TWO COVERING POLYNOMIALS OF A FINITE POSET, WITH APPLICATIONS TO ROOT SYSTEMS AND AD-NILPOTENT IDEALS

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In this note, we introduce two polynomials associated with a finite poset $\mathcal{P}$, study their properties, and then compute these polynomials in the case, where $\mathcal{P}$ is one of the posets considered in [13].

Consider two statistics κ and ι on $\mathcal{P}$. By definition, $\kappa(x)$, $x \in \mathcal{P}$, is the number of elements of $\mathcal{P}$ that are covered by x, and $\iota(x)$ is the number of elements that covers x. The generating function associated with κ (resp. ι) is the *upper* (resp. *lower*) *covering polynomial* of $\mathcal{P}$. That is, $\hat{\mathcal{K}}_{\mathcal{P}}(q) = \sum_{x\in\mathcal{P}} q^{\kappa(x)}$ and $\check{\mathcal{K}}_{\mathcal{P}}(q) = \sum_{x\in\mathcal{P}} q^{\iota(x)}$. The upper covering polynomial, $\hat{\mathcal{K}}_{\mathcal{P}}$, has briefly been considered, without adjective 'upper', in [13, Section 5]. It immediately follows from the definition that $\hat{\mathcal{K}}_{\mathcal{P}}(1) = \check{\mathcal{K}}_{\mathcal{P}}(1)$ and $\hat{\mathcal{K}}'_{\mathcal{P}}(1) = \check{\mathcal{K}}'_{\mathcal{P}}(1)$. The last equality follows from the observation that both values equal the number of edges in the Hasse diagram of $\mathcal{P}$. Hence $\hat{\mathcal{K}}_{\mathcal{P}}(q) - \check{\mathcal{K}}_{\mathcal{P}}(q) = (q-1)^2\mathcal{D}_{\mathcal{P}}(q)$ for some polynomial $\mathcal{D}_{\mathcal{P}}$, which is said to be the *deviation* polynomial of $\mathcal{P}$.

We begin with a simple observation that $\hat{\mathcal{K}}_{\mathcal{P}} \equiv \check{\mathcal{K}}_{\mathcal{P}}$ whenever $\mathcal{P}$ is a distributive lattice or admits an order-reversing involution.

Let Δ be an irreducible root system, Δ^+ a subset of positive roots, and $\Pi \subset \Delta^+$ is the set of simple roots. If Δ is reduced, then $\mathfrak{g}$ denotes the respective simple Lie algebra, with fixed Borel subalgebra $\mathfrak{b}$ corresponding to Δ^+. We determine $\hat{\mathcal{K}}_{\mathcal{P}}$, $\check{\mathcal{K}}_{\mathcal{P}}$, and $\mathcal{D}_{\mathcal{P}}$ in the following three cases:

1) $\mathcal{P} = \Delta^+$;

2) $\mathcal{P} = J^*(\Delta^+)$, the poset of dual order ideals in Δ^+, or $\mathcal{P} = J^*(\Delta^+\setminus\Pi)$, the poset of dual order ideals in $\Delta^+ \setminus \Pi$. In the Lie algebra case, these posets are isomorphic to the poset of all and strictly positive ad-nilpotent ideals in $\mathfrak{b}$. These two posets are also denoted by $\mathfrak{Ad}$ and $\mathfrak{Ad}_0$, respectively;

3) $\mathcal{P}$ is the poset of Abelian ideal in $J^*(\Delta^+)$, denoted $\mathfrak{Ab}$.

Let us briefly describe our results. For Δ^+, it is shown that $\deg\hat{\mathcal{K}}_{\Delta^+} = \deg\check{\mathcal{K}}_{\Delta^+} \leqslant 3$ and the coefficients of q^3 in $\hat{\mathcal{K}}_{\Delta^+}$ and $\check{\mathcal{K}}_{\Delta^+}$ are equal. This readily implies that the deviation polynomial is quite simple; namely, $\mathcal{D}_{\Delta^+}(q) \equiv \mathrm{rk}\,\Delta - 1$.

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Since $\mathfrak{Ad}$ is a distributive lattice, $\hat{\mathcal{K}}_{\mathfrak{Ad}} \equiv \check{\mathcal{K}}_{\mathfrak{Ad}}$. This common polynomial appeared earlier under various guises in different theories in [1, 2, 12, 15]. Although it is known *a posteriori* that $\hat{\mathcal{K}}_{\mathfrak{Ad}}$ is palindromic, no explanation of this phenomenon is available in the context of ad-nilpotent ideals. The common covering polynomial for $\mathfrak{Ad}_0$ is not palindromic. However, the ratio $\hat{\mathcal{K}}'(1)/\hat{\mathcal{K}}(1)$ is determined by similar rules in both cases. We notice that

$$\frac{\mathcal{K}'_{\mathfrak{Ad}}(1)}{\mathcal{K}_{\mathfrak{Ad}}(1)} = \frac{n}{2} = \frac{\#(\Delta^+)}{h} \quad \text{and} \quad \frac{\mathcal{K}'_{\mathfrak{Ad}_0}(1)}{\mathcal{K}_{\mathfrak{Ad}_0}(1)} = \frac{n}{2} \cdot \frac{h-2}{h-1} = \frac{\#(\Delta^+ \setminus \Pi)}{h-1} \ ,$$

where h is the Coxeter number and n is the rank of Δ. (The computation for series $\mathbf{D}_n$ and $\mathbf{E}_n$ in case of $\mathfrak{Ad}_0$ is based on the conjectural identification of the coefficients of $\mathcal{K}_{\mathfrak{Ad}_0}(q)$ with certain numbers computed by F. Chapoton [6].) The first equality stems from the fact that $\mathcal{K}_{\mathfrak{Ad}}$ is palindromic of degree n (although it is not clear why $\mathcal{K}_{\mathfrak{Ad}}$ is palindromic!). The reason for the validity of the second one is totally unclear.

The most interesting case is that of Abelian ideals. Here the upper and lower covering polynomials are usually different. The reason is that although $\mathfrak{Ab}$ is a meet semilattice, it is a distributive lattice if and only if Δ is of type $\mathbf{C}_n$ or $\mathbf{G}_2$. We develop some general theory for computing covering polynomials, which is based on a connection between the Abelian ideals and the minuscule elements of the affine Weyl group of Δ. Let $I \subset \Delta^+$ be an Abelian ideal. Making use of the minuscule element of I, one constructs the *shift vector* $\mathbf{k}_I = (k_0, k_1, \dots, k_n)$ with $k_i \in \{-1, 0, 1, 2\}$. We prove that $\kappa(I) = \#\{i \mid k_i = -1\}$ and $\iota(I) = \#\{j \mid k_j = 1\}$, and describe an inductive procedure for computing all $\mathbf{k}_I$ starting from $I = \varnothing$. The procedure basically asserts that if $k_i = 1$, then $\mathbf{k}_I$ can be replaced with $\mathbf{k}_I -$ (the i-th row of the extended Cartan matrix of Δ), see Section 4 for details.

We also present a method of calculation of $\check{\mathcal{K}}_{\mathfrak{Ab}}$, which exploits the canonical mapping of $\mathfrak{Ab}\setminus\{\varnothing\}$ to the set of long positive roots [11]. For explicit computations with exceptional root systems, we use the general equalities $\hat{\mathcal{K}}_{\mathfrak{Ab}}(1) = 2^n$ [4, 9] and $\hat{\mathcal{K}}'_{\mathfrak{Ab}}(1) = (n+1)2^{n-2}$ [13]; whereas our calculations in the classical cases exploit standard matrix presentations of these Lie algebras and counting certain Ferrers diagrams.

Our computations show that, for many natural posets, the coefficients of $\mathcal{D}_{\mathcal{P}}$ are of the same sign. This includes $\mathfrak{Ab}$, Δ^+, $\Delta^+ \cup \{0\}$, $\Delta^+ \setminus \Pi$. It is likely that there could exist some general condition on $\mathcal{P}$ guaranteeing that $\mathcal{D}_{\mathcal{P}}$ has the coefficients of the same sign.

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1. Definition and basic properties

Let $(\mathcal{P}, \preccurlyeq)$ be a finite poset. Write $\mathcal{H}(\mathcal{P})$ for the Hasse diagram of $\mathcal{P}$ and $\mathcal{E}(\mathcal{P})$ for the set of edges of $\mathcal{H}(\mathcal{P})$. We regard $\mathcal{H}(\mathcal{P})$ as a digraph; if x covers y ($x, y \in \mathcal{P}$), then the edge (x, y) is depicted as $y \to x$ and we say that (x, y) *originates* in y and *terminates* in x.

Let us define two polynomials that encode some properties of the covering relation in $\mathcal{P}$. For any $x \in \mathcal{P}$, $\kappa(x)$ is the number of $y \in \mathcal{P}$ such that y is covered by x, and $\iota(x)$ is the number of $y \in \mathcal{P}$ such that y covers x.

1.1 Definition.
(i) The *upper covering polynomial* of $\mathcal{P}$ is $\hat{\mathcal{K}}_{\mathcal{P}}(q) = \sum_{x\in\mathcal{P}} q^{\kappa(x)}$;
(ii) The *lower covering polynomial* of $\mathcal{P}$ is $\check{\mathcal{K}}_{\mathcal{P}}(q) = \sum_{x\in\mathcal{P}} q^{\iota(x)}$;

It follows that $\hat{\mathcal{K}}_{\mathcal{P}}(0)$ (resp. $\check{\mathcal{K}}_{\mathcal{P}}(0)$) is the number of the minimal (resp. maximal) elements of $\mathcal{P}$. In general, these polynomials are different; they may even have different degree. However, one readily deduce from the definition that

$$\hat{\mathcal{K}}_{\mathcal{P}}(q)|_{q=1} = \check{\mathcal{K}}_{\mathcal{P}}(q)|_{q=1} = \#\mathcal{P} \quad \text{and} \quad \frac{d}{dq}\hat{\mathcal{K}}_{\mathcal{P}}(q)|_{q=1} = \frac{d}{dq}\check{\mathcal{K}}_{\mathcal{P}}(q)|_{q=1} = \#\mathcal{E}(\mathcal{P})\ .$$

Hence $\hat{\mathcal{K}}_{\mathcal{P}}(q) - \check{\mathcal{K}}_{\mathcal{P}}(q) = (q-1)^2\mathcal{D}_{\mathcal{P}}(q)$ for some polynomial $\mathcal{D}_{\mathcal{P}}$. We will say that $\mathcal{D}_{\mathcal{P}}$ is the *deviation polynomial* of $\mathcal{P}$. The following is obvious.

