to \ker \Phi_E \to \ker \tilde{\Phi}_E \to H^0(\hat{A}_2^\times) \to \operatorname{cok} \gamma_1 \to \operatorname{cok} \tilde{\Phi}_E \to 0$$

allows us to compute the 2–rank of $\ker \tilde{\Phi}_E$ in terms of the 2–rank of $\operatorname{cok} \tilde{\Phi}_E$ and previously defined quantities. Recall that γ_E denotes the 2–rank of $H^0(\Gamma(E))$ and $m = \phi(d)/2$. It is also useful to define the quantity

$$\phi_E = \nu_E + \gamma_E$$

where $\operatorname{cok} \tilde{\Phi}_E = (\mathbf{Z}/2)^{\nu_E}$. Putting the information together gives:

PROPOSITION 13.4: *Let $G = C_d \rtimes \mathbf{Z}/2$ be a dihedral group, with $d > 1$ odd. Then*

$L'_0(\mathbf{Z}G)(d) = \Sigma \oplus (\mathbf{Z}/2)^{\phi_E}$
$L'_1(\mathbf{Z}G)(d) = (\mathbf{Z}/2)^{m+\phi_E-1}$
$L'_2(\mathbf{Z}G)(d) = g_2 \cdot \mathbf{Z}/2$
$L'_3(\mathbf{Z}G)(d) = L_3^S(\widehat{A}_2)(d) = g_2 \cdot \mathbf{Z}/4 \oplus (\mathbf{Z}/2)^{m-g_2}$

REMARK 13.5: The signature divisibility this time is given by

$$\Sigma = 8\mathbf{Z} \oplus (4\mathbf{Z})^{m-1} \ .$$

These are the same divisibilities as in the relative group. ∎

14. Appendix: Useful Tables

14.A L–groups of fields and skew fields

We give the L^S to L^K change of K–theory sequences for the antistructures (D, α, u) where D is a (skew) field with center E, and E is either finite, continuous ($\mathbf{R}$ or $\mathbf{C}$) or local (a finite extension field of $\hat{\mathbf{Q}}_p$).

From the tables below, one can read off invariants determining the L–groups in most cases (e.g. discriminant, signature, and Pfaffian). The remaining cases are labelled c, κ, and τ for the Arf invariant, Hasse-Witt invariant or flip respectively (τ in L^S is represented by the automorphism $\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}$ of the hyperbolic plane). Note that L^S–groups are all zero for finite fields or local fields in type U, and that for a division algebra D with centre E the group $L_1^K(D, \alpha, u) = 0$ unless $(D, \alpha, u) = (E, 1, 1)$ and $L_1^K(E, 1, 1) = \mathbf{Z}/2$ detected by the discriminant.

Table 14.1: Finite fields, odd characteristic, Type O

	$L_n^S(E,1,1)$	$L_n^K(E,1,1)$	$H^n(E^\times)$
$n=3$	0	0	$\mathbf{Z}/2$
$n=2$	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$
$n=1$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$
$n=0$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$

For finite fields in type U, both $L^S_n(E,1,1) = L^K_n(E,1,1) = 0$. In characteristic 2, $L^S_n(E,1,1) = L^K_n(E,1,1) = \mathbf{Z}/2$ in each dimension (detected by c in even dimensions, and τ in odd dimensions)

Table 14.2: Local fields, Type OK

	$L^S_n(E,1,1)$	$L^K_n(E,1,1)$	$H^n(E^\times)$
$n=3$	0	0	$\mathbf{Z}/2$
$n=2$	$\mathbf{Z}/2$	0	$H^0(E^\times)$
$n=1$	$H^0(E^\times)$	$\mathbf{Z}/2$	$\mathbf{Z}/2$
$n=0$	$\mathbf{Z}/2\ \langle\kappa\rangle$	$\mathbf{Z}/2\tilde{\times}H^0(E^\times)$	$H^0(E^\times)$

The extension $\mathbf{Z}/2\tilde{\times}H^0(E^\times)$ appearing in this table is split if and only if $-1 \in E^{\times 2}$.

