SUBANALYTIC FUNCTIONS

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0. Introduction.

The main purpose of this paper is to show a strong version of rectylinearization theorem for subanalytic functions (Theorem 2.7). Our original aim of introducing this theorem was the proof of existence of Lipschitz stratification for subanalytic sets [**P**] but it turned out that it can be used also to study arc-analytic functions. Therefore, we have decided to state it separately.

Our approach to subanalytic geometry is based mostly on [**H1**], where the two major techniques used are desingularization and the local flattening theorem. To make our considerations elementary one may use instead Hironaka's desingularization various theorems of Bierstone and Milman [**B-M1**]. Also the local flattening theorem can be replased by a weaker result (Corollary 1.2), where instead of flatness we required only equidimensionality of the morphism. We present a short proof of this result in Section 1. In Section 2 we state and give a proof of our rectylinearization theorem which we use in Section 3 to study the properties of arc-analytic functions.

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1. Local blowings-up.

Let $X \to M$ be a morphism of complex analytic spaces. We asumme for simplicity that M is nonsingular. Let $s_\alpha : W_\alpha \to M$ be a composition of local blowings-up with smooth and nowhere dense centers. By a local blowing-up we mean here a composition of an open imbeding and a blowing-up (see **[H1]** or **[B-M1]** for the exact definitions).Let $E_\alpha \subset: W_\alpha$ denote the union of corresponding exceptional divisors of s_α. By the strict transform (X_α, f_α) of f by s_α we mean the smallest analytic subspace X_α of the fibre product $X \times_M W_\alpha$ which contains $X \times_M (W_\alpha \setminus E_\alpha)$ and the map $f_\alpha : X_\alpha \to W_\alpha$ induced by the projection. Let us denote the other projection $X_\alpha \to X$ by $\tilde{s}_\alpha$. In **[H1]** Hironaka showed the local flattening theorem which we state as follows.

Theorem 1.1. *Let $f : X \to M$ be a morphism of complex analytic spaces and assume that M is nonsingular. Let L and K be compact subsets of X and M respectively. Then, there exists a finite number of analytic morphisms*

$$s_\alpha : W_\alpha \to M,$$

such that:

(1) *each s_α is the composition of a finite sequence of local blowings-up with smooth nowhere dense centers;*

(2) *for each α there is a compact subset K_α of W_α and*

$$\bigcup_\alpha s_\alpha(K_\alpha) = K\,;$$

(3) *the strict transforms $f_\alpha : X_\alpha \to W_\alpha$ are flat at every point $x \in X_\alpha$ corresponding to L (i.e. at every $x \in \tilde{s}_\alpha^{-1}(L)$).*

REMARK. *To have the centers nonsingular we need some kind of desingularization theorem. Since the statement already involves local blowings-up it is enough to use Theorem 4.4 of* [**B-M1**] *(see the end of the proof of Corollary 1.2).*

Hironaka used his flattening theorem to study the properties of subanalytic sets. For this purpose it is enough to have a weaker result with equidimensionality instead of flatness. We present a short proof of this result. The idea of proof is similar to that in [**B**] (one can also use the approach of [**D-D**]).

Corollary 1.2. *Let $f : X \to M$, L, K be as in Theorem 1.1. Then, there exists a finite number of maps*

$$s_\alpha : W_\alpha \to M,$$

satisfying (1) and (2) of Theorem 1.1 and

(3') *the strict transforms $f_\alpha : X_\alpha \to W_\alpha$ satisfy at every point $x \in X_\alpha$ corresponding to L the following equidimensionality property*

$$dim(f_\alpha^{-1}(f_\alpha(x)), x) = dim(X_\alpha, x) - dim M\ .$$

Proof.

Topological preparation.

We start with simple topological preparation which allows us to proceed with induction. Assume that we have constructed a family $t_\beta : V_\beta \to M$, $K_\beta \subset V_\beta$, satisfying the conditions (1) and (2) of the statement. Let $f_\beta : X_\beta \to V_\beta$ denote the corresponding strict transforms. Fix, for each β, a relatively compact subset U_β of V_β which contains K_β. Note that the set $L_\beta \subset X_\beta$ of points of $f_\beta^{-1}(\overline{U_\beta})$ corresponding to L is compact.

Assume that we have found for each $(f_\beta, L_\beta, K_\beta)$ a family $s_{\beta\alpha} : W_{\beta\alpha} \to V_{\beta\alpha}$, $K_{\beta\alpha} \subset W_{\beta\alpha}$, satisfying the statement of Corollary 1.2. Then, it is easy to see that the family of all $t_\beta \circ s_{\beta\alpha}|_{s_{\beta\alpha}^{-1}(U_\beta)}$, $K_{\beta\alpha} \cap s_{\beta\alpha}^{-1}(K_\beta)$ satisfies the statement.

