The Brauer group of torsors and its arithmetic applications

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Introduction

This paper consists of two parts. In the first part we describe the behaviour of the Brauer group with respect to the pull-back map from the variety to a torsor under a torus over any field of characteristic 0. To a variety X endowed with a surjective morphism $\pi : X \to \mathbf{P}^1_k$ with geometrically integral generic fibre one canonically associates certain X-torsors, called vertical torsors. The second part is devoted to the arithmetic of vertical torsors over a number field k. We prove that the Manin obstruction to the Hasse principle and weak approximation on smooth and proper models of such torsors is the only one provided the same property holds for the k-fibres of π, and the number of 'bad' fibres is small. Using the results of the first part we give sufficient conditions for X itself (strictly speaking, for any smooth and proper model of X) to have the afore-mentioned property. To illustrate possible applications we exhibit apparently new classes of conic bundle threefolds and some other varieties with the property that the possible failure of the Hasse principle or weak approximation is accounted for by the Manin obstruction.

Let us now describe the contents of each section in more detail. Let k be a field of characteristic 0, and let X be a variety over k. Denote by $\bar{k}$ an algebraic closure of k, and set $\overline{X} := X \times_k \bar{k}$, $\Gamma := \mathrm{Gal}\,(\bar{k}/k)$. Let $\mathrm{Br}\, X := H^2_{\text{ét}}(X, \mathbf{G}_m)$ be the cohomological Brauer–Grothendieck group of X. In their fundamental paper [CS87a] Colliot-Thélène and Sansuc defined universal torsors and studied their various properties. For a universal torsor $f : Y \to X$ over a variety X such that all invertible functions on $\overline{X}$ are constants, and $\mathrm{Pic}\, \overline{X}$ has no torsion, they prove that the kernel of the map $\mathrm{Br}\, Y \to \mathrm{Br}\, \overline{Y}$ is naturally isomorphic to $\mathrm{Br}\, k$. In this paper we show (Theorem 1.6) that the map $\mathrm{Br}\, X \to \mathrm{Br}\, Y/\mathrm{Br}\, k$ defined by f is canonically identified with the map $\mathrm{Br}\, X \to (\mathrm{Br}\, \overline{X})^\Gamma$ (provided k is such that $H^3(\Gamma, \overline{k}^*) = 0$). We also obtain sufficient conditions that ensure that the unramified Brauer group of Y is the image of the unramified Brauer group of X.

The main achievement of the arithmetic part of the paper, seen from the applications point of view, are the following results (Theorems 2.9 and 2.7, proved by a combination of 'open descent' and fibration methods). Let X be a smooth and projective variety over a number field k, and let $\pi : X \to \mathbf{P}^1_k$ be a dominant morphism with geometrically integral generic fibre. Assume that the sum of degrees of closed points of $\mathbf{P}^1_k$ corresponding to non-split (bad) fibres is at most 3, and that every non-split fibre contains a multiplicity 1 irreducible component splitting into two over an algebraic closure of the residue field. Then, under some mild additional hypotheses, and provided we are not in an explicitly described exceptional case, the possible failure of the Hasse principle and weak approximation on X is accounted for by the Manin obstruction, provided the same is true for smooth k-fibres of π. We also prove a similar result in the case when there are only two non-split fibres, both over k-points, each containing an irreducible component of multiplicity 1 defined over an extension of k of prime degree. This extends the results of [H94], as well as those of [S90], [S96], [CS00], by allowing a few bad fibres in the theorem on the behaviour of the Manin obstruction regarding the passage from k-fibres to the total space.

For an example of application consider the equation

$$a_0(x,t)x_0^2 + a_1(x,t)x_1^2 + a_2(x,t)x_2^2 = 0,$$

where $a_i(x,t) \in k[x,t]$ are pairwise coprime, and $\deg_x a_0 + \deg_x a_1 + \deg_x a_2 \leq 4$. Assume that there are at most three points $t_0 \in \bar{k}$ such that $a_l(x,t_0)$ is identically zero and $a_m(x,t_0)a_n(x,t) \in \bar{k}(x)^{*2}$, for some $\{l,m,n\} = \{0,1,2\}$. Provided we are not in an explicitly described exceptional case the Manin obstruction is shown to be the only obstruction to the Hasse principle and weak approximation for any smooth and projective model of the variety given by the displayed equation (see Proposition 2.10 for a precise statement). An explicit example of a different kind can be found in the end of the paper.

Notation and conventions

All cohomology groups in this paper are Galois or étale cohomology groups. Let G be a finite group, and M be an F-module. Then $Ш^i_\omega(G,M)$ denotes the subgroup of $H^i(G,M)$ consisting of the classes whose image under the restriction map to every cyclic subgroup of G is zero.

For a Γ-module M we write $H^i(k,M)$ for $H^i(\Gamma,M)$. When M, as an abelian group, is of finite type and torsion-free we define $Ш^i_\omega(k,M) := Ш^i_\omega(G,M)$, for $i = 1,2$, where G is the (finite) image of the action of Γ on M. Note that these groups are finite.

Let k' the smallest extension of k such that the action of $\mathrm{Gal}(\bar{k}/k')$ on M is trivial. Write $F = \mathrm{Gal}(k'/k)$. Then $H^1(k',M) = 0$ because $H^1(k',\mathbf{Z}) = 0$. The restriction-inflation sequence for M gives $H^1(k,M) = H^1(G,M)$. There is also an exact sequence

$$0 \to H^2(G, M) \to H^2(k, M) \to H^2(k', M),$$

so that $Ш^i_\omega(k, M)$ is naturally a subgroup of $H^i(k, M)$.

When k is a number field we denote by Ω_k the set of places of k, and by k_v the completion of k at the place v. Let $\mathbf{A}_k$ be the ring of adèles of k. It is an easy corollary of Tchebotarev's density theorem that for a number field k the group $Ш^i_\omega(k, M)$ consists of $\alpha \in H^i(k, M)$ such that the localisation $\alpha_v \in H^i(k_v, M)$ is trivial for almost all places $v \in \Omega_k$.

By a variety X over a field k we understand in this paper a separated scheme of finite type over $\operatorname{Spec} k$. The Brauer group $\operatorname{Br} X = H^2_{\text{ét}}(X, \mathbf{G}_m)$ is equipped with a natural filtration $\operatorname{Br}_0 X \subset \operatorname{Br}_1 X \subset \operatorname{Br} X$, where $\operatorname{Br}_0 X = \operatorname{Im}[\operatorname{Br} k \to \operatorname{Br} X]$ and $\operatorname{Br}_1 X = \operatorname{Ker}[\operatorname{Br} X \to \operatorname{Br} \overline{X}]$. Set $\bar{k}[X]^* = H^0(\overline{X}, \mathbf{G}_m)$. There is a Hochschild–Serre spectral sequence

$$H^p(k, H^q(\overline{X}, \mathbf{G}_m)) \Rightarrow H^{p+q}(X, \mathbf{G}_m).$$

It yields a canonical map $(\operatorname{Br} \overline{X})^\Gamma \to H^2(k, \operatorname{Pic} \overline{X})$. When $\bar{k}[X]^* = \bar{k}^*$ we obtain the exact sequence of low degree terms

$$0 \to (\operatorname{Pic} \overline{X})^\Gamma / \operatorname{Pic} X \to \operatorname{Br} k \to \operatorname{Br}_1 X \to H^1(k, \operatorname{Pic} \overline{X}) \to H^3(k, \bar{k}^*) \to H^3(X, \mathbf{G}_m). \tag{1}$$

Recall that $H^3(k, \bar{k}^*) = 0$ if k is a number field ([CF], 7.11.4), or if $k = k_0(t)$ where k_0 is a number field (see [H94], proof of Thm. 3.5.1). When k is arbitrary, a k-point on X defines a section of the maps $\operatorname{Br} k \to \operatorname{Br} X$ and $H^3(k, \bar{k}^*) \to H^3(X, \mathbf{G}_m)$. If this is the case, (1) shows that $\operatorname{Br}_0 X = \operatorname{Br} k$ and that the map $\operatorname{Br}_1 X \to H^1(k, \operatorname{Pic} \overline{X})$ is surjective.

The unramified Brauer group of the field $k(X)$ with respect to k is a subgroup $\operatorname{Br}_{\text{nr}}(k(X)/k) \subset \operatorname{Br}(k(X))$; it is isomorphic to the Brauer group of any smooth and proper model of X ([G], 6.2).

Recall that for any number field k local class field theory gives an injective map $j_v : \operatorname{Br} k_v \to \mathbf{Q}/\mathbf{Z}$, which is an isomorphism for v finite. Let X be a proper, smooth and geometrically integral k-variety. Then the set of adelic points $X(\mathbf{A}_k)$ is just $\prod_{v \in \Omega_k} X(k_v)$. Assume that $X(\mathbf{A}_k) \neq \emptyset$, and let $\overline{X(k)}$ be the closure of the diagonal image of $X(k)$ in $X(\mathbf{A}_k)$. Set

$$X(\mathbf{A}_k)^{\operatorname{Br}} = \{(P_v) \in X(\mathbf{A}_k) | \sum_{v \in \Omega_k} j_v(A(P_v)) = 0 \text{ for any } A \in \operatorname{Br} X\}.$$

(The sum is well-defined, [CS87a] III, [S01], 5.2). The reciprocity law of global class field theory implies that

$$\overline{X(k)} \subset X(\mathbf{A}_k)^{\operatorname{Br}}.$$

In particular, the condition $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$ is an obstruction to the existence of a k-rational point on X; this is the Manin (or Brauer–Manin) obstruction to the Hasse principle. The condition $X(\mathbf{A}_k)^{\mathrm{Br}} \neq X(\mathbf{A}_k)$ is the Brauer–Manin obstruction to weak approximation. One says that the Brauer–Manin obstruction to weak approximation is the only one if $\overline{X(k)} = X(\mathbf{A}_k)^{\mathrm{Br}}$.

1. Torsors and the Brauer group

In this section k is any field of characteristic zero.

1.1. Torsors under tori and their relative compactifications

By T we shall denote an algebraic k-torus with module of characters $\widehat{T}$. Let $p : X \to \operatorname{Spec} k$ be a smooth and geometrically integral k-variety, and let $f : Y \to X$ be a torsor under T. Equivalently, Y is a smooth and geometrically integral k-variety equipped with a (scheme-theoretically) free action of T such that $Y/T = X$.

We shall frequently use the Leray spectral sequence

$$H^p(X, R^q f_* \mathbf{G}_m) \Rightarrow H^{p+q}(Y, \mathbf{G}_m). \tag{2}$$

By the relative Rosenlicht lemma ([CS87a], Prop. 1.4.2) we have an exact sequence of sheaves on X in the étale topology:

$$1 \to \mathbf{G}_m \to f_* \mathbf{G}_m \to p^* \widehat{T} \to 0. \tag{3}$$

Here $p^*\widehat{T}$ is a sheaf on X obtained from the Γ-module $\widehat{T}$. Note that $H^1(S, \mathbf{Z}) = 0$ for any normal integral scheme S ([SGA 4], IX 3.6 (ii)). This implies in particular that $H^1(\overline{X}, p^*\widehat{T}) = 0$.

