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BACKSTRÖM ALGEBRAS

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Contents

Introduction

Backström orders were introduced in [25], where it was shown that their representations are in correspondence with those of quivers or species. A special class of Backström orders are *nodal orders*, which appeared (without this name) in [11] as such *pure noetherian algebras* that the classification of their finitely generated modules is tame. In [4] the same was proved for the derived categories of nodal algebras. Global analogues of nodal algebras, called *nodal curves* were considered in [6, 14, 15]. Namely, in [6] a sort of tilting theory for such curves was developed, which related them to some quasihereditary finite dimensional algebras. In [14] a criterion was found for a nodal curve to be tame with respect to the classification of vector bundles, and in [15] it was proved that the same class of curves has tame derived categories. It was clear that the tilting theory of [6] can be extended to a general situation, namely, to *Backström curves*, i.e. non-commutative curve having Bachström orders as their localizations. Nodal orders and related gentle algebras appear in studying mirror symmetry, see, for instance, [22].

A finite dimensional analogue of nodal orders, called nodal algebras was introduced in [16, 28]. In the latter paper their structure was completely described. In [29] it was shown that such important classes as gentle and skewed-gentle algebras are nodal. In [7] a tilting theory was developed for

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nodal algebras, which was applied to the study of derived categories of gentle and skewed-gentle algebras.

This paper is devoted to a tilting theory for *Backström rings*, which are a straightforward generalization of Backström orders and algebras. In Section 1 we propose a variant of partial tilting, which generalizes the technique of *minors* from [8]. In Section 2 we introduce *Backström pairs* as the pairs of semi-perfect rings $\boldsymbol{H} \supseteq \boldsymbol{A}$ with common radical and *Backström rings* as the rings $\boldsymbol{A}$ that occur in Backström pairs with hereditary $\boldsymbol{H}$. We construct the *Auslander envelopes* $\widetilde{\boldsymbol{A}}$ of a Backström pair and calculate the global dimension of this envelope. Actually, this global dimension only depends on the global dimension of $\boldsymbol{H}$. In particular, Auslander envelopes for Backström rings are of global dimension at most 2. In Section 3 we apply the tilting technique to show that the derived category of the algebra $\boldsymbol{A}$ is related by recollement to the derived category of its Auslander envelope. It implies that the derived dimension of $\boldsymbol{A}$ in the sense of Rouquier [27] is not greater than that of the Auslander envelope. In Section 4 we consider a recollement between the derived categories of the algebra $\boldsymbol{H}$ and of the Auslander envelope. It is used to calculate the derived dimension of the Auslander envelope, thus obtaining an upper bound for the derived dimension of the smaller algebra. In particular, we prove that the derived dimension of a Backström algebra is at most 2 (exactly 2 if it is not a piecewise hereditary algebra of Dynkin type in the sense of [18]). In Section 5 we establish an equivalence between the category $\mathscr{D}(\widetilde{\boldsymbol{A}}\text{-mod})$ and a bimodule category. Such an equivalence gives a useful instrument for calculations in this derived category (see, for instance, [4, 5, 7, 15]). In Section 6 we consider another partial tilting for the algebra $\boldsymbol{A}$ from a Backström pair, relating its derived category by a recollement to the derived category of an algebra $\boldsymbol{B}$ which looks simpler than the Auslander algebra. In this case we calculate explicitly the kernel of the partial tilting functor $\mathsf{F} : \mathscr{D}(\boldsymbol{B}) \to \mathscr{D}(\boldsymbol{A})$.

s0 1. Partial tilting

Let $\mathscr{T}$ be a triangulated category, $\mathfrak{R} \subseteq \operatorname{Ob}\mathscr{T}$. We denote by $\operatorname{Tri}(\mathfrak{R})$ the smallest strictly full triangulated subcategory containing $\mathfrak{R}$ and closed under coproducts (it means that if a coproduct of objects from $\operatorname{Tri}(\mathfrak{R})$ exists in $\mathscr{T}$, it belongs to $\operatorname{Tri}(\mathfrak{R})$). For a DG-category $\mathcal{R}$ we denote by $\mathscr{D}(\mathcal{R})$ its derived category [19]. The following result is a generalization of [23, Proposition 2.6].

01 **Theorem 1.1.** *Let $\mathfrak{R}$ be a subset of $\operatorname{Ob}\mathscr{D}(\mathcal{A})$, where $\mathcal{A}$ is a Grothendieck category, consisting of compact objects. We consider the DG-category $\mathcal{R}$ with the set of objects $\mathfrak{R}$ and the sets of morphisms $\mathcal{R}(T,R) = \operatorname{RHom}(T,R)$. Define the functor $\mathsf{F} : \mathscr{D}(\mathcal{A}) \to \mathscr{D}(\mathcal{R}^{\mathrm{op}})$ mapping a complex C to the DG-module $\mathsf{F}C = \operatorname{RHom}_{\mathscr{D}(\mathcal{A})}(\_, C)$ restricted onto $\mathfrak{R}$.*

(1) *The restriction of F onto $\operatorname{Tri}(\mathfrak{R})$ is an equivalence $\operatorname{Tri}\mathfrak{R} \xrightarrow{\sim} \mathscr{D}(\mathcal{R}^{\mathrm{op}})$.*

(2) *The functor* F *has a left adjoint* F^* *and a right adjoint* $\mathsf{F}^!$ *such that both adjunction morphisms* $\eta : \mathrm{Id}_{\mathscr{D}(\mathcal{R}^{\mathrm{op}})} \to \mathsf{F}\mathsf{F}^*$ *and* $\zeta : \mathsf{F}\mathsf{F}^! \to \mathrm{Id}_{\mathscr{D}(\mathcal{R}^{\mathrm{op}})}$ *are isomorphisms.*

(3) *There is a* recollement *diagram in the sense of* [3]

e01 (1.1)
$$\operatorname{Ker}\mathsf{F} \underset{\mathsf{I}^!}{\overset{\mathsf{I}^*}{\leftleftarrows}} \xrightarrow{\ \mathsf{I}\ } \mathscr{D}(\mathcal{A}) \underset{\mathsf{F}^!}{\overset{\mathsf{F}^*}{\leftleftarrows}} \xrightarrow{\ \mathsf{F}\ } \mathscr{D}(\mathcal{R}^{\mathrm{op}})$$

where I *is the embedding,* I^* *and* $\mathsf{I}^!$ *are, respectively, its left and right adjoint.*

If $\mathfrak{R}$ generates $\mathscr{D}(\mathcal{A})$, we obtain an equivalence $\mathscr{D}(\mathcal{A}) \simeq \mathscr{D}(\mathcal{R})$, as in [23]. If $\mathfrak{R}$ consists of one object R, we obtain an equivalence $\mathrm{Tri}(R) \simeq \mathscr{D}(\boldsymbol{R}^{\mathrm{op}})$, where $\boldsymbol{R} = \mathrm{RHom}(R, R)$.

Proof. (1) We identify $\mathscr{D}(\mathcal{A})$ with the homotopy category $\mathscr{I}(\mathcal{A})$ of *K-injective complexes*, i.e. such complexes I that $\mathrm{Hom}(C, I)$ is acyclic for every acyclic complex C, and suppose that $\mathfrak{R} \subseteq \mathscr{I}(\mathcal{A})$. Then RHom coincide with Hom within the category $\mathscr{I}(\mathcal{A})$ so, for $C \in \mathscr{I}(\mathcal{A})$, $\mathsf{F}C = \mathrm{Hom}_{\mathscr{I}(\mathcal{A})}(\_, C))$ restricted onto $\mathcal{R}$. The full subcategory of $\mathscr{I}(\mathcal{A})$ consisting of such complexes C that the natural map $\mathrm{Hom}_{\mathscr{I}(\mathcal{A})}(R, C) \to \mathrm{Hom}_{\mathscr{D}(\mathcal{R}^{\mathrm{op}})}(\mathsf{F}R, \mathsf{F}C)$ is bijective for all $R \in \mathfrak{R}$ contains $\mathfrak{R}$, is strictly full, triangulated and closed under coproducts, since all objects from $\mathfrak{R}$ are compact. Therefore, it contains $\mathrm{Tri}(\mathfrak{R})$. Quite analogously, the full subcategory of such complexes C that the natural map $\mathrm{Hom}_{\mathscr{I}(\mathcal{A})}(C, C') \to \mathrm{Hom}_{\mathscr{I}(\mathcal{A})}(\mathsf{F}C, \mathsf{F}C')$ is bijective for every $C' \in \mathrm{Tri}(\mathfrak{R})$ also contains $\mathrm{Tri}(\mathfrak{R})$. Hence the restriction of F onto $\mathrm{Tri}(\mathfrak{R})$ is fully faithful. Moreover, as the functors $\mathrm{Hom}_{\mathcal{R}}(\_, R)$, where R runs through $\mathcal{R}$, generate $\mathscr{D}(\mathcal{R}^{\mathrm{op}})$, the functor F is essentially surjective. Therefore, restricted to $\mathrm{Tri}(\mathfrak{R})$, it gives an equivalence $\mathrm{Tri}(\mathfrak{R}) \to \mathscr{D}(R)$.

(2) and (3) Note that $\mathscr{D}(\mathcal{R}^{\mathrm{op}})$ is cocomplete and compactly generated, hence satisfies the Brown representability theorem [24, Theorem 8.3.3]. Therefore, it is true for $\mathrm{Tri}(\mathfrak{R})$ too. Then [24, Proposition 9.1.19] implies that a Bowsfield localization functor exists for , and $\mathscr{D}(\mathcal{A}) = \mathscr{D}(quasi{-}coherent\mathcal{A})$, where $(X, \mathcal{A})$ is a non-commutative scheme in the array sense of [8] $\mathrm{Tri}(\mathfrak{R}) \subseteq \mathscr{D}(\mathcal{A})$ and [24, Proposition 9.1.18] implies that the embedding $\mathsf{E} : \mathrm{Tri}(\mathfrak{R}) \to \mathscr{D}(\mathcal{R}^{\mathrm{op}})$ has a right adjoint $\Theta : \mathscr{D}(\mathcal{R}^{\mathrm{op}}) \to \mathrm{Tri}(\mathfrak{R})$. Let $\mathsf{F}' : \mathscr{D}(\mathcal{R}^{\mathrm{op}}) \to \mathrm{Tri}(\mathfrak{R})$ be a quasi-inverse to the restriction of F onto $\mathrm{Tri}(\mathfrak{R})$. In particular, F' is a left adjoint to this restriction and the adjunction $\mathsf{F}\mathsf{F}' \to \mathrm{Id}_{\mathscr{D}(\mathcal{R}^{\mathrm{op}})}$ is an isomorphism. Then $\mathsf{F}C = \mathrm{Hom}_{\mathscr{I}(\mathcal{A})}(\_, C)|_{\mathcal{R}} \simeq \mathrm{Hom}_{\mathscr{I}(\mathcal{A})}(\_, \Theta C)|_{\mathcal{R}} = \mathsf{F}\Theta C$. Set $\mathsf{F}^* = \mathsf{E}\mathsf{F}'$. Since $\mathsf{F}'M \in \mathrm{Tri}(\mathfrak{R})$ for every $M \in \mathscr{D}(\mathcal{R}^{\mathrm{op}})$,

$$\mathrm{Hom}_{\mathscr{I}(\mathcal{A})}(\mathsf{F}^*M, C) \simeq \mathrm{Hom}_{\mathrm{Tri}(\mathfrak{R})}(\mathsf{F}'M, \Theta C) \simeq \\ \simeq \mathrm{Hom}_{\mathscr{D}(\mathcal{R}^{\mathrm{op}})}(M, \mathsf{F}\Theta C) \simeq \mathrm{Hom}_{\mathscr{D}(\mathcal{R}^{\mathrm{op}})}(M, \mathsf{F}C),$$

for any $M \in \mathscr{D}(\mathcal{R}^{\mathrm{op}})$ and $C \in \mathscr{I}(\mathcal{A})$. Hence the composition $\mathsf{F}^* = \mathsf{E}\mathsf{F}'$ of F' with the embedding $\mathrm{Tri}(\mathfrak{R}) \to \mathscr{I}(\mathcal{A})$ is a left adjoint to F. If, moreover,