1.2 Lemma.

(i) *If $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$, then $\hat{\mathcal{K}}_{\mathcal{P}} = \hat{\mathcal{K}}_{\mathcal{P}_1} + \hat{\mathcal{K}}_{\mathcal{P}_2}$, and likewise for $\check{\mathcal{K}}$ and $\mathcal{D}$;*
(ii) *If $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$, then $\hat{\mathcal{K}}_{\mathcal{P}} = \hat{\mathcal{K}}_{\mathcal{P}_1}\hat{\mathcal{K}}_{\mathcal{P}_2}$, $\check{\mathcal{K}}_{\mathcal{P}} = \check{\mathcal{K}}_{\mathcal{P}_1}\check{\mathcal{K}}_{\mathcal{P}_2}$, and $\mathcal{D}_{\mathcal{P}} = \hat{\mathcal{K}}_{\mathcal{P}_1}\mathcal{D}_{\mathcal{P}_2} + \check{\mathcal{K}}_{\mathcal{P}_2}\mathcal{D}_{\mathcal{P}_1} = \check{\mathcal{K}}_{\mathcal{P}_1}\mathcal{D}_{\mathcal{P}_2} + \hat{\mathcal{K}}_{\mathcal{P}_2}\mathcal{D}_{\mathcal{P}_1}$.*

We are going to investigate how properties of $\mathcal{P}$ are reflected in $\hat{\mathcal{K}}_{\mathcal{P}}, \check{\mathcal{K}}_{\mathcal{P}}, \mathcal{D}_{\mathcal{P}}$.

1.3 Theorem. *Let $\mathcal{P}$ be a distributive lattice. Then $\hat{\mathcal{K}}_{\mathcal{P}} = \check{\mathcal{K}}_{\mathcal{P}}$. More precisely, if $\mathcal{P} \simeq J(\mathcal{L})$, then the coefficient of q^k equals the number of k-element antichains in $\mathcal{L}$.*

Proof. By Birkhoff's theorem for finite distributive lattices, $\mathcal{P}$ is isomorphic to the poset of order ideals of a unique poset $\mathcal{L}$, i.e., $\mathcal{P} \simeq J(\mathcal{L})$, see e.g. [18, Theorem 3.4.1]. If I is an order ideal of $\mathcal{L}$, then the set of maximal elements of I, $\max(I)$, is an antichain of $\mathcal{L}$. And the same is true for the set of minimal elements of $\mathcal{L} \setminus I$, $\min(\mathcal{L} \setminus I)$. It easily follows from Definiton 1.1 that regarding I as an element of $\mathcal{P}$ we have $\iota(I) = \#\max(I)$ and $\kappa(I) = \#\min(\mathcal{L}\setminus I)$. Conversely, each antichain in $\mathcal{L}$ occurs as both $\max(I)$ and $\min(\mathcal{L}\setminus J)$ for suitable order ideals I, J. This means that both covering polynomials essentially count all the antichains of $\mathcal{L}$ with respect to their cardinality. □

Remark. More generally, the equality $\hat{\mathcal{K}}_{\mathcal{P}} = \check{\mathcal{K}}_{\mathcal{P}}$ holds if $\mathcal{P}$ is a modular lattice. This result of Dilworth (*Ann. Math.* **60**(1954), 359–364) can also be found in [18, Ex. 3.38.5]. This fact was communicated to me by R. Stanley.

In the following sections, it will be more convenient for us to think of a distributive lattice as the poset of *dual order ideals*. Given $\mathcal{L}$, the poset of dual order ideals of $\mathcal{L}$ is denoted by $J^*(\mathcal{L})$. Then $\mathcal{L}$ is restored as the set of meet-irreducibles in $J^*(\mathcal{L})$. If $I \in J^*(\mathcal{L})$, then $\kappa(I) = \#\min(I)$ and $\iota(I) = \#\max(\mathcal{L} \setminus I)$.

1.4 Proposition. *If $\mathcal{P}$ admits an order-reversing bijection, then $\hat{\mathcal{K}}_{\mathcal{P}} = \check{\mathcal{K}}_{\mathcal{P}}$.*

Proof. If $\omega : \mathcal{P} \to \mathcal{P}$ is an order-reversing bijection, then $\kappa(x) = \iota(\omega(x))$ for any $x \in \mathcal{P}$. $\square$

1.5 Examples. 1. Let (W, S) be a finite Coxeter group. Consider W as poset under the Bruhat-Chevalley ordering '$\leqslant$'. It is easily seen that W is not a lattice. But the mapping $w \mapsto ww_0$, where $w_0 \in W$ is the longest element, yields an order-reversing bijection of $(W, \leqslant)$. Hence $\hat{\mathcal{K}}_W(q) = \check{\mathcal{K}}_W(q)$. More generally, the equality also holds for W/W_J, where $J \subset S$ and W_J is the corresponding parabolic subgroup of W.

2. Let $\mathcal{P}$ be an arbitrary poset and $\mathcal{P}^{op}$ the opposite poset. Then $\hat{\mathcal{K}}_{\mathcal{P}} = \check{\mathcal{K}}_{\mathcal{P}^{op}}$ and $\check{\mathcal{K}}_{\mathcal{P}} = \hat{\mathcal{K}}_{\mathcal{P}^{op}}$. Hence $\mathcal{D}_{\mathcal{P}} = -\mathcal{D}_{\mathcal{P}^{op}}$. It then follows from Lemma 1.2 that $\mathcal{D}_{\mathcal{P}\times\mathcal{P}^{op}} = 0$. One may also notice that $\mathcal{P} \times \mathcal{P}^{op}$ admits an order-reversing involution.

To get acquainted with properties of the covering polynomials, let us look at the effect of two simple transformations of a distributive lattice. As we've just shown, if $\mathcal{P} = J(\mathcal{L})$, then $\mathcal{D}_{\mathcal{P}} = 0$. Let $\hat{0}$ and $\hat{1}$ denote the maximal and minimal element of $\mathcal{P}$, respectively. Set $\mathcal{P}' = \mathcal{P} \setminus \{\hat{0}\}$ and $\mathcal{P}'' = \mathcal{P} \setminus \{\hat{1}\}$. It is easy to recognise the effect of this procedure for $\hat{\mathcal{K}}_{\mathcal{P}}$ and $\check{\mathcal{K}}_{\mathcal{P}}$. Let m (resp. l) be the number of maximal (resp. minimal) elements of $\mathcal{L}$. Then

- $\hat{\mathcal{K}}_{\mathcal{P}'} = \mathcal{K}_{\mathcal{P}} - m(q-1) - 1$ and $\check{\mathcal{K}}_{\mathcal{P}'} = \mathcal{K}_{\mathcal{P}} - q^m$. Hence $\hat{\mathcal{K}}_{\mathcal{P}'} - \check{\mathcal{K}}_{\mathcal{P}'} = q^m - mq + (m-1)$ and $\mathcal{D}_{\mathcal{P}'} = q^{m-2} + 2q^{m-3} + \ldots + (m-2)q + (m-1)$.
- $\hat{\mathcal{K}}_{\mathcal{P}''} = \mathcal{K}_{\mathcal{P}} - q^l$ and $\check{\mathcal{K}}_{\mathcal{P}''} = \mathcal{K}_{\mathcal{P}} - l(q-1) - 1$. Hence $\hat{\mathcal{K}}_{\mathcal{P}''} - \check{\mathcal{K}}_{\mathcal{P}''} = -q^l + lq - (l-1)$ and $\mathcal{D}_{\mathcal{P}''} = -\big(q^{l-2} + 2q^{l-3} + \ldots + (l-2)q + (l-1)\big)$.

In both cases, one obtains polynomials having all nonzero coefficients of the same sign.

Our goal in the following sections is to consider the polynomials $\hat{\mathcal{K}}_{\mathcal{P}}$, $\check{\mathcal{K}}_{\mathcal{P}}$, and $\mathcal{D}_{\mathcal{P}}$ for some posets associated with systems of positive roots.

2. Covering polynomials for the root systems

Let Δ be a root system in an n-dimensional real euclidean vector space V. Choose a subsystem of positive roots Δ^+ with the corresponding set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. Write θ for the highest root in Δ^+ and h for the Coxeter number.

We wish to compute the covering polynomials in case $\mathcal{P} = \Delta^+$. The standard root order '$\preccurlyeq$' in Δ^+ is determined by the condition that γ covers μ if and only if $\gamma - \mu \in \Pi$. In view of Lemma 1.2, it suffices to consider only irreducible root systems. In what follows, $[q^m]\mathcal{F}$ stands for the coefficient of q^m in the polynomial $\mathcal{F}(q)$.

2.1 Theorem. *Let Δ be an irreducible root system of rank n. Then*

(i) $\deg \hat{\mathcal{K}}_{\Delta^+} \leqslant 3$ *and* $\deg \check{\mathcal{K}}_{\Delta^+} \leqslant 3$*;*
(ii) $[q^3]\hat{\mathcal{K}}_{\Delta^+} = [q^3]\check{\mathcal{K}}_{\Delta^+}$*;*

(iii) *if Δ is simply-laced, then* $[q]\hat{\mathcal{K}}_{\Delta^+} = [q^3]\hat{\mathcal{K}}_{\Delta^+}$*;*
(iv) $\mathcal{D}_{\Delta^+}(q) \equiv n-1$.

Proof. We provide a uniform proof for parts (i) and (ii) only in the simply-laced case. The remaining cases (including the non-reduced root system $\mathbf{BC}_n$) can be handled in a case-by-case fashion.