Lemma 1.1 *Let $f : Y \to X$ be a torsor under a torus T. Then*
(i) $R^1 f_* \mathbf{G}_m = 0$.
(ii) If $T = \mathbf{G}_m$, then $R^2 f_* \mathbf{G}_m = 0$.

Proof. (i) is established in the proof of ([CS87a], Prop. 2.1.1). For the convenience of the reader we reproduce this argument here in the case $T = \mathbf{G}_m$ alongside with the proof of (ii).

Let x be a geometric point of X, R be the strictly henselian local ring of X at x, and K be the field of fractions of R. The stalk of $R^1 f_* \mathbf{G}_m$ at x is $\operatorname{Pic}(Y \times_X R)$, and the stalk of $R^2 f_* \mathbf{G}_m$ at x is $\operatorname{Br}(Y \times_X R)$. Note that $H^1(R, \mathbf{G}_m) = 0$ for any local ring, hence the scheme $Y \times_X R$ is isomorphic to $\mathbf{G}_{m,R}$. We have $\operatorname{Pic} \mathbf{G}_{m,S} = \operatorname{Pic} S$ for any normal base scheme S. Now $H^1(R, \mathbf{G}_m) = 0$ which implies $\operatorname{Pic}(Y \times_X R) = 0$. Therefore, $R^1 f_* \mathbf{G}_m = 0$.

Recall that a ring of finite type over an excellent ring (resp. the localization of an excellent ring) is excellent ([EGA4], 7.8.3), hence the local ring $\mathcal{O}_{X,x}$ is excellent. Since k is of characteristic zero, Hironaka's resolution of singularities holds for $\mathcal{O}_{X,x}$. By [EGA4], 7.9.5 and 18.8.17, the ring R, and hence also the scheme $\mathbf{G}_{m,R}$, is excellent. Therefore the Brauer group $\operatorname{Br}\mathbf{G}_{m,R}$ fits into the following exact sequence of abelian groups ([G], Cor. 6.2):

$$0 \to \operatorname{Br}\mathbf{A}^1_R \to \operatorname{Br}\mathbf{G}_{m,R} \to H^1(R, \mathbf{Q}/\mathbf{Z}) \to 0.$$

It is known that $\operatorname{Br}\mathbf{A}^1_R = \operatorname{Br} R = 0$ ([M], IV.2.13). Similarly $H^1(R, \mathbf{Z}/n) = 0$ for any $n > 0$ by [M], III.3.11 (because the residue field is algebraically closed), hence $H^1(R, \mathbf{Q}/\mathbf{Z}) = 0$. This proves that $\operatorname{Br}\mathbf{G}_{m,R} = 0$, hence $R^2 f_*\mathbf{G}_m = 0$. QED

From (2), (3) and statement (i) of this lemma we obtain an exact sequence of Γ-modules ([CS87a], Prop. 2.1.1, see also [S01], 1.3):

$$1 \to \bar{k}[X]^*/\bar{k}^* \xrightarrow{f^*} \bar{k}[Y]^*/\bar{k}^* \to \widehat{T} \to \operatorname{Pic}\overline{X} \xrightarrow{f^*} \operatorname{Pic}\overline{Y} \to 0. \tag{4}$$

The map $\widehat{T} \to \operatorname{Pic}\overline{X}$ in this exact sequence is called the *type* of the torsor $f : Y \to X$. The torsor $f : Y \to X$ is called *universal* if its type is an isomorphism. In this case we have $\operatorname{Pic}\overline{Y} = 0$. When $\bar{k}[X]^* = \bar{k}^*$ the type of the torsor $f : Y \to X$ is injective if and only if $\bar{k}[Y]^* = \bar{k}^*$. In particular, this property holds for universal torsors, and implies $\operatorname{Br}_1 Y = \operatorname{Br}_0 Y = \operatorname{Br} k$ by (1).

In this paper we define a relative smooth compactification of a torsor under a torus as follows.

Definition 1.2 *Let $f : Y \to X$ be a torsor under a torus. A relative smooth compactification of $f : Y \to X$ is a geometrically integral variety Z endowed with a smooth and proper morphism $g : Z \to X$ with connected and reduced geometric fibres, together with an embedding $i : Y \hookrightarrow Z$ such that $i(Y)$ is open and dense in Z, and $f = g \circ i$.*

The existence of a relative smooth compactification in characteristic 0 follows from Hironaka's theorem.

Recall [CS77] that an exact sequence of Γ-modules

$$0 \to M \to P \to F \to 0,$$

such that M, P, F are free abelian groups of finite rank, is called a *flasque* resolution of M if the following conditions hold.

(a) P is a permutation Γ-module, that is, P has a $\mathbf{Z}$-basis permuted by Γ.

(b) F is flasque, that is, the Tate cohomology group $H^{-1}(G', F)$ is trivial for any subgroup $G' \subset G$, where G is the image of the action of Γ on F.

Lemma 1.3 *Let $f : Y \to X$ be a torsor under a torus T of type $\tau : \widehat{T} \to \operatorname{Pic} \overline{X}$. Let $g : Z \to X$ be a relative smooth compactification of $f : Y \to X$. Then*

(i) there is a natural injection of Γ-modules $\widehat{T} \to \operatorname{Div}_{\overline{Z} \setminus \overline{Y}}(\overline{Z})$; it defines a flasque resolution of $\widehat{T}$:

$$0 \to \widehat{T} \to \operatorname{Div}_{\overline{Z} \setminus \overline{Y}}(\overline{Z}) \to \operatorname{Div}_{\overline{Z} \setminus \overline{Y}}(\overline{Z})/\widehat{T} \to 0;$$

(ii) we have

$$\operatorname{Ker}[g^* : H^2(k, \operatorname{Pic} \overline{X}) \to H^2(k, \operatorname{Pic} \overline{Z})] = \tau_*(\text{Ш}^2_\omega(k, \widehat{T})).$$

Proof. We prove (i) and (ii) simultaneously. Since Z is regular we have an exact sequence of sheaves on Z:

$$0 \to \mathbf{G}_{m,Z} \to i_*\mathbf{G}_{m,Y} \to \mathcal{D}iv_{Z \setminus Y} \to 0. \tag{5}$$

The sheaf $\mathcal{D}iv_{Z \setminus Y}$ is defined as the direct sum of $j_*\mathbf{Z}$, where j is the embedding into Z of a closed integral subset of codimension 1 contained in $Z \setminus Y$, for all such closed subsets of Z. The sequence (5) is exact since Z is regular, and hence the Weil divisors coincide with Cartier divisors (see [M], II, Example 3.9). Let $\mathcal{D}^+(X)$ (resp. $\mathcal{D}^+(Z)$) be the derived category of bounded below complexes of étale scheaves on X (resp. on Z). The derived functor $\mathbf{R}g_* : \mathcal{D}^+(Z) \to \mathcal{D}^+(X)$ gives an exact triangle in $\mathcal{D}^+(X)$:

$$\mathbf{R}g_*\mathbf{G}_{m,Z} \to \mathbf{R}g_*(i_*\mathbf{G}_{m,Y}) \to \mathbf{R}g_*\mathcal{D}iv_{Z \setminus Y}.$$

In $\mathcal{D}^+(X)$ there is also an exact triangle defined by (3). We have a natural isomorphism $f_*\mathbf{G}_{m,Y} = g_*i_*\mathbf{G}_{m,Y}$, and a canonical morphism $\mathbf{G}_{m,X} \to g_*\mathbf{G}_{m,Z}$ (which is also an isomorphism since g is proper). These morphisms define a morphism of exact triangles in $\mathcal{D}^+(X)$:

$$\begin{array}{ccccc} \mathbf{G}_{m,X} & \to & f_*\mathbf{G}_{m,Y} & \to & p^*\widehat{T} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{R}g_*\mathbf{G}_{m,Z} & \to & \mathbf{R}g_*(i_*\mathbf{G}_{m,Y}) & \to & \mathbf{R}g_*\mathcal{D}iv_{Z \setminus Y} \end{array} \tag{6}$$

Let K be the function field $\overline{k}(X)$, $K_0 = k(X)$. We have the following identifications:

$H^1(\overline{X}, f_*\mathbf{G}_{m,Y}) = \operatorname{Pic} \overline{Y}$ in view of (2) and Lemma 1.1 (i);

$H^1(\overline{Z}, i_*\mathbf{G}_{m,Y}) = \operatorname{Pic} \overline{Y}$ since $R^1 i_*\mathbf{G}_{m,Y} = 0$ ([CS87a], p. 393);

$H^0(\overline{Z}, \mathcal{D}iv_{Z \setminus Y}) = \operatorname{Div}_{\overline{Z} \setminus \overline{Y}}(\overline{Z})$ ([CS87a], (i) on p. 393), likewise

$H^0(\operatorname{Spec} \bar{k}(Z), \mathcal{D}iv_{Z \setminus Y}) = \operatorname{Div}_{Z_K \setminus Y_K}(Z_K)$;

$H^1(\overline{Z}, \mathcal{D}iv_{Z \setminus Y}) = 0$ ([CS87a], (ii) on p. 393).

From (6) we obtain the following commutative diagram of Γ-modules with exact rows and columns, which is the definition of F:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
\bar{k}[Y]^*/\bar{k}^* & \to & \widehat{T} & \to & \mathrm{Pic}\,\overline{X} & \to & \mathrm{Pic}\,\overline{Y} & \to & 0 \\
|| & & \downarrow & & \downarrow & & || & & \\
\bar{k}[Y]^*/\bar{k}^* & \to & \mathrm{Div}_{\overline{Z}\setminus\overline{Y}}(\overline{Z}) & \to & \mathrm{Pic}\,\overline{Z} & \to & \mathrm{Pic}\,\overline{Y} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & F & = & F & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{7}
$$

The arrows in the right hand square are the natural maps. The top row comes from the exact sequence (4); the map $\widehat{T} \to \mathrm{Pic}\,\overline{X}$ is the type τ of the torsor $f : Y \to X$. The injective map $\widehat{T} \to \mathrm{Div}_{\overline{Z}\setminus\overline{Y}}(\overline{Z})$ is the one whose existance was claimed in (i).

Let us now show that F is a flasque Γ-module. Let $k' \subset \overline{k}$ be the smallest extension of k such that $\mathrm{Gal}\,(\bar{k}/k')$ acts trivially on the modules in diagram (7). Call G the finite group $\mathrm{Gal}\,(k'/k)$.

Note that the generic fibre $Z_{K_0} = Z \times_X K_0$ of $g : Z \to X$ is a smooth compactification of the principal homogeneous space $Y_{K_0} = Y \times_X K_0$ of the torus $T_{K_0} = T \times_X K_0$. We identify $\Gamma = \mathrm{Gal}\,(\overline{k}/k)$ with $\mathrm{Gal}\,(K/K_0)$. Then we have a canonical isomorphism of Γ-modules $K[Y_{K_0}]^*/K^* = \widehat{T}$ (see, e.g. [S01], Lemma 2.4.3). This leads to an exact sequence of Γ-modules

$$
0 \to \widehat{T} \to \mathrm{Div}_{\,Z_K \setminus Y_K}(Z_K) \to \mathrm{Pic}\,Z_K \to 0. \tag{8}
$$

The exact sequence (8) will not change if we replace the extension $K = \overline{k}(X)$ of $K_0 = k(X)$ by the extension $L = \overline{k}(Y)$ of $L_0 = k(Y)$. Indeed, since K_0 is integrally closed in L_0 the Galois group $\mathrm{Gal}\,(L/L_0)$ is canonically isomorphic to $\Gamma = \mathrm{Gal}\,(K/K_0)$. On the other hand, we are dealing with Γ-modules which are free abelian groups of finite type that do not change after the passage from K to L. The advantage now is that Y has a canonical L_0-point given by the embedding of the generic point into Y. Hence the Galois module $\mathrm{Pic}\,Z_L = \mathrm{Pic}\,Z_K$ is the Picard module of a smooth compactification of the torus T_L. The Galois group Γ acts on $\mathrm{Pic}\,Z_L$ via its quotient G. By a theorem of Voskresenskiĭ ([V], Thm. 4.6) the G-module $\mathrm{Pic}\,Z_L$ is flasque.