$C \in \mathrm{Tri}(\mathfrak{R})$, we obtain

$$\mathrm{Hom}_{\mathscr{D}(R)}(\mathsf{FF}^*M, \mathsf{F}C) \simeq \mathrm{Hom}_{\mathscr{I}(\mathcal{A})}(\mathsf{F}^*M, C) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(M, \mathsf{F}C).$$

As F is essentially surjective, it implies that $\eta : \mathsf{FF}^* \to \mathrm{Id}_{\mathscr{D}(\mathcal{R}^{\mathrm{op}})}$ is an isomorphism. As all objects from $\mathfrak{R}$ are compact, F respects coproducts, hence has a right adjoin $\mathsf{F}^!$ [24, Theorem 8.4.4]. Now it follows from [8, Corollary 2.3] that ζ is an isomorphism and there is a recollement diagram (1.1) □

Note that $\mathrm{Im}\,\mathsf{F}^* = \mathrm{Tri}(\mathfrak{R})$ by construction, but usually $\mathrm{Im}\,\mathsf{F}^! \neq \mathrm{Tri}(\mathfrak{R})$, though it is equivalent to $\mathrm{Tri}(\mathfrak{R})$.

02 **Corollary 1.2.** *Under conditions and notations of the preceding theorem, suppose that* $\mathrm{Hom}_{\mathscr{D}(\mathcal{A})}(R, T[m]) = 0$ *for* $R, T \in \mathfrak{R}$ *and* $m \neq 0$.[1] *Then the functor* F *induces an equivalence* $\mathrm{Tri}(\mathfrak{R}) \xrightarrow{\sim} \mathscr{D}(\mathcal{R}^{\mathrm{op}})$, *where* $\mathcal{R}$ *is the category with the set of objects* $\mathfrak{R}$ *and the sets of morphisms* $\mathcal{R}(A, B) = \mathrm{Hom}_{\mathscr{D}(\mathcal{A})}(A, B)$.

In this situation we call the functor F a *partial tilting functor.*

s1 2. Backström pairs

Recall [2, 21] that a *semiperfect ring* is such a ring $\boldsymbol{A}$ that $\boldsymbol{A}/\mathrm{rad}\,\boldsymbol{A}$ is a semi-simple artinian ring and idempotents can be lifted modulo $\mathrm{rad}\,\boldsymbol{A}$. Equivalently, as a left (or as a right) $\boldsymbol{A}$-module, $\boldsymbol{A}$ decomposes into a direct sum of modules with local endomorphism rings.

11 **Definition 2.1.** (1) A *Backström pair* is a pair of semiperfect rings $\boldsymbol{H} \supseteq \boldsymbol{A}$ such that $\mathrm{rad}\,\boldsymbol{A} = \mathrm{rad}\,\boldsymbol{H}$. We denote by $\boldsymbol{C}(\boldsymbol{H}, \boldsymbol{A})$ the *conductor* of $\boldsymbol{H}$ in $\boldsymbol{A}$:

$$\boldsymbol{C}(\boldsymbol{H}, \boldsymbol{A}) = \{\, \alpha \in \boldsymbol{A} \mid \boldsymbol{H}\alpha \subseteq \boldsymbol{A} \,\} = \mathrm{Ann}(\boldsymbol{H}/\boldsymbol{A})_{\boldsymbol{A}}$$

(the right subscript $_{\boldsymbol{A}}$ means that we consider $\boldsymbol{H}/\boldsymbol{A}$ as a right $\boldsymbol{A}$-module). Obviously $\boldsymbol{C}(\boldsymbol{H}, \boldsymbol{A}) \supseteq \mathrm{rad}\,\boldsymbol{A}$.

(2) We call a ring $\boldsymbol{A}$ a *(left) Backström ring* if there is a Backström pair $\boldsymbol{H} \supseteq \boldsymbol{A}$, where the ring $\boldsymbol{H}$ is left hereditary. If, moreover, both $\boldsymbol{A}$ and $\boldsymbol{H}$ are finite dimensional algebras over a field $\Bbbk$, we call $\boldsymbol{A}$ a *Backström algebra.*

Note that if e is an idempotent in $\boldsymbol{A}$, then $\mathrm{rad}(e\boldsymbol{A}e) = e(\mathrm{rad}\,\boldsymbol{A})e$, hence, if $\boldsymbol{H} \supseteq \boldsymbol{A}$ is a Backström pair, so is also $e\boldsymbol{H}e \supseteq e\boldsymbol{A}e$. It implies that if P is a finitely generated $\boldsymbol{A}$-module, $\boldsymbol{A}' = \mathrm{End}_{\boldsymbol{A}} P$ and $\boldsymbol{H}' = \mathrm{End}_{\boldsymbol{H}}(\boldsymbol{H} \otimes_{\boldsymbol{A}} P)$, then $\boldsymbol{H}' \supseteq \boldsymbol{A}$ is also a Backström pair. Note that if $\boldsymbol{H}$ is hereditary, so is $\boldsymbol{H}'$, hence $\boldsymbol{A}'$ is a Backström ring. In particular, the notion of Backström ring is Morita invariant. Note also that if $\boldsymbol{H}$ is left hereditary and noetherian, it is also right hereditary, so $\boldsymbol{A}^{\mathrm{op}}$ is also a Backström ring. array

[1] In this situation we say that F is a *partial tilting functor* for $\mathscr{D}(\mathcal{A})$.

ex1 **Example 2.2.** (1) An important example of Backström algebras are *nodal algebras* introduced in [16, 28]. By definition, they are finite dimensional Backström algebras such that there is a Backström pair $\boldsymbol{H} \supseteq \boldsymbol{A}$, where $\boldsymbol{H}$ is a hereditary algebra and $\operatorname{length}_{\boldsymbol{A}}(\boldsymbol{H} \otimes_{\boldsymbol{A}} U) \leqslant 2$ for every simple $\boldsymbol{A}$-module $\boldsymbol{A}$. Their structure was completely described in [28].

ex2 (2) Recall that a $\Bbbk$-algebra $\boldsymbol{A}$ is called *gentle* [1] if $\boldsymbol{A} \simeq \Bbbk\Gamma/J$, where Γ is a finite quiver (oriented graph) and J is an ideal in the path algebra $\Bbbk\Gamma$ such that $(J_+)^2 \supseteq J \supseteq (J_+)^k$ for some k, where J_+ is the ideal generated by all arrows, and the following conditions hold

(a) For every vertex $i \in \operatorname{Ver}\Gamma$ there are at most 2 arrows starting at i and at most 2 arrows ending at i.

(b) If an arrow a starts at i (ends at i) and arrows b_1, b_2 end at i (respectively, start at i), then either $ab_1 = 0$ or $ab_2 = 0$ (respectively, either $b_1a = 0$ or $b_2a = 0$), but not both.

(c) The ideal J is generated by products of arrows of the sort ab.

It is proved in [29] that such algebras are nodal, hence Backström algebras. The same is true for *skewed-gentle* algebras [17] obtained from gentle algebras by blowing some vertices.

(3) *Backström orders* are orders $\boldsymbol{A}$ over a discrete valuation ring such that there is a Backström pair $\boldsymbol{H} \supseteq \boldsymbol{A}$, where $\boldsymbol{H}$ is a hereditary order. They were considered in [25].

ex3 (4) $\Lambda_n = \Bbbk[x_1, x_2, \dots, x_n]/(x_1, x_2, \dots, x_n)^2$ embeds into $\boldsymbol{H} = \prod_{i=1}^n \Bbbk\Gamma_i$, where $\Gamma_i = \bullet \xrightarrow{a_i} \bullet$ (x_i maps to a_i). Obviously, under this embedding $\operatorname{rad}\Lambda_n = \operatorname{rad}\boldsymbol{H}$.

We consider a fixed Backström pair $\boldsymbol{H} \supseteq \boldsymbol{A}$, set $\mathfrak{r} = \operatorname{rad}\boldsymbol{A} = \operatorname{rad}\boldsymbol{H}$ and denote by $\boldsymbol{C}$ the conductor $\boldsymbol{C}(\boldsymbol{H}, \boldsymbol{A})$. Obviously, $\boldsymbol{C}$ is a two-sided $\boldsymbol{A}$-ideal and the biggest left $\boldsymbol{H}$-ideal contained in $\boldsymbol{A}$. Actually, it is even a two-sided $\boldsymbol{H}$-ideal and its definition is left–right symmetric.