(i) For $\hat{\mathcal{K}}_{\Delta^+}$, one has to show that there are at most 3 simple roots that can be subtracted from a positive root. Suppose $\gamma \in \Delta^+$ and $\gamma - \alpha_i \in \Delta^+$ for $\alpha_i \in \Pi$ and $i = 1, 2, \dots, k$. If $(\alpha_1, \alpha_2) \neq 0$, then these two roots generate the root subsystem of type $\mathbf{A}_2$ and γ is the highest weight in the adjoint $\mathbf{A}_2$-module. Therefore the weight $\gamma - \alpha_1 - \alpha_2$ has multiplicity two. This is only possible if $\gamma = \alpha_1 + \alpha_2$. Hence $k = 2$. It is thus proved that in case $k \geqslant 3$, all simple roots that can be subtracted from γ are pairwise orthogonal. In this situation, it was shown in [10, Corollary 3.3] that $k \leqslant 3$.

The argument for $\check{\mathcal{K}}_{\Delta^+}$ is similar.

(ii) Suppose $\kappa(\gamma) = 3$, and let $\alpha_1, \alpha_2, \alpha_3$ be the corresponding simple roots. As is shown in part (i), these roots are pairwise orthogonal. Therefore $\gamma - \alpha_1 - \alpha_2 - \alpha_3 \in \Delta^+$ and the mapping $\gamma \mapsto (\gamma - \alpha_1 - \alpha_2 - \alpha_3)$ sets up a bijection between $\{\gamma \in \Delta^+ \mid \kappa(\gamma) = 3\}$ and $\{\gamma \in \Delta^+ \mid \iota(\gamma) = 3\}$.

(iii) As is well-known, the number of positive roots is $\hat{\mathcal{K}}_{\Delta^+}(1) = nh/2$, where h is the Coxeter number of Δ. By [13, Theorem 1.1], the number of edges of $\mathcal{H}(\Delta^+)$ equals $n(h-2)$ in the simply laced case. That is, $\hat{\mathcal{K}}'_{\Delta^+}(1) = n(h-2)/2$. Writing $\hat{\mathcal{K}}_{\Delta^+}(q) = n + aq + bq^2 + cq^3$ and using the above two equalities, we obtain $a = c$.

(iv) By parts (i) and (ii), $\deg(\hat{\mathcal{K}}_{\Delta^+} - \check{\mathcal{K}}_{\Delta^+}) \leqslant 2$. It is also clear that $[q^0](\hat{\mathcal{K}}_{\Delta^+} - \check{\mathcal{K}}_{\Delta^+}) = n-1$. Since $(q-1)^2$ divides this polynomial, the quotient must be $n-1$. □

Remark. The degree of polynomials $\hat{\mathcal{K}}$ and $\check{\mathcal{K}}$ equals 3 if and only if the Dynkin diagram of Δ has a branching node.

It is not hard to compute both covering polynomials for the posets Δ^+, see Table 1.

Let $\varepsilon_1, \dots, \varepsilon_n$ be an orthonormal basis for V. Recall that the unique nonreduced irreducible root system $\mathbf{BC}_n$ consists of the roots $\pm\varepsilon_i \pm \varepsilon_j$ ($1 \leqslant i < j \leqslant n$), $\pm\varepsilon_i$, $\pm 2\varepsilon_i$ ($1 \leqslant i \leqslant n$). The simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leqslant i \leqslant n-1$, and $\alpha_n = \varepsilon_n$.

The following observation reduces many questions about $\mathbf{BC}_n$ to $\mathbf{B}_{n+1}$ or $\mathbf{C}_{n+1}$.

2.2 Lemma. *The poset of positive roots for $\mathbf{BC}_n$ is isomorphic to the subposet of non-simple positive roots for $\mathbf{B}_{n+1}$ or $\mathbf{C}_{n+1}$. The posets $\Delta^+(\mathbf{B}_{n+1})$ and $\Delta^+(\mathbf{C}_{n+1})$ are isomorphic.*

Δ	$\hat{\mathcal{K}}_{\Delta^+}(q)$	$\check{\mathcal{K}}_{\Delta^+}(q)$
$\mathbf{A}_n$	$n+\binom{n}{2}q^2$	$1+(2n-2)q+\binom{n-1}{2}q^2$
$\mathbf{B}_n, \mathbf{C}_n$	$n+(n-1)q+(n-1)^2q^2$	$1+(3n-3)q+(n-1)(n-2)q^2$
$\mathbf{BC}_n$	$n+nq+n(n-1)q^2$	$1+(3n-2)q+(n-1)^2q^2$
$\mathbf{D}_n$	$n+(n-3)q+(\binom{n}{2}+\binom{n-3}{2})q^2+(n-3)q^3$	$1+(3n-5)q+(\binom{n-1}{2}+\binom{n-3}{2})q^2+(n-3)q^3$
$\mathbf{E}_6$	$6+5q+20q^2+5q^3$	$1+15q+15q^2+5q^3$
$\mathbf{E}_7$	$7+10q+36q^2+10q^3$	$1+22q+30q^2+10q^3$
$\mathbf{E}_8$	$8+21q+70q^2+21q^3$	$1+35q+63q^2+21q^3$
$\mathbf{F}_4$	$4+7q+12q^2+q^3$	$1+13q+9q^2+q^3$
$\mathbf{G}_2$	$2+3q+q^2$	$1+5q$

TABLE 1. The upper and lower covering polynomials for the root systems

Proof. An order-preserving bijection between $\Delta^+(\mathbf{BC}_n)$ and either $\Delta^+(\mathbf{B}_{n+1})\setminus\Pi$ or $\Delta^+(\mathbf{C}_{n+1})\setminus\Pi$ is given as follows:

$$\begin{array}{rcccll}
\mathbf{C}_{n+1} & & \mathbf{BC}_n & & \mathbf{B}_{n+1} & \\
\varepsilon_i-\varepsilon_{j+1} & \longleftarrow & \varepsilon_i-\varepsilon_j & \longmapsto & \varepsilon_i-\varepsilon_{j+1} & (1\leqslant i<j\leqslant n)\\
\varepsilon_i+\varepsilon_j & \longleftarrow & \varepsilon_i+\varepsilon_j & \longmapsto & \varepsilon_i+\varepsilon_{j+1} & (1\leqslant i<j\leqslant n)\\
\varepsilon_i+\varepsilon_{n+1} & \longleftarrow & \varepsilon_i & \longmapsto & \varepsilon_i & (1\leqslant i\leqslant n)\\
2\varepsilon_i & \longleftarrow & 2\varepsilon_i & \longmapsto & \varepsilon_i+\varepsilon_{i+1} & (1\leqslant i\leqslant n)
\end{array}$$

It is easily seen that this extends to an isomorphism between $\Delta^+(\mathbf{B}_{n+1})$ and $\Delta^+(\mathbf{C}_{n+1})$. $\square$

2.3 Example. Consider two modifications of Δ^+.

1. Replace Δ^+ with $\widetilde{\Delta}^+=\Delta^+\cup\{0\}$, where $\{0\}$ is regarded as the unique minimal element in this new poset. Hence $\mathcal{H}(\widetilde{\Delta}^+)$ gains n new edges connecting $\{0\}$ with the simple roots. It is easy to describe the effect of this extension for the covering polynomials. We have

$$\hat{\mathcal{K}}_{\widetilde{\Delta}^+}(q)=\hat{\mathcal{K}}_{\Delta^+}(q)+n(q-1)+1$$
$$\check{\mathcal{K}}_{\widetilde{\Delta}^+}(q)=\check{\mathcal{K}}_{\Delta^+}(q)+q^n\ .$$

It follows that

$$\mathcal{D}_{\widetilde{\Delta}^+}=\mathcal{D}_{\Delta^+}-\frac{q^n-nq+n-1}{(q-1)^2}=-(q^{n-2}+2q^{n-3}+\ldots+(n-2)q)\ .$$

2. Assume that $n\geqslant 2$ and consider $\Delta^+\setminus\Pi$ as subposet of Δ^+. Then the minimal elements of $\Delta^+\setminus\Pi$ are the roots of height 2. Here we obtain $\hat{\mathcal{K}}_{\Delta^+\setminus\Pi}(q)=\hat{\mathcal{K}}_{\Delta^+}(q)-(n-1)q^2-1$. But formulae for $\check{\mathcal{K}}$ depends on the presence of a branching node in the Dynkin diagram, i.e., on the presence of a simple root which is covered by three roots. More

precisely,

$$\check{\mathcal{K}}_{\Delta^+\setminus\Pi}(q)=\check{\mathcal{K}}_{\Delta^+}(q)-\begin{cases} (n-2)q^2+2q, & \text{if } \Delta^+ \text{ does not have a branching node;}\\ q^3+(n-4)q^2+3q, & \text{if } \Delta^+ \text{ has a branching node .}\end{cases}$$

Then

$$\mathcal{D}_{\Delta^+\setminus\Pi}(q)=\begin{cases} n-2, & \text{if } \Delta^+ \text{ does not have a branching node;}\\ q+n-2, & \text{if } \Delta^+ \text{ has a branching node .}\end{cases}$$

Thus, the deviation polynomial of $\Delta^+,\Delta^+\cup\{0\}$, and $\Delta^+\setminus\Pi$ always has the nonzero coefficients of the same sign.

3. Covering polynomials for the poset of ad-nilpotent ideals

Let $\mathfrak{g}$ be the simple complex Lie algebra corresponding to Δ (if Δ is reduced). Fix a triangular decomposition $\mathfrak{g}=\mathfrak{u}^+\oplus\mathfrak{t}\oplus\mathfrak{u}^-$, where $\mathfrak{t}$ is a Cartan subalgebra and the set of $\mathfrak{t}$-roots in $\mathfrak{u}^+$ is Δ^+. Then $\mathfrak{b}=\mathfrak{t}\oplus\mathfrak{u}^+$ is the fixed Borel subalgebra.

