TORIFICATION AND FACTORIZATION OF BIRATIONAL MAPS

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Abstract. Building on work of the fourth author in [71], we prove the weak factorization conjecture for birational maps in characteristic zero: a birational map between complete nonsingular varieties over an algebraically closed field K of characteristic zero is a composite of blowings up and blowings down with smooth centers.

Contents

0. Introduction

We work over an algebraically closed field K of characteristic 0. We denote the multiplicative group of K by K^*.

0.1. **Statement of the main result.** The purpose of this paper is to give a proof for the following weak factorization conjecture of birational maps:

Theorem 0.1.1 (Weak Factorization). *Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between proper nonsingular algebraic varieties X_1 and X_2 over an algebraically closed field K of characteristic zero, and let $U \subset X_1$ be the maximal open set where ϕ is an isomorphism. Then ϕ can be factorized into a sequence of blowings up and blowings down with smooth centers disjoint from U, namely, there exists a sequence of birational maps between proper nonsingular algebraic varieties*

$$X_1 = V_0 \overset{\varphi_1}{\dashrightarrow} V_1 \overset{\varphi_2}{\dashrightarrow} \cdots \overset{\varphi_i}{\dashrightarrow} V_i \overset{\varphi_{i+1}}{\dashrightarrow} V_{i+1} \overset{\varphi_{i+2}}{\dashrightarrow} \cdots \overset{\varphi_{l-1}}{\dashrightarrow} V_{l-1} \overset{\varphi_l}{\dashrightarrow} V_l = X_2$$

where

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1. $\phi = \varphi_l \circ \varphi_{l-1} \circ \cdots \varphi_2 \circ \varphi_1$,
2. φ_i *are isomorphisms on* U, *and*
3. *either* $\varphi_i : V_i \dashrightarrow V_{i+1}$ *or* $\varphi_i^{-1} : V_{i+1} \dashrightarrow V_i$ *is a morphism obtained by blowing up a smooth center disjoint from* U.

Furthermore, there is an index i_0 *such that for all* $i \leq i_0$ *the map* $V_i \dashrightarrow X_1$ *is a projective morphism, and for all* $i \geq i_0$ *the map* $V_i \dashrightarrow X_2$ *is a projective morphism. In particular, if* X_1 *and* X_2 *are projective then all the* V_i *are projective.*

Remark 0.1.2. The reader is referred to section 5 for generalizations of this theorem in various directions.

Another proof of this theorem was independently given by the fourth author in [72]. See section 0.11 for a comparison of the two approaches.

Remark 0.1.3. The assumption on the characteristic of the base field K to be zero is essential in several places of our proof where we use canonical (embedded) resolution of singularities ([24], [69], [7]) and Luna's Étale Slice Theorem.

0.2. **Strong factorization.** If we insist in the assertion above that $\phi_1^{-1}, \dots, \phi_{i_0}^{-1}$ and $\phi_{i_0+1}, \dots, \phi_l$ be regular maps for some i_0, we obtain the following strong factorization conjecture.

Conjecture 0.2.1 (Strong Factorization)**.** *Let the situation be as in Theorem 0.1.1. Then there exists a diagram*

$$\begin{array}{ccccc} & & Y & & \\ & \psi_1 \swarrow & & \searrow \psi_2 & \\ X_1 & & \overset{\phi}{\dashrightarrow} & & X_2 \end{array}$$

where the morphisms ψ_1 *and* ψ_2 *are composites of blowings up of smooth centers disjoint from* U.

0.3. **Early origins of the problem.** The history of the factorization problem of birational maps could be traced back to the Italian school of algebraic geometers, who already knew that the operation of blowing up points on surfaces is a fundamental source of richness for surface geometry: the importance of the strong factorization theorem in dimension 2 (see [73]) cannot be overestimated in the analysis of the birational geometry of algebraic surfaces. We can only guess that Zariski, possibly even members of the Italian school, contemplated the problem in higher dimension early on, but refrained from stating it before results on resolution of singularities were available. The question of strong factorization was explicitly stated by Hironaka as "Question (F′)" in [24], Chapter 0, §6, and the question of weak factorization was raised in [51]. The problem remained largely open in higher dimensions despite the efforts and interesting results of many (see e.g. Crauder [11], Kulikov [40], Schaps [62]). These were summarized by Pinkham [54], where the weak factorization conjecture is explicitly stated.

0.4. **The toric case.** For toric birational maps, the equivariant versions of the weak and strong factorization conjectures was posed in [51] and came to be known as Oda's weak and strong conjectures. Not only is the toric version a special case of the general factorization conjectures, most examples demonstrating the difficulties in higher dimensions are also toric (see Hironaka [23], Sally [60], Shannon [64]). Thus it presented a substantial challenge and combinatorial difficulty. In dimension 3, Danilov's proof [16] was later supplemented by Ewald [19]. Oda's weak conjecture was solved in arbitrary dimension by J. Włodarczyk in [70], and Oda's strong conjecture was proven by R. Morelli in [45] (see also [46], [3]). An important combinatorial notion which Morelli introduced into this study is that of a *cobordism* between fans. The algebro-geometric realization of Morelli's combinatorial cobordism is the notion of a *birational cobordism* introduced in [71] .

0.5. **The local version.** There is a local version of the factorization conjecture, formulated and proved in dimension 2 by Abhyankar ([5], Theorem 3). Christensen [9] posed the problem in general and solved it for some special cases in dimension 3. Here the varieties X_1 and X_2 are replaced by appropriate birational local rings dominated by a fixed valuation, and blowings up are replaced by monoidal transforms subordinate to the valuation. This local conjecture was recently solved by S. D. Cutkosky in its strongest form in a series of papers [12, 13, 14]. In a sense, Cutkosky's result says that there are no local obstructions to solving the global strong factorization conjecture.

0.6. **Birational cobordisms.** Our method is based upon the theory of birational cobordisms [71]. As mentioned above, this theory was inspired by the combinatorial notion of polyhedral cobordisms of R. Morelli [45], which was used in his proof of strong factorization for birational toric maps.

Given a birational map $\phi : X_1 \dashrightarrow X_2$, a *birational cobordism* $B_\phi(X_1, X_2)$ is a variety of dimension $\dim(X_1) + 1$ with an action of the multiplicative group K^*. It is analogous to the usual cobordism $B(M_1, M_2)$ between differentiable manifolds M_1 and M_2 given by a Morse function f. In the differential setting one can construct an action of the additive real group $\mathbb{R}$, where the "time" $t \in \mathbb{R}$ acts as a diffeomorphism induced by integrating the vector field $grad(f)$; hence the multiplicative group $(\mathbb{R}_{>0}, \times) = \exp(\mathbb{R}, +)$ acts as well. The critical points of f are precisely the fixed points of the action of the multiplicative group, and the homotopy type of fibers of f changes when we pass through these critical points. Analogously, in the algebraic setting "passing through" the fixed points of the K^*-action induces a birational transformation. Looking at the action on the tangent space at each fixed point, we obtain a locally toric description of the transformation. This already gives the main result of [71] : a factorization of ϕ into *locally toric birational maps* among varieties with locally toric structures. Such birational transformations can also be interpreted using the work of Brion-Procesi, Thaddeus, Dolgachev-Hu and others (see [8, 67, 68, 18]), which describes the change of Geometric Invariant Theory quotients associated to a change of linearization.

0.7. **Locally toric versus toroidal structures.** Considering the fact that strong factorization has been proven for *toroidal* birational maps ([70], [45], [3]), one might naïvely think that this would already provide a proof for Theorem 0.1.1.

However, in the locally toric structure obtained from a cobordism, the embedded tori chosen may vary from point to point, while a toroidal structure (see below) requires the embedded tori to be induced from one fixed open set. Thus there is still a gap between the notion of locally toric birational maps and that of toroidal birational maps.

0.8. **Torification.** In order to bridge over this gap, we follow ideas introduced in [1], and blow up the birational cobordism $B_\phi(X_1, X_2)$ along *torific ideals*. This operation induces a toroidal structure in a neighborhood of a connected component F of the fixed point set, on which the action of K^* is a *toroidal action* (we say that the blowup *torifies* the action of K^*). Now the birational transformation passing through F is toroidal, and hence we can apply the factorization theorem for toroidal birational maps. As these toroidal structures may have singularities, we use the canonical resolution of singularities at the final stage to complete the proof of Theorem 0.1.1.

At the moment, we can only analyze the effect of blowing up the torific ideals in a neighborhood of one connected component of the fixed point set at a time and fails to carry it out simultaneously for the entire fixed point set in the birational cobordism $B_\phi(X_1, X_2)$. This remains as an obstruction for us to obtain a proof for the strong factorization conjecture. We hope to remove this obstruction in the near future.

0.9. **Relation with the minimal model program.** It is worthwhile to note the relation of the factorization problem to the development of the Mori program. Hironaka [22] used the cone of effective curves to study the properties of birational morphisms. This direction was further developed and given a decisive impact by Mori [47], who, motivated by Kleiman's criterion for ampleness [38], introduced the notion of extremal rays and systematically used it in an attempt to construct minimal models in higher dimension, called *the minimal model program.* Danilov [16] introduced the notion of *canonical and terminal singularities* in conjunction with the toric factorization problem. This was developed by Reid into a general theory of these singularities [56, 57], which appear in an essential way in the minimal model program. The minimal model program is so far proven up to dimension 3 ([48], see also [31, 32, 33, 39, 65]), and for toric varieties in arbitrary dimension (See [58]). In the steps of the minimal model program one is only allowed to contract a divisor into a variety with terminal singularities, or to perform a flip, modifying some codimension 2 loci. This allows a factorization of a given birational morphism into such "elementary operations". An algorithm to factorize birational maps among uniruled varieties, known as *Sarkisov's program,* has been developed and carried out in dimension 3 (see [61, 59, 10], and see [41] for the toric case). Still, we do not know of a way to solve the classical factorization problem using such a factorization.

0.10. **Relation with the toroidalization problem.** In [2], Theorem 2.1, it is proven that given a morphism of projective varieties $X \to B$, there are modifications $m_X : X' \to X$ and $m_B : B' \to B$, with a lifting $X' \to B'$ which has a toroidal structure. The *toroidalization problem* (see [2], [3], [36]) is that of obtaining such m_X and m_B which are composites of blowings up with smooth centers (maybe even with centers supported only over the locus where $X \to B$ is not toroidal).

The proof in [2] relies on the work of De Jong [29] and methods of [1]. The authors of the present paper have tried to use these methods to approach the factorization conjectures, so far without success; one notion we do use in this paper is the torific ideal of [1]. It would be interesting if one could turn this approach on its head and prove a result on toroidalization using factorization.

0.11. **Relation with the proof in** [72]**.** Another proof of the weak factorization theorem was given independently by the fourth author in [72]. The main difference of the two approaches is that in the current paper we are using objects such as torifying ideals defined locally on each quasi-elementary piece of a cobordism. The blowup of a torifying ideal gives the quasi-elementary cobordism a toroidal structure. These toroidal modifications are then pieced together using canonical resolution of singularities. In [72] the objects of study, such as painted semifans, are defined globally on a cobordism. The structure these semifans define on the cobordism is something in between our notions of locally toric and toroidal embedding.

0.12. **Outline of the paper.** In section 1 we discuss locally toric and toroidal structures, and reduce the proof of Theorem 0.1.1 to the case where ϕ is a projective morphism.

Suppose now we have a projective morphism $\phi : X_1 \to X_2$. In section 2 we apply the theory of birational cobordisms to obtain a factorization into locally toric maps. In case X_1 and X_2 are projective, we use a geometric invariant theory analysis, inspired by Thaddeus's work, which shows that the intermediate varieties can be chosen to be projective. In all cases our cobordism B is relatively projective over X_2.