12 **Lemma 2.3.** *Let $R \subseteq S$ be semi-simple algebras, $I = \{\,\alpha \in R \mid S\alpha \subseteq R\,\}$. Then I is a two-sided S-ideal.*

Proof. Obviously, I is a left S-ideal and a two-sided R-ideal. As R is semi-simple, $I = Re$ for some central idempotent $e \in R$. Then $Se \subseteq Re$, so $Se = Re$ and $(1-e)Se = 0$. Hence $eS(1-e)$ is a left ideal in S and $(eS(1-e))^2 = 0$, so $eS(1-e) = 0$ and $I = Se = eS$ is also a right S-ideal. □

13 **Proposition 2.4.** *$\boldsymbol{C}$ is a two-sided $\boldsymbol{H}$-ideal. It is the biggest $\boldsymbol{H}$-ideal contained in $\boldsymbol{A}$. Therefore, it coincides with the set $\{\,\alpha \in \boldsymbol{A} \mid \boldsymbol{H}\alpha \subseteq \boldsymbol{A}\,\}$, or with $\operatorname{Ann}_{\boldsymbol{A}}(\boldsymbol{H}/\boldsymbol{A})$ considered as right $\boldsymbol{A}$-module.*

Proof. It follows from the preceding lemma applied to the algebras $\boldsymbol{A}/\operatorname{rad}\boldsymbol{A}$ and $\boldsymbol{H}/\operatorname{rad}\boldsymbol{H}$. □

In what follows we assume that $\boldsymbol{A} \neq \boldsymbol{H}$, so $\boldsymbol{C} \neq \boldsymbol{A}$. To calculate $\boldsymbol{C}$, we consider a decomposition $\boldsymbol{A} = \bigoplus_{i=1}^m A_i$, where A_i are indecomposable projective $\boldsymbol{A}$-module. Arrange them so that $\boldsymbol{H}A_i \neq A_i$ for $1 \leqslant i \leqslant r$ and $\boldsymbol{H}A_i = A_i$ for $r < i \leqslant m$, and set $A^0 = \bigoplus_{i=1}^r A_i$, $H^0 = \boldsymbol{H}A^0$ and $A^1 = \bigoplus_{i=r+1}^m A_i = \boldsymbol{H}A^1$. Then $\boldsymbol{A} = A^0 \oplus A^1$ and $\boldsymbol{H} = H^0 \oplus A^1$ (possibly, $r = m$, so $A^0 = \boldsymbol{A}$ and $H^0 = \boldsymbol{H}$). Let $A^0 = \boldsymbol{A}e_0$ and $A^1 = \boldsymbol{A}e_1$, where e_0 and e_1 are orthogonal idempotents and $e_0 + e_1 = 1$. Set $A^a_b = e_b\boldsymbol{A}e_a$ and $H^a_b = e_b\boldsymbol{H}e_a$, where $a, b \in \{0, 1\}$. Note that $A^1_b = H^1_b$ and $A^0_1 = H^0_1$. As A^0 and A^1 have no isomorphic direct summands, $A^a_b \subseteq \operatorname{rad}\boldsymbol{A}$ if $a \neq b$. Hence, if we set $\mathfrak{r}^a_a = \operatorname{rad}\boldsymbol{A}^a_a$ $(a = 0, 1)$, and consider the Pierce decomposition of $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{pmatrix} A^0_0 & A^1_0 \\ A^0_1 & A^1_1 \end{pmatrix},$$

the Pierce decomposition of $\mathfrak{r}$ becomes

$$\mathfrak{r} = \begin{pmatrix} \mathfrak{r}^0_0 & A^1_0 \\ A^0_1 & \mathfrak{r}^1_1 \end{pmatrix}.$$

It implies, in particular, that H^0 and H^1 have no isomorphic direct summands and the Pierce decomposition of $\boldsymbol{H}$ is

$$\boldsymbol{H} = \begin{pmatrix} H^0_0 & A^1_0 \\ A^0_1 & A^1_1 \end{pmatrix}.$$

Now one easily sees that an element $a = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ belongs to $\boldsymbol{C}$ if and only if $H^0\alpha \subseteq A^0$. We claim that then $H^0\alpha \subseteq \operatorname{rad} A^0$. Otherwise $H^0\alpha$ contains an idempotent, hence a direct summand of A^0, which is isomorphic to some A_i with $1 \leqslant i < r$. It is impossible, since $\boldsymbol{H}A_i \neq A_i$. Therefore, $\alpha \in \mathfrak{r}^0_0$ and we obtain the following result.

14 **Proposition 2.5.** *The Pierce decomposition of* $\boldsymbol{C}$ *is*

$$\boldsymbol{C} = \begin{pmatrix} \mathfrak{r}^0_0 & A^1_0 \\ A^0_1 & A^1_1 \end{pmatrix},$$

15 **Definition 2.6.** We define the *Auslander envelope* of the Backström pair $\boldsymbol{H} \supseteq \boldsymbol{A}$ as the ring $\widetilde{\boldsymbol{A}}$ of 2×2 matrices of the form

$$\widetilde{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{H} \\ \boldsymbol{C} & \boldsymbol{H} \end{pmatrix}.$$

Using Pierce decompositions of $\boldsymbol{A}, \boldsymbol{H}$ and $\boldsymbol{C}$, we also present $\widetilde{\boldsymbol{A}}$ as the ring of 4×4 matrices

e11 (2.1)
$$\widetilde{\boldsymbol{A}} = \begin{pmatrix} A^0_0 & A^1_0 & H^0_0 & A^1_0 \\ A^0_1 & A^1_1 & A^0_1 & A^1_1 \\ \mathfrak{r}^0_0 & A^1_0 & H^0_0 & A^1_0 \\ A^0_1 & A^1_1 & A^0_1 & A^1_1 \end{pmatrix}$$

Finally, we define $\widetilde{\boldsymbol{H}}$ as the ring of 2×2 matrices

$$\widetilde{\boldsymbol{H}} = \begin{pmatrix} \boldsymbol{H} & \boldsymbol{H} \\ \boldsymbol{C} & \boldsymbol{H} \end{pmatrix}$$

or

$$\widetilde{\boldsymbol{H}} = \begin{pmatrix} H_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & A_1^1 & A_1^0 & A_1^1 \\ \mathfrak{r}_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & A_1^1 & A_1^0 & A_1^1 \end{pmatrix}$$

Obviously, $\operatorname{rad} \widetilde{\boldsymbol{H}} = \operatorname{rad} \widetilde{\boldsymbol{A}}$, so $\widetilde{\boldsymbol{H}} \supset \widetilde{\boldsymbol{A}}$ is also a Backström pair. $\widetilde{\boldsymbol{A}}$ is left noetherian iff $\boldsymbol{A}$ is left noetherian and $\boldsymbol{H}$ is finitely generated as left $\boldsymbol{A}$-module.

In noetherian case one can calculate the global dimensions of $\widetilde{\boldsymbol{A}}$ and $\widetilde{\boldsymbol{H}}$. It turns out that it only depends on $\boldsymbol{H}$.

16 **Theorem 2.7.** *If $\boldsymbol{A}$ and $\boldsymbol{H}$ are left noetherian or left perfect, then*

$$\begin{aligned} \text{l.gl.dim}\, \widetilde{\boldsymbol{A}} &= 1 + \max(1 + \text{pr.dim}_{\boldsymbol{H}}\, \mathfrak{r}^0,\ \text{pr.dim}_{\boldsymbol{H}}\, \mathfrak{r}^1) = \\ &= \begin{cases} 1 + \text{l.gl.dim}\, \boldsymbol{H} & \text{if } \text{pr.dim}_{\boldsymbol{H}}\, \mathfrak{r}^0 \geqslant \text{pr.dim}_{\boldsymbol{H}}\, \mathfrak{r}^1, \\ \text{l.gl.dim}\, \boldsymbol{H} & \text{if } \text{pr.dim}_{\boldsymbol{H}}\, \mathfrak{r}^0 < \text{pr.dim}_{\boldsymbol{H}}\, \mathfrak{r}^1 \end{cases} \end{aligned}$$

and

$$\text{l.gl.dim}\, \widetilde{\boldsymbol{H}} = \text{l.gl.dim}\, \boldsymbol{H}.$$

In particular, if $\boldsymbol{A}$ is a left Backström ring, but is not left hereditary, then $\text{l.gl.dim}\, \widetilde{\boldsymbol{A}} = 2$. For instance, it is the case for nodal (gentle, skewed-gentle) algebras, see Example 2.2(1),(2).

Proof. We recall that if a ring Λ is left perfect or left noetherian and semiperfect, then $\text{l.gl.dim}\, \Lambda = \text{pr.dim}_\Lambda(\Lambda/\operatorname{rad}\Lambda) = 1 + \text{pr.dim}\operatorname{rad}\Lambda$. Obviously, if $\boldsymbol{A}$ and $\boldsymbol{H}$ are left noetherian or left perfect, so is also $\widetilde{\boldsymbol{A}}$. The 4×4 matrix presentation (2.1) of $\widetilde{\boldsymbol{A}}$ implies that the corresponding presentation of $\operatorname{rad} \widetilde{\boldsymbol{A}}$ is

e12 (2.2)
$$\operatorname{rad} \widetilde{\boldsymbol{A}} = \begin{pmatrix} \mathfrak{r}_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & \mathfrak{r}_1^1 & A_1^0 & \mathfrak{r}_1^1 \\ \mathfrak{r}_0^0 & A_0^1 & \mathfrak{r}_0^0 & A_0^1 \\ A_1^0 & \mathfrak{r}_1^1 & A_1^0 & \mathfrak{r}_1^1 \end{pmatrix}$$

An $\widetilde{\boldsymbol{A}}$-module M is given by a quadruple (M', M'', ϕ, ψ), where M' is an A-module, M'' is an H-module, $\psi : M'' \to M'$ is a homomorphism of $\boldsymbol{A}$-module and $\phi : \boldsymbol{C} \otimes_{\boldsymbol{A}} M' \to M''$ is a homomorphism of $\boldsymbol{H}$-modules. We usually write $M = \binom{M'}{M''}$ not mentioning ϕ and ψ. For an $\boldsymbol{H}$-module N we define the $\widetilde{\boldsymbol{A}}$-module $N^+ = \binom{N}{N}$. Then $N \mapsto N^+$ is an exact functor mapping projective modules to projective ones.