An ad-*nilpotent ideal* of $\mathfrak{b}$ is a subspace $\mathfrak{c}\subset\mathfrak{u}^+$ such that $[\mathfrak{b},\mathfrak{c}]\subset\mathfrak{c}$. Combinatorially, the poset of ad-nilpotent ideals in $\mathfrak{b}$ can be defined as the poset of *dual order* (or *upper*) ideals of Δ^+. It will be denoted by $\mathfrak{Ad}$ or $\mathfrak{Ad}(\mathfrak{g})$. If $I\in\mathfrak{Ad}$ is considered as a subset of Δ^+, then $\kappa(I)=\#\min(I)$ and $\iota(I)=\#\max(\Delta^+\setminus I)$. The elements of $\min(I)$ are called *generators* of I. For $\gamma\in\max(\Delta^+\setminus I)$, the passage $I\mapsto I\cup\{\gamma\}$ is called an *extension* of I. Thus, $\kappa(I)$ (resp. $\iota(I)$) is the number of generators (resp. extensions) of I. By Theorem 1.3, $\hat{\mathcal{K}}_{\mathfrak{Ad}}=\check{\mathcal{K}}_{\mathfrak{Ad}}$ and the coefficient of q^k equals the number of k-element antichains in Δ^+. This common polynomial is said to be the covering polynomial, denoted merely $\mathcal{K}_{\mathfrak{Ad}}$. Here $\deg\mathcal{K}_{\mathfrak{Ad}}=\operatorname{rk}\Delta=n$. The polynomial $\mathcal{K}_{\mathfrak{Ad}}$ appears in different contexts, see [1, 2, 12, 15]. It is worth mentioning that there is a uniform expression (and proof) for the number of all ad-nilpotent ideals, i.e., $\mathcal{K}_{\mathfrak{Ad}}(1)$, see [5]. Since $\mathcal{K}_{\mathfrak{Ad}}$ is palindromic, $\mathcal{K}'_{\mathfrak{Ad}}(1)=\frac{n}{2}\mathcal{K}_{\mathfrak{Ad}}(1)$, which yields the expression for the number of edges in $\mathcal{H}(\mathfrak{Ad})$, see [13]. But no uniform description for (the coefficients of) $\mathcal{K}_{\mathfrak{Ad}}$ is known. For future use, we record the following relation between the number of vertices and edges in $\mathcal{H}(\mathfrak{Ad})$:

$$\frac{\#\mathcal{E}(\mathfrak{Ad})}{\#\mathfrak{Ad}}=\frac{\mathcal{K}'_{\mathfrak{Ad}}(1)}{\mathcal{K}_{\mathfrak{Ad}}(1)}=\frac{n}{2}=\frac{\#(\Delta^+)}{h}\ . \tag{3.1}$$

Although there is no Lie algebra associated with the root system $\mathbf{BC}_n$, one can still consider the poset of dual order ideals in $\Delta^+(\mathbf{BC}_n)$, denoted $\mathfrak{Ad}(\mathbf{BC}_n)$. In view of Lemma 2.2, $\#(\mathfrak{Ad}(\mathbf{BC}_n))$ is equal to the number of dual order ideals in $\Delta^+(\mathbf{B}_{n+1})\setminus\Pi$. The latter is known to be equal to $\binom{2n+1}{n}$ [17].

3.2 Proposition. *The covering polynomial of $\mathfrak{Ad}(\mathbf{BC}_n)$ equals $\sum_{k\geqslant 0}\binom{n}{k}\binom{n+1}{k}q^k$.*

Proof. We use a result of Stembridge on trapezoidal antichains [19]. Since the poset $\Delta^+(\mathbf{BC}_n)$ is isomorphic to the trapezoidal poset $\mathcal{T}(n,n+1)$, the coefficient of q^k in the covering polynomial of $\mathfrak{Ad}(\mathbf{BC}_n)$ equals the number of k-element antichains in $\mathcal{T}(n,n+1)$.

By Theorem 5.4 in [19], the latter is the same as the number of k-element antichains in the rectangular poset $\mathcal{R}(n, n+1)$. It is easily seen that the number of k-element antichains in $\mathcal{R}(n, n+1)$ is $\binom{n}{k}\binom{n+1}{k}$. $\square$

Unlike the covering polynomial for the upper ideals in a reduced root system, this polynomial is not palindromic.

3.3 Corollary. *The number of edges of* $\mathcal{H}(\mathfrak{Ad}(\mathbf{BC}_n))$ *is equal to* $(n+1)\binom{2n}{n+1} = n\binom{2n}{n}$.

Proof. $\displaystyle \frac{d}{dq}(\sum_{k\geqslant 0}\binom{n}{k}\binom{n+1}{k}q^k)|_{q=1} = \sum_{k\geqslant 0} k\binom{n}{k}\binom{n+1}{k} = (n+1)\sum_{k\geqslant 0}\binom{n}{k-1}\binom{n}{k} = (n+1)\binom{2n}{n-1}$. $\square$

In fact, the poset $\mathfrak{Ad}(\mathbf{BC}_n)$ occurs as a particular case of the following natural series of examples. An upper ideal $I \subset \Delta^+$ is said to be *strictly positive*, if $I \cap \Pi = \varnothing$. The sub-poset of strictly positive ideals is denoted by $\mathfrak{Ad}_0$ or $\mathfrak{Ad}_0(\mathfrak{g})$. Clearly, $\mathfrak{Ad}_0$ is a distributive lattice whose poset of meet-irreducible elements is isomorphic to $\Delta^+ \setminus \Pi$.

In view of Lemma 2.2, $\mathfrak{Ad}(\mathbf{BC}_n) \simeq \mathfrak{Ad}_0(\mathbf{B}_{n+1}) \simeq \mathfrak{Ad}_0(\mathbf{C}_{n+1})$. This prompts a natural question of what is happening for the other simple Lie algebras. A uniform expression for $\#\mathfrak{Ad}_0(\mathfrak{g})$, i.e., for $\mathcal{K}_{\mathfrak{Ad}_0(\mathfrak{g})}(1)$, is found by Sommers [17]. In our recent context, we may ask what are the covering polynomial for $\mathfrak{Ad}_0(\mathfrak{g})$ and the number of edges of $\mathcal{H}(\mathfrak{Ad}_0(\mathfrak{g}))$? The answer for $\mathfrak{Ad}_0(\mathbf{B}_n)$ and $\mathfrak{Ad}_0(\mathbf{C}_n)$ follows from Lemma 2.2, Proposition 3.2, and Corollary 3.3. The case of $\mathfrak{sl}_{n+1}$ is trivial, because here $\Delta^+(\mathbf{A}_n) \setminus \Pi \simeq \Delta^+(\mathbf{A}_{n-1})$. Hence $\mathcal{K}_{\mathfrak{Ad}_0(\mathbf{A}_n)} = \mathcal{K}_{\mathfrak{Ad}(\mathbf{A}_{n-1})}$. The case of $\mathbf{G}_2$ and $\mathbf{F}_4$ is handled directly. For $\mathbf{D}_n$ and $\mathbf{E}_n$, the answer is not easy to obtain. It seems that the necessary information can be derived from results on the F-triangle associated to generalised associahedra [6, 7]. The coefficients of $\mathcal{K}_{\mathfrak{Ad}_0}(q)$ form "the positive h-vector h^+" in Chapoton's terminology. He computed these h-vectors using a conjectural relation between the F- and H-triangles. (*Warning*: h in these "h-vectors" has nothing in common with the Coxeter number h used above and below.)

The following formulae are conjectural in case of $\mathbf{D}_n$ and $\mathbf{E}_n$. For $\mathbf{D}_n$, one has

$$[q^k]\mathcal{K}_{\mathfrak{Ad}_0} = \frac{1}{n}\binom{n}{k}\Big((k+1)\binom{n-2}{k+1} + (k+2)\binom{n-2}{k} + (k-1)\binom{n-2}{k-1}\Big).$$

The information for the exceptional Lie algebras is gathered in Table 2.

Using these data and above information for classical series, we notice the following fact:

$$\frac{\#\mathcal{E}(\mathfrak{Ad}_0)}{\#(\mathfrak{Ad}_0)} = \frac{\mathcal{K}'_{\mathfrak{Ad}_0}(1)}{\mathcal{K}_{\mathfrak{Ad}_0}(1)} = \frac{n}{2}\cdot\frac{h-2}{h-1} = \frac{\#(\Delta^+ \setminus \Pi)}{h-1}\ . \tag{3.4}$$

This equality has a striking similarity with Eq. (3.1), and it would be interesting to find a conceptual explanation for it.

Δ	$\mathcal{K}_{\mathfrak{Ad}_0}(q)$
$\mathbf{E}_6$	$1+30q+135q^2+175q^3+70q^4+7q^5$
$\mathbf{E}_7$	$1+56q+420q^2+952q^3+770q^4+216q^5+16q^6$
$\mathbf{E}_8$	$1+112q+1323q^2+4774q^3+6622q^4+3696q^5+770q^6+44q^7$
$\mathbf{F}_4$	$1+20q+35q^2+10q^3$
$\mathbf{G}_2$	$1+4q$

TABLE 2. The covering polynomials for $\mathfrak{Ad}_0(\mathfrak{g})$, $\mathfrak{g}$ being exceptional

4. COVERING POLYNOMIALS FOR THE POSET OF ABELIAN IDEALS

There is an interesting subposet of $\mathfrak{Ad}$, where the two covering polynomials are different. An upper ideal $I\subset\Delta^+$ is said to be *Abelian*, if $\gamma'+\gamma''\not\in\Delta^+$ for each pair $\gamma',\gamma''\in I$. Let $\mathfrak{Ab}=\mathfrak{Ab}(\mathfrak{g})$ be the subposet of $\mathfrak{Ad}$ consisting of all Abelian ideals. Clearly $\mathfrak{Ab}$ is a graded meet-semilattice. It follows that $\mathfrak{Ab}$ is a (distributive) lattice if and only if there is a unique maximal Abelian ideal, which happens for $\mathbf{C}_n$ and $\mathbf{G}_2$ only. In all other cases the upper and lower covering polynomials are different.

Our next results rely on the connection, due to D. Peterson, between the Abelian ideals and the so-called *minuscule elements* of the affine Weyl group of $\mathfrak{g}$. Recall the necessary setup.

We have $V=\mathfrak{t}_{\mathbb{R}}=\oplus_{i=1}^n\mathbb{R}\alpha_i$ and $(\ ,\ )$ a W-invariant inner product on V. As usual, $\mu^\vee=2\mu/(\mu,\mu)$ is the coroot for $\mu\in\Delta$. Then $Q^\vee=\oplus_{i=1}^n\mathbb{Z}\alpha_i^\vee$ is the coroot lattice in V.