Table 14.3: Local fields, Type OD

	$L^S_n(D,\alpha,1)$	$L^K_n(D,\alpha,1)$	$H^n(E^\times)$
$n=3$	0	0	$\mathbf{Z}/2$
$n=2$	$\mathbf{Z}/2$	0	$H^0(E^\times)$
$n=1$	$H^0(E^\times)$	0	$\mathbf{Z}/2$
$n=0$	$\mathbf{Z}/2$	$H^0(E^\times)$	$H^0(E^\times)$

In type OD we can always scale the antistructure so that it has $u=+1$. For local fields in type U, we have two–fold periodicity $L^K_n(E,1,1) = L^K_{n+2}(E,1,1)$.

Table 14.4: Local fields, Type U

	$L^S_n(E,1,1)$	$L^K_n(E,1,1)$	$H^n(E^\times)$
$n=1,3$	0	0	0
$n=0,2$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$

Table 14.5: Continuous Fields, $E = \mathbf{R}$, Type O

	$L^S_n(E,1,1)$	$L^K_n(E,1,1)$	$H^n(E^\times)$
$n=3$	0	0	$\mathbf{Z}/2$
$n=2$	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$
$n=1$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$
$n=0$	$4\mathbf{Z}$	$2\mathbf{Z}$	$\mathbf{Z}/2$

Table 14.6: Continuous Fields, $E = \mathbf{C}$, Type O

	$L^S_n(E,1,1)$	$L^K_n(E,1,1)$	$H^n(E^\times)$
$n=3$	0	0	$\mathbf{Z}/2$
$n=2$	$\mathbf{Z}/2$	0	0
$n=1$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$
$n=0$	0	0	0

Table 14.7: Continuous Fields, $E = \mathbf{C}$, Type U

	$L^S_n(E,c,1)$	$L^K_n(E,c,1)$	$H^n(E^\times)$
$n=1,3$	0	0	0
$n=0,2$	$4\mathbf{Z}$	$2\mathbf{Z}$	$\mathbf{Z}/2$

Table 14.8: Continuous Fields, $D = \mathbf{H}$, Type O

	$L^S_n(D,c',1)$	$L^K_n(D,c',1)$	$H^n(E^\times)$
$n=3$	0	0	0
$n=2$	$2\mathbf{Z}$	$2\mathbf{Z}$	0
$n=1$	0	0	0
$n=0$	0	0	0

Here c' denotes the type O involution on the quaternions $\mathbf{H}$. Explicitly, it is given by $c'(i) = i$, $c'(j) = j$ and $c'(k) = -k$. For the usual (type Sp) involution $c(i) = -i$, $c(j) = -j$, we have $L_n(D, c, 1) = L_{n+2}(D, c', 1)$.

14.B The Hasse principle

We will need the groups $CL_n^{O_i}(D, \alpha, u)$ describing the kernel and cokernel of the Hasse principle $L_n^{O_i}(D, \alpha, u) \to L_n^{O_i}(D_A, \alpha, u)$. We will tabulate the associated change of K–theory sequences

$$\ldots \to CL_n^{S}(D, \alpha, u) \to CL_n^{K}(D, \alpha, u) \to H^n(C(E)) \xrightarrow{\delta} CL_{n-1}^{S}(D, \alpha, u) \to \ldots$$

where $C(D) \cong C(E) = E_A^\times/E^\times$ is the idè le class group of the center field E in D. The map δ is the coboundary map in the long exact sequence. There are short exact sequences $(n = 0, 1)$:

$$0 \to H^n(E^\times) \to H^n(E_A^\times) \to H^n(C(E)) \to 0$$

and the maps are induced by the inclusions of fields.

Table 14.9:Type OK

	$CL_n^S(E)$	$CL_n^K(E)$	$H^n(C(E))$
$n = 3$	0	0	$H^1(C(E))$
$n = 2$	$H^1(C(E))$	0	$H^0(C(E))$
$n = 1$	$H^0(C(E))$	$H^1(C(E))$	$H^1(C(E))$
$n = 0$	$\mathbf{Z}/2$	$\mathbf{Z}/2 \tilde{\times} H^0(C(E))$	$H^0(C(E))$

The extension $0 \to \mathbf{Z}/2 \to CL_0^K(D) \to H^0(C(E)) \to 0$ appearing in this table is split if and only if $-1 \in E^{\times 2}$.