In section 3 we construct, for each connected component of the fixed-point set $F \subset B$, an ideal sheaf I_F whose blowing up *torifies* the action of K^* on B in a neighborhood of F. In other words, there is a neighborhood B_F of F such that K^* acts toroidally on the variety obtained by blowing up B_F along I_F.

In section 4 we prove the weak factorization theorem by piecing together the toroidal birational transforms induced by the neighborhoods B_F of a cobordism B near fixed point components F as in section 3. This is done using several applications of canonical resolution of singularities, Luna's Fundamental Lemma and Étale Slice Theorem.

In section 5 we generalize the main theorem to the case of algebraic spaces, complex analytic manifolds, G-equivariant maps, and maps defined over a non-algebraically closed field. We also discuss some problems related to strong factorization.

1. Preliminaries

Definition 1.0.1. Suppose a reductive group G acts on an algebraic variety X. We denote by X/G the geometric quotient representing the space of orbits, and by $X/\!/G$ the space of equivalence classes of orbits, where two orbits are equivalent if their closures intersect.

1.1. **Toric varieties.** Let $N \cong \mathbb{Z}^n$ be a lattice and $\sigma \subset N_{\mathbb{R}}$ a strictly convex rational polyhedral cone. We denote the dual lattice by M and the dual cone by $\sigma^\vee \subset M_{\mathbb{R}}$. The *affine toric variety* $X = X(N, \sigma)$ is

$$X = \operatorname{Spec} K[M \cap \sigma^\vee].$$

For an $m \in M \cap \sigma^\vee$ we denote its image in the semigroup algebra $K[M \cap \sigma^\vee]$ by z^m.

More generally, the toric variety associated to a fan Σ in $N_{\mathbb{R}}$ is denoted by $X(N, \Sigma)$.

If $X_1 = X(N, \Sigma_1)$ and $X_2 = X(N, \Sigma_2)$ are two toric varieties, the embeddings of the torus $T = \operatorname{Spec} K[M]$ in both of them define a birational toric map $X_1 \dashrightarrow X_2$. This map is a morphism if and only if every cone in Δ_1 is contained in a cone of Δ_2

Suppose that K^* acts on an affine toric variety $X = X(N, \sigma)$ as a 1-parameter subgroup of the torus T, corresponding to a lattice point $a \in N$. If $t \in K^*$ and $m \in M$, the action on the monomial z^m is given by

$$t^*(z^m) = t^{(a,m)} \cdot z^m,$$

where $(\cdot, \cdot)$ is the natural pairing on $N \times M$. The K^*-invariant monomials correspond to lattice points $M \cap a^\perp$, hence

$$X/\!/K^* \cong \operatorname{Spec} K[M \cap \sigma^\vee \cap a^\perp],$$

which is again an affine toric variety defined by the lattice $\pi(N)$ and cone $\pi(\sigma)$ where $\pi : N_{\mathbb{R}} \to N_{\mathbb{R}}/\mathbb{R}\cdot a$ is the projection. This quotient is a geometric quotient precisely when $\pi : \sigma \to \pi(\sigma)$ is an isomorphism.

1.2. **Locally toric and toroidal structures.** There is some confusion in the literature between the notion of *toroidal embeddings* and toroidal maps ([35], [2]) and that of *toroidal varieties* (see [15]), which we prefer to call *locally toric varieties* (and morphisms). A crucial issue in this paper is the distinction between the two notions.

Definition 1.2.1. 1. A variety W is *locally toric* if for every closed point $p \in W$ there exists an étale neighborhood $\iota_p : V_p \to W$ of p and an étale morphism $\eta_p : V_p \to X_p$ to a toric variety X_p. We call V_p (together with the maps ι_p, η_p) a toric chart at p.

2. An open embedding $U \subset W$ is *toroidal* if W is locally toric and for every $p \in W$ there exists a toric chart V_p satisfying $\iota_p^{-1}(U) = \eta_p^{-1}(T)$, where $T \subset X_p$ is the torus. We call such charts toroidal. Sometimes we omit the open set U from the notation and simply say that a variety is toroidal.

Definition 1.2.2 ([24],[28])**.** Let $\psi : W_1 \dashrightarrow W_2$ be a rational map defined on a dense open subset U. Denote by Γ_ψ the closure of the graph of ψ_U in $W_1 \times W_2$. We say that ψ is *proper* if the projections $\Gamma_\psi \to W_1$ and $\Gamma_\psi \to W_2$ are both proper.

Definition 1.2.3. 1. A proper birational map $\psi : W_1 \dashrightarrow W_2$ between two locally toric varieties W_1 and W_2 is said to be *locally toric* if for any point $(p, q) \in \Gamma_\psi$ we can find toric charts V_p at p and V_q at q, a lifting of ψ to $\varphi : V_p \dashrightarrow V_q$ and a birational toric map $\nu : X_p \dashrightarrow X_q$ such that the following diagram commutes

$$\begin{array}{ccccc} X_p & \stackrel{\eta_p}{\leftarrow} & V_p & \stackrel{\iota_p}{\to} & W_1 \\ \nu \downarrow & & \varphi \downarrow & & \downarrow \psi \\ X_q & \stackrel{\eta_q}{\leftarrow} & V_q & \stackrel{\iota_q}{\to} & W_2 \end{array}$$

2. If, moreover, $U_1 \subset W_1$ and $U_2 \subset W_2$ are toroidal embeddings, the birational map $\psi : W_1 \dashrightarrow W_2$ is said to be *toroidal* if the charts V_p and V_q above can be chosen toroidal.

Remark 1.2.4. Note that a toroidal birational map $(U_1 \subset W_1) \dashrightarrow (U_2 \subset W_2)$ induces an isomorphism between the open sets U_1 and U_2.

Definition 1.2.5 (cf. [50], p. 198)**.** Let V and X be affine varieties with K^*-actions, and let $\eta : V \to X$ be a K^*-equivariant étale morphism. Then η is said to be *strongly étale* if

(i) the quotient map $V//K^* \to X//K^*$ is étale, and
(ii) the natural map

$$V \to X \underset{X//K^*}{\times} V//K^*$$

is an isomorphism.

Definition 1.2.6. Let W be a locally toric (resp. toroidal) variety with a K^* action. We say that the action is locally toric (resp. toroidal) if at any $p \in W$ there exists a toric (resp. toroidal) chart V_p with a K^* action on V_p such that the étale morphisms $\iota_p : V_p \to \iota_p(V_p)$ and $\eta_p : V_p \to X_p$ are strongly étale, where K^* acts on the toric variety X_p as a subgroup of the torus $T \subset X_p$.

Lemma 1.2.7. *Let W be a nonsingular variety with a K^*-action. Then W has a locally toric structure such that the action is locally toric.*

Proof. Let $p \in W$ be a closed point, G_p the stabilizer of p in K^*, and N_p the normal vector space to the orbit of p at p. We choose an affine open K^*-invariant neighborhood U of p such that the orbit of p is closed in U (see [66]).

By Luna's Étale Slice Theorem there exists a nonsingular locally closed G_p-invariant subvariety $S \subset W$ with $p \in S$ such that, shrinking U if necessary, we have a diagram of fiber squares

$$\begin{array}{ccccccc} K^* \overset{G_p}{\times} N_p & \overset{\eta_p}{\leftarrow} & K^* \overset{G_p}{\times} S & \overset{\iota_p}{\twoheadrightarrow} & U & \subset & W \\ \downarrow & & \downarrow & & \downarrow & & \\ N_p//G_p & \leftarrow & S//G_p & \twoheadrightarrow & U//K^* & & \end{array}$$

where K^* acts on the first factors of $K^* \times^{G_p} N_p$ and $K^* \times^{G_p} S$, the upper horizontal maps are strongly étale, hence the lower horizontal maps are étale as well. Now it suffices to notice that $K^* \times^{G_p} N_p$ is a toric variety on which K^* acts as a subgroup of the torus. ☐

We call the charts constructed in the lemma *Luna's toric charts.*

Remark 1.2.8. If $p \in W$ is a fixed point then $S = K^* \times^{G_p} S$ is a K^*-invariant open neighborhood of p and ι_p is an embedding. If the variety W is proper then every point has a limit fixed point, and it follows that such Zariski local toric charts cover W.

Remark 1.2.9. We remark that if $\eta : V \to X$ is an étale morphism and I a sheaf of ideals on X then we get an étale morphism from the blowup of V along $\eta^{-1}(I)$ to the blowup of X along I. If the morphism ϕ is strongly étale and the ideal I is K^*-equivariant, the morphism of the blowups is also strongly étale when restricted to suitable affine open subsets.

1.3. **Relation with toroidal embeddings of** [35]**.** The results in [35] on toroidal embeddings assume that the embedding $U_W \subset W$ is without self-intersection. This means that the irreducible components of the toroidal divisor $W \setminus U_W$ are normal. If the divisors are not normal, we can take a sequence of blowups with smooth centers which preserves the toroidal structure to make the complement of U_W a divisor with simple normal crossings.

It is shown in [35] that proper birational *allowable* morphisms to a given toroidal embedding X are in one-to-one correspondence with complete subdivisions of the associated polyhedral complex Δ_X. A toroidal morphism $W_1 \to W_2$ is said to be allowable if the charts defining the toroidal structure on the morphism as in Definition 1.2.3 can be chosen so that $\iota_p(V_p) = \psi^{-1}\iota_q(V_q)$. It follows from the results in [2] that every proper birational toroidal morphism $Y \to X$ is allowable: first, the toroidal morphism induces a morphism of polyhedral complexes $\Delta_Y \to \Delta_X$ which is automatically a subdivision; if $Y' \to X$ is the allowable morphism corresponding to this subdivision, the morphism $Y \to X$ factors through $Y \to Y'$; now the latter is an equidimensional proper birational morphism between normal varieties, hence it is an isomorphism.

1.4. **Strong factorization for toroidal birational maps.** Morelli's proof of the strong factorization theorem for proper birational toric maps can be extended to the case of proper birational toroidal maps. This is proved in [3] for toroidal morphisms. The general case of a birational toroidal map $W_1 \dashrightarrow W_2$ follows from this when we take the normalization of the graph of the birational map in the product of the two toroidal embeddings. Then it naturally inherits a toroidal structure, it dominates the original two toroidal embeddings by birational toroidal morphisms, and admits a toroidal resolution of singularities. Now we apply [3] to the first projection to get $W \to W_1$ a sequence of smooth toroidal blowings up and $W \to W_2$ a toroidal morphism. Another application of [3] to the toroidal morphism $W \to W_2$ finishes the proof. We state this result for later reference:

Theorem 1.4.1 (Morelli). *Let $\phi : (U \subset W_1) \dashrightarrow (U \subset W_2)$ be a proper birational toroidal map between nonsingular toroidal embeddings, defined over a perfect field K. Then ϕ can be factorized into a sequence of smooth toroidal blowups followed by a sequence of smooth toroidal blowdowns.*

1.5. **Canonical resolution of singularities and canonical principalization.** In the following, we will use canonical versions of Hironaka's theorems on resolution of singularities and principalization of an ideal, proved in [7, 69].

A canonical (embedded) resolution of singularities $\widetilde{W} \to W$ is a desingularization procedure consisting of a composite of blowings up with smooth centers, and it is canonical in the sense that if $\theta : V \to W$ is a smooth morphism then the ideals blown up are invariant under pulling back by θ; hence θ can be lifted to a smooth morphism $\widetilde{\theta} : \widetilde{V} \to \widetilde{W}$. In particular: **(a)** if $\theta : W \to W$ is an automorphism then it can be lifted to an automorphism $\widetilde{W} \to \widetilde{W}$, and **(b)** the canonical resolution behaves well with respect to étale morphisms: if $V \to W$ is étale, we get an étale morphism of canonical resolutions $\widetilde{V} \to \widetilde{W}$.