Letting $\widehat{V}=V\oplus\mathbb{R}\delta\oplus\mathbb{R}\lambda$, we extend the inner product $(\ ,\ )$ on $\widehat{V}$ so that $(\delta,V)=(\lambda,V)=(\delta,\delta)=(\lambda,\lambda)=0$ and $(\delta,\lambda)=1$. Then

$\widehat{\Delta}=\{\Delta+k\delta\mid k\in\mathbb{Z}\}$ is the set of affine (real) roots;
$\widehat{\Delta}^+=\Delta^+\cup\{\Delta+k\delta\mid k\geqslant 1\}$ is the set of positive affine roots;
$\widehat{\Pi}=\Pi\cup\{\alpha_0\}$ is the corresponding set of affine simple roots.

Here $\alpha_0=\delta-\theta$. For α_i $(0\leqslant i\leqslant n)$, let s_i denote the corresponding reflection in $GL(\widehat{V})$. That is, $s_i(x)=x-2(x,\alpha_i)\alpha_i^\vee$ for any $x\in\widehat{V}$. The affine Weyl group, $\widehat{W}$, is the subgroup of $GL(\widehat{V})$ generated by the reflections s_i, $i=0,1,\dots,n$. If the index of $\alpha\in\widehat{\Pi}$ is not specified, then we merely write s_α. The inner product $(\ ,\ )$ on $\widehat{V}$ is $\widehat{W}$-invariant. The notation $\beta>0$ (resp. $\beta<0$) is a shorthand for $\beta\in\widehat{\Delta}^+$ (resp. $\beta\in-\widehat{\Delta}^+$). The length function on $\widehat{W}$ with respect to $s_0,s_1,\dots,s_p$ is denoted by ℓ. For $w\in\widehat{W}$, we set $N(w)=\{\nu\in\widehat{\Delta}^+\mid w(\nu)<0\}$. Then $\#N(w)=\ell(w)$.

4.1 Definition (Peterson). An element $w\in\widehat{W}$ is said to be *minuscule*, if $N(w)=\{\delta-\gamma\mid\gamma\in I\}$ for some subset $I\subset\Delta$.

Then one can easily show that $I\subset\Delta^+$, I is an Abelian ideal, and this correspondence yields a bijection between the minuscule elements of $\widehat{W}$ and the Abelian ideals. Furthermore, if w is minuscule and $w^{-1}(\alpha)=-\mu+k\delta$ ($\alpha\in\widehat{\Pi}$, $\mu\in\Delta$), then $k\geqslant -1$. (More generally, this holds for elements of $\widehat{W}$ corresponding to arbitrary ad-nilpotent ideals, see [4]). If w is minuscule, then I_w denotes the corresponding Abelian ideal. Conversely, given $I \in \mathfrak{Ab}$, then w_I stands for the corresponding minuscule element. If $I \in \mathfrak{Ab}$ and $\gamma \in \max(\Delta^+ \setminus I)$, then the ideal $I' = I \cup \{\gamma\}$ is not necessarily Abelian. In this section, we are interested only in Abelian extensions, i.e., those with Abelian I'.

4.2 Lemma. *Suppose $w \in \widehat{W}$ is minuscule and $w^{-1}(\alpha) = -\mu + 2\delta$, where $\alpha \in \widehat{\Pi}$ and $\mu \in \Delta$. Then $\mu = \theta$.*

Proof. We have $\alpha = w(2\delta - \mu) = w(2\delta - \theta) + w(\theta - \mu)$. Here $\theta - \mu \in Q^+$. Hence both $2\delta - \theta$ and $\theta - \mu$ do not belong to $N(w)$. Therefore if $\theta \neq \mu$, then one obtains a contradiction with the fact that α is simple. □

For a minuscule w, consider the vector $\mathbf{k} = \mathbf{k}_w = (k_0, k_1, \dots, k_n)$, where k_i is defined by the equality $w^{-1}(\alpha_i) = -\mu_i + k_i\delta$ ($\mu_i \in \Delta$). Recall that $k_i \geqslant -1$ for each i.

4.3 Proposition.

(i) $k_i \leqslant 2$ *for each* i*;*

(ii) *There is at most one index i such that $k_i = 2$. The corresponding simple root α_i is necessarily long.*

(iii) $k_0 \leqslant 1$*, that is,* $k_0 \neq 2$*.*

Proof. (i) If $w^{-1}(\alpha_i) = -\mu_i + k_i\delta$ and $k_i \geqslant 3$, then $w(2\delta - \mu_i) = -(k_i - 2)\delta - \alpha_i < 0$. Hence w is not minuscule.

(ii) If $w^{-1}(\alpha_i) = -\mu_i + 2\delta$, then $\mu_i = \theta$ by Lemma 4.2. Hence such i is unique and $\|\alpha_i\| = \|\theta\|$, i.e., α_i is long.

(iii) Suppose $w^{-1}(\alpha_0) = -\mu_0 + 2\delta$. Then $\mu_0 = \theta$ and $w(2\delta - \theta) = \delta - \theta$. However, it is shown in [11, Prop. 2.5] that $w(2\delta - \theta) \in \Delta^+$ for any non-trivial minuscule element w. □

We shall say that $\mathbf{k}_w$ is the *shift vector* of w or I_w. If $w = w_I$, then we also use the notation $\mathbf{k}_I$ for this vector.

4.4 Theorem. *Let $(k_0, k_1, \dots, k_n)$ be the shift vector of $I \in \mathfrak{Ab}$. Then $\kappa(I) = \#\{i \mid k_i = -1\}$ and $\iota(I) = \#\{i \mid k_i = 1\}$.*

Proof. 1. It is shown in [11, Theorem 2.2] that $\gamma \in I$ is a generator if and only if $w_I(\delta - \gamma) = -\alpha_i \in \widehat{\Pi}$. That is, $k_i = -1$ for the corresponding coordinate i.

2. If $w^{-1}(\alpha_i) = -\mu_i + \delta$, i.e., $k_i = 1$, then one easily sees that $\mu_i \in \Delta^+$, $s_i w_I$ is again minuscule, and the corresponding Abelian ideal is $\tilde{I} = I \cup \{\mu_i\}$. Conversely, if $I \to I \cup \{\gamma\}$ is an Abelian extension of I, then $w_{\tilde{I}} = s_i w_I$ for some $i \in \{0, 1, \dots, n\}$ and $w_{\tilde{I}}(\gamma) = -\alpha_i$ [11, Theorem 2.4]. Then $w_I(\delta - \gamma) = \alpha_i$, i.e., $k_i = 1$. □

As a consequence of this theorem, one obtains a method for inductive computing the shift vector. The minuscule elements ($\sim$ Abelian ideals) can be constructed recursively.

One starts with the minuscule element $1 \in \widehat{W}$ (or the empty ideal). The corresponding shift vector is $(1, 0, \dots, 0)$. The inductive step consists in replacing $w = w_I$ with $s_i w$ for some i. However one have to be careful while choosing s_i, otherwise $s_i w$ may fail to be minuscule.

4.5 Proposition. *Suppose $w \in \widehat{W}$ is minuscule. Then $s_i w$ is again minuscule if and only if $(\mathbf{k}_w)_i = 1$. In this case,*

$$\mathbf{k}_{s_i w} = \mathbf{k}_w - (\textit{the } i\textit{-th row of the extended Cartan matrix of } \Delta).$$

Proof. The first claim is essentially proved in the second part of the above theorem. The second claim follows from the assumption $(\mathbf{k}_w)_i = 1$ and the equalities:

$$(s_i w)^{-1}(\alpha_j) = w^{-1}(\alpha_j) - (\alpha_j, \alpha_i^\vee) w^{-1}(\alpha_i); \quad j = 0, 1, \dots, n.$$

Recall that the extended Cartan matrix is the $(n+1) \times (n+1)$ matrix with the entries $c_{ji} = (\alpha_j, \alpha_i^\vee)$, $0 \leqslant i, j \leqslant n$. □

Remark. Let $\theta = \sum_{i \geqslant 1} c_i \alpha_i$. Set also $c_0 = 1$. Then $\sum_{i \geqslant 0} c_i \alpha_i = \delta$. Since δ is $\widehat{W}$-invariant, the definition of k_i's implies that $\sum_{i \geqslant 0} c_i k_i = 1$ for any shift vector. Hence $\mathbf{k}_I$ is fully determined by $k_1, \dots, k_n$. Let $z = z_I \in V$ be the unique point such that $(z_I, \alpha_i) = k_i$, $i = 1, \dots, n$. Then $z \in Q^\vee$. (Again, this is true in the context of arbitrary ad-nilpotent ideals, see [5]). Note that $k_0 = 1 - (z, \theta)$. The constraints of Proposition 4.3 show that $-1 \leqslant (z, \alpha_i) \leqslant 2$ for $i = 1, \dots, n$ and $0 \leqslant (z, \theta) \leqslant 2$. However, a stronger result is valid. It was shown by Kostant [9] that the mapping $I \in \mathfrak{Ab} \mapsto z_I \in V$ sets up a bijection between the Abelian ideals and the points $z \in Q^\vee$ such that $-1 \leqslant (z, \gamma) \leqslant 2$ for each $\gamma \in \Delta^+$.

Let Δ_l^+ denote the set of long positive roots and $\Pi_l := \Delta_l^+ \cap \Pi$. In [11], we constructed a disjoint partition of $\overset{o}{\mathfrak{Ab}} := \mathfrak{Ab} \setminus \{\varnothing\}$ parametrised by Δ_l^+. In other words, there is a natural surjective mapping $\tau : \overset{o}{\mathfrak{Ab}} \to \Delta_l^+$. Given $I \in \overset{o}{\mathfrak{Ab}}$ and the corresponding minuscule element $w \in \widehat{W}$, we set $\tau(I) = w(2\delta - \theta)$. By [11, Prop. 2.5], it is an element of Δ_l^+. Then $\mathfrak{Ab}_\mu = \tau^{-1}(\mu)$.

Remark. Using the above definition of the shift vector of an Abelian ideal and Lemma 4.2, one observes that $k_i = 2$ if and only if $I \in \mathfrak{Ab}_{\alpha_i}$.