Table 14.10:Type OD

	$CL_n^S(E)$	$CL_n^K(E)$	$H^n(C(E))$
$n = 3$	0	0	$H^1(C(E))$
$n = 2$	$H^1(C(E))$	0	$H^0(C(E))$
$n = 1$	$H^0(C(E))$	$\ker\{\delta\colon H^1(C(E)) \to \mathbf{Z}/2\}$	$H^1(C(E))$
$n = 0$	$\mathbf{Z}/2$	$H^0(C(E))$	$H^0(C(E))$

Table 14.11:Type U

	$CL_n^S(E)$	$CL_n^K(E)$	$H^n(C(E))$
$n=1,3$	0	0	0
$n=0,2$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$

14.C The relative groups $L_n^{X_1}(\mathbf{Z}G\to\widehat{\mathbf{Z}}_2G)$

We now suppose that G is a 2–hyperelementary group and give the tables for calculating the relative groups $L_n^{X_1}(\mathbf{Z}G\to\widehat{\mathbf{Z}}_2G)$ and $L_n^{X_0}(\mathbf{Z}G\to\widehat{\mathbf{Z}}_2G)$. Recall that by excision these split up according to the way $\mathbf{Q}G$ splits into simple algebras with involution. Then if $G=C\rtimes P$ where $C=C_d$ we can compute the d–component in terms of the map γ_n defined earlier. In particular, a summand of $\gamma_n(d)$ is determined by a single algebra (D,α,u) with centre field E and ring of integers $A\subset E$. Restricted to this summand it is the natural map

$$\gamma_n(d)\colon \prod_{\ell\nmid 2d} L_n^S(\widehat{A}_\ell)\times L_n^S(E_\infty)\longrightarrow CL_n^S(D)$$

and in the domain the terms

$$L_n^S(\widehat{A}_\ell)=L_n^S(\widehat{A}_\ell/Rad)$$

are just L–groups of finite fields. Thus all the terms in the domain and range are given in the previous tables for fields. The maps $\gamma_n(d)$ are also easy to relate to number theory. In particular, note that mapping a term $H^n(\hat{A}_{2d'}^\times)$ or $H^n(E_\infty^\times)$ to $H^n(C(E))$ is the map induced by the inclusion into $H^n(E_A^\times)$ followed by the projection $H^n(E_A^\times)\to H^n(C(E))$. The symbol Σ in the tables denotes a free abelian group of signatures at infinite primes.

Table 14.12: Type $OK(\mathbf{R})$ or Type $OD(\mathbf{R})$

	$\prod_{\ell\nmid 2d} L_n^S(\widehat{A}_\ell)\times L_n^S(E_\infty)$	$CL_n^S(D)$	$\ker\gamma_n^S(d)$	$\operatorname{cok}\gamma_n^S(d)$
$n=3$	0	0	0	0
$n=2$	$H^1(\hat{A}_{2d'}^\times)\times H^1(E_\infty^\times)$	$H^1(C(E))$	0	$H^1(\hat{A}_{2d}^\times)/H^1(A^\times)$
$n=1$	$H^0(\hat{A}_{2d'}^\times)\times H^0(E_\infty^\times)$	$H^0(C(E))$	$\ker\ \Phi$	$\operatorname{cok}\Phi\oplus H^0(\Gamma(E))$
$n=0$	$0\times\oplus\, 4\mathbf{Z}$	$\mathbf{Z}/2$	Σ	0