By *canonical principalization of an ideal sheaf* in a nonsingular variety we mean "the canonical embedded resolution of singularities of the subscheme defined by the ideal sheaf making it a divisor with normal crossings"; i.e., a composite of blowings up with smooth centers such that the total transform of the ideal is a strict divisor of normal crossings. Embedded resolution of singularities of a non-reduced subscheme is discussed in Section 11 of [7].

Now let $\phi : W_1 \dashrightarrow W_2$ be a proper birational map between nonsingular varieties, and $U \subset W_1$ an open set on which ϕ restricts to an isomorphism. By *elimination of indeterminacies* of ϕ we mean a morphism $e : W_1' \to W_1$, obtained by a sequence of blowups with smooth centers disjoint from U, such that the birational map $\phi \circ e$ is a morphism. Elimination of indeterminacies can be reduced to principalization of an ideal sheaf as follows.

We may assume that ϕ^{-1} is a morphism; otherwise we replace W_2 by the closure of the graph of ϕ. If ϕ^{-1} is a projective morphism, i.e., a blowup of an ideal sheaf I, then we can take for W_1' the canonical principalization of I. If ϕ^{-1} is not projective, we use Chow's lemma (Corollary 2, p. 504, [25]): there exists an ideal sheaf I on W_1 such that the blowup of W_1 along I factors through W_2. Hence a canonical principalization of I also factors through W_2. Although it is not explicitly stated by Hironaka, the ideal I is supported in the complement of the open set U: the blowup of I consists of a sequence of *permissible* blowups (Definition 4.4.3, p. 537, [25]), each of which is supported in the complement of U.

1.6. **Reduction to projective morphisms.** We start with a birational map

$$\phi : X_1 \dashrightarrow X_2$$

between proper nonsingular algebraic varieties X_1 and X_2 defined over K and restricting to an isomorphism on an open set U.

Lemma 1.6.1 (Hironaka). *There is a commutative diagram*

$$\begin{array}{ccc} X_1' & \stackrel{\phi'}{\to} & X_2' \\ g_1 \downarrow & & \downarrow g_2 \\ X_1 & \stackrel{\phi}{\dashrightarrow} & X_2 \end{array}$$

such that g_1 and g_2 are composites of blowings up with smooth centers disjoint from U, and ϕ' is a projective birational morphism.

Proof. By Hironaka's theorem on elimination of indeterminacies, there is a morphism $g_2 : X_2' \to X_2$ which is a composite of blowings up with smooth centers disjoint from U, such that the birational map $h := \phi^{-1} \circ g_2 : X_2' \to X_1$ is a morphism:

$$\begin{array}{ccc} & & X_2' \\ & h \swarrow & \downarrow g_2 \\ X_1 & \stackrel{\phi}{\dashrightarrow} & X_2 \end{array}.$$

By the same theorem, there is a morphism $g_1 : X_1' \to X_1$ which is a composite of blowings up with smooth centers disjoint from U, such that $\phi' := h^{-1} \circ g_1 : X_1' \to X_2'$ is a morphism. Since the composite $h \circ \phi' = g_1$ is projective, it follows that ϕ' is projective. $\square$

Thus we may replace $X_1 \dashrightarrow X_2$ by $X_1' \to X_2'$ and assume from now on that ϕ is a projective morphism.

2. Birational Cobordisms

2.1. Definitions.

Definition 2.1.1 ([71]). Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between two algebraic varieties X_1 and X_2 over K, isomorphic on an open set U. An algebraic variety B is called a *birational cobordism* for ϕ and denoted by $B_\phi(X_1, X_2)$ if it satisfies the following conditions.

1. The multiplicative group K^* acts effectively on $B = B_\phi(X_1, X_2)$.
2. The sets
$$\begin{array}{rcl} B_- & := & \{x \in B : \ \lim_{t\to 0}\ t(x) \text{ does not exist in } B\} \\ \text{and } B_+ & := & \{x \in B : \ \lim_{t\to \infty} t(x) \text{ does not exist in } B\} \end{array}$$
are nonempty Zariski open subsets of B.
3. There are isomorphisms
$$B_-/K^* \stackrel{\sim}{\to} X_1 \quad \text{and} \quad B_+/K^* \stackrel{\sim}{\to} X_2.$$
4. Considering the rational map $\psi : B_- \dashrightarrow B_+$ induced by the inclusions $B_- \cap B_+ \subset B_-$ and $B_- \cap B_+ \subset B_+$, the following diagram commutes:
$$\begin{array}{ccc} B_- & \stackrel{\psi}{\dashrightarrow} & B_+ \\ \downarrow & & \downarrow \\ X_1 & \stackrel{\phi}{\dashrightarrow} & X_2 \end{array}.$$

We say that B *respects the open set* U if U is contained in the image of $B_- \cap B_+/K^*$.

Definition 2.1.2 ([71]). Let $B = B_\phi(X_1, X_2)$ be a birational cobordism, and let $F \subset B^{K^*}$ be a connected component of the fixed-point set. We define

$$\begin{array}{rcl} F^+ & = & \{x \in B |\lim_{t\to 0}\ t(x) \in F\} \\ F^- & = & \{x \in B |\lim_{t\to \infty} t(x) \in F\} \\ F^\pm & = & F^+ \cup F^- \\ F^* & = & F^\pm \setminus F \end{array}$$

Definition 2.1.3 ([71]). A *nonsingular* birational cobordism $B = B_\phi(X_1, X_2)$ is said to be *elementary* if the fixed-point set B^{K^*} is connected.

Definition 2.1.4. A nonsingular birational cobordism $B = B_\phi(X_1, X_2)$ is said to be *quasi-elementary* if for any two connected components $F_1 \neq F_2 \subset B^{K^*}$ we have $F_1^\pm \cap F_2^\pm = \emptyset$.

Definition 2.1.5 ([71])**.** Let $B = B_\phi(X_1, X_2)$ be a birational cobordism. We define a relation $\prec$ among connected components of B^{K^*} as follows: let $F_1 \neq F_2 \subset B^{K^*}$ be two connected components, and set $F_1 \prec F_2$ if $F_1^+ \cap F_2^- \neq \emptyset$.

Definition 2.1.6 (cf. [45],[71])**.** We say that a birational cobordism $B = B_\phi(X_1, X_2)$ is *collapsible* if the relation $\prec$ is a (strict) pre-order, namely, it can be extended to a strict order.

2.2. **The main example.** We now recall a fundamental example of an elementary birational cobordism in the toric setting, discussed in [71]:

Example 2.2.1. Let $B = \mathbb{A}^n = \operatorname{Spec} K[z_1, \cdots, z_n]$ and let $t \in K^*$ act by

$$t(z_1, \dots, z_i, \dots, z_n) = (t^{\alpha_1} z_1, \dots, t^{\alpha_i} z_i, \dots, t^{\alpha_n} z_n).$$

We regard $\mathbb{A}^n$ as a toric variety defined by a lattice $N \cong \mathbb{Z}^n$ and a regular cone $\sigma \in N_\mathbb{R}$ generated by the standard basis

$$\sigma = \langle v_1, \dots, v_n \rangle.$$

The dual cone $\sigma^\vee$ is generated by the dual basis $v_1^*, \dots, v_n^*$, and we identify $z^{v_i^*} = z_i$. The K^*-action then corresponds to a one-parameter subgroup

$$a = (\alpha_1, \dots, \alpha_n) \in N.$$

We have the obvious description of the sets B_+ and B_-:

$$\begin{array}{rcl} B_+ & = & \{(z_1, \cdots, z_n); z_i \neq 0 \text{ for some } i \text{ with } \alpha_i = (v_i^*, a) > 0\}, \\ B_- & = & \{(z_1, \cdots, z_n); z_i \neq 0 \text{ for some } i \text{ with } \alpha_i = (v_i^*, a) < 0\}. \end{array}$$

We define the upper boundary and lower boundary of σ to be

$$\begin{array}{rcl} \partial_+\sigma & = & \{x \in \sigma; x + \epsilon \cdot (-a) \notin \sigma \text{ for } \epsilon > 0\}, \\ \partial_-\sigma & = & \{x \in \sigma; x + \epsilon \cdot a \notin \sigma \text{ for } \epsilon > 0\}). \end{array}$$

Then we obtain the description of B_+ and B_- as the toric varieties corresponding to the fans $\partial_+\sigma$ and $\partial_-\sigma$ in $N_\mathbb{R}$.

Let $\pi : N_\mathbb{R} \to N_\mathbb{R}/\mathbb{R} \cdot a$ be the projection. Then $B/\!/K^*$ is again an affine toric variety defined by the lattice $\pi(N)$ and cone $\pi(\sigma)$. Similarly, one can check that the geometric quotients B_-/K^* and B_+/K^* are toric varieties defined by fans $\pi(\partial_+\sigma)$ and $\pi(\partial_-\sigma)$. Since both $\pi(\partial_+\sigma)$ and $\pi(\partial_-\sigma)$ are subdivisions of $\pi(\sigma)$, we get a diagram of birational toric maps

$$\begin{tikzcd} B_-/K^* \arrow[rr, dashed, "\varphi"] \arrow[dr] & & B_+/K^* \arrow[dl] \\ & B/\!/K^* & \end{tikzcd}$$

More generally, one can prove that if Σ is a fan in $N_\mathbb{R}$ with lower boundary $\partial_-\Sigma$ and upper boundary $\partial_+\Sigma$ relative to an element $a \in N$, then the toric variety corresponding to Σ, with the K^*-action given by the one-parameter subgroup $a \in N$, is a birational cobordism between the two toric varieties corresponding to $\pi(\partial_-\Sigma)$ and $\pi(\partial_+\Sigma)$ as fans in $N_\mathbb{R}/\mathbb{R} \cdot a$.

For the details, we refer the reader to [45], [71] and [3]. $\square$

2.3. **Construction of a cobordism.** It was shown in [71] that cobordisms exist for any birational map $X_1 \dashrightarrow X_2$. We give a very special case of this construction.

Theorem 2.3.1. *Let $\phi : X_1 \to X_2$ be a projective birational morphism between proper nonsingular algebraic varieties, which is an isomorphism on an open set U. Then there is a proper nonsingular algebraic variety $\overline{B}$ with a K^* action, satisfying the following properties:*

1. *There exist closed embeddings $\iota_1 : X_1 \hookrightarrow \overline{B}^{K^*}$ and $\iota_2 : X_2 \hookrightarrow \overline{B}^{K^*}$ with disjoint images.*
2. *The open subvariety $B = \overline{B} \smallsetminus (\iota_1(X_1) \cup \iota_2(X_2))$ is a birational cobordism between X_1 and X_2 respecting the open set U.*
3. *There is a vector bundle $E \to X_2$ with a K^* action, and a closed K^*-equivariant embedding $\overline{B} \subset \mathbb{P}(E)$.*

Proof. Let $J \subset \mathcal{O}_{X_2}$ be an ideal sheaf which does not vanish on U and such that $\phi : X_1 \to X_2$ is the blowup morphism of X_2 along J. Let I_0 be the ideal of the point $0 \in \mathbb{P}^1$. Consider $W_0 = X_2 \times \mathbb{P}^1$ and let $p : W_0 \to X_2$ and $q : W_0 \to \mathbb{P}^1$ be the projections. Let $I = (p^{-1}J + q^{-1}I_0)\mathcal{O}_{W_0}$. Let W be the blowup of W_0 along I.