One of the main results of [11] is that each $\mathfrak{Ab}_\mu$ has a unique maximal and unique minimal element, and that the maximal elements of $\mathfrak{Ab}$ are exactly the maximal elements of $\mathfrak{Ab}_\alpha$, $\alpha \in \Pi_l$. Let $I \in \overset{o}{\mathfrak{Ab}}$ and $\tau(I) = \mu$. By [11, Prop. 3.2], if $I \to I'$ is an extension, then $\tau(I') \preccurlyeq \tau(I)$. Hence I has an extension outside $\mathfrak{Ab}_\mu$ only if $\mu \notin \Pi$. Now we make that analysis more precise by showing that the number of possible Abelian extensions of $I \to I'$ such that $I' \notin \mathfrak{Ab}_\mu$ depend only on μ and not on I.

4.6 Theorem. *For any $\mu \in \Delta_l^+ \setminus \Pi$ and any $I \in \mathfrak{Ab}_\mu$, the number of Abelian extensions $I \to I'$ such that $I' \notin \mathfrak{Ab}_\mu$ equals the number of $\alpha \in \Pi$ such that $(\mu, \alpha) > 0$. In particular, this number does not depend on I, and if Δ is simply laced, then it is equal to $\kappa(\mu)$.*

Proof. 1. Suppose $(\alpha, \mu) > 0$ and $\mu' := s_\alpha(\mu) = \mu - n_\alpha \alpha$. Here $n_\alpha \in \mathbb{N}$ and $n_\alpha = 1$ if and only if α is long. Anyway, $\mu'' := \mu - \alpha$ is again a root, and we will work with the sum $\mu = \alpha + \mu''$. Then $w_I^{-1}(\mu'' + \alpha) = 2\delta - \theta$ and, as shown in the proof of Theorem 2.6 in [11], one has $w_I^{-1}(\mu'') = \delta - \gamma''$ and $w_I^{-1}(\alpha) = \delta - \gamma'$ for some γ', γ'' such that $\gamma' + \gamma'' = \theta$. This shows that $w = s_\alpha w_I$ is minuscule, $I_w = I \cup \{\gamma'\}$, and $\tau(I_w) = s_\alpha w_I(2\delta - \theta) = s_\alpha(\mu) = \mu'$.

2. Conversely, suppose $I \in \mathfrak{Ab}_\mu$ and $I \to I'$ is an extension. Then $I' = I \cup \{\gamma\}$ for some $\gamma \in \Delta^+$ and $w_{I'} = s_\alpha w_I$, where $\alpha \in \Pi$ is determined by the equality $w_I(\delta - \gamma) = \alpha$. The condition $\tau(I') \neq \mu$ means $s_\alpha(\mu) \neq \mu$, i.e., $(\alpha, \mu) \neq 0$. To compute the sign, we notice that $(\alpha, \mu) = (w_I^{-1}(\alpha), w_I^{-1}(\mu)) = (\delta - \gamma, 2\delta - \theta) = (\gamma, \theta)$, which cannot be negative. □

4.7 Corollary. *The ideals having a unique Abelian extension are the following:*

(a) $\varnothing$*;*

(b) The minimal elements of posets $\mathfrak{Ab}_\mu$*, where* $\mu \notin \Pi$ *and the inequality* $(\alpha, \mu) > 0$ *holds for a unique* $\alpha \in \Pi$*;*

(c) The ideals having a unique Abelian extension inside $\mathfrak{Ab}_\alpha$*,* $\alpha \in \Pi_l$*.*

It was noticed in [11] that each $\mathfrak{Ab}_\mu$ is a *minuscule poset*, i.e., there is a simple Lie algebra $\mathfrak{l}$ and a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{l}$ with Abelian nilpotent radical $\mathfrak{p}^u$ such that $\mathfrak{Ab}_\mu$ is isomorphic to the poset of Abelian $\mathfrak{p}$-ideals in $\mathfrak{p}^u$. The construction of $\mathfrak{l}$ as a subalgebra of $\mathfrak{g}$ is given in [11, Section 5]. Since the structure of the minuscule posets is well known, Theorem 4.6 provides an effective tool for computing the lower covering polynomial in the small rank cases, e.g. for the exceptional Lie algebras.

We say that $\gamma \in \Delta^+$ is *commutative*, if the upper ideal generated by γ is Abelian. Clearly, the set of commutative roots forms an upper ideal. A uniform description of this ideal (and its cardinality) is given in [13, Theorem 4.4]. In particular, if the Dynkin diagram has no branching node then the number of commutative roots is $n(n+1)/2$.

In the following assertion, we gather some information that is helpful in practical computations of the covering polynomials.

4.8 Proposition.

(i) $[q^0]\hat{\mathcal{K}}_{\mathfrak{Ab}} = 1$, $[q^0]\check{\mathcal{K}}_{\mathfrak{Ab}} = \#(\Pi_l)$;

(ii) $[q]\hat{\mathcal{K}}_{\mathfrak{Ab}} =$ *the number of commutative roots;*

(iii) $\deg \hat{\mathcal{K}}_{\mathfrak{Ab}} \leqslant \deg \check{\mathcal{K}}_{\mathfrak{Ab}}$. *If these degrees are equal, say to* m*, then* $[q^m]\hat{\mathcal{K}}_{\mathfrak{Ab}} \leqslant [q^m]\check{\mathcal{K}}_{\mathfrak{Ab}}$*;*

(iv) $\hat{\mathcal{K}}_{\mathfrak{Ab}}(1) = \check{\mathcal{K}}_{\mathfrak{Ab}}(1) = 2^n$*;*

(v) $\hat{\mathcal{K}}'_{\mathfrak{Ab}}(1) = \check{\mathcal{K}}'_{\mathfrak{Ab}}(1) = (n+1)2^{n-2}$.

Proof. (i) These are the numbers of minimal and maximal elements of $\mathfrak{Ab}$.
(ii) Obvious.
(iii) Let I be an Abelian ideal with m generators, say $\{\gamma_1, \dots, \gamma_m\}$. Then $I \setminus \{\gamma_1, \dots, \gamma_m\}$ has at least m extensions. Then take $m = \deg \hat{\mathcal{K}}_{\mathfrak{Ab}}$.
(iv) This is Peterson's theorem on $\#(\mathfrak{Ab})$, see e.g. [4, Theorem 2.9];
(v) This is the number of edges of $\mathcal{H}(\mathfrak{Ab})$, which is computed in [13, Theorem 4.1]. □

Let m be the maximal size of an antichain of commutative elements. Then $\deg \check{\mathcal{K}}_{\mathfrak{Ab}} \leqslant m$. But this bound may not be sharp, since the ideal having a prescribed antichain as the set of either generators or minimal elements in the complement can be non-Abelian. However, in case $\mathbf{E}_7$ and $\mathbf{E}_8$ this bound does give the precise value for both degrees, and we obtain $\deg \mathcal{K}_{\mathfrak{Ab}(\mathbf{E}_7)} = \deg \mathcal{K}_{\mathfrak{Ab}(\mathbf{E}_8)} = 4$, where $\mathcal{K}$ is either $\hat{\mathcal{K}}$ or $\check{\mathcal{K}}$.

Also, it follows from a result of Sommers [17, Theorem 6.4] that the number of generators of an Abelian ideal is at most the maximal number of mutually orthogonal roots in Π. This provides an upper bound for $\deg \hat{\mathcal{K}}_{\mathfrak{Ab}}$. Since it is easy to find an Abelian ideal with such a number of generators, one actually obtains the exact value of the degree. For instance, $\deg \hat{\mathcal{K}}_{\mathfrak{Ab}(\mathbf{D}_n)} = \left[\frac{n}{2}\right] + 1$ and $\deg \hat{\mathcal{K}}_{\mathfrak{Ab}(\mathbf{E}_6)} = 3$. If $\deg \check{\mathcal{K}}_{\mathfrak{Ab}} \leqslant 3$, then both covering polynomials can be computed using Proposition 4.8. This applies to $\mathbf{F}_4$ and $\mathbf{E}_6$. For $\mathbf{E}_7$ and $\mathbf{E}_8$, it suffices to determine one more value (or coefficient) of $\hat{\mathcal{K}}_{\mathfrak{Ab}}$ and $\check{\mathcal{K}}_{\mathfrak{Ab}}$.

4.9 Example. $\mathfrak{g} = \mathbf{E}_8$.

(1) We use Corollary 4.7 to compute the coefficient $[q]\check{\mathcal{K}}_{\mathfrak{Ab}(\mathbf{E}_8)}$. Here the number of roots μ such that $\kappa(\mu) = 1$ is 21 (see Section 2). The posets $\mathfrak{Ab}_{\alpha_i}$, $\alpha_i \in \Pi$, have the cardinalities 1,2,3,4,5,6,8,6. Furthermore, for $\mathbf{E}_8$, each poset $\mathfrak{Ab}_\mu$ is totally ordered. Hence the contribution from part (c) of the corollary is $0+1+2+3+4+5+7+5 = 27$. Thus, $[q]\check{\mathcal{K}}_{\mathfrak{Ab}(\mathbf{E}_8)} = 1 + 21 + 27 = 49$.

(2) Since each $\mathfrak{Ab}_\mu$ is a chain, any ideal I has at most one extension inside its own poset $\mathfrak{Ab}_\mu$. Because $\kappa(\mu) \leqslant 3$ for all $\mu \in \Delta^+$, Theorem 4.6 shows that

$$[q^4]\check{\mathcal{K}}_{\mathfrak{Ab}(\mathbf{E}_8)} = \sum_{\kappa(\mu)=3} \#(\mathfrak{Ab}_\mu) - 1 \ .$$

We have $\#\{\mu \in \Delta^+ \mid \kappa(\mu) = 3\} = [q^3]\hat{\mathcal{K}}_{\Delta^+(\mathbf{E}_8)} = 21$. Of these 21 roots, there are

11 roots with $\#(\mathfrak{Ab}_\mu) = 1$ (these are exactly the roots μ with $(\theta, \mu) \neq 0$, see [11, 5.1];

5 roots with $\#(\mathfrak{Ab}_\mu) = 2$;

3 roots with $\#(\mathfrak{Ab}_\mu) = 3$;

2 roots with $\#(\mathfrak{Ab}_\mu) = 4$.

Hence $[q^4]\check{\mathcal{K}}_{\mathfrak{Ab}(\mathbf{E}_8)} = 17$.