The G-module $\mathrm{Div}_{\overline{Z}\setminus\overline{Y}}(\overline{Z})$ is the direct sum of the module of 'vertical' divisors $\mathrm{Div}_{\,v}$, that is, the divisors on $\overline{Z}$ that do not intersect the generic fibre Z_K, and the module of 'horizontal' divisors, that is, the divisors on $\overline{Z}$ that are Zariski closures of the divisors on Z_K with support in $Z_K \setminus Y_K$. The G-module of horizontal divisors is thus identified with $\mathrm{Div}_{\,Z_K\setminus Y_K}(Z_K)$. Restriction to the generic fibre defines a map of

G-modules $\mathrm{Pic}\,\overline{Z} \to \mathrm{Pic}\,Z_K$. This map is surjective because Z is regular. Its kernel obviously contains $\mathrm{Pic}\,\overline{X}$, hence it factors through a surjective map $F \to \mathrm{Pic}\,Z_K$.

The composition of the map $\widehat{T} \to \mathrm{Div}_{\overline{Z}\setminus\overline{Y}}(\overline{Z})$ from diagram (7) with the map $\mathrm{Div}_{\overline{Z}\setminus\overline{Y}}(\overline{Z}) \to \mathrm{Div}_{Z_K\setminus Y_K}(Z_K)$ given by the restriction of divisors to the generic fibre Z_K, is no other than the second map from (8). Indeed, the right hand vertical arrow in (6) gives rise to a commutative diagram

$$\begin{array}{ccc} H^0(\overline{X}, p^*\widehat{T}) & \to & H^0(\mathrm{Spec}\,\bar{k}(X), p^*\widehat{T}) \\ \downarrow & & \downarrow \\ H^0(\overline{Z}, \mathcal{D}iv_{Z\setminus Y}) & \to & H^0(\mathrm{Spec}\,\bar{k}(Z), \mathcal{D}iv_{Z\setminus Y}) \end{array}$$

which is precisely what we need:

$$\begin{array}{ccc} \widehat{T} & = & \widehat{T} \\ \downarrow & & \downarrow \\ \mathrm{Div}_{\overline{Z}\setminus\overline{Y}}(\overline{Z}) & \to & \mathrm{Div}_{Z_K\setminus Y_K}(Z_K) \end{array}$$

Putting everything together we obtain a commutative diagram of G-modules with exact rows

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \widehat{T} & = & \widehat{T} & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Div}_v & \to & \mathrm{Div}_{\overline{Z}\setminus\overline{Y}}(\overline{Z}) & \to & \mathrm{Div}_{Z_K\setminus Y_K}(Z_K) & \to & 0 \\ & & || & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Div}_v & \to & F & \to & \mathrm{Pic}\,Z_K & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \qquad (9)$$

Since Div_v is a permutation module, it is flasque by Shapiro's lemma. Now the bottom line of the diagram shows that F is flasque. This establishes (ii).

In particular, we have $H^{-1}(C, F) = 0$ for any cyclic subgroup $C \subset G$. By the periodicity of Tate cohomology of cyclic groups we obtain $H^1(C, F) = 0$ for any cyclic subgroup $C \subset G$.

We now go back to diagram (7). The just established property of F, and the fact that $\mathrm{Div}_{\overline{Z}\setminus\overline{Y}}(\overline{Z})$ is a permutation G-module, and hence

$$H^1(k, \mathrm{Div}_{\overline{Z}\setminus\overline{Y}}(\overline{Z})) = Ш^2_\omega(k, \mathrm{Div}_{\overline{Z}\setminus\overline{Y}}(\overline{Z})) = 0,$$

enable us to conclude that $Ш^2_\omega(k, \widehat{T})$ coincides with the injective image of $H^1(k, F)$ in $H^2(k, \widehat{T})$ (cf. [CS87b], the proof of Prop. 9.5 (ii)). The statement (ii) now follows from diagram (7). QED

Remark 1.4 We have associated a canonical flasque resolution of $\widehat{T}$ to any relative smooth compactification of a torsor under T. This generalizes the construction due to Voskresenskiĭ (see [V]) and Colliot-Thélène–Sansuc [CS77] corresponding to the case when the base scheme X is the spectrum of a field.

One can give a shorter proof under the assumption $\bar{k}[Y]^* = \bar{k}^*$ as follows. Define $F = \mathrm{Pic}\,\overline{Z}/\mathrm{Pic}\,\overline{X}$, then formally define the left hand column of (7) by taking the kernels of the restriction maps to $\mathrm{Pic}\,\overline{Y}$. Now observe that the restriction to Z_K gives a map $F \to \mathrm{Pic}\,Z_K$ with kernel Div_v. Indeed, this kernel is generated by the classes of divisors from Div_v. If $(\phi) \in \mathrm{Div}_v$ for some non-constant function $\phi \in \bar{k}(Z)^*$, then the restriction to Z_K shows, since Z_K is proper and does not intersect with (ϕ), that $\phi \in \bar{k}(X)^*$. But this is impossible because then (ϕ) has a non-trivial intersection with $\overline{Y}$.

The proof we have given explains where the map $\widehat{T} \to \mathrm{Div}_{\overline{Z}\setminus\overline{Y}}\overline{Z}$ comes from.

1.2. The pull-back of the Brauer group to a torsor

In connection with arithmetic applications of torsors under tori we are interested in describing the cokernel of the map $f^* : \mathrm{Br}\,X \to \mathrm{Br}\,Y$, especially when Y is a universal torsor. Arguably a more essential question is the computation of the cokernel of the map $f^*_{\mathrm{nr}} : \mathrm{Br}_{\mathrm{nr}}\,(k(X)/k) \to \mathrm{Br}_{\mathrm{nr}}\,(k(Y)/k)$. Colliot-Thélène and Sansuc pointed out that when k is algebraically closed the map f^*_{nr} is an isomorphism. (This follows from the fact that in this situation Y is birationally equivalent to $X \times_k \mathbf{P}^r_k$.)

Our first result concerns the cokernel of $f^* : \mathrm{Br}\,X \to \mathrm{Br}\,Y$ over an algebraically closed field.

Theorem 1.5 *Let k be an algebraically closed field of characteristic* 0. *Let X be a smooth and geometrically integral variety over k such that $k[X]^* = k^*$ and* $\mathrm{Pic}\,X$ *is a free abelian group of finite type. Let $f : Y \to X$ be a torsor under a k-torus such that $k[Y]^* = k^*$ and* $\mathrm{Pic}\,Y$ *is a free abelian group of finite type (for example, a universal torsor). Then the map $f^* : \mathrm{Br}\,X \to \mathrm{Br}\,Y$ is an isomorphism.*

Note that after [CS87a] the injectivity of $f^* : \mathrm{Br}\,X \to \mathrm{Br}\,Y$ is known to be true for any torsor under a torus, but the surjectivity does not hold in general. A well known example is the torsor $\mathbf{G}^2_{m,k} \to \mathbf{G}_{m,k}$ under $\mathbf{G}_{m,k}$. Here $\mathrm{Br}\,\mathbf{G}_{m,k} = 0$, but $\mathrm{Br}\,\mathbf{G}^2_{m,k} = \mathbf{Q}/\mathbf{Z}$.

Proof of Theorem 1.5. Let X be a smooth and geometrically connected variety over an *algebraically closed* field k, such that $k[X]^* = k^*$ and $\mathrm{Pic}\,X$ is torsion-free. Let $f : Y \to X$ be a torsor under T. We denote its type by τ_Y. The assumptions of Theorem 1.5 are equivalent to the condition that τ_Y is injective and $\tau_Y(\widehat{T})$ is a primitive sublattice of $\mathrm{Pic}\,X$.

Let us choose a $\mathbf{Z}$-basis in $\widehat{T} \simeq \mathbf{Z}^n$, and factor $f : Y \to X$ into a composition of torsors under $\mathbf{G}_m$:

$$Y = Y_0 \to Y_1 \to \ldots \to Y_n = X,$$

where each Y_i is a k-variety. Then we have $H^0(Y_i, \mathbf{G}_m) = k^*$. It is also clear that $\mathrm{Pic}\, Y_i$ is a torsion free abelian group of finite type. It is therefore enough to prove Theorem 1.5 when $T = \mathbf{G}_m$. We assume this from now on.

The spectral sequence (2) and Lemma 1.1 imply that $\mathrm{Br}\, Y = H^2(X, f_*\mathbf{G}_m)$. Now the exact sequence (3) together with the remark that follows it, and the fact that $\mathrm{Br}\, Y$ is a torsion group lead to an exact sequence

$$0 \to \mathrm{Br}\, X \to \mathrm{Br}\, Y \to H^2(X, \mathbf{Z})_{\mathrm{tors}}. \tag{10}$$

The Kummer exact sequence gives rise to the exact sequence

$$0 \to k[X]^*/k[X]^{*n} \to H^1(X, \mathbf{Z}/n) \to (\mathrm{Pic}\, X)[n] \to 0.$$

Now our assumptions imply $H^1(X, \mathbf{Z}/n) = 0$. Using the exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\times n} \mathbf{Z} \to \mathbf{Z}/n \to 0$$

we see that $H^2(X, \mathbf{Z})[n] = 0$ for any $n > 0$, hence $H^2(X, \mathbf{Z})_{\mathrm{tors}} = 0$. It now follows from (10) that the map $f^* : \mathrm{Br}\, X \to \mathrm{Br}\, Y$ is an isomorphism. Theorem 1.5 is proved. QED

Over general fields the situation is more complicated.