In particular, in the proof we may proceed locally on M. This means that it is enough to show the corollary for a small neighbourhood U_x of each point $x \in K$, a compact subneighbourhood K_x of U_x and $L_x = L \cap f^{-1}(K_x)$.

Induction

Let $n = dimM$, The proof is by induction on

$$s = max_{x \in L} \left\{ dim(f^{-1}(f(x)), x) + n - dim(X, x) \right\}.$$

Since, by topological preparation, our problem is local on M, we may assume that M is an open neighbourhood U of the origin in $\mathbf{C}^n$. Fix $x_0 \in L \cap f^{-1}(0)$ and assume that X is of pure dimension m in a neighbourhood of x_0. By localizing X about x_0 we may assume that f is induced by a projection

$$\begin{array}{ccc} x_0 = 0 \in X & \longrightarrow & U \times \mathbf{C}^k \\ & & \downarrow \\ & & U \end{array}$$

Let $r = dim(f^{-1}(0), 0)$.

If $r = m - n$ there is nothing to prove. Since always $r \geq m - n$, we may assume that $s = r + n - m > 0$.

Let $r < k$. Then, there exists a linear projection $p : \mathbf{C}^k \to \mathbf{C}^r$ such that $(f, p)|_{(X,0)} : (X, 0) \to (U \times \mathbf{C}^r, 0)$ is finite. So its image Z is an m-dimensional analytic subset of a neighbourhood of the origin in $U \times \mathbf{C}^r$. Since $s = r + n - m > 0$, we have $(Z, 0) \neq (U \times \mathbf{C}^r, 0)$, and therefore there exists a nonzero function $F \in \mathcal{O}_{(U \times \mathbf{C}^r, 0)}$ which vanishes identically on $(Z, 0)$. Let us write it in a neighbourhood of the origin as

$$F(y, t) = \sum_\beta \Delta_\beta(y) t^\beta, \quad \text{where } (y, t) \in U \times \mathbf{C}^r.$$

Let $\sigma : U' \to U$ be the blowing-up of the ideal I generated by all Δ_β (so by the finite number of them). Consider the following diagram

$$\begin{array}{ccc} Z & \longleftarrow & Z \times_U U' \subset U' \times \mathbf{C}^r \\ \downarrow & & \downarrow \\ U & \xleftarrow{\sigma} & U' \end{array}$$

Let Z' denote the strict transform of Z by σ. At each point $y_0 \in U'$ the ideal $(\sigma^* I)_{y_0}$ is invertible (i.e. principal and generated by some $\Delta_{\beta_0} \circ \sigma$). Let us denote by y' the coordinates near y_0. Then, near y_0

$$F(\sigma(y')) = \sum_\beta \Delta_\beta(\sigma(y'))t^\beta = \Delta_{\beta_0}(\sigma(y')) \cdot F'(y', t)\,,$$

where

$$F'(y', t) = \sum_\beta \Delta'_\beta(y')t^\beta \text{ and } \Delta'_\beta = (\Delta_\beta \circ \sigma)/(\Delta_{\beta_0} \circ \sigma)\,.$$

It is easy to see that $Z' \subset F'^{-1}(0)$ and F' does not vanishes identically on any fiber of the projection $U' \times \mathbf{C}^r \to U'$.

From the above follows that if we consider the analogous construction for f, then for the strict transforms f', X' of f, X by σ and every point $x' \in X'$ corresponding to x_0 we get

(1.1) $$dim(f'^{-1}(f'(x')), x') + n - dim(X', x') < s\,.$$

Observe that if $r = k$, then we simply apply the above construction without choosing the projection p (so $Z = X$) to obtain (1.1). By the construction, (1.1) holds also for all $x' \in X'$ corresponding to a neighbourhood of x_0 in L.

If X is not of pure dimension, then it is near L the union of finitely many pure dimensional analytic spaces X_i. In this case we apply the above arguments to each $X_i \to M$ separately, let us say by blowing up the ideal I_i. If we blow up the product $\prod I_i$, then each

$(\sigma^* I_i)$ becomes invertible and consequently we get (1.1) for all the points corresponding to a neighbourhood of x_0 in L. Since $L \cap f^{-1}(0)$ is compact we may cover it by a finite number of open subsets V_i, such that we get (1.1) for the points of corresponding of V_i by blowing-up the ideal J_i. Then, by the same argument as above, the blowing-up of the ideal $\prod J_i$ gives (1.1) for the points corresponding to a neighbourhood of $L \cap f^{-1}(0)$ in L.

The only thing which lacks above is the smoothness of the centers. In fact we do not need the blowing-up of a given ideal I but the compositions of local blowings-up with smooth centers σ such that $\sigma^* I$ is locally invertible. This can be achieved by using Desingularization II of Hironaka **[H5]** or Theorem 4.4 of **[B-M1]** together with Lemma 4.7 (ibid.). This ends the proof.