(3) Then using Proposition 4.8(iv),(v), we compute $\check{\mathcal{K}}_{\mathfrak{Ab}(\mathbf{E}_8)}(q) = 8 + 49q + 87q^2 + 95q^3 + 17q^4$.

5. Computing the covering polynomials for $\mathfrak{Ab}(\mathfrak{g})$, $\mathfrak{g}$ being classical

In this section, we prove 4 theorems for all classical series. All proofs are based on an explicit presentation of the upper ideal of commutative roots and understanding which ideals inside it are really Abelian. To this end, one has to know the generators of all maximal Abelian ideals, which are determined in [16] (see also [14, Table 1]).

5.1 Theorem. *If* $\mathfrak{g} = \mathfrak{sl}_{n+1}$*, then*

(i) $\hat{\mathcal{K}}_{\mathfrak{Ab}}(q) = \sum_{k\geqslant 0}\binom{n+1}{2k}q^k$;

(ii) $\check{\mathcal{K}}_{\mathfrak{Ab}}(q) = \sum_{k\geqslant 0}\left(\binom{n}{2k+1}+\binom{n}{2k-2}\right)q^k$;

(iii) $\mathcal{D}_{\mathfrak{Ab}}(q) = -\sum_{k\geqslant 0}\binom{n-1}{2k+1}q^k$.

Proof. The formula for $\hat{\mathcal{K}}$ (but not for $\check{\mathcal{K}}$!) is implicit in [14, Section 3]. We recall the necessary setup and then deduce the expressions (i) and (ii). Then part (iii) is obtained via formal manipulations.

For $\mathfrak{g} = \mathfrak{sl}_{n+1}$, each positive root is commutative. The root $\varepsilon_i - \varepsilon_j \in \Delta^+(\mathfrak{sl}_{n+1})$ is identified with the pair (i,j). Suppose $\mathfrak{a} \in \mathfrak{Ab}(\mathfrak{sl}_{n+1})$ and $\kappa(\mathfrak{a}) = k$. That is, $\mathfrak{a}$ has k generators (minimal roots). If $\min(\mathfrak{a}) = \{(a_1,b_1),\dots,(a_k,b_k)\}$, where $a_1 < a_2 < \dots < a_k$, then we actually have $1 \leqslant a_1 < a_2 < \dots < a_k < b_1 < \dots < b_k \leqslant n+1$. Thus, any $2k$-element subset of $[1, n+1]$ gives rise to an Abelian ideal with $\kappa(\mathfrak{a}) = k$ and vice versa. This yields (i).

The edges of $\mathcal{H}(\mathfrak{Ab})$ originating in $\mathfrak{a}$ bijectively correspond to the maximal roots γ in $\Delta^+ \setminus \mathfrak{a}$ such that $\{\gamma\} \cup \mathfrak{a}$ is again an Abelian ideal. The set of these maximal roots always contains $\{(a_1+1, b_2-1), \dots, (a_{k-1}+1, b_k-1)\}$; furthermore, if $a_k+1 < b_1$, then two more roots are required: $(1, b_1-1), (a_k+1, n+1)$. From this we deduce that $\iota(\mathfrak{a}) = k$ if and only if one of the following two conditions hold:

($\diamondsuit_1$) $\#\min(\mathfrak{a}) = k+1$ and $a_{k+1}+1 = b_1$.

($\diamondsuit_2$) $\#\min(\mathfrak{a}) = k-1$ and $a_{k-1}+1 < b_1$.

In case ($\diamondsuit_1$) the ideal is determined by a sequence of $2k+1$ integers

$1 \leqslant a_1 < \dots < a_k < a_{k+1} = b_1 - 1 < b_2 - 1 < \dots < b_{k+1} - 1 \leqslant n.$

Hence there are $\binom{n}{2k+1}$ such possibilities.

In case ($\diamondsuit_2$) the ideal is determined by a sequence of $2k-2$ integers

$1 \leqslant a_1 < \dots < a_{k-1} < b_1 - 1 < b_2 - 1 < \dots < b_{k-1} - 1 \leqslant n.$

Hence there are $\binom{n}{2k-2}$ such possibilities. This proves (ii).

(iii) It follows from parts (i) and (ii) that

$$\hat{\mathcal{K}}_{\mathfrak{Ab}}(q) - \check{\mathcal{K}}_{\mathfrak{Ab}}(q) = -\sum_{k\geqslant 0}\left(\binom{n}{2k+1} - \binom{n+1}{2k} + \binom{n}{2k-2}\right)q^k = -\sum_{k\geqslant 0}\left(\binom{n-1}{2k+1} - 2\binom{n-1}{2k-1} + \binom{n-1}{2k-3}\right)q^k = -(q-1)^2\sum_{k\geqslant 0}\binom{n-1}{2k+1}q^k. \quad \square$$

5.2 Theorem. *If* $\mathfrak{g} = \mathfrak{sp}_{2n}$*, then* $\hat{\mathcal{K}}_{\mathfrak{Ab}}(q) = \check{\mathcal{K}}_{\mathfrak{Ab}}(q) = \sum_{k\geqslant 0}\binom{n+1}{2k}q^k$.

Proof. The commutative roots are $\{\varepsilon_i + \varepsilon_j \mid 1 \leqslant i \leqslant j \leqslant n\}$. Since these roots form the unique maximal Abelian ideal, the poset $\mathfrak{Ab}(\mathfrak{sp}_{2n})$ is a distributive lattice whose subset of meet-irreducibles is isomorphic to the set of commutative roots. Hence $\hat{\mathcal{K}}_{\mathfrak{Ab}}(q) = \check{\mathcal{K}}_{\mathfrak{Ab}}(q)$. The explicit form of this polynomials stems from the description given in [14, Section 3]: the ideals with k generators (or with k extensions) are in a bijection with the the sequences $1 \leqslant a_1 < a_2 < \dots < a_k \leqslant b_1 < \dots < b_k \leqslant n$. Hence there are $\binom{n+1}{2k}$ possibilities for them. $\square$

The poset of commutative roots for $\mathfrak{sp}_{2n}$ is represented by the triangular Ferrers diagram with row lengths $(n, n-1, \dots, 1)$ (= triangle 'of size n'). In the following two theorems, a subposet of the poset of commutative roots for $\mathfrak{so}_{2n+1}$ and $\mathfrak{so}_{2n}$ appears to be such a triangle, which allows us to exploit the formula of Theorem 5.2.

5.3 Theorem. *If* $\mathfrak{g} = \mathfrak{so}_{2n+1}$, *then*

(i) $\hat{\mathcal{K}}_{\mathfrak{Ab}}(q) = \sum_{k\geqslant 0} \binom{n+1}{2k} q^k$;
(ii) $\check{\mathcal{K}}_{\mathfrak{Ab}}(q) = \sum_{k\geqslant 0} \big(\binom{n-1}{2k+1} + \binom{n}{2k-1} + \binom{n-1}{2k-2}\big) q^k$;
(iii) $\mathcal{D}_{\mathfrak{Ab}}(q) = -\sum_{k\geqslant 0} \binom{n-2}{2k+1} q^k$.

Proof. Here the commutative roots are $\{\varepsilon_i + \varepsilon_j \mid 1 \leqslant i < j \leqslant n\} \cup \{\varepsilon_1 - \varepsilon_i \mid 2 \leqslant i \leqslant n\} \cup \{\varepsilon_1\}$.
Graphically, this set is represented by the skew Ferrers diagram with row lengths $(2n-1, n-2, n-3, \dots, 1)$, see the sample Figure for $\mathfrak{so}_{13}$, where the leftmost (resp. rightmost) box is $\alpha_1 = \varepsilon_1 - \varepsilon_2$ (resp. $\theta = \varepsilon_1 + \varepsilon_2$). The central box in the upper row is ε_1, and lowest box is $\varepsilon_{n-1} + \varepsilon_n$.

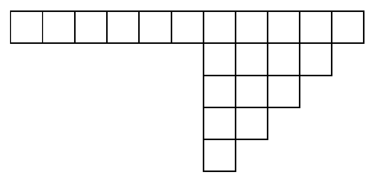


FIGURE 1. The (po)set of commutative roots for $\mathfrak{so}_{13}$

We have used the convention that the largest element appears in the northeast corner and the smaller elements appear to the south and west. This Ferrers diagram consists of the tail of length n and the triangle 'of size $n-1$'. The triangle itself represents an Abelian ideal, and the structure of the set of ideals sitting inside this triangle is the same as for $\mathfrak{sp}_{2n-2}$.
(i) We wish to compute the number of Abelian ideals with k generators.

- The number of such ideals inside the triangle is equal to $\binom{n}{2k}$.
- Suppose that $\mathfrak{a}$ has the tail of length m, $m \geqslant 1$, i.e., $\mathfrak{a}$ has the generator $\varepsilon_1 - \varepsilon_{n+2-m}$ in the upper row. Then the rest of this row (to the right) is also in the ideal, and the ideal is determined by its part lying in the triangle of size $n-2$, in the rows from 2 to $n-1$. The condition of being Abelian means that $\mathfrak{a}$ cannot have elements from $m-1$ leftmost columns of the triangle. (Formally: if $\varepsilon_1 - \varepsilon_{n+2-m} \in \mathfrak{a}$, then for all other roots $\varepsilon_i + \varepsilon_j \in \mathfrak{a}$, $2 \leqslant i \leqslant j$, we must have $j \leqslant n+1-m$.) Hence as a degree of freedom for further constructing $\mathfrak{a}$ we have a triangle of size $n-1-m$, where an ideal with $k-1$ generators have to be chosen. The symplectic case shows that the number of such possibilities equals $\binom{n-m}{2k-2}$.

Thus, the total number of Abelian ideals with k generators equals

$$\binom{n}{2k} + \sum_{m\geqslant 1}\binom{n-m}{2k-2} = \binom{n}{2k} + \binom{n}{2k-1} = \binom{n+1}{2k}.$$

(ii) We wish to compute the number of all Abelian ideals with k extensions. Here the argument is similar in the spirit, but more tedious.

• Suppose $\mathfrak{a}$ lies in the triangle. The difficulty here is that $\mathfrak{a}$ may have an extension that does not fit in the triangle (namely, if the length of the first row equals $n-1$). Therefore the symplectic formula does not immediately apply.