Table 14.13: Type $OK(\mathbf{C})$ or Type $OD(\mathbf{C})$

	$\prod_{\ell \nmid 2d} L^s_n(\widehat{A}_\ell) \times L^s_n(E_\infty)$	$CL^s_n(D)$	$\ker \gamma^s_n(d)$	$\operatorname{cok} \gamma^s_n(d)$
$n=3$	0	0	0	0
$n=2$	$H^1(\hat{A}^\times_{2d'}) \times H^1(E^\times_\infty)$	$H^1(C(E))$	0	$H^1(\hat{A}^\times_{2d})/H^1(A^\times)$
$n=1$	$H^0(\hat{A}^\times_{2d'}) \times H^0(E^\times_\infty)$	$H^0(C(E))$	$\ker\ \Phi$	$\operatorname{cok}\Phi \oplus H^0(\Gamma(E))$
$n=0$	0×0	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$

Table 14.14: Type $OD(\mathbf{H})$

	$\prod_{\ell \nmid 2d} L^s_n(\widehat{A}_\ell) \times L^s_n(D_\infty)$	$CL^s_n(D)$	$\ker \gamma^s_n(d)$	$\operatorname{cok} \gamma^s_n(d)$
$n=3$	0	0	0	0
$n=2$	$H^1(\hat{A}^\times_{2d'}) \times \oplus 2\mathbf{Z}$	$H^1(C(E))$	Σ	$H^1(\hat{A}^\times_{2d})/H^1(A^\times)$
$n=1$	$H^0(\hat{A}^\times_{2d'}) \times 0$	$H^0(C(E))$	$\ker\ \Phi'$	$\operatorname{cok}\Phi' \oplus H^0(\Gamma(E))$
$n=0$	0×0	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$

Table 14.15: Type U

	$\prod_{\ell \nmid 2d} L^s_n(\widehat{A}_\ell) \times L^s_n(E_\infty)$	$CL^s_n(E)$	$\ker \gamma^s_n(d)$	$\operatorname{cok} \gamma^s_n(d)$
$n=1,3$	0	0	0	0
$n=0,2$	$0 \times \oplus\ 4\mathbf{Z}$	0	$\oplus\ 4\mathbf{Z}$	0

Since the L^s–groups are all zero in type GL, this completes the L^s–tables.

14.D The relative groups $L_n^{X_0}(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G)$

We now give the relative group tables for the L^p–groups. Some additional notation is defined as it appears.

Table 14.16: Type $OK(\mathbf{R})$

	$\prod_{\ell \nmid 2d} L_n^K(\widehat{A}_\ell) \times L_n^K(E_\infty)$	$CL_n^K(E)$	$\ker \gamma_n^K(d)$	$\operatorname{cok} \gamma_n^K(d)$
$n=3$	0	0	0	0
$n=2$	0	0	0	0
$n=1$	$H^1(\hat{A}_{2d'}^\times) \times H^1(E_\infty)$	$H^1(C(E))$	0	$H^1(\hat{A}_{2d}^\times)/H^1(A^\times)$
$n=0$	$H^0(\hat{A}_{2d'}^\times) \times \oplus\, 2\mathbf{Z}$	$\mathbf{Z}/2\tilde{\times} H^0(C(E))$	$\Sigma \oplus \ker \Phi'$	$\operatorname{cok} \Phi \oplus H^0(\Gamma(E))$

Table 14.17: Type $OD(\mathbf{R})$

	$\prod_{\ell \nmid 2d} L_n^K(\widehat{A}_\ell) \times L_n^K(E_\infty)$	$CL_n^K(D)$	$\ker \gamma_n^K(d)$	$\operatorname{cok} \gamma_n^K(d)$
$n=3$	0	0	0	0
$n=2$	0	0	0	0
$n=1$	$H^1(\hat{A}_{2d'}^\times) \times H^1(E_\infty)$	$\ker \delta$	0	$\ker \Delta'$
$n=0$	$H^0(\hat{A}_{2d'}^\times) \times \oplus\, 2\mathbf{Z}$	$H^0(C(E))$	$\Sigma \oplus \ker \Phi'$	$\operatorname{cok} \Phi \oplus H^0(\Gamma(E))$

Here the map Δ' is the map

$$\Delta'\colon \frac{H^1(\hat{A}_{2d}^\times) \oplus H^1(E_\infty^\times)}{H^1(A^\times)} \longrightarrow \{\pm 1\}$$

defined by $\Delta'(\langle -1 \rangle_\ell) = -1$ if and only if $\widehat{D}_\ell$ is nonsplit.