We claim that X_1 and X_2 lie in the nonsingular locus of W. For $X_2 \cong X_2 \times \{\infty\} \subset X_2 \times \mathbb{P}^1$ this is clear. Since X_1 is nonsingular, embedded in W as the strict transform of $X_2 \times \{0\} \subset X_2 \times \mathbb{P}^1$, to prove that X_1 lies in the nonsingular locus, it suffices to prove that X_1 is a Cartier divisor in W. We look at local coordinates. Let $A = \Gamma(V, \mathcal{O}_V)$ for some affine open subset $V \subset X_2$, and let $y_1, \dots, y_m$ be a set of generators of J on V. Then on the affine open subset $V \times \mathbb{A}^1 \subset X_2 \times \mathbb{P}^1$ with coordinate ring $A[x]$, the ideal I is generated by $y_1, \dots, y_m, x$. The charts of the blowup containing the strict transform of $\{x = 0\}$ are of the form

$$\operatorname{Spec} A[x]\left[\frac{y_1}{y_i}, \dots, \frac{y_m}{y_i}, \frac{x}{y_i}\right] = \operatorname{Spec} A\left[\frac{y_1}{y_i}, \dots, \frac{y_m}{y_i}, \frac{x}{y_i}\right].$$

The strict transform of $\{x = 0\}$ is defined by $\frac{x}{y_i}$, hence it is Cartier.

Let $\overline{B} \to W$ be a canonical resolution of singularities. Then conditions 1 and 2 are clearly satisfied. Condition 3 follows from the fact that the projection $X_2 \times \mathbb{P}^1 \to X_2$ composed with the blowup of the ideal I and canonical resolution of singularities is a K^*-equivariant projective morphism. □

We refer the reader to [71] for more details.

We call a variety $\overline{B}$ as in the theorem a *compactified, relatively projective cobordism.*

2.4. **Collapsibility and Projectivity.** Let $B = B_\phi(X_1, X_2)$ be a birational cobordism. We seek a criterion for collapsibility of B.

Let $\mathcal{C}$ be the set of connected components of $B_\phi(X_1, X_2)^{K^*}$, and let $\chi : \mathcal{C} \to \mathbb{Z}$ be a function. We say that χ is strictly increasing if $F \prec F' \Rightarrow \chi(F) < \chi(F')$. The following lemma is obvious:

Lemma 2.4.1. *Assume there exists a strictly increasing function χ. Then $\prec$ is a strict pre-order, and B is collapsible. Conversely, suppose B is collapsible. Then there exists a strictly increasing function χ.* □

Remark 2.4.2. It is evident that every strictly increasing function can be replaced by one which induces a strict total order. However, it will be convenient for us to consider arbitrary strictly increasing functions.

Let χ be a strictly increasing function, and let $a_0 < a_1 \cdots < a_k \in \mathbb{Z}$ be the values of χ.

Definition 2.4.3. We denote

1. $F_{a_i} = \cup\{F|\chi(F) = a_i\}$.
2. $F^+_{a_i} = \cup\{F^+|\chi(F) = a_i\}$.
3. $F^-_{a_i} = \cup\{F^-|\chi(F) = a_i\}$.
4. $F^\pm_{a_i} = \cup\{F^\pm|\chi(F) = a_i\}$.
5. $F^*_{a_i} = \cup\{F^*|\chi(F) = a_i\}$.
6. $B_{a_i} = B \smallsetminus (\ \bigcup\{F^-|\chi(F) < a_i\} \quad \cup \quad \bigcup\{F^+|\chi(F) > a_i\}\ )$.

The following is an immediate extension of Proposition 1 of [71] .

Proposition 2.4.4. 1. *B_{a_i} is a quasi-elementary cobordism.*
2. *For $i = 0, \ldots, k-1$ we have $(B_{a_i})_+ = (B_{a_{i+1}})_-$.*

The following is an analogue of Lemma 1 of [71] in the case of the cobordisms we have constructed.

Proposition 2.4.5. *Let $E \to X_2$ be a locally free sheaf with a K^* action, and let $\overline{B} \subset \mathbb{P}(E)$ be a compactified, relatively projective cobordism embedded K^*-equivariantly. Then there exists a strictly increasing function χ for the cobordism $B = \overline{B} \smallsetminus (X_1 \cup X_2)$. In particular, the cobordism is collapsible.*

Proof. Since K^* acts trivially on X_2, there exists a direct sum decomposition

$$E = \bigoplus_{b \in \mathbb{Z}} E_b$$

where E_b is the subbundle on which the action of K^* is given by the character $t \mapsto t^b$. (Proof: If $K^d \times V \to V$ is a trivialization of E over an invariant affine open subset $V \subset X_2$ then the action of K^* on E in this trivialization is given by a group homomorphism $K^* \to \Gamma(V, GL_d(K))$. Since K^* is reductive, we can diagonalize all these matrices simultaneously.) Denote by $b_0, \ldots, b_k$ the characters which figure in this representation, and let $d_i = \operatorname{rank} E_{b_i}$. Note that there are disjoint embeddings $\mathbb{P}(E_{b_i}) \subset \mathbb{P}(E)$.

Let $p \in B$ be a fixed point lying in the fiber $\mathbb{P}(E_q)$ over $q \in X_2$. We choose a basis $(x_{b_0,1}, \ldots, x_{b_0,d_0}, \ldots x_{b_k,1}, \ldots, x_{b_k,d_k})$ of E_q where $x_{b_i,j} \in E_{b_i}$ and use the following lemma:

Lemma 2.4.6. *Suppose $p \in \mathbb{P}(E_q)^{K^*}$ is a fixed point with homogeneous coordinates*

$$(p_{b_0,1}, \ldots, p_{b_0,d_0}, \ldots, p_{b_k,1}, \ldots, p_{b_k,d_k}).$$

Then there is an i_p such that $p_{b_i,j} = 0$ whenever $i \neq i_p$. In particular, $p \in \mathbb{P}(E_{b_{i_p}}) \subset \mathbb{P}(E)$. □

If $F \subset B^{K^*}$ is a connected component of the fixed point set, then it follows from the lemma that $F \subset \mathbb{P}(E_{b_i})$ for some i. We define

$$\chi(F) = b_i.$$

To check that χ is strictly increasing, consider a point $p \in B$ such that $\lim_{t\to 0} t \cdot p \in F_1$ and $\lim_{t\to\infty} t \cdot p \in F_2$ for some fixed point components F_1 and F_2. Let the coordinates of p in the fiber over $q \in X_2$ be $(p_{b_0,1}, \ldots, p_{b_0,d_0}, \ldots, p_{b_k,1}, \ldots, p_{b_k,d_k})$. Now

$$\lim_{t\to 0} t \cdot p \in \mathbb{P}(E_{b_{min}}),$$
$$\lim_{t\to\infty} t \cdot p \in \mathbb{P}(E_{b_{max}}),$$

where

$$b_{min} = \min\{b_i : p_{b_i,j} \neq 0 \text{ for some } j\},$$
$$b_{max} = \max\{b_i : p_{b_i,j} \neq 0 \text{ for some } j\}.$$

Thus, if F_1 and F_2 are distinct, the orbit of p is not constant and

$$\chi(F_1) = b_{min} < b_{max} = \chi(F_2).$$

□

2.5. **Geometric invariant theory and projectivity.** In this section we use geometric invariant theory, and ideas (originating in symplectic geometry) developed by M. Thaddeus and others (see e.g. [68]), in order to obtain a result about relative projectivity of quotients.

We continue with the notation of the last section. Consider the vector bundle E and its decomposition according to the character. Let $\{b_j\}$ be the characters of the action of K^* on E, and $\{a_i\}$ the subset of those b_j that are images of χ. If we use the Veronese embedding $\overline{B} \subset \mathbb{P}(\mathrm{Sym}^2(E))$ and replace E by $\mathrm{Sym}^2(E)$, we may assume that a_i are even, in particular $a_{i+1} > a_i + 1$ (this is a technical condition which comes up handy in what follows).

Denote by $\rho_0(t)$ the action of $t \in K^*$ on E. For any $r \in \mathbb{Z}$ consider the "twisted" action $\rho_r(t) = t^{-r} \cdot \rho_0(t)$. Note that the induced action on $\mathbb{P}(E)$ does not depend on the "twist" r. Considering the decomposition $E = \bigoplus E_{b_j}$, we see that $\rho_r(t)$ acts on E_{b_j} by multiplication by $t^{b_j - r}$.

We can apply geometric invariant theory in its relative form (see, e.g., [53], [27]) to the action $\rho_r(t)$ of K^*. Recall that a point $p \in \mathbb{P}(E)$ is said to be semistable with respect to ρ_r, written $p \in (\mathbb{P}(E), \rho_r)^{ss}$, if there is a positive n and a local section $s \in \mathrm{Sym}^n(E)$ such that $s(p) \neq 0$, and such that s is invariant under the action induced by ρ_r on $\mathrm{Sym}^n(E)$. The main result of geometric invariant theory implies that

$$\mathcal{P}roj_{X_2 \times \mathbb{P}^1} \bigoplus_{n \geq 0}^{\infty} (\mathrm{Sym}^n(E))^{\rho_r} = (\mathbb{P}(E), \rho_r)^{ss} // K^*.$$

We denote by $(\overline{B}, \rho_r)^{ss} = \overline{B} \cap (\mathbb{P}(E), \rho_r)^{ss}$.

The numerical criterion of semistability (see [50]) immediately implies the following:

Lemma 2.5.1. *For $0 < i < k$ we have*

1. $(\overline{B}, \rho_{a_i})^{ss} = B_{a_i}$;
2. $(\overline{B}, \rho_{a_i+1})^{ss} = (B_{a_i})_+$;
3. $(\overline{B}, \rho_{a_i-1})^{ss} = (B_{a_i})_-$;

Thus we obtain:

Proposition 2.5.2. *The morphisms $(B_{a_i})_+/K^* \to X_2$, $(B_{a_i})_-/K^* \to X_2$ and $B_{a_i}//K^* \to X_2$ are projective.*

2.6. **The main result of** [71]**.** Let $B \subset \overline{B}$ be a collapsible nonsingular birational cobordism. Then we can write B as a union of quasi-elementary cobordisms $B = \cup_i B_{a_i}$, with $(B_{a_i})_+ = (B_{a_{i+1}})_-$. By Lemma 1.2.7 each B_{a_i} has a locally toric structure such that the action of K^* is locally toric.

Lemma 2.6.1. *Let B_{a_i} be a nonsingular quasi-elementary cobordism. For each $p \in B_{a_i}$ there exists a toric chart V_p at p such that we have a digram of fiber squares*

$$\begin{array}{ccccccc} X_p & \stackrel{\eta_p}{\leftarrow} & V_p & \stackrel{\iota_p}{\twoheadrightarrow} & \iota_p(V_p) & \subset & B_{a_i} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \pi \\ X_p//K^* & \leftarrow & V_p//K^* & \twoheadrightarrow & \iota_p(V_p)//K^* & \subset & B_{a_i}//K^* \end{array}$$

which gives B_{a_i} and $B_{a_i}//K^*$ locally toric structures.

Proof. Choosing Luna's toric charts, we get that the left and the center squares are fiber squares. To show that we also have a fiber square on the right, we need to choose V_p so that its image $\iota_p(V_p)$ is saturated with respect to the projection $\pi : B_{a_i} \to B_{a_i}//K^*$; that means, if a point $q \in \iota_p(V_p)$ has a limit in B_{a_i} then this limit also lies in $\iota_p(V_p)$. We do this as follows.