$\square$

Real case.

Let $f : X \to M$ be a morphism of real analytic spaces and assume again that M is nonsingular. Let L,K be compact subsets of X and M respectively. Apply Corollary 1.2 to a complexification $\tilde{f} : \tilde{X} \to \tilde{M}$ of f. It follows from the proof of Corollary 1.2 that in this case the local blowings-up in the statement can be choosen as complexifications of local real blowings-up.

In fact, for $x \in L$ the ideal $I \subset \mathcal{O}_y$, where $y = f(x)$, which we have to blow up can be choosen real (i.e. generated by complexifications of real analytic functions). Therefore we get a blowing-up $\tilde{\sigma} : \tilde{U}' \to \tilde{U}$ which is a complexification of a real analytic blowing-up $\sigma : U' \to U$. Let $\tilde{f}' : \tilde{X}' \to \tilde{U}'$ be the strict transform of $\tilde{f}$ by $\tilde{\sigma}$. Then the set of points of $\tilde{X}'$ corresponding to L is contained in the real part of $\tilde{X}'$ (i.e. invariant under autoconjugation of $\tilde{X}'$). This is certainly the case for the points outside $\tilde{f}'^{-1}(\tilde{E})$ (where $\tilde{E}$

is the exceptional divisor of $\tilde{\sigma}$), and therefore, since $\overline{\tilde{X}' \setminus \tilde{f}'^{-1}(\tilde{E})} = \tilde{X}'$ (set theoretically), holds everywhere.

Thus, from the proof of Corallary 1.2 we can derive its real version (which can be also obtained from 4.17 of [**H1**]).

Corollary 1.3. *Let $\tilde{f} : \tilde{X} \to \tilde{M}$ be a complexification of a morphism of real analytic spaces $f : X \to M$ and assume that M is nonsingular. Let L, K be compact subsets of X and M respectively. Then, one can choose $\tilde{s}_\alpha : \tilde{W}_\alpha \to \tilde{M}$ satisfying the statement of Corollary 1.2 as complexifications of the compositions of finite sequences of real-analytic local blowings-up with smooth nowhere dense centers $s_\alpha : W_\alpha \to M$.*

We can choose also K_α as subsets of W_α.

2. Rectylinearization of subanalytic functions.

In this section we shall prove a version of Rectylinearization Theorem for subanalytic functions (Theorem 2.7).

DEFINITION 2.1. *Let U be an open subset of $\mathbf{R}^n$. We call a function $f : U \to \mathbf{R}$ subanalytic if the closure of its graph Γ_f in $\mathbf{R}^n \times \mathbf{P}^1$ (where $\mathbf{P}^1 = \mathbf{R} \cup \{\infty\}$ is the real projective line) is subanalytic.*

One may also define subanalytic functions as those whose graph is subanalytic only in $\mathbf{R}^n \times \mathbf{R}$ (see for example [**D-L-S**]). In this way one gets a broader class of functions for which our rectylinearization (Theorem 2.7) does not hold. Note that a function f is subanalytic in our sense if and only if f and f^{-1} is subanalytic in the weaker one (see also

[**K1**] the classes SUB(M) and SUBB(M)). Note that both classes coincide for f locally bounded on $\mathbf{R}^n$.

The sum and the product of subanalytic functions is again a subanalytic function ([**K1**]) (on the intersection of the domains) and the same is true for the quotient on the set where the denominator is not zero. Each subanalytic function $f : U \to \mathbf{R}$ is analytic on an open dense subanalytic subset of $\overline{U}$. This shows that the notion of the partial derivatives of f makes sense (on the obvious domain) and it is again a subanalytic function [**K1**].

Let us, throughout this section, denote the standard projections $\mathbf{R}^n \times \mathbf{P}^1 \to \mathbf{R}^n$, $\mathbf{R}^n \times \mathbf{P}^1 \to \mathbf{P}^1$ by π_1 and π_2 respectively.

DEFINITION 2.2. *Let M be an analytic manifold (over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$) and let f be an analytic function on M. We say that f is* <u>*locally normal crossings*</u> *if each point of M admits a coordinate neighbourhood U, with coordinates $x = (x_1, x_2, \dots, x_n)$, such that*

$$f(x) = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} g(x), \tag{2.1}$$

where $x \in U$, $\alpha_i \in \mathbf{N}$ and g is analytic and nowhere vanishes in U.

We say that f is <u>*normal crossings*</u> *if (2.1) can be achieve in one coordinate system ($M = U \subset \mathbf{K}^n$).*

Note that the function which is identically equal to zero <u>is not normal crossings</u>. By this convention, if $\prod f_i$ is (locally) normal crossings, then each of f_i is (locally) normal crossings.