Theorem 1.6 *Let k be a field of characteristic 0 such that $H^3(k, \bar{k}^*) = 0$. Let X be a smooth and geometrically integral k-variety such that $\bar{k}[X]^* = \bar{k}^*$ and $\mathrm{Pic}\, \overline{X}$ is a free abelian group of finite type.*

(a) *Let $f : Y \to X$ be a torsor under a k-torus T, such that $\bar{k}[Y]^* = \bar{k}^*$ and $\mathrm{Pic}\, \overline{Y}$ is a free abelian group of finite type (for example, a universal torsor). Let $\tau : \widehat{T} \to \mathrm{Pic}\, \overline{X}$ be the type of $f : Y \to X$. Let P_τ be the intersection of the image of $(\mathrm{Br}\, \overline{X})^\Gamma \to H^2(k, \mathrm{Pic}\, \overline{X})$ with $\tau_*(H^2(k, \widehat{T}))$ in $H^2(k, \mathrm{Pic}\, \overline{X})$. Then there is an exact sequence of abelian groups*

$$0 \to \mathrm{Br}\, X/\mathrm{Br}_1\, X \to \mathrm{Br}\, Y/\mathrm{Br}_1\, Y \to P_\tau \to 0.$$

(b) *Let $f : Y \to X$ be a universal torsor. Then there is a natural isomorphism $\mathrm{Br}\, Y/\mathrm{Br}\, k = (\mathrm{Br}\, \overline{X})^\Gamma$ such that the canonical map $\mathrm{Br}\, X \to (\mathrm{Br}\, \overline{X})^\Gamma$ identifies with the map $\mathrm{Br}\, X \to \mathrm{Br}\, Y/\mathrm{Br}\, k$ defined by f^*.*

Note that, in the assumptions of this theorem, the surjectivity of the map $f^* : \mathrm{Br}\, X \to \mathrm{Br}\, Y$ for a universal torsor $f : Y \to X$ is equivalent to the surjectivity of the map $\mathrm{Br}\, X \to (\mathrm{Br}\, \overline{X})^\Gamma$.

The following particular case of Theorem 1.6 (b) is valid over any field of characteristic 0.

Corollary 1.7 *Let k be a field of characteristic* 0*. Let X be a smooth and geometrically integral k-variety such that $\bar{k}[X]^* = \bar{k}^*$, $\operatorname{Pic}\overline{X}$ is a free abelian group of finite type, and $(\operatorname{Br}\overline{X})^\Gamma = 0$. If $f : Y \to X$ is a universal torsor, then $\operatorname{Br} Y = \operatorname{Br} k$.*

This corollary follows from Theorem 1.5 and the exact sequence (1) applied to Y. It improves an earlier similar result of Colliot-Thélène and Sansuc valid for smooth compactifications of universal torsors over projective varieties.

Proof of Theorem 1.6. The natural map of Γ-modules $f^* : \operatorname{Br}\overline{X} \to \operatorname{Br}\overline{Y}$ is an isomorphism by Theorem 1.5. Hence $(\operatorname{Br}\overline{X})^\Gamma \to (\operatorname{Br}\overline{Y})^\Gamma$ is also an isomorphism.

We now exploit the functoriality of the Hochschild–Serre spectral sequence

$$H^p(\Gamma, H^q(\overline{X}, \mathbf{G}_m)) \Rightarrow H^{p+q}(X, \mathbf{G}_m)$$

with respect to the morphism $f : Y \to X$. In view of our assumption $H^3(k, \overline{k}^*) = 0$ this spectral sequence gives rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \to & \operatorname{Br} X/\operatorname{Br}_1 X & \to & (\operatorname{Br}\overline{X})^\Gamma & \to & H^2(k, \operatorname{Pic}\overline{X}) \\ & & \downarrow & & || & & \downarrow \\ 0 & \to & \operatorname{Br} Y/\operatorname{Br}_1 Y & \to & (\operatorname{Br}\overline{Y})^\Gamma & \to & H^2(k, \operatorname{Pic}\overline{Y}) \end{array} \qquad (11)$$

The statement (a) now follows from this diagram combined with exact sequence (4). Taking into account that $\operatorname{Pic}\overline{Y} = 0$ for a universal X-torsor Y one obtains (b). Theorem 1.6 is proved. QED

The following theorem, in which the torsor is replaced by a relative smooth compactification, seems to be more useful in applications.

Theorem 1.8 *We keep the notation and conditions of Theorem* 1.6 (a)*. Let Q_τ be the intersection of the image of $(\operatorname{Br}\overline{X})^\Gamma \to H^2(k, \operatorname{Pic}\overline{X})$ with $\tau_*(\text{Ш}^2_\omega(k, \widehat{T}))$ in $H^2(k, \operatorname{Pic}\overline{X})$. Then for any relative smooth compactification $Z \to X$ of $Y \to X$ the following sequence of abelian groups is exact:*

$$0 \to \operatorname{Br} X/\operatorname{Br}_1 X \to \operatorname{Br} Z/\operatorname{Br}_1 Z \to Q_\tau \to 0.$$

Proof. The beginning of the proof is similar to the proof of Theorem 1.6. Let $g : Z \to X$ be any relative smooth compactification of $f : Y \to X$. This time we use the functoriality of the Hochschild–Serre spectral sequence

$$H^p(\Gamma, H^q(\overline{X}, \mathbf{G}_m)) \Rightarrow H^{p+q}(X, \mathbf{G}_m)$$

with respect to the morphism $g : Z \to X$.

The natural map $g^* : (\operatorname{Br}\overline{X})^\Gamma \to (\operatorname{Br}\overline{Z})^\Gamma$ is an isomorphism. Indeed, by [G] the map $i^* : \operatorname{Br}\overline{Z} \to \operatorname{Br}\overline{Y}$ is injective (since $Y \subset Z$ is an open and dense embedding of regular varieties). The composed map $i^*g^* = f^* : \mathrm{Br}\,\overline{X} \to \mathrm{Br}\,\overline{Y}$ is an isomorphism by Theorem 1.5. Thus $g^* : \mathrm{Br}\,\overline{X} \to \mathrm{Br}\,\overline{Z}$ is also an isomorphism. All these maps are homomorphisms of Γ-modules, hence $g^* : (\mathrm{Br}\,\overline{X})^\Gamma \to (\mathrm{Br}\,\overline{Z})^\Gamma$ is an isomorphism.

It is clear that $H^0(\overline{Z}, \mathbf{G}_m) = \overline{k}^*$ since the same property is true for Y. In view of our assumption that $H^3(k, \overline{k}^*) = 0$ the Hochschild–Serre spectral sequence now gives rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Br}\,X/\mathrm{Br}_1\,X & \to & (\mathrm{Br}\,\overline{X})^\Gamma & \to & H^2(k, \mathrm{Pic}\,\overline{X}) \\ & & \downarrow & & || & & \downarrow \\ 0 & \to & \mathrm{Br}\,Z/\mathrm{Br}_1\,Z & \to & (\mathrm{Br}\,\overline{Z})^\Gamma & \to & H^2(k, \mathrm{Pic}\,\overline{Z}) \end{array} \tag{12}$$

The proof is now completed by an easy diagram chase and an application of Lemma 1.3. QED

For a proper variety X we obtain, as a corollary, a result about the cokernel of the map $f^*_{\mathrm{nr}} : \mathrm{Br}\,X = \mathrm{Br}_{\mathrm{nr}}\,(k(X)/k) \to \mathrm{Br}_{\mathrm{nr}}\,(k(Y)/k)$.

Corollary 1.9 *In the assumptions of Theorem* 1.6 (a) *assume further that X is proper. Then we have an exact sequence*

$$0 \to \mathrm{Br}\,X/\mathrm{Br}_1\,X \to \mathrm{Br}_{\mathrm{nr}}\,(k(Y)/k)/\mathrm{Br}_{\,\mathrm{nr},1}(Y) \to Q_\tau \to 0,$$

where

$$\mathrm{Br}_{\,\mathrm{nr},1}(Y) := \mathrm{Ker}[\mathrm{Br}_{\mathrm{nr}}\,(k(Y)/k) \to \mathrm{Br}_{\mathrm{nr}}\,(\bar{k}(Y)/\bar{k})].$$

The corollary follows from Theorem 1.8 because in this case Z is a smooth and proper model of Y, hence $\mathrm{Br}\,Z = \mathrm{Br}_{\mathrm{nr}}\,(k(Y)/k)$.

Corollary 1.10 *In the assumptions of Theorem* 1.6 (a) *assume further that X is proper and Y is a universal torsor. Then* $\mathrm{Br}_{\mathrm{nr}}\,(k(Y)/k) = f^*_{\mathrm{nr}}(\mathrm{Br}_{\mathrm{nr}}\,(k(X)/k))$ *whenever* $(\mathrm{Br}\,\overline{X})^\Gamma = 0$ *or* $\text{Ш}^2_\omega(k, \mathrm{Pic}\,\overline{X}) = 0$.

More precisely, in the case when X is a proper variety over a number field k the unramified Brauer group of a universal torsor $f : Y \to X$ can be described as follows:

$$\mathrm{Br}_{\mathrm{nr}}\,(k(Y)/k)/f^*_{\mathrm{nr}}(\mathrm{Br}_{\mathrm{nr}}\,(k(X)/k)) = \bigcap \mathrm{Ker}\big[(\mathrm{Br}\,\overline{X})^\Gamma/\mathrm{Br}\,X \to (\mathrm{Br}\,\overline{X_v})^{\Gamma_v}/\mathrm{Br}\,X_v\big].$$

Here $\Gamma_v = \mathrm{Gal}\,(\overline{k}_v/k_v)$, $X_v = X \times_k k_v$, $\overline{X_v} = X \times_k \overline{k_v}$, and the intersection is taken over the places v of k such that the Γ-module $\mathrm{Pic}\,\overline{X}$ is unramified at v (the inertia group at v acts trivially). Such are almost all the places of k. It would be interesting to give an example of a proper and smooth variety X satisfying the assumptions of Theorem 1.6 (a) such that the group described in the last displayed formula is non-zero.

Remark 1.11 If $Y(k) \neq \emptyset$, then the existence of a section of the map $H^3(k, \bar{k}^*) \to H^3(Y, \mathbf{G}_m)$ makes the assumption $H^3(k, \bar{k}^*) = 0$ in Theorem 1.6 redundant. The same is true for Theorem 1.8 if $Z(k) \neq \emptyset$.

2. Arithmetic of vertical torsors

In this section we cross the fibration method of [H94] and [H97] with the descent method of [S90], [S96] and [CS00].

2.1. Main theorem

We shall consider a geometrically integral variety X equipped with a surjective morphism f to $\mathbf{P}^1_k$. Let η be the generic point of $\mathbf{P}^1_k$, $K = k(\eta) = k(\mathbf{P}^1_k)$. Let $X_K := X \times_{\mathbf{P}^1_k} K$ be the generic fibre of f, $X_{\bar{k}(\eta)} = X \times_K \bar{k}(\eta)$, and let $X_{\overline{K}} := X \times_K \overline{K}$ be the generic geometric fibre.

Recall that a k-scheme V is called *split* if it contains a non-empty geometrically integral open k-subscheme (which need not be dense). By a further restriction such a subscheme can be assumed to be smooth.

The *rank* of f, denoted by $\mathrm{rk}\, f$, is defined as the sum of the degrees of the closed points $P \in \mathbf{P}^1_k$ such that the fibre $X_P = f^{-1}(P)$ is not split over the residue field $k(P)$ of P. Let $U \subset X$ be a dense open subset such that

(a) the restriction of f to U is a smooth surjective morphism with integral closed fibres;

(b) for any closed point $P \in \mathbf{P}^1_k$ the fibre U_P is geometrically integral if and only if X_P is split;

(c) the generic fibre U_K coincides with X_K.