If p lies in $F^{\pm}_{a_i}$ then it has a unique limit p' in F_{a_i}. Since a toric chart at p' also covers p, we may replace p by p' and assume that either p is fixed or p has no limit in B_{a_i}. Now we replace the existing V_p by

$$V_p \setminus \iota_p^{-1}\pi^{-1}\pi(B_{a_i} \setminus \iota_p V_p).$$

This is an open subset of V_p because π is closed; and it is a K^*-equivariant étale neighborhood of p by the choice of p. ☐

Lemma 2.6.2. *Let B_{a_i} be a nonsingular quasi-elementary cobordism. Then $B_{a_i}//K^*$, $(B_{a_i})_-/K^*$, $(B_{a_i})_+/K^*$ are locally toric varieties and we have a diagram of locally toric maps*

$$\begin{array}{ccccc} (B_{a_i})_-/K^* & & \stackrel{\varphi_i}{\dashrightarrow} & & (B_{a_i})_+/K^* \\ & \searrow & & \swarrow & \\ & & B_{a_i}//K^* & & \end{array}$$

which in toric charts is described by the main example in Section 2.2.

If B_{a_i} is toroidal then all these varieties and maps are also toroidal.

Proof. We cover B_{a_i} with charts as in the previous lemma, giving a locally toric structure on $B_{a_i}//K^*$. When we restrict these charts to $(B_{a_i})_- \subset B_{a_i}$, we get a diagram of fiber squares

$$\begin{array}{ccccccc} (X_p)_- & \leftarrow & (V_p)_- & \twoheadrightarrow & \iota_p((V_p)_-) & \subset & (B_{a_i})_- \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \pi \\ (X_p)_-/K^* & \leftarrow & (V_p)_-/K^* & \twoheadrightarrow & \iota_p((V_p)_-)/K^* & \subset & (B_{a_i})_-/K^* \end{array}$$

giving a locally toric structure on $(B_{a_i})_-/K^*$. The morphism $(B_{a_i})_-/K^* \to B_{a_i}//K^*$ induced by $(B_{a_i})_- \hookrightarrow B_{a_i}$ is given in these toric charts by the toric morphism $(X_p)_-/K^* \to X_p//K^*$ as in the main example. Similar construction applies for $(B_{a_i})_+//K^*$.

If $U_{B_{a_i}} \subset B_{a_i}$ is a toroidal embedding, the charts in the previous lemma can be chosen toroidal. Hence we have a corresponding fiber square of open subsets

$$\begin{array}{ccccccc} T & \leftarrow & U_{V_p} & \twoheadrightarrow & \iota_p(U_{V_p}) & \subset & U_{B_{a_i}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T//K^* & \leftarrow & U_{V_p}//K^* & \twoheadrightarrow & \iota_p(U_{V_p})//K^* & \subset & U_{B_{a_i}}//K^* \end{array}$$

giving $B_{a_i}//K^*$ a toroidal structure. Restricting this diagram to $(B_{a_i})_\pm$ we get toroidal structures on $(B_{a_i})_\pm/K^*$ such that the birational maps in the statement of the lemma are toroidal. ☐

When we compose the birational transformations obtained from each B_{a_i} we get a slight refinement of the main result in [71] .

Theorem 2.6.3. *Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between proper nonsingular algebraic varieties X_1 and X_2 over an algebraically closed field K of characteristic zero, and let $U \subset X_1$ be the maximal open set where ϕ is an isomorphism. Then there exists a sequence of birational maps between proper nonsingular algebraic varieties*

$$X_1 = W_0 \overset{\varphi_1}{\dashrightarrow} W_1 \overset{\varphi_2}{\dashrightarrow} \cdots \overset{\varphi_i}{\dashrightarrow} W_i \overset{\varphi_{i+1}}{\dashrightarrow} W_{i+1} \overset{\varphi_{i+2}}{\dashrightarrow} \cdots \overset{\varphi_{l-1}}{\dashrightarrow} W_{l-1} \overset{\varphi_l}{\dashrightarrow} W_l = X_2$$

where

1. $\phi = \varphi_l \circ \varphi_{l-1} \circ \cdots \varphi_2 \circ \varphi_1$,
2. *φ_i are isomorphisms on U, and*
3. *For each i, the map φ_i is locally toric, and étale locally equivalent to the map φ described in 2.2.*

Furthermore, there is an index i_0 such that for all $i \leq i_0$ the map $W_i \dashrightarrow X_1$ is a projective morphism, and for all $i \geq i_0$ the map $W_i \dashrightarrow X_2$ is a projective morphism. In particular, if X_1 and X_2 are projective then all the W_i are projective.

Remark 2.6.4. For the projectivity claim 2, we take the first i_0 terms in the factorization to come from Hironaka's elimination of indeterminacies in Lemma 1.6.1, which is projective over X_1, whereas the last terms come from $\overline{B}$, which is projective over X_2, and the geometric invariant theory considerations as in Proposition 2.5.2.

3. Torification

We wish to replace the locally toric factorization of Theorem 2.6.3 by a toroidal factorization. This amounts to replacing B with a locally toric K^*-action by some B' with a toroidal K^*-action. We call such a procedure *torification.* The basic idea, which goes back at least to Hironaka, is that if one blows up an ideal, the exceptional divisors provide the resulting variety with useful extra structure. The ideal we construct, called a *torific ideal*, is closely related to the torific ideal of [1].

3.1. **Construction of a torific ideal.**

Definition 3.1.1. Let V be an algebraic variety with a K^*-action, $p \in V$ a closed point, $G_p \subset K^*$ the stabilizer of p. Fix an integer α. Then we define

$$J_{\alpha,p} \subset \mathcal{O}_{V,p}$$

to be the ideal generated by the semi-invariant functions $f \in \mathcal{O}_{V,p}$ of G_p-character α, that is, for $t \in G_p$ and a generator $f \in J_{\alpha,p}$ we have

$$t^*(f) = t^\alpha f.$$

Lemma 3.1.2. *Let V be a nonsingular variety with a K^*-action, and consider*

$$\mathcal{O}_{V,p} \hookrightarrow \hat{\mathcal{O}}_{V,p}.$$

We lift the action of G_p to $\hat{\mathcal{O}}_{V,p}$ and define

$$\hat{J}_{\alpha,p} \subset \hat{\mathcal{O}}_{V,p}$$

to be the ideal generated by semi-invariants $f \in \hat{\mathcal{O}}_{V,p}$ of G_p-character α. Then $\hat{J}_{\alpha,p}$ is generated by the image of $J_{\alpha,p}$.

Proof. Using Luna's Étale Slice Theorem we choose G_p-semi-invariant coordinates $z_1, \dots, z_n$ at p. Then $\hat{\mathcal{O}}_{V,p} \cong K[\![z_1, \dots, z_n]\!]$, and the claim is obvious. $\square$

For the rest of this section, we let B be a *nonsingular quasi-elementary* cobordism, $B = B_{a_i}$ for some i according to our previous notation, and let $F_0 = B^{K^*}$. Recall the notation F_0^* above. This is a constructible set in B.

Proposition 3.1.3. *There exists a unique coherent K^*-equivariant ideal sheaf I_α on B, such that for all $p \in B \smallsetminus F_0^*$ we have $(I_\alpha)_p = J_{\alpha,p}$.*

Definition 3.1.4. The sheaf I_α is called *the α-torific ideal sheaf* of the action of K^* on B.

Remark 3.1.5. Notice that the collection of ideals $J_{\alpha,p}$ for $p \in B$ do not define a coherent sheaf of ideals in general. As an example, let $B = \mathbb{A}^2$, and let the action of K^* be

$$t \cdot (x, y) = (tx, t^{-1}y).$$

Then at $p = (0, 0)$, the stabilizer is $G_p = K^*$, and $J_{1,p} = (x)$. Any other point $q \in B \smallsetminus \{p\}$ has a trivial stabilizer, hence $J_{1,q}$ is trivial. These germs do not form a coherent ideal sheaf on B. In this case, the ideal sheaf generated by x is the sheaf I_1 of the proposition.

Proof of 3.1.3. First we prove the uniqueness of I_α. Clearly the stalks of I_α are uniquely defined at all points $p \in B \smallsetminus F_0^*$. Now if $p \in F_0^*$ then p has a unique limit fixed point $p' \in F_0$. Since I_α is uniquely determined at p', hence also near p' by coherence, it follows from K^*-equivariance that I_α is uniquely defined at p.

To prove the existence of I_α we cover B with toric charts as in Lemma 2.6.1. With such charts, it follows that F_0^* defined in B, V_p and X_p correspond to each other. By Lemma 3.1.2 we are reduced to the case where B is toric, and the proposition follows from the lemma below. $\square$

Lemma 3.1.6. *Let $B = X(N, \sigma)$ be an affine toric variety on which K^* acts as a subgroup of the torus. Then I_α is the ideal generated by all monomials on which K^* acts by character α.*

Proof. By abuse of notation we will identify a lattice point $m \in M$ with the corresponding monomial z^m. We will also use the same letter α to denote a character of a subgroup of K^*.

For any $p \in B$ the ideal $J_{\alpha,p}$ is generated by all monomials regular at p on which G_p acts by character α. To see this, write a function $f \in \mathcal{O}_{B,p}$ as a power series in the monomials. Then f is semi-invariant of character α if and only if each term is semi-invariant of character α.

Let $p \in B \smallsetminus F_0^*$, and let τ be the smallest face of σ such that p lies in the affine open toric subvariety $X(N, \tau)$. Then the (Laurent) monomials regular at p are the monomials in $M \cap \tau^\vee$, and the monomials invertible at p are the ones in $M \cap \tau^\perp$.

If $m \in M \cap \sigma^\vee$ is a monomial regular on B on which K^* acts by character α then clearly m is regular at p and $G_p \subset K^*$ acts on it by character α. Conversely, let $m \in M \cap \tau^\vee$ be a monomial regular at p, on which G_p acts by character α. We show that there exists a monomial $m' \in M \cap \tau^\perp$ (i.e., invertible at p) such that $m + m'$ is regular on B and K^* acts on it by character α. This is done in two steps:

STEP 1. There exists $m' \in M \cap \tau^\perp$ such that $m + m'$ has K^*-character α. Since G_p is the subgroup of K^* acting trivially on the monomials $M \cap \tau^\perp$, we have an exact sequence

$$M \cap \tau^\perp \to \hat{K}^* \to \hat{G}_p \to 0,$$

where $\hat{H}$ denotes the character group of H. Thus, we may replace m by $m+m'$ and assume that K^* acts on m by character α.

STEP 2. There exists $m' \in M \cap \tau^\perp$ such that $m+m' \in M \cap \sigma^\vee$. Since the monomial m is K^*-semi-invariant, there exists an affine open K^*-invariant neighborhood of p on which m is regular, and since $p \notin F_0^*$, this neighborhood can be chosen of the form $\pi^{-1}(V)$ where $\pi : B \to B//K^*$ is the projection and $V \subset B//K^*$ is an affine open toric subvariety. Let $m' \subset M \cap \sigma \cap a^\perp$ be a monomial on $B//K^*$ such that V is the nonvanishing locus of m'. Then m' as a monomial on B is invertible at p, has K^*-character 0, and replacing m' by a multiple if necessary, we have that $m+m'$ is regular on B. □

3.2. **The torifying property of the torific ideal.** Let B be a quasi-elementary nonsingular cobordism. Choose $a_1, \dots, a_m \in \mathbb{Z}$ a finite set of integers containing all characters of the G_p-action on the tangent space of B at p for all $p \in B$. Let

$$I = I_{a_1} \cdots I_{a_m}$$

be the product of the a_i-torific ideals, and let $B^{tor} \to B$ be the normalized blowup of B along I. Since I is K^*-equivariant, we can lift the action of K^* to B^{tor}. Denote by $D \subset B^{tor}$ the total transform of the support of I, and $U_{B^{tor}} = B^{tor} \smallsetminus D$.

Proposition 3.2.1. 1. *The cobordism B^{tor} is quasi-elementary.*
2. *The embedding $U_{B^{tor}} \subset B^{tor}$ is toroidal and K^* acts toroidally on this embedding.*

Definition 3.2.2. We call I a *torific* ideal and $B^{tor} \to B$ a *torific* blowup.