By [**B-M1**] and [**H1**] one may describe the continuous subanalytic functions as follows.

Proposition 2.3. *Let U be an open subset of $\mathbf{R}^n$ and let $f : U \to \mathbf{R}$ be a continuous subanalytic function. Then there exist a real analytic n-dimensional manifold N and* *a proper surjective real analytic mapping $\phi : N \to U$ such that $f \circ \phi$ is locally normal crossings (on these components of N where it does not vanish identically).*

We sketch very briefly the proof of Proposition 2.3.

By the resolution of singularities there exist a real analytic manifold M and a proper real analytic map $\psi : M \to \mathbf{R}^n \times \mathbf{P}^1$ whose image is just $\overline{\Gamma_f}$ (Theorem 0.1 of [**B-M1**] or Proposition 8.1 of [**H1**]). The function $f \circ \pi_1 \circ \psi$ is analytic on $\psi^{-1}(U)$ and now we may apply to it the rectylinearization theorem for analytic functions (Desingularization II [**H2**] or Theorem 4.4 of [**B-M1**]). This gives the result.

DEFINITION 2.4. *A subset X of $\mathbf{R}^n$ is a (closed) quadrant if there is a partition of $\{1, 2, \dots, n\}$ into disjoint subsets I_0, I_- and I_+, such that*

$$X = \{x \in \mathbf{R}^n;\ x_i = 0 \text{ if } i \in I_0,\ x_i \le 0 \text{ if } i \in I_- \text{ and } x_i \ge 0 \text{ if } i \in I_+\}\,.$$

DEFINITION 2.5. *We call a familly of real analytic morphisms*

$$s_\alpha : V_\alpha \to \mathbf{R}^n\,, \alpha \in I$$

locally finite if for each compact $K \subset \mathbf{R}^n$, $s_\alpha^{-1}(K) = \emptyset$ for all but finite number of α.

The rectylinearization of subanalytic sets may be obtain by maps of special form, namely the compositions of local blowings-up (see Theorem 7.1 of [**H1**]). For closed subanalytic subsets of $\mathbf{R}^n$ it can be expressed as follows.

Theorem 2.6. *Let X be a closed subanalytic subset of $\mathbf{R}^n$. Then there exist a locally finite collection of real analytic morphisms*

$$s_\alpha : V_\alpha \to \mathbf{R}^n$$

such that each of them is the composition of a finite sequence of local blowings-up with smooth centers and:

(1) *each V_α is isomorphic to $\mathbf{R}^n$ and there are compact subsets K_α of V_α such that $\bigcup_\alpha s_\alpha(K_\alpha)$ is a neighbourhood of X in $\mathbf{R}^n$;*

(2) *for each α, $s_\alpha^{-1}(X)$ is a union of quadrants in $\mathbf{R}^n$.*

We will show Theorem 2.6 during the proof of Theorem 2.7 (see Remark 2.9). It is interesting to ask whether the rectylinearization of subanalytic functions (Proposition 2.3) may be also obtained by local blowings-up. The following simple example shows that generally this is not possible. The function $f(x) = |x|^{1/r}$, $r \in \mathbf{N}$ is a well defined subanalytic (even semialgebraic) function on $\mathbf{R}$ but any blowing-up can not change $\mathbf{R}$. Note that the obvious substitution $x = \pm y^{2r}$ makes f analytic and normal crossings.

In **[B-M2]** Bierstone and Milman have develloped a method which shows that the compositions of local blowings-up and substitution of powers give rectylinearization of any continuous subanalytic function. We shall prove below that in fact in this compositions it is enough to substitute powers only at the last step after all local blowings-up. This result we shall use in the next section to give a new proof of Theorem 1.4 of **[B-M2]** and in **[P]** to construct a Lipschitz stratification of subanalytic sets.

Theorem 2.7. *(Rectylinearization of subanalytic functions)*

Let U be an open subset of $\mathbf{R}^n$ and let $f : U \to \mathbf{R}$ be a continuous subanalytic function. Then there exist a locally finite collection Ψ of real analytic morphisms

$$\phi_\alpha : W_\alpha \to \mathbf{R}^n$$

such that:

(1) *each W_α is isomorphic to $\mathbf{R}^n$ and there are compact subsets K_α of W_α such*

that $\bigcup_\alpha \phi_\alpha(K_\alpha)$ *is a neighbourhood of* $\overline{U}$*;*

(2) *for each* α *there exist* $r_i \in \mathbf{N}$, $i = 1, \dots, n$, *such that*

$$\phi_\alpha = \sigma_\alpha \circ \psi_\alpha\,,$$

where σ_α *is the composition of finite sequence of local blowings-up with smooth centers and*

$$\psi_\alpha(x) = (\varepsilon_1 x_1^{r_1}, \varepsilon_2 x_2^{r_2}, \dots, \varepsilon_n x_n^{r_n}), \quad \text{for some } \varepsilon_i = -1 \text{ or } 1\,; \tag{2.2}$$