We denote by L^i and by R^i respectively, the i-th column of the presentations (2.1) and (2.2). Then $R^1 = (\mathfrak{r}^0)^+$ and $R^2 \simeq R^4 \simeq (\mathfrak{r}^1)^+$, where $\mathfrak{r}^a = \mathfrak{r}e_a$. Note that if P is a projective $\boldsymbol{H}$-module, then $P^+ = \binom{P}{P}$ is a projective $\widetilde{\boldsymbol{A}}$-module. Hence, if

$$\cdots \to F_k \to \cdots \to F_1 \to F_0 \to N \to 0$$

is a minimal projective resolution of an $\boldsymbol{H}$-module N,

$$\cdots \to F_k^+ \to \cdots \to F_1^+ \to F_0^+ \to N^+ \to 0$$

is a minimal projective resolution of N^+, so $\operatorname{pr.dim}_{\widetilde{\boldsymbol{A}}} N^+ = \operatorname{pr.dim}_{\boldsymbol{H}} N$. In particular, $\operatorname{pr.dim}_{\widetilde{\boldsymbol{A}}} R^1 = \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^0$ and $\operatorname{pr.dim}_{\widetilde{\boldsymbol{A}}} R^2 = \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^1$. For the module R^3 we have an exact sequence

e13 (2.3)
$$0 \to (\mathfrak{r}^0)^+ \to R^3 \to \begin{pmatrix} H^0/\mathfrak{r}^0 \\ 0 \end{pmatrix} \to 0.$$

Note that $H^0/\mathfrak{r}^0$ is a semi-simple $\boldsymbol{A}$-module and $e_1(H^0/\mathfrak{r}^0) = 0$, hence it contains the same simple direct summands as $A^0/\mathfrak{r}^0$. The same is true for $\begin{pmatrix} H^0/\mathfrak{r}^0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} A^0/\mathfrak{r}^0 \\ 0 \end{pmatrix} = L^1/R^1$. Hence

$$\operatorname{pr.dim}_{\widetilde{\boldsymbol{A}}} \begin{pmatrix} H^0/\mathfrak{r}^0 \\ 0 \end{pmatrix} = 1 + \operatorname{pr.dim}_{\widetilde{\boldsymbol{A}}} R^1 = 1 + \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^0.$$

Therefore, the exact sequence (2.3) shows that $\operatorname{pr.dim}_{\widetilde{\boldsymbol{A}}} R^3 = 1 + \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^0$ and

$$\operatorname{pr.dim}_{\widetilde{\boldsymbol{A}}} \operatorname{rad} \widetilde{\boldsymbol{A}} = \max(1 + \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^0, \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^1),$$

which gives the necessary result for $\widetilde{\boldsymbol{A}}$.

On the other hand, R^3 is a projective $\widetilde{\boldsymbol{H}}$-module, whence $\operatorname{l.gl.dim} \widetilde{\boldsymbol{H}} = \operatorname{l.gl.dim} \boldsymbol{H}$. □

s2 3. The structure of derived categories

In what follows we denote by $\mathscr{D}(\boldsymbol{A})$ the derived category $\mathscr{D}(\boldsymbol{A}\text{-Mod})$. We denote by $\mathscr{D}_f(\boldsymbol{A})$ the full subcategory of $\mathscr{D}(\boldsymbol{A})$ consisting of complexes quasi-isomorphic to complexes of finitely generated projective modules. If $\boldsymbol{A}$ is left noetherian, it coincides with the derived category of the category $\boldsymbol{A}$-mod of finitely generated $\boldsymbol{A}$-modules. We also use the usual superscripts $^+, ^-, ^b$. By $\operatorname{Perf}(\boldsymbol{A})$ we denote the full subcategory of *perfect complexes* from $\mathscr{D}(\boldsymbol{A})$, i.e. complexes quasi-isomorphic to finite complexes of finitely generated projective modules. It coincides with the full subcategory of *compact objects* in $\mathscr{D}(\boldsymbol{A})$ [27]. If $\boldsymbol{A}$ is left noetherian, an $\boldsymbol{A}$-module M belongs to $\operatorname{Perf}(\boldsymbol{A})$ if and only if it is finitely generated and of finite projective dimension.

There are close relations between the categories $\mathscr{D}(\boldsymbol{A})$, $\mathscr{D}(\boldsymbol{H})$ and $\mathscr{D}(\widetilde{\boldsymbol{A}})$ based on the construction [8].

Let $\boldsymbol{P} = \left(\begin{smallmatrix}\boldsymbol{A}\\ \boldsymbol{C}\end{smallmatrix}\right)$. It is a projective $\widetilde{\boldsymbol{A}}$-module and $\operatorname{End}_{\widetilde{\boldsymbol{A}}} \boldsymbol{P} \simeq \boldsymbol{A}^{\mathrm{op}}$, so it can be considered as a right [3] $\boldsymbol{A}$-module. Consider the functors

$$\begin{aligned}
\mathsf{F} &= \operatorname{Hom}_{\widetilde{\boldsymbol{A}}}(\boldsymbol{P}, _) \simeq \boldsymbol{P}^{\vee} \otimes_{\widetilde{\boldsymbol{A}}} _ : \widetilde{\boldsymbol{A}}\text{-Mod} \to \boldsymbol{A}\text{-Mod},\\
\mathsf{F}^{*} &= \boldsymbol{P} \otimes_{\boldsymbol{A}} _ : \boldsymbol{A}\text{-Mod} \to \widetilde{\boldsymbol{A}}\text{-Mod},\\
\mathsf{F}^{!} &= \operatorname{Hom}_{\boldsymbol{A}}(\boldsymbol{P}^{\vee}, _) : \boldsymbol{A}\text{-Mod} \to \widetilde{\boldsymbol{A}}\text{-Mod},
\end{aligned}$$

where $\boldsymbol{P}^{\vee} = \operatorname{Hom}_{\widetilde{\boldsymbol{A}}}(\boldsymbol{P}, \widetilde{\boldsymbol{A}}) \simeq (\boldsymbol{A}\ \boldsymbol{H})$ is the dual right projective $\widetilde{\boldsymbol{A}}$-module. The functor F is exact, F^{*} is its left adjoint and $\mathsf{F}^{!}$ is its right adjoint. Moreover, the adjunction morphisms $\mathsf{F}\mathsf{F}^{*} \to \operatorname{Id}_{\boldsymbol{A}\text{-Mod}}$ and $\operatorname{Id}_{\boldsymbol{A}\text{-Mod}} \to \mathsf{F}\mathsf{F}^{!}$ are isomorphisms [8, Theorem 4.3]. The functors F^{*} and $\mathsf{F}^{!}$ are fully faithful and F is *essentially surjective*, i.e. every $\boldsymbol{A}$-module is isomorphic to $\mathsf{F}M$ for some $\widetilde{\boldsymbol{A}}$-module M. $\operatorname{Ker}\mathsf{F}$ is a Serre subcategory of $\widetilde{\boldsymbol{A}}$-Mod equivalent to $\bar{\boldsymbol{H}}$-Mod, where $\bar{\boldsymbol{H}} = \boldsymbol{H}/\boldsymbol{C} \simeq \widetilde{\boldsymbol{A}}/\left(\begin{smallmatrix}\boldsymbol{A} & \boldsymbol{H}\\ \boldsymbol{C} & \boldsymbol{C}\end{smallmatrix}\right)$. The embedding functor $\mathsf{I} : \operatorname{Ker}\mathsf{F} \to \widetilde{\boldsymbol{A}}$-Mod has a left adjoint I^{*} and a right adjoint $\mathsf{I}^{!}$ and we obtain a recollement diagram

$$\operatorname{Ker}\mathsf{F} \;\underset{\mathsf{I}^{!}}{\overset{\mathsf{I}^{*}}{\leftleftarrows}}\!\xrightarrow{\ \mathsf{I}\ }\; \widetilde{\boldsymbol{A}}\text{-Mod} \;\underset{\mathsf{F}^{!}}{\overset{\mathsf{F}^{*}}{\leftleftarrows}}\!\xrightarrow{\ \mathsf{F}\ }\; \boldsymbol{A}\text{-Mod}$$

As the functor F is exact, it extends to the functor between the derived categories $\mathsf{DF} : \mathscr{D}(\widetilde{\boldsymbol{A}}\text{-Mod}) \to \mathscr{D}(\boldsymbol{A}\text{-Mod})$ acting on complexes componentwise. The derived functors LF^{*} and $\mathsf{RF}^{!}$ are, respectively, its left and right adjoints, the adjunction morphisms $\operatorname{Id}_{\mathscr{D}(\boldsymbol{A}\text{-Mod})} \to \mathsf{DF}\cdot\mathsf{LF}^{*}$ and $\mathsf{DF}\cdot\mathsf{LF}^{*} \to \operatorname{Id}_{\mathscr{D}(\boldsymbol{A}\text{-Mod})}$ again are isomorphisms and we have a recollement diagram

$$\operatorname{Ker}\mathsf{DF} \;\underset{\mathsf{RI}^{!}}{\overset{\mathsf{LI}^{*}}{\leftleftarrows}}\!\xrightarrow{\ \mathsf{DI}\ }\; \mathscr{D}(\widetilde{\boldsymbol{A}}) \;\underset{\mathsf{RF}^{!}}{\overset{\mathsf{LF}^{*}}{\leftleftarrows}}\!\xrightarrow{\ \mathsf{DF}\ }\; \mathscr{D}(\boldsymbol{A})$$

(It also follows from Corollary 1.2.) Here $\operatorname{Ker}\mathsf{DF} = \mathscr{D}_{\bar{\boldsymbol{H}}}(\widetilde{\boldsymbol{A}}\text{-Mod})$, the full subcategory of complexes whose cohomologies are $\bar{\boldsymbol{H}}$-modules, i.e. are annihilated by the ideal $\left(\begin{smallmatrix}\boldsymbol{A} & \boldsymbol{H}\\ \boldsymbol{C} & \boldsymbol{C}\end{smallmatrix}\right)$. Note that, as a rule, it is not equivalent to $\mathscr{D}(\bar{\boldsymbol{H}}\text{-Mod})$. From the definition of F it follows that

$$\operatorname{Ker}\mathsf{DF} = \boldsymbol{P}^{\perp} = \left\{\, \mathcal{C} \in \mathscr{D}(\boldsymbol{A}) \mid \operatorname{Hom}_{\mathscr{D}(\boldsymbol{A})}(\boldsymbol{P}, \mathcal{C}[k] \,\right\} = 0 \ \text{ for all } \ k.$$

Obviously, DF maps $\mathscr{D}^{\sigma}(\widetilde{\boldsymbol{A}})$ to $\mathscr{D}^{\sigma}(\boldsymbol{A})$ for $\sigma \in \{+,-,b\}$, LF^{*} maps $\mathscr{D}^{-}(\boldsymbol{A})$ to $\mathscr{D}^{-}(\widetilde{\boldsymbol{A}})$ and $\mathsf{RF}^{!}$ maps $\mathscr{D}^{+}(\boldsymbol{A})$ to $\mathscr{D}^{+}(\widetilde{\boldsymbol{A}})$. If $\widetilde{\boldsymbol{A}}$ is noetherian, DF maps $\mathscr{D}_{f}(\widetilde{\boldsymbol{A}})$ to $\mathscr{D}_{f}(\boldsymbol{A})$ and LF^{*} maps $\mathscr{D}_{f}(\boldsymbol{A})$ to $\mathscr{D}_{f}(\widetilde{\boldsymbol{A}})$. Finally, both DF and LF^{*} map compact objects (i.e. perfect complexes) to compact ones, since they have right adjoints. On the contrary, usually LF^{*} does not map $\mathscr{D}^{b}(\boldsymbol{A})$ to $\mathscr{D}^{b}(\widetilde{\boldsymbol{A}})$. For instance, it is definitely so if $\operatorname{l.gl.dim}\widetilde{\boldsymbol{A}} < \infty$ while $\operatorname{l.gl.dim}\boldsymbol{A} = \infty$ as in Example 2.2(4).