Proof. We take toric charts as in Lemma 2.6.1. Then

$$\iota_p^*(I) = \eta_p^*(I^{X_p}),$$

where I^{X_p} is a torific ideal in X_p. The blowups of these ideals provide a locally toric structure on B^{tor}, such that the total transforms of the ideals agree on the overlaps. Thus we are reduced to the case where $B = X(N, \sigma)$ is an affine toric variety.

To prove the first claim, let $\pi : B \to B//K^*$ be the projection, and note that the torific ideal I is generated by K^*-semi-invariants of a fixed character. Dividing all generators of I by a semi-invariant of the same character, we may assume that $B^{tor} \to B$ is the normalized blowup of a fractional ideal sheaf $\pi^{-1}J$. Now clearly $B^{tor}//K^*$ is the normalized blowup of $B//K^*$ along J. Since $\pi : B \to B//K^*$ is affine, so is $B^{tor} \to B^{tor}//K^*$, hence B^{tor} is quasi-elementary.

For the second claim, note that since the ideal I is toric, $T \subset B^{tor}$ is again toric (with K^* acting as a subgroup of the torus), and the divisor D consists of a subset of the toric divisors. We have to show that we can "remove" the toric divisors not in D so that the embedding and the K^*-action remain toroidal. For this, it suffices to show that if $Y \subset B^{tor}$ is any affine open toric subvariety intersecting an irreducible toric divisor E not in D, we can write

$$Y \cong Y' \times \mathbb{A}^1$$
$$E \cong Y' \times \{0\}$$

such that the K^*-action on Y is a product of a K^*-action on Y' with the trivial K^*-action on $\mathbb{A}^1$.

Let $\sigma = \langle v_1, \dots, v_m \rangle$, $\sigma^\vee = \langle v_1^*, \dots, v_m^*, \pm v_{m+1}^*, \dots, \pm v_n^* \rangle$, and let K^* act on z_i by character a_i. The only toric divisors of B^{tor} that do not lie in the total transform of I are among the strict transforms of the divisors $\{z_i = z^{v_i^*} = 0\} \subset B$. Consider the divisor

$\{z_1 = 0\}$. The ideal I_{a_1} contains z_1. If I_{a_1} is principal then the strict transform of $\{z_1 = 0\}$ is a component of D. Otherwise, we choose monomial generators for I_{a_1} corresponding to lattice points $v_1^*, m_1, \ldots, m_l$ in $M \cap \sigma^\vee$. We may assume that m_i do not contain v_1^*, i.e., all m_i lie in the face $v_1^\perp \cap \sigma^\vee = \langle v_2^*, \ldots, \pm v_n^* \rangle$ of $\sigma^\vee$. To study the strict transform of $\{z_1 = 0\}$ in B^{tor} we first blow up I_{a_1}, then the rest of the I_β, and then normalize.

Let Y be an affine chart of the blowup of B along I_{a_1} (not necessarily normal), obtained by inverting one of the generators of I_{a_1}, and let E be the strict transform of $\{z_1 = 0\}$ in Y. Then E is nonempty if and only if Y is the chart of the blowup where we invert one of the m_i, say m_1. Hence the coordinate ring of Y is generated by monomials

$$v_1^* - m_1, m_2 - m_1, \ldots, m_l - m_1, v_2^*, \ldots, \pm v_n^*.$$

Since the coefficient of v_1^* in $v_1^* - m_1$ is one, and the other generators lie in $v_1^\perp$, we have

$$\begin{aligned} Y &= \operatorname{Spec} K\left[\frac{z_1}{z^{m_1}}, \frac{z^{m_2}}{z^{m_1}}, \ldots, \frac{z^{m_l}}{z^{m_1}}, z_2, \ldots, z_n^{\pm 1}\right] \\ &= \operatorname{Spec} K\left[\frac{z_1}{z^{m_1}}\right] \times \operatorname{Spec} K\left[\frac{z^{m_2}}{z^{m_1}}, \ldots, \frac{z^{m_l}}{z^{m_1}}, z_2, \ldots, z_n^{\pm 1}\right] \\ &= \mathbb{A}^1 \times Y', \end{aligned}$$

where the strict transform E of $\{z_1 = 0\}$ is defined by z_1/z^{m_1} on which K^* acts trivially.

It remains to show that if we blow up the ideals I_β for $\beta \neq a_1$ pulled back to Y, this product structure is preserved. We define the ideals I_β^Y on Y generated by all monomials on which K^* acts by character β. The lemma below shows that I_β^Y is equal to the pullback of I_β. Hence we may blow up I_β^Y instead of the pullback of I_β. Since K^* acts on $v_1^* - m_1$ by character 0, the ideals I_β^Y are generated by monomials in the second term of the product. Thus, blowing up I_β^Y preserves the product structure, and so does normalization. $\square$

Lemma 3.2.3. *Let Y be a chart of the blowup of $X(N, \sigma)$ along I_α for some $\alpha \in \mathbb{Z}$. Then the ideal I_β^Y on Y generated by all monomials of K^*-character β is equal to the pullback of I_β.*

Proof. Clearly the pullback of I_β lies in I_β^Y. For the converse, let the monomial generators of the coordinate ring of Y be $m_2 - m_1, \ldots, m_l - m_1, v_1^*, \ldots, \pm v_n^*$ for some generators m_i of I_α. Thus a monomial on Y can be written as a linear combination

$$m = b_1(m_2 - m_1) + \ldots + b_l(m_l - m_1) + c_1 v_1^* + \ldots + c_n v_n^*$$

for some integers $b_i, c_j \geq 0$ for $i = 1, \ldots, l$, $j = 1, \ldots, m$. If m happens to be a generator of I_β^Y, i.e., K^* acts on m by character β, then also K^* acts on $m' = c_1 v_1^* + \ldots + c_n v_n^*$ by character β, and m' is a generator of I_β. $\square$

The proposition and Lemma 2.6.2 imply that B_-^{tor}/K^* and B_+^{tor}/K^* are toroidal embeddings, and the birational map $B_-^{tor}/K^* \dashrightarrow B_+^{tor}/K^*$ is toroidal. With the help of the previous lemma, we can prove a little more about this birational map:

Corollary 3.2.4. *The toroidal map $B_-^{tor}/K^* \dashrightarrow B_+^{tor}/K^*$ is in fact an isomorphism of toroidal embeddings.*

Proof. Considering toric charts we may assume that $B = X(N, \sigma)$, where $\sigma \subset N_{\mathbb{R}}$ is a cone of full dimension. Let Y be a chart of the blowup of B along I, and $X(N, \tau)$ the normalization of Y (hence $\tau^\vee$ is the saturation of the semigroup of monomials of Y in M). By the previous lemma the ideals I_β^Y are principal for all $\beta \in \mathbb{Z}$. Let $m_1, \ldots, m_l$ be the primitive points of $\tau^\vee$, i.e., the first lattice points on the 1-dimensional faces of $\tau^\vee$. We claim that there is at most one m_i such that $(m_i, a) > 0$. Indeed, if there are two such points, say m_1 and m_2, then for some integers α_1, α_2 we have that $\alpha_1 m_1$ and $\alpha_2 m_2$ are both monomials of Y, and

$$(\alpha_1 m_1, a) = (\alpha_2 m_2, a) = \beta,$$

hence I^Y_β is not principal. The same argument shows that there is at most one m_i such that $(m_i, a) < 0$. Note also that there is at least one m_i such that $(m_i, a) > 0$ (resp. $(m_i, a) < 0$); otherwise, we have $a \in \tau \subset \sigma$ (resp. $a \in -\tau \subset -\sigma$) and B is not a birational cobordism. Thus we can write

$$\tau^\vee = \rho + \langle m_1\rangle + \langle m_2\rangle,$$

where ρ is a cone in $a^\perp$, $(m_1, a) > 0$ and $(m_2, a) < 0$. Now

$$X(N, \tau)_+/K^* = \operatorname{Spec} K[(\tau^\vee + \langle -m_1\rangle) \cap a^\perp \cap M] = \operatorname{Spec} K[\rho \cap M],$$
$$X(N, \tau)_-/K^* = \operatorname{Spec} K[(\tau^\vee + \langle -m_2\rangle) \cap a^\perp \cap M] = \operatorname{Spec} K[\rho \cap M].$$

□

Example 3.2.5. Consider $B = \mathbb{A}^3 = \operatorname{Spec} K[z_1, z_2, z_3]$, where $t \in K^*$ acts as

$$t \cdot (z_1, z_2, z_3) = (t^2 z_1, t z_2, t^{-1} z_3).$$

We have the following generators of the torific ideals I_α:

$$\begin{aligned} I_2 &= \{z_1, z_2^2\} \\ I_1 &= \{z_2, z_1 z_3\} \\ I_{-1} &= \{z_3\}. \end{aligned}$$

If we regard $B = X(N, \sigma)$ as the toric variety corresponding to the regular cone

$$\sigma = \langle v_1, v_2, v_3\rangle \subset N_{\mathbb{R}},$$

then B^{tor} is described by the fan covered by the following three maximal cones

$$\begin{aligned} \sigma_1 &= \langle v_1, v_3, 2v_1 + v_2\rangle \\ \sigma_2 &= \langle v_1 + v_2, v_2 + v_3, v_2\rangle \\ \sigma_3 &= \langle v_3, 2v_1 + v_2, v_1 + v_2, v_2 + v_3\rangle. \end{aligned}$$

The dual cone $\sigma_2^\vee$ has the product description

$$\begin{aligned} \sigma_2^\vee &= \langle v_2^* - (v_1^* + v_3^*), v_1^*, v_3^*\rangle \\ &= \langle v_2^* - (v_1^* + v_3^*)\rangle \times \langle v_1^*, v_3^*\rangle. \end{aligned}$$

Thus from the description of the original toric structure $X(N, \sigma_2) = \operatorname{Spec} k[z_2/z_1 z_3, z_1, z_3]$, and even if we remove the divisor $\{z_2/z_1 z_3 = 0\}$, it still has the toroidal embedding structure

$$X(N, \sigma_2) \smallsetminus (\{z_1 = 0\} \cup \{z_3 = 0\}) \subset X(N, \sigma_2).$$

As $z_2/z_1 z_3$ is invariant, the action of K^* is toroidal. For example, at $0 \in X(N, \sigma_2)$ we have a toric chart

$$\begin{aligned} \mathbb{G}_m \times \mathbb{A}^2 &\to \mathbb{A}^1 \times \mathbb{A}^2 \cong X(N, \sigma_2) \\ (x_1, x_2, x_3) &\mapsto (x_1 - 1, x_2, x_3). \end{aligned}$$

Globally, the divisors corresponding to the new 1-dimensional rays

$$D_{\langle 2v_1 + v_2\rangle}, D_{\langle v_2 + v_3\rangle}, D_{\langle v_1 + v_2\rangle},$$

together with $D_{\langle v_3\rangle}$ coming from I_{-1}, are obtained through the blowup of the torific ideals. Considering

$$U_{B^{tor}} = B^{tor} \smallsetminus (D_{\langle v_3\rangle} \cup D_{\langle 2v_1+v_2\rangle} \cup D_{\langle v_2+v_3\rangle} \cup D_{\langle v_1+v_2\rangle})$$

we obtain a toroidal structure $U_{B^{tor}} \subset B^{tor}$ compatible with the K^*-action. $\square$

4. A proof of the weak factorization theorem

4.1. Lifting the toroidal structures. In Theorem 2.6.3 we have constructed a factorization of the given birational map ϕ into locally toric birational maps

$$\phi : X_1 = W_1 \overset{\varphi_1}{\dashrightarrow} W_2 \overset{\varphi_2}{\dashrightarrow} \cdots W_j \overset{\varphi_j}{\dashrightarrow} W_{j+1} \cdots W_{l-1} \overset{\varphi_{l-1}}{\dashrightarrow} W_l = X_2$$

where $W_j = (B_{a_j})_-/K^*$ and $W_{j+1} = (B_{a_j})_+/K^*$, and the birational map $\varphi_j : W_j \dashrightarrow W_{j+1}$ is induced by the elementary cobordism B_{a_j} associated to the value a_j of a strictly increasing function $\varphi : \mathcal{C} \to \mathbb{Z}$.