(3) *for each* α

$$\phi_\alpha(W_\alpha) \subset \overline{U}$$

and $f \circ \phi_\alpha$ *extends from* $\phi_\alpha^{-1}(U)$ *on* W_α *to one of the following functions:*

(a) the function identically equal to zero;

(b) a normal crossings ;

(c) the inverse of a normal crossings (this can happen only if $\phi_\alpha(0) \in \overline{U} \setminus U$*);*

(4) *if* $\phi_\alpha = \sigma_\alpha \circ \psi_\alpha \in \Psi$ *and* $\phi_\alpha(0) \in U$, *then* $\phi_\alpha(W_\alpha) \subset U$ *and for each* ψ *like in (2.2) (i.e. with all possible* ε_i*, but fixed* r_i*) the composition* $\sigma_\alpha \circ \psi \in \Psi$.

Proof. Let $f : U \to \mathbf{R}$ be as above and let us assume for simplicity that U is relatively compact. By the resolution of singularities (or Theorem 0.1 of [**B-M1**]) there exist a real analytic manifold N and a proper real analytic map $\phi : N \to \mathbf{R}^n \times \mathbf{P}^1$ whose image is just $\overline{\Gamma_f}$.

We apply Corollary 1.3 to $\pi_1 \circ \phi : N \to \mathbf{R}^n$, $L = N$ and K a compact neighbourhood of $\overline{U}$, and let $s_\alpha : W_\alpha \to \mathbf{R}^n$, $K_\alpha \subset W_\alpha$ be the maps and compact sets satisfying the statement of this corollary. Take one of them $s = s_\alpha : W = W_\alpha \to \mathbf{R}^n$. Let $\tilde{(\pi_1 \circ \phi)}' : \tilde{N}' \to \tilde{W}$ be a strict transform of a complexification of $(\pi_1 \circ \phi)$ which is, by assumption, a finite map.

Consider $N' = \overline{N \times_{\mathbf{R}^n} (W \setminus E)} \subset \tilde{N}'$, where E denote the union of the exceptional divisors. Let $(\pi_1 \circ \phi)' : N' \to W$, $s' : N' \to N$ be the map induced by tse standard projections.

Then, the image Γ' of N' in $W \times \mathbf{P}^1$ by the map $((\pi_1 \circ \phi)', \pi_2 \circ \phi \circ s')$ is contained in a proper analytic subset of $W \times \mathbf{P}^1$ (since $\tilde{(}\pi_1 \circ \phi)'$ is finite) whose projection on W is finite. Moreover, in our case, Γ' is just the closure of the graph of

$$g = f \circ s : s^{-1}(U) \to \mathbf{R},$$

so the fibres of the projection $\Gamma' \to W$ over $s^{-1}(U)$ are just single points (at all points of W they are finite).

Take any $x_0 \in K_\alpha$. Choosing coordinates near x_0 we may assume that it is the origin in $\mathbf{R}^n$. By the above g satisfies near an equation of the type

$$(g^k + \sum_{i=1}^{k} a_i \cdot g^{k-i}) \cdot (g^{-l} + \sum_{j=1}^{l} \tilde{a}_j \cdot g^{j-l}) \equiv 0, \tag{2.3}$$

where all a_i, $\tilde{a}_j$ are analytic in a neighbourhood of the origin and $\tilde{a}_j(0) = 0$ for all j. By Topological preparation (Section 1), it is enough to show the statement for g restricted to a small neighbourhooud of x_0.

We consider two different cases: the first when $x_0 \in s^{-1}(U)$, so g is well defined and continuous in a neighbourhood of x_0, and the second $x_0 \in s^{-1}(\overline{U} \setminus U)$.

Case 1. $x_0 \in K_\alpha \cap s^{-1}(U)$

Then (2.3) has a form

$$g^k + \sum_{i=1}^{k} a_i \cdot g^{k-i} \equiv 0,$$

and we may assume that the discriminant $\Delta(x)$ of

$$F(t, x) = t^k + \sum_{i=1}^{k} a_i(x) \cdot t^{k-i}, \; t \in \mathbf{R} \tag{2.4}$$

is not identically equal to zero.

Now we apply Theorem 4.4 of **[B-M1]** to make, after using again local blowings-up with smooth centers, Δ and the first not identically equal to zero a_i (i.e. either a_0 or a_1) normal crossings. Note that the discriminant of a complexification of F is a complexification of Δ and consequently is also normal crossings (in a neighbourhood of the origin).

For $\delta = (\delta_1, \dots, \delta_n) \in \mathbf{R}_+^n$ we denote $U_\delta = \{x \in \mathbf{C}^n : |x_i| < \delta_i\}$. The following observation is due to Sussman §5 of **[S]**.