Note that such a dense open subset exists if and only if $f : X \to \mathbf{P}^1_k$ locally in étale topology has a section at every closed point $P \in \mathbf{P}^1_k$. Equivalently, every closed fibre of f contains an irreducible component of multiplicity 1. We define

$$M_U = \mathrm{Ker}\,[\mathrm{Pic}\,\overline{U} \to \mathrm{Pic}\, X_{\bar{k}(\eta)}].$$

A Γ-module M such that $M = M_U$ for a dense open subset $U \subset X$ satisfying (a), (b) and (c) will be called a *vertical module associated to* f. A U-torsor of type $M_U \hookrightarrow \mathrm{Pic}\,\overline{U}$ is called *vertical*. It is easy to see that M_U is a torsion-free abelian group ([S96], Prop. 3.2.3).

Theorem 2.1 *Let X be a smooth, projective and geometrically integral variety over a number field k, and let $f : X \to \mathbf{P}^1_k$ be a dominant morphism with geometrically integral generic fibre. Suppose that the following conditions hold:*

(1) $\mathrm{rk}\, f \leq 2$;

(2) $\mathrm{Pic}\, X_{\overline{K}}$ *is torsion-free,* $\mathrm{Br}\, X_{\overline{K}}$ *is finite, and X_K has a $\bar{k}(\eta)$-point;*

(3) *either* $\mathrm{Br}\, X_{\bar{k}(\eta)} = 0$*, or there exists a vertical module M associated to f such that* $Ш^2_\omega(k, M) = 0$.

Then if the Manin obstruction is the only obstruction to the Hasse principle or to weak approximation for the k-fibres of f in a dense open subset of $\mathbf{P}^1_k$, then the same property holds for X.

This theorem should be compared with Thm. A of [CS00]. Under additional cohomological assumptions we replace the arithmetic hypothesis that the k-fibres of f satisfy the Hasse principle or weak approximation by a weaker condition.

Let us describe the idea of proof of Theorem 2.1. As in [S96] and [CS00] we can lift an adelic point $\{P_v\}$ of $U(\mathbf{A}_k)^{\mathrm{Br}}$ to an adelic point on some vertical torsor $Y \to U$. The difficulty, pointed out in [H97], 4.5, is that in our situation there is no reason why Y should satisfy the Hasse principle: the hypothesis that the Manin obstruction to the Hasse principle is the only one for the k-fibres of f is too weak for this. However, using Proposition 2.3 below and the results of Section 1 we show that our condition (3) implies that $\{P_v\}$ lifts to an adelic point $\{Q_v\} \in Y(\mathbf{A}_k)^{\mathrm{Br}\, Z}$, where Z is a relative smooth compactification of Y. On the other hand, we show that on Z the Manin obstruction to the Hasse principle is the only one: this is a consequence of the hypothesis $\mathrm{rk}\, f \leq 2$ and the results of [H97] (which replace Thm. 2.1 of [S96]). Therefore the assumption $U(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies that $Y(k)$ is not empty, hence $U(k)$ is non-empty. As $U(\mathbf{A}_k)^{\mathrm{Br}}$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}}$ (by [CS00], Prop. 1.1) the statement regarding the Hasse principle follows. The argument for weak approximation is similar.

Remark 2.2 If we just want to deal with the case $\mathrm{Br}\, X_{\bar{k}(\eta)} = 0$, the results of Section 1 are not needed. Indeed, this assumption implies $\mathrm{Br}\, \overline{X} = 0$ and we can apply [H97], 4.5.

The key observation for the proof of Theorem 2.1 is the following statement.

Proposition 2.3 *Let X be a smooth and geometrically integral variety over a number field k, such that $H^0(\overline{X}, \mathbf{G}_m) = \overline{k}^*$ and $\mathrm{Pic}\, \overline{X}$ is an abelian group of finite type. Let T be a k-torus, and τ be an injection of Γ-modules $\widehat{T} \to \mathrm{Pic}\, \overline{X}$. Suppose that $\{M_v\} \in X(\mathbb{A}_k)^{\mathrm{Br}_1 X}$. Then there exists a torsor $f : Y \to X$ under T of type τ, and a point $\{P_v\} \in Y(\mathbb{A}_k)^{\mathrm{Br}_1 Y}$ such that $M_v = f(P_v)$ for all places v.*

Proof. Let S be the k-group of multiplicative type dual to the Γ-module $\mathrm{Pic}\, \overline{X}$, $S \to T$ be the surjective k-homomorphism dual to τ, and S_1 be its kernel. We denote by Id the canonical isomorphism $\widehat{S} \to \mathrm{Pic}\, \overline{X}$. Under our assumptions the main theorem of the descent theory ([CS87a], [S01], 6.1.2) states that an adelic point on X, which is orthogonal to $\mathrm{Br}_1 X$ with respect to the Brauer–Manin pairing, can be lifted to an adelic point $\{N_v\}$ on some universal torsor $Z \to X$ of type Id. Then the quotient $Y = Z/S_1$ is an X-torsor under T of type τ (this trivially follows from the functoriality of type with respect to the structure group).

Let us prove that $Z \to Y$ is a universal torsor. Since Z is a universal X-torsor and $H^0(\overline{X}, \mathbf{G}_m) = \overline{k}^*$, we see from (4) that $H^0(\overline{Z}, \mathbf{G}_m) = \overline{k}^*$ and $\mathrm{Pic}\, \overline{Z} = 0$. The same exact sequence (4) considered for the torsor $Z \to Y$ now shows that $\widehat{T} \to \mathrm{Pic}\, \overline{Y}$ is an isomorphism, hence Z is a universal Y-torsor.

Now let $\{P_v\}$ be the image of $\{N_v\}$ on Y. The inverse main theorem of the descent theory states that the image of an adelic point on a universal torsor is orthogonal to $\mathrm{Br}_1 Y$. This is exactly what we wanted to prove. QED

Note that this statement is non-trivial in the following sense: in general the natural map $\mathrm{Br}_1 X \to \mathrm{Br}_1 Y$ is not surjective.

We shall need the following small refinement of [H97], Thm. 3.2.1:

Proposition 2.4 *Let F be a closed subset of $\mathbf{A}^n_k$ of codimension at least 2, $L = k(\mathbf{A}^n_k)$. Let V be a quasi-projective k-variety equipped with a surjective morphism $p : V \to \mathbf{A}^n_k \setminus F$ with split fibres and geometrically integral generic fibre V_L. Let X_L be a smooth and projective model of V_L. Assume the following conditions:*

(a) *The restriction of p to a general line in $\mathbf{A}^n_k$ has a section.*

(b) $\mathrm{Br}\, X_{\overline{L}}$ *is finite, and* $\mathrm{Pic}\,(X_{\overline{L}})$ *torsion-free, where $\overline{L}$ is an algebraic closure of L, and $X_{\overline{L}} = X_L \times_L \overline{L}$.*

(c) *There exists a non-empty open subset Ω of $\mathbf{A}^n_k$ such that for each $\theta \in \Omega(k)$, the Manin obstruction to the Hasse principle (resp. to weak approximation) is the only one for smooth and projective models of the fibre V_θ.*

Then we have the following statements:

(*i*) *The Manin obstruction to the Hasse principle (resp. to weak approximation) is the only one for a smooth and projective model X of V.*

(*ii*) *Assume further that $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$. Let Ω' be any non-empty open subset of $\mathbf{A}^n_k$. Then there exists a k-point θ of Ω' such that a smooth and projective model of the fibre V_θ has a k-point.*

In particular, if V_L is smooth and projective, and $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$, then $p(V(k))$ is Zariski dense in $\mathbf{A}^n_k$.

Proof. (*i*) is precisely [Ha97], Thm. 3.2.1. We prove (*ii*) using the same induction argument. We can find a k-variety V' equipped with a k-morphism $p' : V' \to \mathbf{A}^n_k \setminus F$ such that p' is projective and its generic fibre is smooth and L-birational to V_L. We can also suppose that p' satisfies the same assumptions as p (see [H97], proof of Thm. 3.2.1). In the case $n = 1$ it follows immediately from [H97], proof of Prop. 3.1.1 (see also [H94], proof of Thm. 4.2.1) that if $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$, then the image of $V'(k)$ by p' is Zariski dense in $\mathbf{A}^1_k$. This proves (*ii*) for $n = 1$ via Nishimura's lemma.

Assume that (*ii*) is proved for any integer less than n. As in [H97], proof of Thm. 3.2.1, we can find a k-point M_0 in $\mathbf{A}^n_k$ such that the k-morphism $\pi : V' \setminus p'^{-1}(M_0) \to \mathbf{P}^{n-1}_k$ obtained by composing p' with the map $\mathbf{A}^n_k \setminus \{M_0\} \to \mathbf{P}^{n-1}_k$, has the property that the restriction of π to $\pi^{-1}(\mathbf{A}^{n-1}_k)$ satisfies the assumptions of the proposition. (Recall that V' is k-birationally equivalent to V.) If $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$, then the induction hypothesis implies that for any non-empty Zariski open subset $\mathcal{U} \subset \mathbf{P}^{n-1}_k$ there exists a k-point m of $\mathcal{U}$ (corresponding to an affine line D passing through M_0) such that a smooth and projective model Z of $\pi^{-1}(m) = {p'}^{-1}(D)$ has a k-point. This implies that $Z(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$. Applying the case $n = 1$ to the restriction of p' to ${p'}^{-1}(D)$ (this is possible by [H97], Lemma 3.2.2), we obtain that each non-empty open subset of D contains a k-rational point θ such that a smooth and projective model of the fibre ${p'}^{-1}(\theta)$ has a k-point. This proves (ii) for p', hence for p by Nishimura's lemma. QED

Proof of Theorem 2.1. The case $\operatorname{rk} f \leq 1$ is known ([H97], Prop. 3.1.1, here condition (3) is not required). We now assume that $\operatorname{rk} f = 2$.

If $\operatorname{Br} X_{\bar{k}(\eta)} = 0$, then we take $U = X$. In this case $\operatorname{Br} \overline{U}$ is trivial since it is a subgroup of $\operatorname{Br} X_{\bar{k}(\eta)}$ by Grothendieck's theorem.

If $\operatorname{Br} X_{\bar{k}(\eta)} \neq 0$ we choose a vertical module M associated to f with $\text{Ш}^2_\omega(k, M) = 0$, which is possible by condition (3). In this case we define U to be the dense open subset of X such that $M = M_U$.

We observe that $\bar{k}[U]^* = \bar{k}^*$. Indeed, the generic fibre $U_{\bar{k}(\eta)} = X_{\bar{k}(\eta)}$ of $\overline{U} \to \mathbf{P}^1_{\bar{k}}$ is proper and geometrically integral, hence every regular function on $U_{\bar{k}(\eta)}$ comes from a rational function on $\mathbf{P}^1_{\bar{k}}$. However, no such function can be regular on $\overline{U}$ because of surjectivity of the morphism $\overline{U} \to \mathbf{P}^1_{\bar{k}}$.

We now claim that condition (2) implies the finiteness of $\operatorname{Br} U_{\bar{k}(\eta)}$. Indeed, $\operatorname{Br} \bar{k}(\eta) = 0$ by Tsen's Theorem; $H^1(\bar{k}(\eta), \operatorname{Pic} X_{\overline{K}})$ is finite because $\operatorname{Pic} X_{\overline{K}}$ is torsion-free; and $\operatorname{Br} X_{\overline{K}}$ is finite. The exact sequence (1) now shows that $\operatorname{Br} X_{\bar{k}(\eta)} = \operatorname{Br} U_{\bar{k}(\eta)}$ is finite. Since $\operatorname{Br} \overline{U}$ injects into $\operatorname{Br} U_{\bar{k}(\eta)}$ we conclude that $\operatorname{Br} \overline{U}$ is also finite.