Let $B^{tor}_{a_j} \to B_{a_j}$ be a torific blowup. Write $W^{tor}_{j-} = B^{tor}_{a_j-}/K^*$ and $W^{tor}_{j+} = B^{tor}_{a_j+}/K^*$. We have natural birational morphisms

$$f_{j-} : W^{tor}_{j-} \to W_j \text{ and } f_{j+} : W^{tor}_{j+} \to W_{j+1}.$$

Lemma 4.1.1. *The morphisms f_{j-} and f_{j+} can be obtained by blowing up ideals I_{j-} and I_{j+} whose zero sets are disjoint from the open set $U \in X_1$ on which $\phi : X_1 \dashrightarrow X_2$ is an isomorphism. In particular, U is canonically an open set in $W^{tor}_{j\pm}$.*

Proof. We discuss f_{j-}, since the case of f_{j+} is identical. First we claim that the morphism f_{j-} is projective.

We take a toric chart V_p as in the proof of Proposition 3.1.3 so that the torific ideals on B_{a_j} and X_p agree when pulled back to V_p. Hence blowing up these ideals provides a toric chart for $B^{tor}_{a_j}$, and we are reduced to the case where B_{a_j} is toric.

If $B_{a_j} = X(N, \sigma)$ is an affine toric variety then blowing up a torific ideal corresponds to a projective subdivision of the cone σ. This subdivision restricts to a projective subdivision of $\partial_-\sigma$, and hence also of $\pi(\partial_-\sigma)$. Now it follows that the morphism f_{j-} corresponding to this subdivision of $\pi(\partial_-\sigma)$ is projective.

Since f_{j-} is projective, it is isomorphic to a blowup of an ideal sheaf. Next we describe this ideal sheaf explicitly. Take an f_{j-}-ample divisor A^{tor}_{j-} on W^{tor}_{j-}. Since W_j has only quotient singularities (the cones in $\pi(\partial_-\sigma)$ are simplicial), the divisor $A_{j-} := {f_{j-}}_*(A^{tor}_{j-})$ is $\mathbb{Q}$-Cartier. Therefore, by replacing A^{tor}_{j-} with a sufficiently divisible multiple, we may assume from the beginning that both A^{tor}_{j-} and A_{j-} are Cartier divisors. Then we have

$$A^{tor}_{j-} = f^*_{j-}(A_{j-}) - E^{tor}_{j-}$$

for some effective Cartier divisor E^{tor}_{j-} consisting of exceptional divisors for f_{j-} so that $\mathcal{O}_{W^{tor}_{j-}}(-E^{tor}_{j-})$ is f_{j-}-ample and that f_{j-} is the blowup of the ideal

$$I_{j-} := {f_{j-}}_*\mathcal{O}_{W^{tor}_{j-}}(-E^{tor}_{j-}).$$

Note that the torific ideals are trivial at the points $p \in B \smallsetminus F^*_0$ whose stabilizers are trivial, hence on the locus $B_+ \cap B_-$. Therefore, over $U = (B_+ \cap B_-)/K^*$ the birational morphism $f_{j-} : W^{tor}_{j-} \to W_j$ is an isomorphism, and hence I_{j-} is trivial over U. $\square$

By Lemma 2.6.2 we have the toroidal structures

$$U^{tor}_{j-} \subset W^{tor}_{j-} \text{ and } U^{tor}_{j+} \subset W^{tor}_{j+}$$

induced from $U_j \subset B^{tor}_{a_j}$, and the birational map

$$W^{tor}_{j-} \overset{\varphi^{tor}_j}{\dashrightarrow} W^{tor}_{j+}$$

is toroidal with respect to these toroidal structures (in fact an isomorphism of the toroidal embeddings by Corollary 3.2.4).

Let $W^{res}_j \to W_j$ be a canonical resolution of singularities [7].

Proposition 4.1.2. *Let $I^{res}_{j-} = I_{j-}\mathcal{O}_{W^{res}_j}$ and $I^{res}_{j+} = I_{j+}\mathcal{O}_{W^{res}_{j+1}}$. Let $W^{can}_{j-} \to W^{res}_j$ be a canonical principalization of I^{res}_{j-}, and similarly let $W^{can}_{j+} \to W^{res}_{j+1}$ be a canonical principalization of I^{res}_{j+}.*

Denote by $h_{j-} : W^{can}_{j-} \to W^{tor}_{j-}$ and $h_{j+} : W^{can}_{j+} \to W^{tor}_{j+}$ the induced morphisms. Let $U^{can}_{j-} = h^{-1}_{j-}(U^{tor}_{j-})$ and $U^{can}_{j+} = h^{-1}_{j+}(U^{tor}_{j+})$.

$$\begin{array}{ccccccc}
W^{can}_{j-} & \overset{h_{j-}}{\to} & W^{tor}_{j-} & \overset{\varphi^{tor}_j}{\dashrightarrow} & W^{tor}_{j+} & \overset{h_{j+}}{\leftarrow} & W^{can}_{j+} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
W^{res}_j & \to & W_j & \overset{\varphi_j}{\dashrightarrow} & W_{j+1} & \leftarrow & W^{res}_{j+1}
\end{array}$$

Then we have

(i) *$U^{can}_{j-} \subset W^{can}_{j-}$ and $U^{can}_{j+} \subset W^{can}_{j+}$ are toroidal embeddings, and*
(ii) *the morphisms h_{j-} and h_{j+} are toroidal with respect to these toroidal structures.*

Proof. We discuss h_{j-}, since the case of h_{j+} is identical. Observe that canonical resolutions can be "pulled back" by étale morphisms and that all the relevant ideals above have the corresponding ideals on the toric charts so that their pull-backs by $(i_p)_\pm/K^*$ and $(\eta_p)_\pm/K^*$ coincide. Therefore, in order to show the assertion, we can use the locally toric charts as in the proof of Proposition 3.1.3.

Let $B_{a_j} = \operatorname{Spec} K[z_1, \dots, z_m, z^{\pm 1}_{m+1}, \dots, z^{\pm 1}_n]$, and let σ be the cone in $N_{\mathbb{R}}$ describing its toric structure. Let $a \in N$ be the one-parameter subgroup corresponding to the action of K^* and let $\pi : N_{\mathbb{R}} \to N_{\mathbb{R}}/\mathbb{R} \cdot a$ be the projection.

By the Main Example of Section 2.2, $W_j = (B_{a_j})_-/K^*$ corresponds to the fan $\pi(\partial_-\sigma)$. We have three subdivisions of $\pi(\partial_-\sigma)$: first $\Sigma^{tor} \to \pi(\partial_-\sigma)$ induced from that of σ by the torification (blowing up of the torific ideal), second $\Sigma^{res} \to \pi(\partial_-\sigma)$ obtained from the process of taking the canonical resolution of singularities of W_j, and third $\Sigma^{can} \to \Sigma^{res}$ obtained from the canonical principalization of the ideal I^{res}_{j-}. (Remark that the canonical resolution of singularities and the canonical principalization blow up nonsingular toric subvarieties, hence the blowups are toric.)

$$\begin{array}{ccc}
\Sigma^{can} & \longrightarrow & \Sigma^{tor} \\
\big\downarrow & & \big\downarrow \\
\Sigma^{res} & \longrightarrow & \pi(\partial_-\sigma)
\end{array}$$

Recall that the toroidal structure on $W^{tor}_{j-} = X(N, \Sigma^{tor})$ is obtained by removing some of the toric divisors from $X(N, \Sigma^{tor})$. We have to show that we can also remove the inverse images of these divisors from $X(N, \Sigma^{can})$. For this it suffice to show the following:

> Let v_1 be a generator of σ, corresponding to the divisor $E \subset X(N, \sigma)$ whose strict transform is removed from the torification of $X(N, \sigma)$. Let $\tau \in \Sigma^{tor}$ be any cone containing $\pi(v_1)$. Recall that there is a factorization $\tau = \tau' \times \langle v_1 \rangle$

induced from the product structure described in the proof of Proposition 3.2.1, which descends also to the product structure on the quotient by K^*. Then for any (new) ray $w \in \Sigma^{can}$, we have

$$w \subset \tau \Longrightarrow w \in \tau'.$$

In fact, the above implies that both Σ^{can} and Σ^{tor} have a product structure near v_i and the map between them preserves this structure.

Using the notation in the proof of Proposition 3.2.1, τ has its dual generated by

$$< v_1^* - m_1, m_2 - m_1, \cdots, m_l - m_1, v_2^*, \cdots, \pm v_n^* > \cap a^\perp,$$

where $(v_1^* - m_1, \cdot) = 0$ defines the face τ' of τ. Therefore, it is sufficient to show

$$(v_1^* - m_1, w) = \mathrm{ord}_{E_w}(z_1 \cdot z^{-m_1}) = 0,$$

where $\mathrm{ord}_{E_w}(r)$ denotes the order of the rational function r on the divisor E_w (exceptional for h_{j-}) corresponding to the ray w.

Now consider the family of automorphisms θ_c, parameterized by $c \in K$, of $K[z_1, \cdots, z_n^{\pm 1}]$ defined by

$$\theta_c(z_1) = z_1 + c \cdot z^{m_1}, \theta_c(z_i) = z_i \text{ for } i = 2, \ldots, n.$$

Observe that these automorphisms preserve the torific ideals because z_1 and m_1 have the same characters. It follows that θ_c induces a family of automorphisms of W_{j-}^{tor} mapping the exceptional divisor E_{j-}^{tor} to itself. Since the irreducible components of the exceptional divisor are discrete, they are preserved by these automorphisms, and therefore the automorphisms also preserve I_{j-}.

By the canonicity of the resolutions the automorphisms θ_c induce automorphisms of W_{j-}^{can}, which preserve the divisor E_w exceptional for h_{j-} by the same argument as above. Then

$$\begin{aligned}
\mathrm{ord}_{E_w}(z_1 \cdot z^{-m_1}) &= \mathrm{ord}_{E_w}(\theta_c(z_1 \cdot z^{-m_1})) \\
&= \mathrm{ord}_{E_w}((z_1 + c \cdot z^{m_1}) \cdot z^{-m_1}) \\
&= \mathrm{ord}_{E_w}(z_1 \cdot z^{-m_1} + c).
\end{aligned}$$

Since the above equality holds for any $c \in K$, we conclude

$$\mathrm{ord}_{E_w}(z_1 \cdot z^{-m_1}) = 0.$$

□

4.2. **Conclusion of the proof of Theorem 0.1.1.** Since $X_1 = W_0$ and $X_2 = W_l$ are nonsingular, we have $W_0^{res} = W_0$ and $W_l^{res} = W_l$. For each $j = 0, \ldots, l-1$ we have obtained a diagram

$$\begin{tikzcd}
 & & W_{j-}^{can} \arrow[r, dashed, "\varphi_j^{can}"] \arrow[dll, "r_{j-}"'] \arrow[d, "h_{j-}"] & W_{j+}^{can} \arrow[d, "h_{j+}"] \arrow[drr, "r_{j+}"] & & \\
W_j^{res} \arrow[d] & & W_{j-}^{tor} \arrow[r, dashed, "\varphi_j^{tor}"] \arrow[dll, "f_{j-}"'] & W_{j+}^{tor} \arrow[drr, "f_{j+}"] & & W_{j+1}^{res} \arrow[d] \\
W_j \arrow[rrrrr, dashed, "\varphi_j"] & & & & & W_{j+1}
\end{tikzcd}$$

where

(4.2.1) the canonical principalizations r_{j-} and r_{j+} are composites of blowings up with smooth centers,

(4.2.2) h_{j-} and h_{j+} are toroidal by Proposition 4.1.2, and

(4.2.3) φ_j^{tor} is an isomorphism of toroidal embeddings by Corollary 3.2.4.