Lemma 2.8. *Let $G(t,x) = t^k + \sum_{i=1}^k a_i \cdot t^{k-i}$ be a complex analytic function defined in $\mathbf{C} \times U_\delta$ and assume that the discriminant $\Delta(x)$ of G is normal crossings. Then, there exist positive integers r_i such that for any $\varepsilon_i = 1$ or -1*

$$G(t, \varepsilon_1 y_1^{r_1}, \varepsilon_2 y_2^{r_2}, \dots, \varepsilon_n y_n^{r_n}) = \prod_{i=1}^k (t - b_i(y)) \tag{2.5}$$

in $\mathbf{C} \times U_{\delta'}$, where $\delta' = (\delta_1^{1/r_1}, \dots, \delta_n^{1/r_n})$ and b_i are complex analytic functions (depending maybe on ε_i).

Moreover, all the differences of b_i are normal crossings and if the first not identically equal to zero a_i is normal crossings, then so are all not identically equal to zero b_i.

REMARK. *We need to consider various ε_i since we are interested in the real domain and the images of all real maps $y \to (\varepsilon_1 y_1^{r_1}, \varepsilon_2 y_2^{r_2}, \dots, \varepsilon_n y_n^{r_n})$ (with fixed r_i but all possible $\varepsilon_i = 1$ or -1) cover the neighbourhood of the origin in $\mathbf{R}^n$.*

Proof. By assumption, the projection of the zero set Z of G on U_δ is finite. Fix $i = 1, \dots, n$ and take a point $x^0 \in H_i = \{x_i = 0, x_j \neq 0 \text{ for } j \neq i\}$. Then, by the Puiseux Lemma (with parameter) and the assumption on the discriminant we may find r_i

such that the substitution

$$x_i = \varepsilon_i y_i^{r_i},$$

$$x_j = y_j \text{ if } j \neq i,$$

for all possible ε_i, gives (2.5) near x^0. Such r_i does not depend on the choice of $x_0 \in H_i$ (in the sense that if it is good at one point it is so at the others). Take such r_i (for each i) and fix also all ε_i. Then, b_i satisfying (2.5) are well defined as a (not ordered) set of analytic functions in the complement of a subset of (complex) codimension 2. Since such a subset has to be simply connected, b_i are in fact well defined bounded complex analytic functions outside this set. By the Hartogs Theorem we may extend b_i on the whole $U_{\delta'}$ of the origin which proves the first part of the lemma.

Since the discriminant of $G(t, \varepsilon_1 y_1^{r_1}, \varepsilon_2 y_2^{r_2}, \dots, \varepsilon_n y_n^{r_n})$ is normal crossings so are the differences of b_i. The product of all b_i equals $(-1)^k a_0(\varepsilon_1 y_1^{r_1}, \varepsilon_2 y_2^{r_2}, \dots, \varepsilon_n y_n^{r_n})$, and consequently, if a_0 is normal crossings so are the differences of b_i. If $a_0 \equiv 0$, then so is exactly one of b_i and the product of the rest of b_i equals $(-1)^{k-1} a_1(\varepsilon_1 y_1^{r_1}, \varepsilon_2 y_2^{r_2}, \dots, \varepsilon_n y_n^{r_n})$, which is by assumption normal crossings. This end the proof of the lemma.

$\square$

Apply the lemma to a complexification of F. Then there exist positive integers r_i such that in a neighbourhood of the origin

$$F(t, \varepsilon_1 y_1^{r_1}, \varepsilon_2 y_2^{r_2}, \dots, \varepsilon_n y_n^{r_n}) = \prod_{i=1}^{k_1} (t - b_i(y)) \prod_{j=1}^{(k-k_1)/2} (t - c_j(y))(t - \overline{c}_j(y)),$$

where b_i are real analytic functions and c_j are complex-valued real analytic functions. Fix $c = c_j = d + i \cdot e$, where $d = Re(c)$, $e = Im(c)$. Then

$$(t - c)(t - \overline{c}) = (t - d)^2 + e^2,$$

and since Δ is normal crossings so is $e = \frac{1}{2i}(c - \overline{c})$. Thus we have got the real-analytic version of Lemma 2.8.

Lemma 2.9. *Let $F(t,x) = t^k + \sum_{i=1}^k a_i \cdot t^{k-i}$ be a real analytic function ($t \in \mathbf{R}$, x from a neighbourhood of the origin in $\mathbf{R}^n$) and assume that the discriminant $\Delta(x)$ of F is normal crossings. Then, there exist positive integers r_i such that for any $\varepsilon_i = 1$ or -1 and in some neighbourhood of the origin*

$$F(t, \varepsilon_1 y_1^{r_1}, \varepsilon_2 y_2^{r_2}, \dots, \varepsilon_n y_n^{r_n}) = \prod_{i=1}^{k_1}(t - b_i(y)) \cdot \prod_{j=1}^{(k-k_1)/2} ((t - d_j(y))^2 + e_j^2(y)),$$

where b_i, d_i, and e_i are real analytic functions (depending maybe on ε_i).