We also note that $\operatorname{Pic} \overline{U}$ is torsion-free, because $\operatorname{Pic} U_{\bar{k}(\eta)}$ and $M = \operatorname{Ker}[\operatorname{Pic} \overline{U} \to \operatorname{Pic} U_{\bar{k}(\eta)}]$ are torsion-free.

Let $\{Q_v\}$ be an adelic point of $X(\mathbf{A}_k)^{\mathrm{Br}}$, and Σ be a finite set of places of k containing all the places where we want to approximate. By [CS00], Prop. 1.1, there exists an adelic point $\{M_v\}$ in $U(\mathbf{A}_k)^{\mathrm{Br}}$. If we deal with weak approximation, then the points M_v for $v \in \Sigma$ can be chosen to be as close as we wish to the corresponding points Q_v.

Let τ be the natural injection of Γ-modules $M \hookrightarrow \operatorname{Pic} \overline{U}$. By Proposition 2.3 there exist a vertical torsor $Y \to U$ and an adelic point $\{P_v\} \in Y(\mathbf{A}_k)^{\mathrm{Br}_1 Y}$ that is mapped to $\{M_v\}$ by the structure map $Y \to U$. Let $g : Z \to U$ be a relative smooth compactification of $Y \to U$. Recall that $\operatorname{Br} \overline{U}$ injects into $\operatorname{Br} X_{\bar{k}(\eta)}$, thus condition (3) implies that $Q_\tau = 0$ in the notation of Theorem 1.8 (with X replaced by U). Therefore by Theorem 1.8 the map $\operatorname{Br} U \to \operatorname{Br} Z/\operatorname{Br}_1 Z$ is surjective. Let α be an arbitrary element of $\operatorname{Br} Z$. Then we can write $\alpha = \beta + \gamma$, where $\beta \in \operatorname{Im}[\operatorname{Br} U \to \operatorname{Br} Z]$ and $\gamma \in \operatorname{Br}_1 Z$. Since $\{M_v\} \in U(\mathbf{A}_k)^{\mathrm{Br}}$, the projection formula gives that $\sum_{v\in\Omega} j_v(\beta(P_v)) = 0$. On the other hand, $\sum_{v\in\Omega} j_v(\gamma(P_v)) = 0$ since $\{P_v\} \in Y(\mathbf{A}_k)^{\mathrm{Br}_1 Y}$ and $\operatorname{Br}_1 Z \subset \operatorname{Br}_1 Y$. This proves that $\{P_v\} \in Y(\mathbf{A}_k)^{\mathrm{Br}\, Z}$.

The assumption $\operatorname{rk} f = 2$ allows us to apply [S96], Cor. 1.4 (see also [CS00]). This yields an open subset $W \subset \mathbf{A}^n_k$ with $\operatorname{codim}(\mathbf{A}^n_k \backslash W) \geq 2$, equipped with a morphism

$W \to \mathbf{P}^1_k$ such that Y is k-birationally equivalent to $Y' := U \times_{\mathbf{P}^1_k} W$. Moreover, the following properties hold: the projection $p : Y' \to W$ is surjective, its generic fibre is projective, and all fibres of p are split. Furthermore, the restriction of p to a general line in $\mathbf{A}^n_{\bar{k}}$ has a section because we assumed that the generic fibre of f had a $\bar{k}(\eta)$-point. Let $Y'_{\overline{L}}$ be the generic geometric fibre of p over the algebraic closure $\overline{L}$ of $L = k(\mathbf{A}^n_k)$. Then the assumptions that $\mathrm{Pic}\, X_{\overline{K}}$ is torsion-free and $\mathrm{Br}\, X_{\overline{K}}$ is finite imply that $\mathrm{Pic}\, Y'_{\overline{L}}$ is torsion-free and $\mathrm{Br}\, Y'_{\overline{L}}$ is finite. Let V be a smooth and projective compactification of Y. By [H97], Thm. 3.2.1, on V the Brauer–Manin obstruction to the Hasse principle (or to weak approximation if we assumed the same property for the k-fibres of f) is the only one. Since $\{P_v\} \in Y(\mathbf{A}_k)^{\mathrm{Br}\, Z} \subset V(\mathbf{A}_k)^{\mathrm{Br}}$ we can use Proposition 2.4 (ii). Thus $p(Y'(k))$ is Zariski dense in W, which implies $U(k) \neq \emptyset$. If we deal with weak approximation, then there exists a k-point P on Y that is arbitrary close to P_v for $v \in \Sigma$. Projecting the point P to U, we obtain a k-point on X, which is as close to Q_v for $v \in \Sigma$ in the k_v-topology as we wish. QED

There is a natural way to construct an open set U with properties (a), (b) and (c). For any closed point P of $\mathbf{P}^1_k$ the fibre X_P contains an irreducible component of multiplicity 1. (Indeed, X_K has a $\bar{k}(\eta)$-point and X is proper, thus f has a $\bar{k}$-section. Since X is regular, the intersection point of this section with X_P must be regular on X_P ([S96], Lemma 1.1 (b)). In particular, this point belongs to an irreducible component of multiplicity 1.) For every closed point P choose an irreducible component $E_P \subset X_P$ of multiplicity 1. Whenever there is a geometrically irreducible component we choose it as E_P. Let F_P be the union of all the remaining components of X_P, and let G_P be the singular locus of E_P. Define $U \subset X$ as the complement to the (finite) union of the F_P and the G_P for all the closed points $P \in \mathbf{P}^1_k$. This dense open subset $U \subset X$ satisfies properties (a), (b) and (c). It is clearly enough to consider so constructed open subsets to get all possible vertical modules.

Let $\Delta \subset \mathbf{P}^1_k$ be the union of the closed points P such that the fibre X_P is not split (equivalently, the fibre U_P is not geometrically integral). For each $P \in \Delta$ we denote by k_P the integral closure of $k(P)$ in the function field $k(E_P)$ of E_P.

The inclusion $\mathrm{Pic}\, \mathbf{P}^1_{\bar{k}} \hookrightarrow M = M_U$ gives rise to the exact sequence

$$0 \to \mathbf{Z} \to M \to N \to 0. \tag{13}$$

Here the Γ-module N is isomorphic to the module of characters of the torus

$$\prod_{P \in \Delta} \mathbf{R}_{k(P)/k}(\mathbf{R}^1_{k_P/k(P)} \mathbf{G}_m),$$

where $\mathbf{R}_{k(P)/k}$ is the Weil restriction of scalars from $k(P)$ to k, and $\mathbf{R}^1_{k_P/k(P)}\mathbf{G}_m$ is the kernel of the norm map $\mathbf{R}_{k_P/k(P)}\mathbf{G}_m \to \mathbf{G}_m$. For any finite extension k'/k we write $\mathbf{Z}[k'/k]$ for the induced Γ-module $\mathbf{Z}[\Gamma/\Gamma_{k'}]$, where $\Gamma_{k'} = \mathrm{Gal}\,(\bar{k}/k')$. Then the Γ-module N is the direct sum of the natural 'diagonal' quotients of $\mathbf{Z}[k_P/k]$ by $\mathbf{Z}[k(P)/k]$, for $P \in \Delta$.

2.2. Variants and applications

We now give a sufficient condition for the vanishing of $Ш^2_\omega(k, M)$ in the situation of Theorem 2.1. For any finite field extension k'/k we write $H^1(k'/k, \mathbf{Q}/\mathbf{Z})$ for the kernel of the restriction map $H^1(k, \mathbf{Q}/\mathbf{Z}) \to H^1(k', \mathbf{Q}/\mathbf{Z})$. If k' and k'' are subfields of $\bar{k}$, then $k'k''$ is the smallest subfield of $\bar{k}$ containing k' and k''.

Proposition 2.5 *Let X be a smooth, projective, and geometrically integral variety over a number field k, and let $f : X \to \mathbf{P}^1_k$ be a dominant morphism with geometrically integral generic fibre. Assume that two k-fibres of f, say X_P and X_Q, are not split, and all the other closed fibres are split. Assume also that the fibre X_P (resp. X_Q) contains a multiplicity 1 component E_P (resp. E_Q) such that if k_P (resp. k_Q) is the integral closure of k in $k(E_P)$ (resp. in $k(E_Q)$), then*

(i) $Ш^2_\omega(k, \mathbf{Z}[k_P/k]/\mathbf{Z}) = Ш^2_\omega(k, \mathbf{Z}[k_Q/k]/\mathbf{Z}) = 0$, and

(ii) for some k-embeddings $k_P \subset \bar{k}$, $k_Q \subset \bar{k}$, the following natural map is surjective:

$$H^1(k_P/k, \mathbf{Q}/\mathbf{Z}) \oplus H^1(k_Q/k, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^1(k_P k_Q/k, \mathbf{Q}/\mathbf{Z}).$$

Then $Ш^2_\omega(k, M) = 0$ for the vertical module M constructed from E_P and E_Q. Moreover, (i) and (ii) hold as long as k_P and k_Q are cyclic extensions of k.

Proof. From the exact sequence (13) we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^1(k, N) & \longrightarrow & H^1(k, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^2(k, M) & \longrightarrow & H^2(k, N) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_{v \in \Omega_k} H^1(k_v, N) & \longrightarrow & \bigoplus_{v \in \Omega_k} H^1(k_v, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & \bigoplus_{v \in \Omega_k} H^2(k_v, M) & \longrightarrow & \bigoplus_{v \in \Omega_k} H^2(k_v, N)
\end{array}$$

From the description of N given above it is clear that $N = \mathbf{Z}[k_P/k]/\mathbf{Z} \oplus \mathbf{Z}[k_Q/k]/\mathbf{Z}$. Thus $Ш^2_\omega(k, N) = 0$ by condition (i). Hence any non-zero $\alpha \in Ш^2_\omega(k, M)$ is the image of an element $\beta \in H^1(k, \mathbf{Q}/\mathbf{Z}) = \mathrm{Hom}(\Gamma, \mathbf{Q}/\mathbf{Z})$. We have

$$H^1(k, N) = H^1(k_P/k, \mathbf{Q}/\mathbf{Z}) \oplus H^1(k_Q/k, \mathbf{Q}/\mathbf{Z}).$$

The restriction of the differential $H^1(k, N) \to H^2(k, \mathbf{Z}) = H^1(k, \mathbf{Q}/\mathbf{Z})$ from (13) to the subgroup $H^1(k_P/k, \mathbf{Q}/\mathbf{Z}) \subset H^1(k, N)$ is the natural inclusion of $H^1(k_P/k, \mathbf{Q}/\mathbf{Z})$ into $H^1(k, \mathbf{Q}/\mathbf{Z})$ (and the same for Q). Indeed, the exact sequence of Γ-modules

$$0 \to \mathbf{Z} \to \mathbf{Z}[k_P/k] \to \mathbf{Z}[k_P/k]/\mathbf{Z} \to 0,$$

where the second arrow is the diagonal embedding, naturally maps to the exact sequence (13). The desired fact now follows from the commutative diagram where the left hand arrow is the natural inclusion:

$$\begin{array}{ccc} H^1(k,N) & \to & H^2(k,\mathbf{Z}) \\ \uparrow & & || \\ H^1(k,\mathbf{Z}[k_P/k]/\mathbf{Z}) & \hookrightarrow & H^2(k,\mathbf{Z}) \end{array}$$

Because of the assumption (ii) a character of Γ belongs to the image of $H^1(k,N)$ if and only if its restriction to $\Gamma' := \mathrm{Gal}\,(\bar{k}/k_Pk_Q) = \mathrm{Gal}\,(\bar{k}/k_P) \cap \mathrm{Gal}\,(\bar{k}/k_Q)$ is trivial. Since $\alpha \neq 0$ the restriction of β to Γ' is non-trivial. By Tchebotarev's density theorem (cf. [H94], Prop. 2.2.1), there exist infinitely many places v of k such that the image β_v of β in $H^1(k_v,\mathbf{Q}/\mathbf{Z})$ is not zero, but k_v is a direct summand of $k_Pk_Q \otimes_k k_v$. For such a place v we have $H^1(k_v,N)=0$, hence β_v does not come from $H^1(k_v,N)$. This contradicts the fact that $\alpha_v = 0$ for almost all v.