Table 14.18: Type $OK(\mathbf{C})$

	$\prod_{\ell \nmid 2d} L_n^K(\widehat{A}_\ell) \times L_n^K(E_\infty)$	$CL_n^K(E)$	$\ker \gamma_n^K(d)$	$\operatorname{cok} \gamma_n^K(d)$
$n=3$	0	0	0	0
$n=2$	0	0	0	0
$n=1$	$H^1(\hat{A}_{2d'}^\times) \times H^1(E_\infty)$	$H^1(C(E))$	0	$H^1(\hat{A}_{2d}^\times)/H^1(A^\times)$
$n=0$	$H^0(\hat{A}_{2d'}^\times) \times 0$	$\mathbf{Z}/2\tilde{\times} H^0(C(E))$	$\ker \Phi$	$\mathbf{Z}/2\tilde{\times}(\operatorname{cok} \Phi \oplus H^0(\Gamma(E)))$

Table 14.19: Type $OD(\mathbf{C})$

	$\prod_{\ell\nmid 2d} L_n^K(\widehat{A}_\ell)\times L_n^K(E_\infty)$	$CL_n^K(D)$	$\ker\gamma_n^K(d)$	$\operatorname{cok}\gamma_n^K(d)$
$n=3$	0	0	0	0
$n=2$	0	0	0	0
$n=1$	$H^1(\hat{A}_{2d'}^\times)\times H^1(E_\infty)$	$\ker\delta$	0	$\ker\Delta$
$n=0$	$H^0(\hat{A}_{2d'}^\times)\times 0$	$H^0(C(E))$	$\ker\Phi$	$\operatorname{cok}\Phi\oplus H^0(\Gamma(E))$

The map Δ has the same definition as Δ' but $H^1(E_\infty^\times)$ is missing from the domain.

Table 14.20:Type $OD(\mathbf{H})$

	$\prod_{\ell\nmid 2d} L_n^K(\widehat{A}_\ell)\times L_n^K(D_\infty)$	$CL_n^K(D)$	$\ker\gamma_n^K(d)$	$\operatorname{cok}\gamma_n^K(d)$
$n=3$	0	0	0	0
$n=2$	$0\times\oplus 2\mathbf{Z}$	0	Σ	0
$n=1$	$H^1(\hat{A}_{2d'}^\times)\times 0$	$\ker\delta$	0	$\ker\Delta'$
$n=0$	$H^0(\hat{A}_{2d'}^\times)\times 0$	$H^0(C(E))$	$\ker\ \Phi'$	$\operatorname{cok}\Phi'\oplus H^0(\Gamma(E))$

Table 14.21:Type $U(\mathbf{C})$

	$\prod_{\ell\nmid 2d} L_n^K(\widehat{A}_\ell)\times L_n^K(D_\infty)$	$CL_n^K(D)$	$\ker\gamma_n^K(d)$	$\operatorname{cok}\gamma_n^K(d)$
$n=1,3$	0	0	0	0
$n=0,2$	$0\times\oplus\ 2\mathbf{Z}$	$H^0(C(E))$	Σ	0

Table 14.22: Type $U(\mathbf{GL})$

	$\prod_{\ell\nmid 2d} L_n^K(\widehat{A}_\ell)\times L_n^K(D_\infty)$	$CL_n^K(D)$	$\ker\gamma_n^K(d)$	$\operatorname{cok}\gamma_n^K(d)$
$n=1,3$	0	0	0	0
$n=0,2$	0×0	$H^0(C(E))$	0	$\mathbf{Z}/2$

14.E Finite 2–groups

Here complete calculations already appear in [**31**, §3, p.80]. To compare our results with the tables there note that $\Gamma(E)$ and $\Gamma^*(E)$ have odd order (Weber's Theorem) for all the centre fields appearing in $\mathbf{Q}G$ and $g_2(E) = 1$. Hence Φ and Φ' are injective with $\operatorname{cok}\Phi_E$ of 2-rank $1 + r_2$ (resp. $\operatorname{cok}\Phi'_E$ of 2-rank $1 + r_1$). As above, the degree $[E, \mathbf{Q}] = r_1 + 2r_2$, where r_1 denotes the number of real places of E and r_2 the number of complex places.