Moreover, all the differences of b_i and all e_i are normal crossings and if the first not identically equal to zero a_i is normal crossings so are all not identically equal to zero b_i.

We continue the proof of Theorem 2.7. Take r_i satisfying the statement of Lemma 2.9 for F given by (2.4) and fix all ε_i. The function

$$h(y) = g(\varepsilon_1 y_1^{r_1}, \varepsilon_2 y_2^{r_2}, \dots, \varepsilon_n y_n^{r_n})$$

is continuous, real-valued, subanalytic and satisfies the equation

$$\prod_{i=1}^{k_1}(h - b_i) \cdot \prod_{j=1}^{(k-k_1)/2} ((h - d_j)^2 + e_j^2) \equiv 0 .$$

Since all e_j are normal crossings $((h-d_j)^2 + e_j^2)$ can vanish only on the union of coordinate hyperplanes $\mathcal{H} = \{y_1 \cdot y_2 \cdot \ldots \cdot y_n = 0\}$. Therefore, since h is continuous,

$$\prod_{i=1}^{k_1}(h - b_i) \equiv 0 .$$

This does not necessarily mean that h equals one of b_i, but this happens on $\mathcal{H}_+ = \{y_i \geq 0 \, i = 1, 2, \dots, n\}$ since on $\mathcal{H}_+ \setminus \mathcal{H}$ all b_i are distinct. Therefore, if we substitute $y_i = z_i^2$, $i = 1, 2, \dots, n$ (the images of $z \to (\varepsilon_1 z_1^{2r_1}, \varepsilon_2 z_2^{2r_2}, \dots, \varepsilon_n z_n^{2r_n})$ with various ε_i cover the neighbourhood of the origin in $\mathbf{R}^n$) we get

$$h(z_1^2, z_2^2, \dots, z_n^2) = b_i(z_1^2, z_2^2, \dots, z_n^2)$$

for each z from a neighbourhood of the origin in $\mathbf{R}^n$ and some b_i. This ends the proof of Case 1.

REMARK 2.10. Thoerem 2.6 follows from Case 1 applied to $f(x) = dist(x, X)$.

Case 2. $(x_0 \in K_\alpha \cap s^{-1}(\overline{U} \setminus U)$

By Theorem 2.6 we may assume that $s^{-1}(\overline{U})$ is a union od quadrants. As in Case 1 we make the first not identically equal to zero a_i and $\tilde{a}_j$ and the discriminants of $F(t,x) = t^k + \sum_{i=1}^{k} a_i(x)\cdot t^{k-i}$ and $\tilde{F}(t,x) = t^{-l} + \sum_{j=1}^{l} \tilde{a}_j(x)\cdot t^{j-l}$ normal crossings. We may also do it simultaneously for $x_1 \cdot x_2 \cdot \ldots \cdot x_n$ so $s^{-1}(\overline{U})$ remains to be a union of quadrants. By Lemma 2.9 applied to F and $\tilde{F}$, the function h we get satisfies an equation of the type

$$\prod_{i=1}^{k_1}(h - b_i)\cdot\prod_{j=1}^{l_1}(h^{-1} - b_j) \equiv 0\,.$$

The rest of the proof is the same as in Case 1. This ends the proof of Theorem 2.7.

□

3. Arc-analytic functions.

Let M be a real analytic manifold and let $dimM = n$. We call a function $f : M \to \mathbf{R}$ arc-analytic if $f \circ \gamma$ is analytic for every analytic arc $\gamma : I \to M$ (where $I = (0,1) \subset \mathbf{R}$). This notion was introduced by Kurdyka in relation with arc-symetric semi-algebraic sets **[K2]** (he considered only functions with semi-algebraic graphs), but in fact appeared before in the works of Kuo on blow-analytic equivalence **[Ku]**. He considered the maps $f : X \to Y$ of real analytic spaces, which he called blow-analytic i.e. which become analytic after composition with modifications $X' \to X$. He also gave a nontrivial example $f : \mathbf{R}^2 \to \mathbf{R}$, defined by $f(x,y) = x^2y/(x^2+y^2)$, $f(0,0) = 0$. It is obvious that the blow-analytic functions are arc-analytic and it is interesting to ask whether the converse is true (maybe with additional assumptions). This question was first posed by Kurdyka. Namely, he asked whether an arc-analytic function with a semi-algebraic graph is blow-analytic. The affirmative answer was given in **[B-M1]** Theorem 1.1 by Bierstone and Milman, where they also gave a characterization (Theorem 4.4 of **[B-M2]**) of arc-analytic functions with subanalytic graph as "weakly" blow-analytic i.e. such functions which become analytic after compositions with local blowings-up. Here we present a proof of this result based on our rectylinearization of subanalytic functions (Theorem 2.7).