Now suppose that k_P/k and k_Q/k are cyclic. Then $\text{Ш}^2_\omega(k,N) = 0$ by [CS77], Prop. 2 and Prop. 6, [CS87b], Prop. 9.1. Since Γ/Γ' is a subgroup of the finite abelian group $\mathrm{Gal}\,(k_P/k) \times \mathrm{Gal}\,(k_Q/k)$, any character of Γ/Γ' extends to this group. Therefore the condition (ii) is also satisfied. QED

Remark 2.6 Condition (i) holds as long as the degrees $[k_P : k]$ and $[k_Q : k]$ are prime numbers. It also holds if the Galois closures of k_P/k and k_Q/k are metacyclic (*loc. cit.*).

Here is a particular case of Theorem 2.1 with conditions that are easy to check.

Theorem 2.7 *Let X be a smooth, projective, and geometrically integral variety over a number field k, and let $f : X \to \mathbf{P}^1_k$ be a dominant morphism with geometrically integral generic fibre. Suppose that the following conditions hold:*

(1) *two k-fibres of f, say X_P and X_Q, are not split, and all the other closed fibres are split;*

(2) $\mathrm{Pic}\, X_{\overline{K}}$ *is torsion-free,* $\mathrm{Br}\, X_{\overline{K}}$ *is finite, and X_K has a $\bar{k}(\eta)$-point;*

(3) *the fibre X_P (resp. X_Q) contains a multiplicity* 1 *component E_P (resp. E_Q) such that if k_P (resp. k_Q) is the integral closure of k in $k(E_P)$ (resp. in $k(E_Q)$), then $[k_P : k]$ (resp. $[k_Q : k]$) is a prime number.*

Then if the Manin obstruction is the only obstruction to the Hasse principle or to weak approximation for the k-fibres of f in a dense open subset of $\mathbf{P}^1_k$, then the same property holds for X.

Proof. In view of the previous remark we only need to check condition (ii) of Proposition 2.5. We fix k-embeddings of k_P and k_Q into $\bar{k}$. Let $p = [k_P : k]$, $q = [k_Q : k]$. If k_P and k_Q are Galois (hence cyclic) extensions of k, we can apply Proposition 2.5. If k_Q is not Galois and $p \neq q$, then a non-trivial cyclic subextension of k_Pk_Q/k is either k_P, or is linearly disjoint from k_P and k_Q. The latter case is impossible because neither p^2 nor q^2 divides $[k_P k_Q : k] = pq$. Thus Proposition 2.5 still applies in this case. From now on we assume that $p = q$. We need the following lemma:

Lemma 2.8 *Let D be a subgroup of the symmetric group $\mathcal{S}_p$ not isomorphic to $\mathbf{Z}/p$. Then* $\mathrm{Hom}(D, \mathbf{Z}/p) = 0$.

Proof. Let $\varphi : D \to \mathbf{Z}/p$ be a non-zero homomorphism. If $\sigma \in D$ is not a p-cycle, then the order of σ is prime to p, hence $\varphi(\sigma) = 0$. In particular, D contains a p-cycle c such that $\varphi(c) \neq 0$. Suppose that D contains an element σ which does not belong to the cyclic group $< c >$. There exists an integer m, $0 < m < p$, such that σc^m has a fixed point. Therefore σc^m is not a p-cycle, hence $\varphi(\sigma c^m) = 0$. If σ is not a p-cycle, then $\varphi(\sigma) = 0$ and we have a contradiction with $\varphi(c) \neq 0$. Thus any non-trivial element of D is a p-cycle. This leads to a contradiction because if σ is a p-cycle not in $< c >$, then σc is not a p-cycle. QED.

We resume the proof of Theorem 2.7. If k_P is Galois and k_Q is not Galois, then the Galois closure k' of k_Q has a non-abelian Galois group $D \subset \mathcal{S}_p$. Since $\mathrm{Hom}(D, \mathbf{Z}/p) = 0$ by Lemma 2.8, k_P and k' are linearly disjoint, that is, $\mathrm{Gal}\,(k_P k'/k) = \mathbf{Z}/p \times D$. Thus the only normal subgroup of $\mathrm{Gal}\,(k_P k'/k)$ of index p is D. This means that the only cyclic subextension of $k_P k_Q/k$ is k_P, and we conclude by applying Proposition 2.5.

Finally, if neither k_P nor k_Q is Galois, the assumption (ii) of Proposition 2.5 fails only if k_P and k_Q are linearly disjoint, and $k_P k_Q$ contains a cyclic extension E/k of degree p. Then $Ek_P = Ek_Q = k_P k_Q$, and $\mathrm{Gal}\,(Ek'/k) = \mathbf{Z}/p \times D$. Define $H_P = \mathrm{Gal}\,(Ek'/k_P)$, $H_Q = \mathrm{Gal}\,(Ek'/k_Q)$. Then $\mathrm{Gal}\,(Ek'/E) = D$, hence $H_P \cap H_Q \subset D$. The projection $H_P \to D$ is an isomorphism (H_P does not contain $\mathbf{Z}/p$ since k_P is not contained in k'). Therefore, $\mathrm{Hom}(H_P, \mathbf{Z}/p) = 0$ by Lemma 2.8, so that the projection $H_P \to \mathbf{Z}/p$ is zero. Hence we obtain $H_P = D$, which is a contradiction because H_P is not normal in $\mathrm{Gal}\,(Ek'/k)$ (recall that k_P/k is not Galois). Theorem 2.7 is proved. QED

We have the following variant of Theorem 2.1 applicable in some cases when $\mathrm{rk}\, f = 3$.

Theorem 2.9 *Let X be a smooth, projective and geometrically integral variety over a number field k, and let $X \to \mathbf{P}^1_k$ be a dominant morphism with geometrically integral generic fibre. Assume that* $\mathrm{rk}\, f \leq 3$ *and every non-split fibre X_P contains a multiplicity* 1 *component E_P such that the integral closure k_P of $k(P)$ in $k(E_P)$ is a quadratic extension of $k(P)$. Assume also the hypothesis* (2) *of Theorem* 2.1*. Then we have the same conclusion as in Theorem* 2.1 *unless* $\mathrm{rk}\, f = 3$ *and there exists a biquadratic field extension $k(\sqrt{a}, \sqrt{b})$ and non-split closed fibres X_{P_1} and X_{P_2} with $k(P_1) = k$, $k(P_2) = k(\sqrt{a})$, $k_{P_1} = k(\sqrt{a})$ and $k_{P_2} = k(\sqrt{a}, \sqrt{b})$.*

Proof. The beginning of the proof is similar to that of Theorem 2.1, so we only indicate the places where this proof is different. We construct $U \subset X$ from the components E_P, then the variety W is the punctured affine cone over the complement to a closed subset of codimension 2 in a smooth projective quadric of dimension 4 (see [S90] or [CS00]). We have a vertical torsor Y such that $Y(\mathbf{A}_k) \neq \emptyset$ and Y is k-birational to $Y' := U \times_{\mathbf{P}^1_k} W$. Thus $W(k) \neq \emptyset$ by the Minkowski–Hasse Theorem. Hence W is k-isomorphic to an open subset of the affine space whose complementary set is of codimension at least 2.

To conclude the proof it is sufficient to check that $Ш^2_\omega(k, M) = 0$ for the vertical module $M = M_U$. Set $n = \mathrm{rk}\, f$ and denote by l_i, l'_i, $1 \leq i \leq n$, the two conjugate components corresponding to the E_P's. The Weyl group $W(\mathrm{B}_n)$ of the root system B_n is the semi-direct product of $(\mathbf{Z}/2)^n$ (permutations interchanging l_i with l'_i) and $\mathcal{S}_n$ (permutations of the indices of the l_i and l'_i). This action gives M and N the structure of a $W(\mathrm{B}_n)$-module. In [KST], Def. 1.2, the $W(R)$-modules $Q(R)$ and $M(R)$ are defined for root systems $R = \mathrm{B}_n$ and C_n. Using the exact sequence (13) we see that $N = Q(\mathrm{B}_n)$ and $M = M(\mathrm{C}_n)$ as W-modules (see [KST], p. 29).

The action of Γ on the set of the l_i and l'_i defines a homomorphism $\Gamma \to W(\mathrm{B}_n)$. Let G be its image. Thm. 1.22 of [KST] describes $Ш^2_\omega(k, M(\mathrm{C}_n)) = Ш^2_\omega(G, M(\mathrm{C}_n))$ in terms of the subgroup $G \subset W(\mathrm{B}_n)$. (Note that by [KST], Prop. 1.19, we have $Ш^2_\omega(G, M(\mathrm{C}_n)) = Ш^2_\omega(G, Q(\mathrm{C}_n))$.) It follows from Cor. 4.14 of [KST] that the assumption $n \leq 3$ implies that $Ш^2_\omega(k, M(\mathrm{C}_n)) \neq 0$ only in the exceptional case described in the statement of the theorem. QED

2.3. Examples: conic bundles over the plane

Let V be the conic bundle over $\mathbf{A}^2_k$ defined in $\mathbf{P}^2_k \times \mathbf{A}^2_k$ by the equation

$$a_0(x,t)X_0^2 + a_1(x,t)X_1^2 + a_2(x,t)X_2^2 = 0, \quad (X_0, X_1, X_2) \in \mathbf{P}^2_k, \quad (x,t) \in \mathbf{A}^2_k, \quad (14)$$

where the $a_i(x,t)$ are non-zero polynomials in x and t. By a simple change of variables one can arrange that the $a_i(x,t)$ are pairwise coprime; we assume this from now on. Then for any given value of t at most one $a_i(x, t_0)$ vanishes. Let us now introduce some notation.

For $i = 0, 1, 2$ let $d_i := \deg_t a_i(x,t)$, and let $b_i(x)$ be the leading coefficient of $a_i(x,t)$ as a polynomial in t. Then we can write

$$a_i(x,t) = b_i(x)t^{d_i} + \text{terms of lower degree in } t.$$

Let Δ_l be the union of closed points $P \in \mathbf{A}^1_k$ such that

$$a_l(x,P) = 0 \quad \text{and} \quad a_m(x,P)a_n(x,P) \in \overline{k(P)}^{*2} \setminus k(P)^{*2},$$

where $\{l,m,n\} = \{0,1,2\}$. Let $\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2$, and let N be the cardinality of $\Delta(\bar{k})$.