Now we claim that φ_j^{can} is toroidal. For this we don't need the stronger statement that a composition of toroidal maps is toroidal. It suffices to notice that we choose toroidal charts in which φ_j^{tor} is toroidal and then perform canonical resolution of singularities and canonical principalization of an ideal sheaf in the toroidal charts (recall that these operations behave well with respect to étale morphisms since they are canonical). This way we get toroidal charts for the map φ_j^{can}.

Applying Theorem 1.4.1 to the toroidal map φ_j^{can} we see that φ_j^{can} is a composite of toroidal blowings up and blowings down with smooth centers. Thus we get a factorization

$$\phi : X_1 = W_1^{res} \dashrightarrow W_2^{res} \dashrightarrow \cdots W_j^{res} \dashrightarrow W_{j+1}^{res} \cdots W_{l-1}^{res} \dashrightarrow W_l^{res} = X_2,$$

where all W_i^{res} are nonsingular, and the birational maps are composed of a sequence of blowings up followed by a sequence of blowings down. We do not touch the open subset $U \subset X_1$ on which ϕ is an isomorphism. The projectivity over X_2 follows from Proposition 2.5.2 and the construction. This completes the proof of Theorem 0.1.1. $\square$

5. Generalizations and Problems

In this section we generalize the theorem of weak factorization in the following settings:

(i) $\phi : X_1 \dashrightarrow X_2$ is a birational map between complete nonsingular algebraic spaces X_1 and X_2.

(ii) $\phi : X_1 \dashrightarrow X_2$ is a bimeromorphic map between compact complex manifolds X_1 and X_2.

(iii) X_1 and X_2 are varieties with a group action and the birational map $\phi : X_1 \dashrightarrow X_2$ is equivariant.

(iv) The base field K is not algebraically closed.

(v) The complements of U in X_1 and X_2 are normal crossings divisors.

We finish by briefly discussing two open problems.

5.1. Algebraic and analytic spaces.

We start with the first two cases.

Theorem 5.1.1. *Let $\phi : X_1 \dashrightarrow X_2$ be either*

(i) *a birational map between complete nonsingular algebraic spaces X_1 and X_2 over an algebraically closed field K of characteristic zero; or*

(ii) *a bimeromorphic map between compact complex analytic manifolds X_1 and X_2 defined on an open dense subset,*

and let $U \subset X_1$ be the maximal open set where ϕ is an isomorphism. Then ϕ can be factorized, as in Theorem 0.1.1, into a sequence of blowings up and blowings down with smooth centers disjoint from U.

Proof. We note that the definitions of "locally toric" and "toroidal" in Section 1 make sense in the case of algebraic spaces and complex analytic spaces. For analytic spaces we take the chart maps ι_p and η_p to be open embeddings. The definitions of locally toric and toroidal behave well under étale morphisms (both going up and going down), so we have no problems with algebraic spaces.

Examining the proof of Theorem 0.1.1, we only have to show that the following constructions are valid in the case of algebraic and analytic spaces:

1. Elimination of indeterminacies. This is proved via the analytic "Chow's lemma" as before. In fact, Hironaka [25] only states it in the analytic case.
2. Construction of locally toric charts in the case of a K^* action on a smooth space. For analytic spaces, as we noted in Remark 1.2.8, it suffices to construct such charts at fixed points. At a fixed point we take local semi-invariant parameters defining an open embedding of some K^*-invariant neighborhood of the fixed point into $\mathbb{A}^n$. In case of algebraic spaces, every point has an étale K^*-invariant neighborhood (i.e., a strongly étale morphism) in the form of a nonsingular variety, hence we can use the construction as before.
3. Canonical resolution of singularities and canonical principalization of an ideal sheaf. These work in both algebraic and analytic cases (see [7]).
4. Morelli's theorem for toroidal morphisms. This works as before because the construction is combinatorial and does not depend on the structure of the space.

□

5.2. **Equivariant factorization and Galois descent.** Next we consider the G-equivariant case. We state the theorem for varieties, as the analytic case relies on the delicate details in Hironaka's flatenning theorem [25].

Theorem 5.2.1. *Let G be a group acting on proper nonsingular varieties X_1 and X_2, defined over an algebraically closed field K of characteristic zero. Let $\phi : X_1 \dashrightarrow X_2$ be a G-equivariant birational map and let $U \subset X_1$ be the maximal open set where ϕ is an isomorphism (in particular, U is G-invariant). Then ϕ can be factorized, as in Theorem 0.1.1, into a sequence of G-equivariant blowings up and blowings down with smooth centers disjoint from U.*

Proof. We have to check the validity of the same four constructions as in the previous proof in the G-equivariant case.

1. Elimination of indeterminacies was reduced to the canonical principalization of an ideal sheaf I, and we need the ideal I to be invariant. We can use the construction in [55], Theorem 5.2.2, which is evidently invariant as it only uses the "universal flattener".
2. Choice of charts. Note that we do not have to choose the charts G-invariant since no construction depends on the choice of charts.
3. Canonical resolution of singularities and canonical principalization are G-equivariant by canonicity.
4. For the G-equivariant version of Morelli's theorem we use the method from [4]. A G-equivariant toroidal morphism $\phi : W_1 \to W_2$ induces a G-equivariant morphism of the polyhedral complexes $\phi_\Delta : \Delta_{W_1} \to \Delta_{W_2}$. Since the automorphism group of Δ_{W_1} is finite, this action factors through a finite group G_Δ. Let $\Delta_{W_2'}$ be the barycentric subdivision of Δ_{W_2} corresponding to the blowup of all (smooth) closures of toroidal strata. It is easy to construct a G-equivariant toroidal ideal I giving rise to the blowup $W_2' \to W_2$ as in [4]. Let $W_1' \to W_1$ be the canonical (hence G-equivariant and toroidal) principalization of the ideal $\phi^{-1}I$. Then we have the corresponding G-equivariant morphism of polyhedral complexes
$$\Delta_{W_1'} \to \Delta_{W_2'}$$

induced by ϕ_Δ, such that if a cone σ in $\Delta_{W'_i}$ is mapped to itself by some $g \in G$ then g acts as an identity map on σ. It follows that the quotients $\Delta_{W'_i}/G_\Delta$ are again polyhedral complexes, and the projections $\Delta_{W'_i} \to \Delta_{W'_i}/G_\Delta$ map cones isomorphically to cones. Thus, smooth star subdivisions of $\Delta_{W'_i}/G_\Delta$ give rise to G-equivariant smooth star subdivisions of $\Delta_{W'_i}$, and it suffices to apply Morelli's theorem to the morphism

$$\Delta_{W'_1}/G_\Delta \to \Delta_{W'_2}/G_\Delta.$$

□

Remark 5.2.2. The same argument evidently works for a G-equivariant projective bimeromorphic morphism $\phi : X_1 \to X_2$ of compact complex manifolds, or algebraic spaces. Inspecting Hironaka's argument in [25], it seems that no unnecessary choices are made and thus the same argument should work for arbitrary G-equivariant bimeromorphic maps of compact complex manifolds. However, since Hironaka's argument is delicate and intricate, we leave the verification of this fact to the experts in analytic geometry.

Corollary 5.2.3. *Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between proper nonsingular varieties X_1 and X_2 defined over a field K of characteristic zero such that $K \subset \overline{K}$ is finite, and let $U \subset X_1$ be the maximal open set where ϕ is an isomorphism. Then ϕ can be factorized, as in Theorem 0.1.1 into a sequence of blowings up and blowings down with smooth centers disjoint from U, defined over K.*

Proof. We take $G = Gal(\overline{K}/K)$ and apply the previous theorem to the G-equivariant map $(X_1)_{\overline{K}} \dashrightarrow (X_2)_{\overline{K}}$. □

5.3. **Factorization with a normal crossings divisor.** We now assume that in any of the previous situations, the open set U is given such that $X_1 \setminus U$ and $X_2 \setminus U$ are strict normal crossings divisors (this can always be acheived by blowing up X_1 and X_2). The following result was commissioned by J. Denef, F. Loeser and W. Veys:

Theorem 5.3.1. *The factorization in Theorems 5.1.1 and 5.2.1 can be carried out in such a way that $V_i \setminus U$ are strict normal crossings divisors, and the blowups $\varphi_i^\pm$ are allowable with respect to these divisors.*

An argument for this theorem is already given in [72] in the context of that paper, so we will limit ourselves here to sketching a proof.

1. Elimination of indeterminacies evidently holds for this context.
2. In the construction of the compactified cobordism $\overline{B} \to X_2$ we make the inverse image of $X_2 \setminus U$ a strict normal crossings divisor.
3. Charts on B_{a_i} can be chosen so that the inverse image of $X_2 \setminus U$ is part of the locally toric divisor. Therefore $U \subset W_{j\pm}^{tor}$ is toroidal. (This toroidal structure enriches the one given by the support of the torific ideal.)
4. The anonical resolution of singularities $W_j^{res} \to W_j$ is taken so that $W_j^{res} \setminus U$ is a strict normal crossings divisor.
5. The canonical principalization $W_{j\pm}^{can} \to W_j^{res}$ is taken admissible with respect to the divisor. The morphism $W_{j\pm}^{can} \to W_{j\pm}^{tor}$ is toroidal with respect to the toridal structure given by the open set U, since it enriches the toroidal structure given by the torific ideal (see [2]).
6. Toroidal factorization is admissible automatically, since it is obtained by blowing up and down toroidal strata.

Remark 5.3.2. The statement of Theorem 5.3.1 is motivated by applications to various questions of birational invariance. For instance, Willem Veys pointed out that using this statement, one can give a slightly more geometric proof of Batyrev's theorem, showing that the hodge numbers of birational Calabi Yau varieties coincide. The crucial point is that one can easily see that Batyrev's *stringy E-function* of a nonsingular variety with a strict normal crossings divisor is invariant under an admissible blowing up.

5.4. **Strong factorization.**

Problem 5.4.1. *Can one extend the torific ideal to all of B in such a way that the strong factorization conjecture follows?*

In the construction of the torific ideal in 3.6 and the analysis of its blowup in 3.2 and 4.1, the assumption of the cobordism B_{a_j} being quasi-elementary is essential. Actually we can extend the ideal to the one over the entire cobordism B by taking its Zariski closure, but its behavior along $B - B_{a_j}$ seems to be subtle. Other extensions exixt, and seem to be promising.

5.5. **Toroidalization.**

Problem 5.5.1 (Toroidalization conjecture)**.** *Let $\phi : X \to Y$ be a surjective proper morphism between complete nonsingular varieties. Do there exist sequences of blowups with smooth centers $\nu_X : \tilde{X} \to X$ and $\nu_Y : \tilde{Y} \to Y$ so that the induced map $\tilde{\phi} : \tilde{X} \dashrightarrow \tilde{Y}$ is a toroidal morphism?*

We note that the toroidalization conjecture concerns not only birational morphisms ϕ but also generically finite morphisms or morphisms with $\dim X > \dim Y$. The solution to the above conjecture would immediately lead to that of the strong factorization conjecture. At present the authors know a complete proof only in dimension 2, where the generically finite case already requires a simple but nontrivial analysis.

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