If $\operatorname{l.gl.dim}\boldsymbol{H}$ is finite, so is $\operatorname{l.gl.dim}\widetilde{\boldsymbol{A}}$, thus this recollement can be considered as a sort of categorical resolution of the category $\mathscr{D}(\boldsymbol{A})$. In any case, it is useful for studying the categories $\boldsymbol{A}$-Mod and $\mathscr{D}(\boldsymbol{A})$ if we know the structure of the categories $\widetilde{\boldsymbol{A}}$-Mod and $\mathscr{D}(\widetilde{\boldsymbol{A}})$. For instance, it is so if we are interesting in the *derived dimension*, i.e. the dimension of the category $\mathscr{D}^b_f(\boldsymbol{A})$ in the sence of Rouquier [27]. We recall the definition.

21 **Definition 3.1.** Let $\mathscr{T}$ be a triangular category and $\mathfrak{M}$ be a set of objects from $\mathscr{T}$.

(1) We denote by $\langle\,\mathfrak{M}\,\rangle$ the smallest full subcategory of $\mathscr{T}$ containing $\mathfrak{m}$ and closed under direct sums, direct summands and shifts (not closed under cones, so not a triangular subcategory).
(2) If $\mathfrak{N}$ is another subset of $\mathscr{T}$, we denote by $\mathfrak{M} \dagger \mathfrak{N}$ the set of objects C from $\mathscr{T}$ such that there is an exact triangle $A \to B \to C \xrightarrow{+}$, where $A \in \mathfrak{M}$, $B \in \mathfrak{N}$.
(3) We define $\langle\,\mathfrak{M}\,\rangle_k$ recursively, setting $\langle\,\mathfrak{M}\,\rangle_1 = \langle\,\mathfrak{M}\,\rangle$ and $\langle\,\mathfrak{M}\,\rangle_{k+1} = \langle\,\langle\,\mathfrak{M}\,\rangle \dagger \langle\,\mathfrak{M}\,\rangle_k\,\rangle$.
(4) The *dimension* $\dim \mathscr{T}$ of $\mathscr{T}$ is the smallest k such that there is a finite set of objects $\mathfrak{M}$ such that $\langle\,\mathfrak{M}\,\rangle_{k+1} = \mathscr{T}$ (if it exists). We call the dimension $\dim \mathscr{D}^b_f(\boldsymbol{A})$ the *derived dimension* of the ring $\boldsymbol{A}$ and denote it by der.dim $\boldsymbol{A}$.

As the functor F is exact and essentially surjective, the next result is evident.

22 **Proposition 3.2.** der.dim $\boldsymbol{A} \leqslant$ der.dim $\widetilde{\boldsymbol{A}}$. *Namely, if* $\mathscr{D}^b_f(\widetilde{\boldsymbol{A}}) = \langle\,\mathfrak{M}\,\rangle_{k+1}$, *then* $\mathscr{D}^b_f(\boldsymbol{A}) = \langle\,\mathsf{DF}(\mathfrak{M})\,\rangle_{k+1}$.

s3

4. Semi-orthogonal decomposition

There is another recollement diagram for $\mathscr{D}(\widetilde{\boldsymbol{A}})$ related to the projective module $\boldsymbol{Q} = \binom{\boldsymbol{H}}{\boldsymbol{H}}$ with $\operatorname{End}_{\widetilde{\boldsymbol{A}}} \boldsymbol{Q} \simeq \boldsymbol{H}^{\mathrm{op}}$. Namely, we set

$$\begin{aligned}
\mathsf{G} &= \operatorname{Hom}_{\widetilde{\boldsymbol{A}}}(\boldsymbol{P}, _\,) \simeq \boldsymbol{P}^\vee \otimes_{\widetilde{\boldsymbol{A}}} _ : \widetilde{\boldsymbol{A}}\text{-Mod} \to \boldsymbol{A}\text{-Mod},\\
\mathsf{G}^* &= \boldsymbol{P} \otimes_{\boldsymbol{A}} _ : \boldsymbol{A}\text{-Mod} \to \widetilde{\boldsymbol{A}}\text{-Mod},\\
\mathsf{G}^! &= \operatorname{Hom}_{\boldsymbol{A}}(\boldsymbol{P}^\vee, _\,) : \boldsymbol{A}\text{-Mod} \to \widetilde{\boldsymbol{A}}\text{-Mod},
\end{aligned}$$

where $\mathsf{G}^\vee = \operatorname{Hom}_{\widetilde{\boldsymbol{A}}}(\boldsymbol{Q}, \widetilde{\boldsymbol{A}}) \simeq (\boldsymbol{C}\ \boldsymbol{H})$,

$$\begin{aligned}
\mathsf{DG} &: \mathscr{D}(\widetilde{\boldsymbol{A}}) \to \mathscr{D}(\boldsymbol{A}) \quad \text{be } \mathsf{G} \text{ applied compnentwise},\\
\mathsf{LG}^* &: \mathscr{D}(\boldsymbol{A}) \to \mathscr{D}(\widetilde{\boldsymbol{A}}) \quad \text{be the left adjoint of } \mathsf{G}^*,\\
\mathsf{RG}^! &: \mathscr{D}(\boldsymbol{A}) \to \mathscr{D}(\widetilde{\boldsymbol{A}}) \quad \text{be the right adjoint of } \mathsf{G}^!.
\end{aligned}$$

We also set $\bar{\boldsymbol{A}} = \boldsymbol{A}/]qC \simeq \widetilde{\boldsymbol{A}}/\left(\begin{smallmatrix}\boldsymbol{C} & \boldsymbol{H}\\ \boldsymbol{C} & \boldsymbol{H}\end{smallmatrix}\right)$. Then we have a recollement diagram

$$\operatorname{Ker}\mathsf{DF} \;\underset{\mathsf{RJ}^!}{\overset{\mathsf{LJ}^*}{\leftleftarrows}}\!\xrightarrow{\ \mathsf{DJ}\ } \mathscr{D}(\widetilde{\boldsymbol{A}}) \;\underset{\mathsf{RG}^!}{\overset{\mathsf{LG}^*}{\leftleftarrows}}\!\xrightarrow{\ \mathsf{DG}\ } \mathscr{D}(\boldsymbol{H})\ [m]$$

Since the $\widetilde{\boldsymbol{A}}$-ideal $\begin{pmatrix} \boldsymbol{C} & \boldsymbol{H} \\ \boldsymbol{C} & \boldsymbol{H} \end{pmatrix}$ is projective as right $\widetilde{\boldsymbol{A}}$-module, [8, Theorem 4.6] implies that $\operatorname{Ker} \mathsf{DG} \simeq \mathscr{D}(\bar{\boldsymbol{A}})$.

As usually, this recollement diagram gives *semi-orthogonal decompositions* [8, Corollary 2.6]

dec (4.1) $$\mathscr{D}(\widetilde{\boldsymbol{A}}) = (\operatorname{Ker} \mathsf{DG}, \operatorname{Im} \mathsf{LG}^*) = (\operatorname{Im} \mathsf{RG}^!, \operatorname{Ker} \mathsf{DG})$$

with $\operatorname{Ker} \mathsf{DG} \simeq \mathscr{D}(\bar{\boldsymbol{A}})$ and $\operatorname{Im} \mathsf{LG}^* \simeq \operatorname{Im} \mathsf{RG}^! \simeq \mathscr{D}(\boldsymbol{H})$ (though usually $\operatorname{Im} \mathsf{LG}^* \neq \operatorname{Im} \mathsf{RG}^!$). Recall that a semi-orthogonal decomposition $\mathscr{T} = (\mathscr{T}_1, \mathscr{T}_2)$ means that $\operatorname{Hom}_{\mathscr{T}}(T_2, T_1) = 0$ if $T_i \in \mathscr{T}_i$ and for every object $T \in \mathscr{T}$ there is an exact triangle $T_1 \to T_2 \to T \xrightarrow{+}$.

31 **Lemma 4.1.** *If $\mathscr{T} = (\mathscr{T}_1, \mathscr{T}_2)$ is a semi-orthogonal decomposition of a triangulated category $\mathscr{T}$, then*

e31 (4.2) $$\dim \mathscr{T} \leqslant \dim \mathscr{T}_1 + \dim \mathscr{T}_2 + 1.$$

m. First we show that, for any subsets $\mathfrak{M}, \mathfrak{N}$ of objects in $\mathscr{T}$,

e32 (4.3) $$\begin{aligned} \langle \mathfrak{M} \rangle_{k+1} \dagger \mathfrak{N} &\subseteq \langle \mathfrak{M} \rangle \dagger \langle \langle \mathfrak{M} \rangle_k \dagger \mathfrak{N} \rangle \subseteq \\ &\subseteq \underbrace{\langle \mathfrak{M} \rangle \dagger \langle \langle \mathfrak{M} \rangle \dagger \langle \langle \mathfrak{M} \rangle \dagger \dots \langle \langle \mathfrak{M} \rangle}_{k+1} \dagger \mathfrak{N} \rangle \dots \rangle \rangle \end{aligned}$$

Indeed, let $C \in \langle \mathfrak{M} \rangle_{k+1} \dagger \mathfrak{N}$, i.e there is an exact triangle $A \to B \to C \xrightarrow{+}$, where $A \in \langle \mathfrak{M} \rangle_{k+1}$, $B \in \mathfrak{N}$. There is an exact triangle $A_1 \to A \to A_2 \xrightarrow{+}$, where $A_1 \in \langle \mathfrak{M} \rangle_k$, $A_2 \in \langle \mathfrak{M} \rangle$. The octahedron axiom implies that there ar exact triangles $A_1 \to B \to B' \xrightarrow{+}$ and $A_2 \to B' \to C \xrightarrow{+}$. Therefore, $B' \in \langle \mathfrak{M} \rangle_k \dagger \mathfrak{N}$ and $C \in \langle \mathfrak{M} \rangle \dagger \langle \langle \mathfrak{M} \rangle_k \dagger \mathfrak{N} \rangle$.