Proposition 2.10 *Let V be the subvariety of $\mathbf{P}^2_k \times \mathbf{A}^2_k$ defined by equation* (14). *In the above notation assume the following conditions:*

(i) $\deg_x a_0(x,t) + \deg_x a_1(x,t) + \deg_x a_2(x,t) \leq 4$;

(ii) $N = 1$; *or* $N = 2$ *and we are not in the exceptional case* $\mathbf{E}_2$; *or* $N = 3$, $d_0 \equiv d_1 \equiv d_2 \pmod 2$, *and we are not in the exceptional case* $\mathbf{E}_3$.

Then the Manin obstruction to the Hasse principle and weak approximation is the only one for any smooth and projective model of V.

The exceptional cases are defined as follows:

$\mathbf{E}_2$: $d_l \not\equiv d_m \equiv d_n \pmod 2$, where $\{l,m,n\} = \{0,1,2\}$, and there exists a biquadratic extension $k(\sqrt{a},\sqrt{b})$ and a closed point $P \in \mathbf{A}^1_k$ with residue field $k(P) = k(\sqrt{a})$ such that we have

(1) $-ab_m(x)b_n(x) \in k(x)^{*2}$, and

(2) $a_p(x,P) = 0$, $-ba_q(x,P)a_r(x,P) \in k(P)(x)^{*2}$, where $\{p,q,r\} = \{0,1,2\}$;

$\mathbf{E}_3$: there exists a biquadratic extension $k(\sqrt{a},\sqrt{b})$ and closed points $P_1, P_2 \in \mathbf{A}^1_k$ with residue fields $k(P_1) = k$, $k(P_2) = k(\sqrt{a})$, such that we have

(1) $a_l(x,P_1) = 0$, $-aa_m(x,P_1)a_n(x,P_1) \in k(x)^{*2}$, where $\{l,m,n\} = \{0,1,2\}$, and

(2) $a_p(x,P_2) = 0$, $-ba_q(x,P_2)a_r(x,P_2) \in k(P_2)(x)^{*2}$, where $\{p,q,r\} = \{0,1,2\}$.

Proof. Consider the morphism $V \to \mathbf{A}^1_k$ given by the projection to the coordinate t. This morphism naturally extends to a surjective morphism $V' \to \mathbf{P}^1_k$, where the fibre at ∞ is given in $\mathbf{P}^2_k \times \mathbf{A}^1_k$ by the equation:

$$b_0(x)X_0^2 + b_1(x)X_1^2 + b_2(x)X_2^2 = 0$$

if the d_i have the same parity. In the opposite case we can assume that $d_p \not\equiv d_q \equiv d_r \pmod 2$ for some $\{p,q,r\} = \{0,1,2\}$. Then the fibre at ∞ is given by

$$b_q(x)X_q^2 + b_r(x)X_r^2 = 0.$$

Let $P \in \Delta$, say $P \in \Delta_0$. Then (14) shows that the fibre V_P contains an irreducible component of multiplicity 1 defined over the quadratic extension of $k(P)$ given by the square root of $a_1(x,P)a_2(x,P)$. The closed fibres V_Q at the closed points $Q \notin \Delta \cup \{\infty\}$ are geometrically integral.

The restricted morphism $V'_{\text{smooth}} \to \mathbf{P}^1_k$ is still surjective. It extends to a morphism $f : X \to \mathbf{P}^1_k$, where X is a smooth and projective model of V (by Hironaka's theorem). Split fibres of $V'_1 \to \mathbf{P}^1_k$ give rise to split fibres of $X \to \mathbf{P}^1_k$.

The generic fibre $X_{k(t)}$ of $f : X \to \mathbf{P}^1_k$ is a conic bundle surface over $\mathbf{P}^1_{k(t)}$. In particular, $X_{k(t)}$ is a rational surface, defined over $K = k(t)$, hence its geometric Brauer group $\operatorname{Br} X_{\overline{K}}$ is zero and its geometric Picard group $\operatorname{Pic} X_{\overline{K}}$ is torsion-free. Moreover, $X_{k(t)}$ has a $\bar{k}(t)$-point since any smooth conic over $\bar{k}(t)$ has a $\bar{k}(t)$-point by Tsen's theorem. In view of condition (i) each smooth k-fibre X_m of f is a conic bundle over $\mathbf{P}^1_k$ with at most 5 degenerate fibres. Then by the results of [CSS87], [CT] and [SS] the Manin obstruction to the Hasse principle and weak approximation is the only one for X_m.

To apply Theorem 2.9 it remains to check that for $N = 2$ or $N = 3$ we are not in the exceptional case of that theorem. This follows from the description of the fibres at the points of $\Delta \cup \{\infty\}$ given in the beginning of this proof. QED.

Remark 2.11 One can give a variant of the previous proposition, where $a_0(x,t)$ and $a_1(x,t)$ are in $k[t]$, and $a_2(x,t)$ is a product of two irreducible polynomials of degrees 2 and 4 in x. The arithmetic condition on the k-fibres is ensured by a theorem of Swinnerton-Dyer [SD], see [S01], Thm. 7.4.1. However, to satisfy the irreducibility condition one would need a slightly stronger version of Theorem 2.1, namely, a version only requiring that the k-fibres in a Hilbertian subset of $\mathbf{P}^1_k(k)$ have the property that the Manin obstruction is the only one. Such a version can be easily proved along the same lines as Theorem 2.1 (cf. [H97], p. 158).

2.4. Examples: equations of norm type

Let C be a geometrically integral, smooth and proper curve, and let $\pi : C \to \mathbf{P}^1_k$ be a finite morphism of degree n. Write $\pi^{-1}(\mathbf{A}^1_k) = \operatorname{Spec} A$, then A is a finitely generated $k[t]$-module. Since π is flat, A is projective, and hence is free because $k[t]$ is principal. Thus we can choose a basis $\omega_1, \dots, \omega_n$ of the $k[t]$-module A. Let $L = k(C)$; it is also the field of fractions of A. Let $N_t(z_1, \dots, z_n)$ be the norm of $z_1\omega_1 + \dots + z_n\omega_n$ for the finite field extension $L(z_1, \dots, z_n)/k(t)(z_1, \dots, z_n)$. It is clear that $N_t(z_1, \dots, z_n)$ is homogeneous of degree n in variables $z_1, \dots, z_n$.

Let $g(t) \in k[t]$ be a polynomial with leading coefficient $c \neq 0$. Consider the affine k-variety V defined by the equation in variables $z_1, \dots, z_n, t$:

$$N_t(z_1, \dots, z_n) = g(t). \tag{15}$$

Set formally $g(\infty) = 0$ if n does not divide $\deg g$, and $g(\infty) = c$ otherwise. For a closed point $P \in \mathbf{P}^1_k$ we denote by m_P the highest common factor of the ramification indices of of π at closed geometric points in the fibre C_P (for example, $m_P = 1$ if π is unramified at some point over P). Define Δ as the union of closed points $P \in \mathbf{P}^1_k$ such that $g(P) = 0$, and there is no $k(P)$-point $Q \in C_P$ such that π is unramified at Q.

Proposition 2.12 *Let V be an affine variety defined by equation* (15). *In the above notation assume that*

(i) Δ consists of at most two k-points of $\mathbf{P}^1_k$;

(ii) for any $P \in \Delta$ there is a closed point $Q \in C_P$ such that π is unramified at Q, and the residue field $k(Q)$ has prime degree over $k(P) = k$;

(iii) for any closed point $P \in \mathbf{P}^1_k$ we have $m_P = 1$.

Then the Manin obstruction to the Hasse principle and weak approximation is the only one for a smooth and projective model of V.

Proof. The morphism $V_{\text{smooth}} \to \mathbf{A}^1_k$ given by the coordinate t extends to a morphism $f : X \to \mathbf{P}^1_k$, where X is a smooth and projective model of V. Let $K = k(t)$. The generic fibre X_K of f is a smooth compactification of a principal homogeneous space V_K of the K-torus $T_K = \mathbf{R}^1_{L/K}\mathbf{G}_m$. In particular, X_K is a rational variety (over $\overline{K}$), and V_K has a $\bar{k}(t)$-point because $H^1(\bar{k}(t), T_K) = 0$ by Steinberg's Theorem (recall that $\bar{k}(t)$ is a C_1-field). Thus condition (2) of Theorem 2.7 is satisfied.

Let P be a closed point of $\mathbf{P}^1_k$ such that $g(P) \neq 0$. Condition (iii) implies that the fibre V_P is geometrically integral. Indeed, $V_P \times_{k(P)} \overline{k(P)}$ is the product of an affine space and an affine variety given by

$$u_1^{m_1} \ldots u_r^{m_r} = a,$$

where $a \neq 0$. By condition (iii) the highest common factor of the multiplicities m_i is 1. Hence the $\overline{k(P)}$-variety given by the displayed equation is integral.

If P is a closed point of $\mathbf{P}^1_k$ not in Δ but such that $g(P) = 0$, then the fibre V_P is split. Indeed, it follows from the definition of Δ that in this case the $k(P)$-variety V_P contains a geometrically irreducible component of multiplicity 1. Now (i) shows that condition (1) of Theorem 2.7 holds.

By (ii) the fibre X_P for $P \in \Delta$ contains an irreducible component E_P of multiplicity 1 such that the integral closure k_P of k in $k(E_P)$ is of prime degree over k. We use these components to construct a vertical module M. It is a theorem of Colliot-Thélène and Sansuc that (smooth compactifications of) the principal homogeneous spaces of k-tori have the property that the Manin obstruction to the Hasse principle and weak approximation is the only one (see, e.g. [S01], Thm. 6.3.1). For $m \in k$ such that $g(m) \neq 0$ and $\pi : C \to \mathbf{P}^1_k$ is not ramified at m, the fibre V_m is a principal homogeneous space of a k-torus. We can now conclude by applying Theorem 2.7. QED

Remark 2.13 The assumption (iii) of Proposition 2.12 can be satisfied only if the degree n of π is at least 3.

Explicit example. Let C be the smooth and proper model of the curve given by $x^4 + p(t)x + a = 0$, where the degree of the non-constant polynomial $p(t)$ is divisible by 3, and $a \in k^*$. Then $\pi : C \to \mathbf{P}^1_k$ is unramified at a k-point over ∞. We apply Proposition 2.12 to study weak approximation on the variety V given by

$$N_t(z_1, \ldots, z_4) = ct(t-1),$$

where $c \in k^*$. (V has a smooth k-point $t = z_i = 0$.) One checks easily that conditions (i) and (iii) of Proposition 2.12 are satisfied. Condition (ii) is satisfied as long as we assume that none of the polynomials $x^4 + p(0)x + a = 0$ and $x^4 + p(1)x + a = 0$ is irreducible. (The weirdness of these conditions is partly due to the fact that we need an example not covered by the methods already described in literature. For example, we could replace 4 by 3, and require the degree of $p(x)$ to be even, but then the smooth k-fibres satisfy the Hasse principle and weak approximation, so that we would be able to conclude using previously known results.)

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