In [**31**] the basic antistructures on the simple components of $\mathbf{Q}G$ are labelled Γ_N, F_N, R_N, H_N, UI and UII. These have type $OK(\mathbf{C})$, $OK(\mathbf{C})$, $OK(\mathbf{R})$, $OD(\mathbf{H})$, $U(\mathbf{C})$ and $U(\mathbf{GL})$ respectively in our notation. In our tables, the distinction between Γ_N and F_N is whether $-1 \in E^{\times 2}$. Let ζ_N denote a primitive $2^N th$ root of unity. The centres E for the type O factors are $\mathbf{Q}(\zeta_{N+1})$, $\mathbf{Q}(\zeta_{N+2} - \zeta_{N+2}^{-1})$, $\mathbf{Q}(\zeta_{N+2} + \zeta_{N+2}^{-1})$, $\mathbf{Q}(\zeta_N + \zeta_N^{-1})$ so that (r_1, r_2) equals $(0, 2^{N-1})$, $(0, 2^{N-1})$, $(2^N, 0)$, $(2^{N-2}, 0)$ respectively. Therefore using Tables 14.12–14.14 and 14.16–14.20 we can list the contribution of the type O components to $L_n^{X_1}(R \to \widehat{R}_2)$ or $L_n^{X_0}(R \to \widehat{R}_2)$. The contributions from type U components are already easily read off from Tables 14.15, 14.21 and 14.22 . Note that only the *rank* and not the divisibilities in the signature groups are given in the tables.

Table 14.23: $L_n^{X_1}(R \to \widehat{R}_2)$ **in Type** O

O	Γ_N	F_N	R_N	H_N
$n = 3$	0	0	0	$\mathbf{Z}^{r_1}$
$n = 2$	0	0	0	0
$n = 1$	$(\mathbf{Z}/2)^{r_2+1}$	$(\mathbf{Z}/2)^{r_2+1}$	$\mathbf{Z}^{r_1} \oplus \mathbf{Z}/2$	$(\mathbf{Z}/2)^{r_1+1}$
$n = 0$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$

Table 14.24: $L_n^{X_0}(R \to \widehat{R}_2)$ **in Type** O

O	Γ_N	F_N	R_N	H_N
$n = 3$	0	0	0	$\mathbf{Z}^{r_1}$
$n = 2$	0	0	0	0
$n = 1$	0	0	$\mathbf{Z}^{r_1}$	$(\mathbf{Z}/2)^{r_1-1}$
$n = 0$	$(\mathbf{Z}/2)^{r_2+2}$	$\mathbf{Z}/4 \oplus (\mathbf{Z}/2)^{r_2}$	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^{r_1+1}$

The divisibilities for L^p are determined in [**28, 2.8**] to be $\Sigma = 8\mathbf{Z} \oplus (4\mathbf{Z})^{r_1-1}$ in type R_N (for 2–power cyclotomic extensions E, the quantity $r_E = 0$), and $\Sigma = \oplus\, 2\mathbf{Z}$ in type H_N. Those for L' are the same, by the Rothenberg sequence tables.

Table 14.25: $H^n(K_1(R \to \widehat{R}_2))$ in Type O

O	Γ_N	F_N	R_N	H_N
$n=3$	0	0	0	$(\mathbf{Z}/2)^{r_1}$
$n=2$	$(\mathbf{Z}/2)^{r_2+1}$	$(\mathbf{Z}/2)^{r_2+1}$	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^{r_1+1}$
$n=1$	0	0	0	$(\mathbf{Z}/2)^{r_1}$
$n=0$	$(\mathbf{Z}/2)^{r_2+1}$	$(\mathbf{Z}/2)^{r_2+1}$	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^{r_1+1}$

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