Now, let $\langle \mathfrak{M} \rangle_{k+1} = \mathscr{T}_1$ and $\langle \mathfrak{N} \rangle_{l+1} = \mathscr{T}_2$. Then, for every $T \in \mathscr{T}$ there is an exact triangle $T_1 \to T_2 \to T \xrightarrow{+}$, where $T_1 \in \langle \mathfrak{M} \rangle_{k+1}$, $T_2 \in \langle \mathfrak{N} \rangle_{l+1}$. But, according to (4.3), $\langle \mathfrak{M} \rangle_{k+1} \dagger \langle \mathfrak{N} \rangle_{l+1} \subseteq \langle \mathfrak{M} \cup \mathfrak{N} \rangle_{k+l+2}$, so $\mathscr{T} = \langle \mathfrak{M} \cup \mathfrak{N} \rangle_{k+l+2}$ and $\dim \mathscr{T} \leqslant k + l + 1$. □

As $\bar{\boldsymbol{A}}$ is semi-simple, any indecomposable object from $\mathscr{D}(\bar{\boldsymbol{A}})$ is just a shifted simple module, so $\mathscr{D}^b_f(\bar{\boldsymbol{A}}) = \langle \bar{\boldsymbol{A}} \rangle$ and $\dim \mathscr{D}^b(\boldsymbol{A}) = 1$. If $\boldsymbol{H}$ is hereditary, every indecomposable object from $\mathscr{D}^b_f(\boldsymbol{H})$ is a shift of a module. For every module M there is an exact sequence $0 \to P' \to P \to M \to 0$ with projective modules P, P' and, since $\boldsymbol{H}$ is semiperfect, every indecomposable projective $\boldsymbol{H}$-module is a direct summand of $\boldsymbol{H}$. Hence $\mathscr{D}^b_f(\boldsymbol{H}) = \langle \boldsymbol{H} \rangle_2$ and $\operatorname{der.dim} \boldsymbol{H} \leqslant 1$.

32 **Corollary 4.2.** $\operatorname{der.dim} \boldsymbol{A} \leqslant \operatorname{der.dim} \boldsymbol{H} + 1$. *In particular, if $\boldsymbol{A}$ is a Backström ring,* $\operatorname{der.dim} \boldsymbol{A} \leqslant 2$.

A finite dimensional hereditary algebra is said to be *of Dynkin type* if it has finitely many isomorphism classes of indecomposable modules. Such algebras correspond to Dynkin quivers [10]. We say that a Backström algebra $\boldsymbol{A}$ is *of Dynkin type* if there is a Backström pair $\boldsymbol{H} \supseteq \boldsymbol{A}$, where $\boldsymbol{H}$ is a hereditary algebra of Dynkin type. For instance, it is so if $\boldsymbol{A}$ is a gentle or skewed-gentle algebra [29], or if it is the algebra Λ_n from Example 2.2(4). In[m] this case $\mathscr{D}^b_f(\boldsymbol{H}) = \langle M_1, M_2, \dots, M_m \rangle_1$, where $M_1, M_2, \dots, M_m$ are all pairwise non-isomorphic indecomposable $\boldsymbol{H}$-modules, so $\operatorname{der.dim} \boldsymbol{H} = 0$.

33 **Corollary 4.3.** *If $\boldsymbol{A}$ is a Backström algebra of Dynkin type (for instance, gentle or skewed-gentle), then* $\operatorname{der.dim} \boldsymbol{A} \leqslant 1$.

In [9] it was proved that $\operatorname{der.dim} \boldsymbol{A} = 0$ for a finite dimensional algebra if and only if $\boldsymbol{A}$ is a *piecewise hereditary algebra of Dynkin type* (or, equivalently, an *iterated tilted algebra of Dynkin type*) [18], i.e. it is derived equivalent to a hereditary algebra of Dynkin type.

s4 5. Relation to bimodule category

Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories, $\mathcal{U}$ be an $\mathcal{A}$-$\mathcal{B}$-bimodule, i.e. an additive functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathbf{Ab}$. Recall [12] that the *bimodule category* or the *category of elements of the bimodule* $\mathcal{U}$ is the category $\mathrm{El}(\mathcal{U})$ with the set of objects $\bigcup_{A\in\mathcal{A},\, B\in\mathcal{B}} \mathcal{U}(A,B)$ and the set of morphisms $u \to v$, where $u \in \mathcal{U}(A,B)$, $v \in \mathcal{U}(A',B')$ being the set of pairs

$$\{\, (\alpha,\beta) \mid \alpha : A' \to A,\ \beta : B \to B',\ u\alpha = \beta v \,\}.$$

Here we write, as usually $u\alpha$ and βv, respectively, instead of $\mathcal{U}(\alpha, 1_B)u$ and $\mathcal{U}(1_A, \beta)v$. Bimodule categories appear when there is a semi-orthogonal decomposition of a triangulated category.

41 **Theorem 5.1.** *Let $(\mathcal{A},\mathcal{B})$ be a semi-orthogonal decomposition of a triangulated category $\mathcal{C}$. Consider the $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{U}$ such that $\mathcal{U}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$ $(A \in \mathcal{A},\ B \in \mathcal{B})$. For every $f : A \to B$ fix a cone Cf such that there is an exact triangle $A \xrightarrow{f} B \xrightarrow{f_1} Cf \xrightarrow{f_2} A[1]$. The map $f \mapsto Cf$ induces an equivalence of categories $\mathsf{C} : \mathrm{El}(\mathcal{U}) \xrightarrow{\sim} \mathcal{C}/\mathcal{J}$, where $\mathcal{J}$ is the ideal of $\mathcal{C}$ consisting of morphisms η such that there are factorizations $\eta = \eta'\xi = \gamma''$, where the source of η' is in $\mathcal{A}$ and the target of η'' is in $\mathcal{B}$. Moreover, $\mathcal{J}^2 = 0$, so C induces a bijection between isomorphism classes of objects in $\mathrm{El}(\mathcal{U})$ and in $\mathcal{C}$.*[2]

Proof. As $(\mathcal{A},\mathcal{B})$ is a semi-orthogonal decomposition of $\mathcal{C}$, every object from $\mathcal{C}$ occur in an exact triangle $A \xrightarrow{B}\!\!\to C \xrightarrow{+}$, where $A \in \mathcal{A}$, $B \in \mathcal{B}$, so f is an object from $\mathrm{El}(\mathcal{U})$ and $C \simeq Cf$. Let $f' : A' \to B'$ be another object of $\mathrm{El}(\mathcal{U})$ and $(\alpha,\beta) : f \to f'$ in $\mathrm{El}(\mathcal{U})$. Fix a commutative diagram

e41 (5.1)
$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{f_1} & Cf & \xrightarrow{f_2} & A[1] \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\alpha[1]} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{f'_1} & Cf' & \xrightarrow{f'_2} & A'[1]
\end{array}$$

[2] This theorem is actually a partial case of [13, Theorem 1.1].

It exists, though is not unique. Let γ' be another morphism making the diagram (5.1) commutative an set $\eta = \gamma - \gamma'$. Then $\eta f_1 = 0$, hence η factors through f_2, and $f_2'\eta = 0$, hence η factors through f_1'. Thus $\eta \in \mathcal{J}$. On the other hand, if $\eta : Cf \to Cf'$ is in $\mathcal{J}$, the decomposition $\eta = \eta'\xi$ implies that $\eta f_1 = \eta'\xi f_1 = 0$ and the decomposition $\eta = \zeta\eta''$ implies that $\alpha[1]\eta = \alpha[1]\zeta\eta'' = 0$, hence the morphism $\gamma' = \gamma + \eta$ makes the diagram (5.1) commutative. Therefore, the class $C(\alpha,\beta)$ of γ modulo $\mathcal{J}$ is uniquely defined, so the maps $f \mapsto Cf$ and $(\alpha,\beta) \mapsto C(\alpha,\beta)$ define a functor $\mathsf{C} : \mathrm{El}(\mathcal{U}) \to \mathcal{C}/\mathcal{J}$.

Let now $\gamma : Cf \to Cf'$ be any morphism. Then $f_2'\gamma f_1 = 0$, so $\gamma f_1 = \beta f$ for some $\beta : B \to B'$. Hence there is a morphism $\alpha : A \to A'$ making the diagram (5.1) commutative, i.e. defining a morphism $(\alpha,\beta) : f \to f'$ such that $\gamma \equiv C(\alpha,\beta) \pmod{\mathcal{J}}$. If (α',β') is another such morphism, $f_1'(\beta-\beta') = 0$, so $\beta-\beta' = f'\xi$ for some $\xi : B \to A$. But $\xi = 0$, so $\beta = \beta'$. In the same way $\alpha = \alpha'$. Hence the functor C is fully faithful. As we have already notices, it is essentially surjective, therefore defines an equivalence $\mathrm{El}(\mathcal{U}) \simeq \mathcal{C}/\mathcal{J}$.

The equality $\mathcal{J}^2 = 0$ follows immediately from the definition and the conditions of the theorem. □

We apply Theorem 5.1 to Backström pairs $\boldsymbol{H} \supseteq \boldsymbol{A}$ such that $\boldsymbol{A}$ is left noetherian and $\boldsymbol{H}$ is left hereditary and finitely generated as left $\boldsymbol{A}$-module. For instance, it is so in the case of Backström algebras. Then the ring $\widetilde{\boldsymbol{A}}$ is also noetherian and $\boldsymbol{C}$ is projective as left $\boldsymbol{H}$-module. According to (4.1), $(\mathrm{Ker}\,\mathsf{DG}, \mathrm{Im}\,\mathsf{LG}^*)$ is a semi-orthogonal decomposition of $\mathscr{D}(\widetilde{\boldsymbol{A}})$. Moreover, both G and G^* map finitely generated modules to finitely generated, so the same is valid if we consider their restrictions onto $\mathscr{D}_f(\widetilde{\boldsymbol{A}})$ and $\mathscr{D}_f(\boldsymbol{H})$. Note also that G^* is exact, so LG^* can be applied to complexes componentwise. The $\widetilde{\boldsymbol{A}}$-module G^*M can be identified with the module of columns $M^2 = \binom{M}{M}$ with the action of $\widetilde{\boldsymbol{A}}$ given by the matrix multiplication. It gives an equivalence of $\mathscr{D}(\boldsymbol{H})$ with $\mathrm{Im}\,\mathsf{LG}^*$. As $\boldsymbol{H}$ is left hereditary, every complex from $\mathscr{D}(\boldsymbol{H})$ is equivalent to a direct sum of shifted modules (see [20, Section 2.5]). other hand $\mathrm{Ker}\,\mathsf{DG} \simeq \mathscr{D}(\bar{\boldsymbol{A}})$ and $\bar{\boldsymbol{A}}$ is semi-simple, so every complex from $\mathscr{D}(\bar{\boldsymbol{A}})$ is isomorphic to a direct sum of shifted simple $\boldsymbol{A}$-modules. So, to calculate the bimodule C, we only have to calculate $\mathrm{Ext}^i_{\widetilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^2)$, where . Note also that a projective resolution of $\bar{\boldsymbol{A}}$ is $P : 0 \to \boldsymbol{C}^2 \xrightarrow{\varepsilon} \boldsymbol{P} \to 0$, since $\boldsymbol{C}$ is a projective $\boldsymbol{H}$-module, so $\boldsymbol{C}^2$ is a projective $\widetilde{\boldsymbol{A}}$-module. Hence, we only have to calculate $\mathrm{Hom}_{\widetilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^2)$ and $\mathrm{Ext}^1_{\widetilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^2)$.