Theorem 3.1. *A function $f : M \to \mathbf{R}$ is arc-analytic and has a subanalytic graph if and only if there is a locally finite family of analytic morphisms $\{\pi_j : M_j \to M\}$ and compacts $K_j \subset M_j$ such that:*

(1) $\bigcup \pi_j(K_j) = M$;

(2) *Each π_j is a composite of finitely many local blowings-up with smooth centers;*

(3) *Each $f \circ \pi_j$ is analytic .*

Proof. The "if" is obvious. Since the problem is local we may assume that $M = U \subset \mathbf{R}^n$ is a small neighbourhood of the origin. By simple argument (see Lemma 6.8 of [**B-M2**]) each arc-analytic function with subanalytic graph is continuous. Let $\Psi = \{\phi_\alpha = \sigma_\alpha \circ \psi_\alpha\}$ be a family satisfying the statement of Theorem 2.7 for (f, U). Fix $\sigma = \sigma_\alpha$ and the corresponding $r_i \in \mathrm{N}$. Then $h = f \circ \sigma$ is arc-analytic and for each ψ like in (2.2) $h \circ \psi$ is normal crossings (in particular analytic). We shall show that it follows that h is analytic.

Fix ψ (and so the ε_i's) for a moment. Let

$$h \circ \psi(y) = \sum a_\beta y^\beta .$$

Therefore, on $\mathcal{H}_{\{\varepsilon_i\}} = \{x \in \mathbf{R}^n; \varepsilon_i \cdot x_i \geq 0 \text{ for } i = 1, 2, \dots, n\}$

$$h(x) = \sum a_\beta \prod_{i=1}^{n} (\varepsilon_i \cdot x_i)^{\beta_i / r_i} .$$

Suppose that for some β^0 one of coordinates β_i^0 is not divisible by r_i and $a_{\beta^0} \neq 0$. Then, for generic $c = (c_1, \dots, \hat{c}_i, \dots, c_n)$, such that $\varepsilon_j \cdot c_j \geq 0$ for each $j \neq i$,

$$\gamma(t) = h(c_1, \dots, c_{i-1}, t, c_{i+1}, \dots, c_n)$$

has the Puiseaux expansion with nonzero coefficient at $t^{\beta_i^0 / r_i}$. This contradicts the assumption that h is arc-analytic. Therefore, h is analytic on the quadrant $\mathcal{H}_{\{\varepsilon_i\}} = \{x \in \mathbf{R}^n; \varepsilon_i \cdot x_i \geq 0\}$ (i.e. extends to an analytic function on a neighbourhood of $\mathcal{H}_{\{\varepsilon_i\}}$).

Denote by $\{h_k\}$ the family of the obtained analytic functions (one for each quadrant $\mathcal{H}_{\{\varepsilon_i\}}$). We claim that they give one analytic function. If not, then would exist two points $q_1, q_2 \in \mathbf{R}^n$ such that in a neighbourhood of q_1 we have $h \equiv h_{k_1}$, $h(q_1) \neq h_{k_2}(q_1)$ and in a neighbourhood of q_2 we have $h \equiv h_{k_2}$ and $h(q_2) \neq h_{k_1}(q_2)$. But we may join q_1 and q_2

by an analytic arc (simply a line) and since h is arc-analytic we get a contradiction. This ends the proof of the theorem.

$\square$

REMARK. *The assumption on subanalycity of the graph is essential. One may construct an arc-analytic function whose graph is not subanalytic [**B-M-P**], [**K3**] or even an arc-analytic function which is not continuous [**B-M-P**].*

We call a function $f : M \to \mathbf{R}$ <u>weakly arc-analytic</u> if for each analytic $\gamma : [0,1] \to M$ the composition $f \circ \gamma$ is analytic in a neighbourhod of 0. By the same argument as in Lemma 6.8 of [**B-M2**], a weakly arc-analytic function with a subanalytic graph has to be continuous. Therefore by the proof of Theorem 3.1 we get:

Proposition 3.2. *A function $f : M \to \mathbf{R}$ is weakly arc-analytic and has a subanalytic graph if and only if there is a locally finite family of analytic morphisms $\{\pi_j : M_j \to M\}$ and compacts $K_j \subset M_j$ such that:*

(1) $\bigcup \pi_j(K_j) = M$;

(2) *Each π_j is a composite of finitely many local blowings-up with smooth centers;*

(3) *Each M_j is isomorphic to $\mathbf{R}^n$ and $f \circ \pi_j$ is continuous on $\mathbf{R}^n$ and analytic and normal crossing on each quadrant.*

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