Theorem 5.2. (1) $\mathrm{Hom}_{\widetilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^2) \simeq \mathrm{Ann}_M \boldsymbol{C} = \{\, u \in M \mid \boldsymbol{C}u = 0 \,\}$.

(2) $\mathrm{Ext}^1_{\widetilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^2) \simeq \mathrm{Hom}_{\boldsymbol{H}}(\boldsymbol{C}, M)/(M/\mathrm{Ann}_M \boldsymbol{C})$, *where the quotient* $M/\mathrm{Ann}_M \boldsymbol{C}$ *embeds in* $\mathrm{Hom}_{\boldsymbol{H}}(\boldsymbol{C}, M)$ *if we consider an element* $u \in M$ *as the homomorphism* $\mu_c : c \mapsto cu$.

Proof. (1) $\mathrm{Hom}_{\widetilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^2)$ is identified with the set of homomorphisms $\varphi : \boldsymbol{P} \to M^2$ such that $\varphi\varepsilon = 0$. A homomorphism $\varphi : \bar{\boldsymbol{A}} \to M^2$ is uniquely defined by an element $u \in M$ such that $\varphi\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}u\\0\end{smallmatrix}\right)$. Namely, $\varphi\left(\begin{smallmatrix}a\\c\end{smallmatrix}\right) = \left(\begin{smallmatrix}au\\cu\end{smallmatrix}\right)$. Obviously, $\varphi\varepsilon = 0$ if and only if $\boldsymbol{C}u = 0$, i.e. $u \in \operatorname{Ann}_M \boldsymbol{C}$.

(2) $\operatorname{Ext}^1_{\widetilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^2) \simeq \operatorname{Hom}_{\widetilde{\boldsymbol{A}}}(\boldsymbol{C}^2, M^2)/\operatorname{Hom}_{\widetilde{\boldsymbol{A}}}(\boldsymbol{P}, M^2)\varepsilon$. As the functor G^* is fully faithful, $\operatorname{Hom}_{\widetilde{\boldsymbol{A}}}(\boldsymbol{C}^2, M^2) \simeq \operatorname{Hom}_{\boldsymbol{H}}(\boldsymbol{C}, M)$. Namely, $\psi : \boldsymbol{C} \to M$ induces $\psi_2 : \boldsymbol{C}^2 \to M$ mapping $\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)$ to $\begin{pmatrix}\psi(a)\\\psi(b)\end{pmatrix}$. Let $\varphi : \boldsymbol{P} \to M^2$ correspond, as above, to an element $u \in M$. Then $\varphi\varepsilon\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right) = \left(\begin{smallmatrix}au\\bu\end{smallmatrix}\right)$, so it equals μ_u and $\operatorname{Hom}_{\widetilde{\boldsymbol{A}}}(\boldsymbol{P}, M^2)\varepsilon$ is identified with $M/\operatorname{Ann}_M \boldsymbol{C}$ embedded in $\operatorname{Hom}_{\boldsymbol{H}}(\boldsymbol{C}, M)$. □

s5 6. Partial tilting for Backström pairs

Let $\boldsymbol{H} \supseteq \boldsymbol{A}$ is a Backström pair. Consider the ring $\boldsymbol{B}$ of triangular matrices of the form

$$\boldsymbol{B} = \begin{pmatrix}\bar{\boldsymbol{A}} & \bar{\boldsymbol{H}}\\ 0 & \boldsymbol{H}\end{pmatrix}.$$

Let $e_1 = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$, $B^1 = \boldsymbol{B}e_1$ and $B^2 = \boldsymbol{B}e_2$ are projective $\boldsymbol{B}$-modules given by the first and the second column of $\boldsymbol{B}$, i.e.

$$B^1 = \begin{pmatrix}\bar{\boldsymbol{A}}\\ 0\end{pmatrix}, \quad B^2 = \begin{pmatrix}\bar{\boldsymbol{H}}\\ \boldsymbol{H}\end{pmatrix}.$$

A $\boldsymbol{B}$-module M is defined by a triple $\begin{pmatrix}M_1\\M_2\,\chi_M\end{pmatrix}$, where $M_1 = e_1M$ is an $\bar{\boldsymbol{A}}$-module, $M_2 = e_2M$ is an $\boldsymbol{H}$-module and $\chi_M : M_2 \to M_1$ is an $\boldsymbol{A}$-homomorphism such that $\operatorname{Ker}\chi_M \supseteq \boldsymbol{C}M_2$. We write an element $u \in M$ as a column $\left(\begin{smallmatrix}u_1\\u_2\end{smallmatrix}\right)$, where $u_i = e_iu$. Then

$$\begin{pmatrix}a & b\\ 0 & c\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}au_1 + \chi_M(bu_2)\\ cu_2\end{pmatrix}.$$

A homomorphism $\alpha : M \to N$ is defined by two homomorphisms $\alpha_1 : M_1 \to N_1$ and $\alpha_2 : M_2 \to N_2$ such that $\alpha_1\chi_M = \chi_N\alpha_2$. We write $\alpha = \left(\begin{smallmatrix}\alpha_1\\\alpha_2\end{smallmatrix}\right)$.

51 **Proposition 6.1.** $\operatorname{l.gl.dim}\boldsymbol{B} = \max\{\operatorname{l.gl.dim}\boldsymbol{H}, \operatorname{w.dim}\bar{\boldsymbol{H}}_{\boldsymbol{H}} + 1\}$.
In particular, if $\boldsymbol{H}$ is left hereditary and $\bar{\boldsymbol{H}}$ is not projective as right $\boldsymbol{H}$-module, then $\operatorname{l.gl.dim}\boldsymbol{B} = 2$.

Proof. [26, Theorem 5] shows that $\operatorname{l.gl.dim}\boldsymbol{B} \leqslant n$ if and only if

$$\operatorname{l.gl.dim}\boldsymbol{H} \leqslant n \ \text{ and } \ R^n\operatorname{Hom}_{\bar{\boldsymbol{A}}}(\bar{\boldsymbol{H}} \otimes_{\boldsymbol{H}} \_\,, \_) = 0.$$

As the ring $\bar{\boldsymbol{A}}$ is semi-simple,

$$R^n\operatorname{Hom}_{\bar{\boldsymbol{A}}}(\bar{\boldsymbol{H}} \otimes_{\boldsymbol{H}} \_\,, \_) = \operatorname{Hom}_{\bar{\boldsymbol{A}}}(\operatorname{Tor}^{\boldsymbol{H}}_n(\bar{\boldsymbol{H}}, \_), \_),$$

it implies the first claim. The second claim follows, since $\operatorname{Tor}^{\boldsymbol{H}}_1(\bar{\boldsymbol{A}}, \bar{\boldsymbol{A}}) = 0$ if and only if $\bar{\boldsymbol{A}}_{\boldsymbol{H}}$ is projective. □

Note that $\bar{\boldsymbol{H}}_{\boldsymbol{H}}$ is projective if and only if the rings $\boldsymbol{H}$ and $\boldsymbol{A}$ are of the form

$$\boldsymbol{H} = \begin{pmatrix}\bar{\boldsymbol{H}} & 0\\ A_1^0 & A_1^1\end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix}\bar{\boldsymbol{A}} & 0\\ A_1^0 & A_1^1\end{pmatrix}.$$

If $\boldsymbol{H}$ is left hereditary, then A_1^1 is left hereditary and A_1^0 is a projective left A_1^1-module. Then $\boldsymbol{A}$ is also[m] hereditary, as well as $\boldsymbol{B}$.

We denote by R the $\boldsymbol{B}$-module given by the triple $\begin{pmatrix} \boldsymbol{H}/\boldsymbol{A} & \\ \boldsymbol{H} & \pi \end{pmatrix}$, where $\pi : \boldsymbol{H} \to \boldsymbol{H}/\boldsymbol{A}$ is the natural surjection.

52 **Proposition 6.2.** (1) $\operatorname{End}_{\boldsymbol{B}} R \simeq \boldsymbol{A}^{\mathrm{op}}$.

(2) $\mathrm{pr.dim}_{\boldsymbol{B}}\, R = 1$.

(3) $\operatorname{Ext}^1_{\boldsymbol{B}}(R, R) = 0$.

Recall that the conditions (2) and (3) mean that R is a *partial tilting* $\boldsymbol{B}$*-module.*

Proof. (1) The minimal projective resolution of R is the complex

$$P : \quad 0 \to B^1 \xrightarrow{\varepsilon} B^2 \to 0,$$

where ε is the embedding, which gives (2). Any endomorphism γ of R induces a commutative diagram

$$\begin{array}{ccc} B^1 & \longrightarrow & B^2 \\ {\scriptstyle\gamma_1}\downarrow & & \downarrow{\scriptstyle\gamma_2} \\ B^1 & \longrightarrow & B^2 \end{array}$$

As $\operatorname{End}_{\boldsymbol{B}} B^2 \simeq \boldsymbol{H}^{\mathrm{op}}$, α_2 is given by multiplication with an element $h \in \boldsymbol{H}$ on the right. If there is a commutative diagram as above, then $h \in \boldsymbol{A}$, which proves (1). Finally, a homomorphism $\alpha : B^1 \to R$ maps the generator $\binom{1}{0}$ of B^1 to an element $\binom{\bar{h}}{0} \in R$. If h is a preimage of $\bar{h}$ in $\boldsymbol{H}$, then α extends to the homomorphism $B^2 \to R$ that maps the generator $\binom{0}{1}$ of B^2 to $\binom{0}{h} \in R$. It implies (3). $\square$

Together with Theorem 1.1, it gives the following result. [m]

53 **Theorem 6.3.** (1) *The functor* $\mathsf{F} = \operatorname{RHom}_{\boldsymbol{B}}(R, \_)$ *induces an equivalence* $\operatorname{Tri}(R) \to \mathscr{D}(\boldsymbol{A})$.

(2) $\operatorname{Ker} \mathsf{F}$ *consists of complexes* C *such that the map* $\chi_{H^k(C)}$ *is bijective for all* k.

(3) *There is a recollement diagram*

$$\operatorname{Ker}\mathsf{F} \;\underset{\mathsf{I}^!}{\overset{\mathsf{I}^*}{\leftleftarrows}}\!\xrightarrow{\ \mathsf{I}\ }\; \mathscr{D}(\boldsymbol{B}) \;\underset{\mathsf{F}^!}{\overset{\mathsf{F}^*}{\leftleftarrows}}\!\xrightarrow{\ \mathsf{F}\ }\; \mathscr{D}(\boldsymbol{A})$$

where F^* *and* $\mathsf{F}^!$ *are, respectively, left and right adjoints to* F, I *is the embedding,* I^* *and* $\mathsf{I}^!$ *are, respectively, left and right adjoints to* I *and both adjunction morphisms* $\eta : \mathsf{F}\mathsf{F}^* \to \mathrm{Id}_{\mathscr{D}(R)}$ *and* $\zeta : \mathrm{Id}_{\mathscr{D}(R)} \to \mathsf{F}\mathsf{F}^!$ *are isomorphisms.*

Actually, the claim (2) means that a complex C is in $\operatorname{Ker}\mathsf{F}$ if and only if its cohomologies are direct sums of modules of the form $(U,U,1_U)$, where U is a simple $\bar{\boldsymbol{H}}$-module.

Proof. (1) and (3) follow from Theorem 1.1, since the complex P is perfect, hence compact, and isomorphic to R in $\mathscr{D}(\boldsymbol{B})$. To find $\operatorname{Ker}\mathsf{F}$, consider a complex

$$C:\ \dots\to C^{k-1}\xrightarrow{d^{k-1}} C^k\xrightarrow{d^k} C^{k+1}\to\dots,$$

where C^k is defined by a triple $\begin{pmatrix} C_1^k \\ C_2^k\ \chi_k\end{pmatrix}$ and $d^k=\begin{pmatrix} d_1^k \\ d_2^k\end{pmatrix}$, where $d_1^k\chi_k=\chi_{k+1}d_2^k$ for all k. Note that $C_i=(C_i^k,d_i^k)$ $(i=1,2)$ are complexes, (χ_k) is a homomorphism of complexes and $H^k(C)=(H^k(C^1),H^k(C^2),\bar\chi_k)$, where $\bar\chi_k=\chi_{H^k(C)}$ is induced by χ_k. A homomorphism $P\to C[k]$, where P is the complex $0\to B^1\xrightarrow{\varepsilon}B^2\to 0$, is a pair of homomorphisms $\alpha:B^2\to C^k,\beta:B^1\to C^{k-1}$ such that $\alpha_1\pi=\chi_k\alpha_2$, $\beta_2=0$, $d_i^k\alpha_i=0$ $(i=1,2)$ and $d_1^{k-1}\beta_1=\alpha_1|_{\bar{\boldsymbol{A}}}$. Let $\alpha_2(1)=x\in C_2^k$ and $\beta_1(1)=y\in C_1^{k-1}$. These values completely define α and β. The conditions for α and β mean that $d_2^kx=0$ and $d_1^{k-1}y=\chi_kx$.

This morphism is homotopic to zero if and only if there are maps $\sigma:B^2\to C^{k-1}$ and $\tau:B^1\to C^{k-2}$ such that $\alpha=d^{k-1}\sigma[\mathrm{m}]$ and $\beta=\sigma\varepsilon+d^{k-2}\tau$. Again σ is defined by the element $z=\sigma_2(1)\in C_2^{k-1}$ and τ is defined by the element $t=\tau_1(1)\in C_1^{k-2}$; note also that $\tau_2=0$ and $\sigma_1\pi=\chi_{k-1}\sigma_2$.

Suppose that any homomorphism $P\to C$ is homotopic to zero. Let $\bar x\in H^k(C_2)$ be such that $\bar\chi_k(\bar x)=0$ and $x\in\operatorname{Ker}d_2^k$ be a representative of $\bar x$. Then $\chi_k(x)=dy$ for some $y\in C_1^{k-1}$, so the pair (x,y) defines a homomorphism $P\to C[k]$. Therefore, there must be $z\in C_2^{k-1}$ such that $d_2^{k-1}y=0$, thus $\bar x=0$ and $\bar\chi_k$ is injective. Let now $\bar y\in H^{k-1}(C_2)$ and $y\in C_2^{k-1}$ be its representative. Then the pair $(0,y)$ defines a homomorphism $P\to C[k]$, so there must be an element $z\in C_2^{k-1}$ such that $d_2^{k-1}z=0$ and $y=\chi_{k-1}z+d_1^{k-1}t$ for some t. Hence $\bar y=\bar\chi_{k-1}(\bar z)$ and $\bar\chi_{k-1}$ is surjective. As it holds for all k, we have that all maps $\bar\chi_k$ are bijective.

On the contrary, suppose that all $\bar\chi_k$ are bijective. If (x,y) defines a homomorphism $P\to C[k]$, then $\chi_k(x)=d_1^{k-1}y$, so $\bar\chi_k(x)=0$. Therefore, $\bar x=0$, i.e. $x=d_2^{k-1}z$ for some $z\in C_2^{k-1}$ and $\chi_kx=d_1^k\chi_{k-1}z$. Then $d(y-\chi_{k-1}z)=0$, hence there is an element $z'\in C_2^{k-1}$ such that the cohomology class of $y-\chi_{k-1}z$ equals $\bar\chi_k\bar{z'}$ where $z'\in C_2^{k-1}$ and $d_2^{k-1}z'=0$, i.e. $y-\chi_{k-1}z=\chi_{k-1}(z')+dt$ for some t. Then $x=d_2^{k-1}(z+z')$ and $y-\chi_{k-1}(z+z')=0$, so this homomorphism is homotopic to zero. □

As usually, we identify the category $\boldsymbol{A}$-Mod with the full subcategory of $\mathscr{D}(\boldsymbol{A})$ consisting of the complexes C concentrated in degree 0. The following result shows how the partial tilting functor F behaves with respect to modules.

54 **Corollary 6.4.** *Let a $\boldsymbol{B}$-module M be given by the triple* $\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\,\chi_M\right)$.

(1) $\mathsf{F}M \in \boldsymbol{A}\text{-Mod}$ *if and only if* χ_M *is surjective.*
(2) $\mathsf{F}M \in \boldsymbol{A}\text{-Mod}[1]$ *if and only if* χ_M *is injective.*

Proof. $H^i\,\mathrm{RHom}(R,M) \simeq \mathrm{Ext}^i_{\boldsymbol{B}}(R,M)$, so $M \in \boldsymbol{A}$-Mod if and only if $\mathrm{Ext}^1_{\boldsymbol{B}}(R,M) = 0$, i.e. any homomorphism $B^1 \to M$ factors through ε. One easily sees that it is just when χ_M is surjective. Analogously, $\mathsf{F}M \in \boldsymbol{A}$-Mod[1] if and only if $\mathrm{Hom}_{\boldsymbol{B}}(R,M) = 0$, which means that χ_M is injective. □

References

[1] Assem, I., Skowroński, A. Iterated tilted algebras of type $\tilde{\mathbb{A}}_n$. Math. Z. 195 (1987) 269–290.

[2] Bass, H. Finitistic dimension and a homological generalization of semi-primary rings. Trans. Amer. Math. Soc. 95 (1960) 466–488.

[3] Beilinson A. A, Bernstein J., Deligne P. Faisceaux pervers. Asterisque 100 (1982).

[4] Burban, I., Drozd, Y. Derived categories of nodal algebras. J. Algebra 272 (2004) 46–94.

[5] Burban, I., Drozd, Y. Derived categories for nodal rings and projective configurations. Noncommutative Algebra and Geometry. Lecture Notes in Pure and Applied Mathematics 243 (2005), 23-46.

[6] Burban, I., Drozd, Y. Tilting on non-commutative rational projective curves. Math. Ann. 351 (2011) 665–709.

[7] Burban, I., Drozd, Y. On the derived categories of gentle and skew-gentle algebras: homological algebra and matrix problems. arXiv:1706.08358.

[8] Burban, I., Drozd, Y., Gavran, V. Minors of non-commutative schemes. European J. of Math. 3 (2017) 311341.

[9] Chen, Xiao-Wu, Ye, Yu, Zhang, Pu. Algebras of derived dimension zero. Commun. Algebra 36, (2008) 1-10.

[10] Dlab, V., Ringel, C. M. Indecomposable representations of graphs and algebras. Mem. Am. Math. Soc. 173 (1976).

[11] Drozd Yu. A. Finite modules over pure Noetherian algebras. Proc. Steklov Inst. Math. 4 (1991) 97–108.

[12] Drozd Yu. A. Reduction algorithm and representations of boxes and algebras. Comtes Rendue Math. Acad. Sci. Canada 23 (2001) 97-125.

[13] Drozd, Yu. Matrix problems, triangulated categories and stable homotopy types. São Paulo J. Math. Sci. 4 (2010) 209–249.

[14] Drozd, Yu. A., Voloshyn, D. E. Vector bundles over noncommutative nodal curves. Ukr. Math. J. 64, No.2 (2012) 185–199.

[15] Drozd, Yu. A., Voloshyn, D. E. Derived categories of nodal curves. Ukr. Math. J. 64, No.7 (2012) 1033–1040.

[16] Drozd Yu. A., Zembyk, V. V. Representations of nodal algebras of type **A**. Algebra Discrete Math. 15, No.2 (2013) 179–200.

[17] Geiß, Ch., de la Peña, J. A. Auslander-Reiten components for clans. Bol. Soc. Mat. Mex., III. Ser. 5, No. 2 (1999) 307–326.

[18] Happel, D. Triangulated Categories in the Representation Theory of Finite Dimensional Algebras. London Math. Soc. Lectute Note Series 119. Cambridge University Press, 1988.

[19] Keller, B. Deriving DG-categories. Annales scientifiques de lE.N.S. 27 (1994), p. 63–102.

[20] Keller, B. Derived categories and tilting. Handbook of Tilting Theory. London Math. Soc. Lectute Note Series 332. Cambridge University Press, 2007.
[21] Lambek, J. Lectures on rings and modules. AMS Publishing, 1966.
[22] Lekili, Y., Polishchuk, A. Auslander orders over nodal stacky curves and partially wrapped Fukaya categories. arXiv:1705.06023.
[23] Lunts, V. A. Categorical resolution of singularities. J. Algebra 323 (2010) 2977–3003.
[24] Neeman, A. Triangulated Categories. Ann. of Math. Studies 148. PrincetonUniversity Press, Princeton (2001).
[25] Ringel, C. M., Roggenkamp, K. W. Indecomposable representations of orders and Dynkin diagrams. Comtes Rendue Math. Acad. Sci. Canada 1 (1979) 91–94.
[26] Palmer, I., Roos, J.-E. Explicit formulae for the global homological dimensions of trivial extensions of rings. J. Algebra 27 (1973) 380–413.
[27] Rouquier, R. Dimensions of triangulated categories. J. K-Theory 1 (2008) 193–256.
[28] Zembyk, V.V. Structure of finite dimensional nodal algebras. Ukrainian Math. J. 66 (2014) 415–419.
[29] Zembyk, V. V. Skewed-gentle algebras are nodal. Ukr. Mat. Zh. 67 (2015) 574–576.

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