Let $p \leqslant n-1$ be the length of the first row of $\mathfrak{a}$. Then $\mathfrak{a}$ certainly have the extension in the first row. The rest of $\mathfrak{a}$ (in the second row and below) sits in the triangle of size $p-1$ and must have $k-1$ extensions. By the symplectic formula, the number of possibilities here is $\binom{p}{2k-2}$. Hence, the total number of possibilities for the ideals inside the triangle equals $\sum_{p\leqslant n-1}\binom{p}{2k-2} = \binom{n}{2k-1}$.

• Suppose an Abelian ideal $\mathfrak{a}$ has the tail of length m, $m \geqslant 1$. Let s be the length of the second row of $\mathfrak{a}$. Then, as explained in the proof of part (i), $s \leqslant n-m-1$. Here one has to distinguish two cases.

(1) If $s = n-m-1$, then $\mathfrak{a}$ has no extensions in the first two rows. Hence all k extensions must occur in row no. 3 and below. This part of $\mathfrak{a}$ sits in the triangle of size $n-m-2$. Therefore one has $\binom{n-m-1}{2k}$ possibilities for constructing an ideal.

(2) If $s < n-m-1$, then $\mathfrak{a}$ has extensions in both the first and second row. Hence the lower part of $\mathfrak{a}$, in row no. 3 and below, must have $k-2$ extensions. Since this lower part sits inside the triangle of size $s-1$, one has $\binom{s}{2k-4}$ possibilities. Altogether, we obtain $\sum_{s\leqslant n-m-2}\binom{s}{2k-4} = \binom{n-m-1}{2k-3}$ variants.

Thus, the total number of Abelian ideals with k extensions equals

$$\binom{n}{2k-1} + \sum_{m\geqslant 1}\binom{n-m-1}{2k} + \sum_{m\geqslant 1}\binom{n-m-1}{2k-3} = \binom{n}{2k-1} + \binom{n-1}{2k+1} + \binom{n-1}{2k-2}.$$

(iii) This follows from (i) and (ii) via a straightforward calculation. □

5.4 Theorem. *If* $\mathfrak{g} = \mathfrak{so}_{2n}$, *then*

(i) $\hat{\mathcal{K}}_{\mathfrak{Ab}}(q) = \sum_{k\geqslant 0}\left(\binom{n+2}{2k} - 4\binom{n-1}{2k-2}\right)q^k = \sum_{k\geqslant 0}\left(\binom{n}{2k} + \binom{n-1}{2k-1} + \binom{n-2}{2k-1} + \binom{n-2}{2k-4}\right)q^k$;

(ii) $\check{\mathcal{K}}_{\mathfrak{Ab}}(q) = \sum_{k\geqslant 0}\left(\binom{n}{2k+1} + \binom{n}{2k-2}\right)q^k$;

(iii) $\mathcal{D}_{\mathfrak{Ab}}(q) = -\sum_{k\geqslant 0}\left(\binom{n-2}{2k+1} + \binom{n-3}{2k}\right)q^k$.

Proof. Here the commutative roots are

$\{\varepsilon_i+\varepsilon_j \mid 1 \leqslant i < j \leqslant n\} \cup \{\varepsilon_1-\varepsilon_i \mid 2 \leqslant i \leqslant n\} \cup \{\varepsilon_i-\varepsilon_n \mid 2 \leqslant i \leqslant n-1\}$. Graphically, this set is represented by the skew Ferrers diagram with row lengths $(2n-2, n-1, n-2, \dots, 2)$, see the sample Figure for $\mathfrak{so}_{12}$, where the leftmost (resp. rightmost) box is $\alpha_1 = \varepsilon_1 - \varepsilon_2$ (resp. $\theta = \varepsilon_1+\varepsilon_2$). The two boxes in the lowest row are $\alpha_{i-1} = \varepsilon_{n-1}-\varepsilon_n$ and $\alpha_i = \varepsilon_{n-1}+\varepsilon_n$, respectively. The roots $\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_1 - \varepsilon_{n-1}$ form the tail of this diagram.

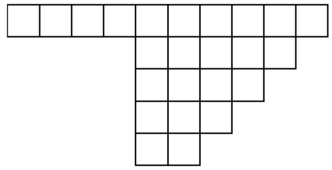


FIGURE 2. The (po)set of commutative roots for $\mathfrak{so}_{12}$

We follow here the same convention as in the proof of Theorem 5.3. However, the notable distinction of this diagram from Figure 1 is that here the roots in the two central columns are incomparable.

(i) We wish to compute the number of Abelian ideals with k generators.

$(\heartsuit_1)$ Suppose that $\mathfrak{a}$ has the tail of length m, $m \geqslant 1$, i.e., $\mathfrak{a}$ has the generator $\varepsilon_1 - \varepsilon_{n-m}$ in the upper row. Then the rest of this row (to the right) is also in the ideal, and the ideal is determined by its part lying in rows from 2 to $n-1$. The condition of being Abelian means that $\mathfrak{a}$ cannot have elements from $m+2$ leftmost columns in this lower part. (Formally: if $\varepsilon_1 - \varepsilon_{n-m} \in \mathfrak{a}$, then for all other roots $\varepsilon_i + \varepsilon_j \in \mathfrak{a}$, $2 \leqslant i \leqslant j$, we must have $j \leqslant n-1-m$.) Hence as a degree of freedom for further constructing $\mathfrak{a}$ we obtain a triangle of size $n-3-m$, where an ideal with $k-1$ generators have to be chosen. The symplectic case shows that the number of possibilities equals $\binom{n-m-2}{2k-2}$. Thus, in this case we have $\sum_{m\geqslant 1}\binom{n-m-2}{2k-2} = \binom{n-2}{2k-1}$ possibilities.

$(\heartsuit_2)$ Suppose that an ideal $\mathfrak{a}$ has no tail, i.e., $\varepsilon_1 - \varepsilon_{n-1} \notin \mathfrak{a}$. Consider all the relevant variants.

1. $\varepsilon_1 - \varepsilon_n, \varepsilon_1 + \varepsilon_n \in \mathfrak{a}$. These two roots are generators of $\mathfrak{a}$, so that we have to choose an ideal with $k-2$ generators in the triangle of size $n-3$. This yields $\binom{n-2}{2k-4}$ possibilities.
2. $\varepsilon_1 - \varepsilon_n \notin \mathfrak{a}$. Then have to choose an ideal with k generators in the triangle of size $n-1$. This yields $\binom{n}{2k}$ possibilities.
3. $\varepsilon_1 + \varepsilon_n \notin \mathfrak{a}$. This part is the same as previous one, and we obtain $\binom{n}{2k}$ possibilities.
4. In items (2) and (3), we have counted twice the ideals that contain neither $\varepsilon_1 - \varepsilon_n$ nor $\varepsilon_1 + \varepsilon_n$, i.e., the ideals with k generators that fit in the triangle of size $n-2$. Therefore $\binom{n-1}{2k}$ must be subtracted.

Thus, if $\mathfrak{a}$ has no tail, one obtains the sum $\binom{n-2}{2k-4} + \binom{n}{2k} + \binom{n-1}{2k-1}$.
Combining $(\heartsuit_1)$ and $(\heartsuit_2)$ yields the coefficients of q^k presented as the sum of four summands. It is a good exercise to transform this sum into the second expression in the formulation.

(ii) Counting the ideals with k extensions is even more tedious. Our method yields here 10 terms, which sum luckily up to the two summands in the formulation. We only list all the possibilities for the Ferrers diagram and the corresponding number of ideals:

- the length of the first row is $\leqslant n-3$ — $\binom{n-2}{2k-1}$
- the length of the first row is $n-2$ — $\binom{n-2}{2k-4}$
- $\varepsilon_1-\varepsilon_n \notin \mathfrak{a}$, $\varepsilon_1+\varepsilon_n \in \mathfrak{a}$ — $\binom{n-2}{2k}+\binom{n-2}{2k-3}$
- $\varepsilon_1-\varepsilon_n \in \mathfrak{a}$, $\varepsilon_1+\varepsilon_n \notin \mathfrak{a}$ — $\binom{n-2}{2k}+\binom{n-2}{2k-3}$
- $\varepsilon_1 \pm \varepsilon_n \in \mathfrak{a}$ — $\binom{n-3}{2k}+\binom{n-3}{2k-3}$
- The ideal has the tail (of length $\geqslant 1$) — $\binom{n-3}{2k+1}+\binom{n-3}{2k-2}$.

(iii) This follows from (i) and (ii) via a straightforward calculation. □

Finally, we present the table with complete information about the covering and deviation polynomials for $\mathfrak{Ab}(\mathfrak{g})$.

$\mathfrak{g}$	$\hat{\mathcal{K}}_{\mathfrak{Ab}(\mathfrak{g})}$	$\check{\mathcal{K}}_{\mathfrak{Ab}(\mathfrak{g})}$	$-\mathcal{D}_{\mathfrak{Ab}(\mathfrak{g})}$
$\mathbf{A}_n$	$\sum_{k\geqslant 0}\binom{n+1}{2k}q^k$	$\sum_{k\geqslant 0}\left(\binom{n}{2k+1}+\binom{n}{2k-2}\right)q^k$	$\sum_{k\geqslant 0}\binom{n-1}{2k+1}q^k$
$\mathbf{B}_n$	$\sum_{k\geqslant 0}\binom{n+1}{2k}q^k$	$\sum_{k\geqslant 0}\left(\binom{n-1}{2k+1}+\binom{n}{2k-1}+\binom{n-1}{2k-2}\right)q^k$	$\sum_{k\geqslant 0}\binom{n-2}{2k+1}q^k$
$\mathbf{C}_n$	$\sum_{k\geqslant 0}\binom{n+1}{2k}q^k$	$\sum_{k\geqslant 0}\binom{n+1}{2k}q^k$	0
$\mathbf{D}_n$	$\sum_{k\geqslant 0}\left(\binom{n+2}{2k}-4\binom{n-1}{2k-2}\right)q^k$	$\sum_{k\geqslant 0}\left(\binom{n}{2k+1}+\binom{n}{2k-2}\right)q^k$	$\sum_{k\geqslant 0}\left(\binom{n-2}{2k+1}+\binom{n-3}{2k}\right)q^k$
$\mathbf{E}_6$	$1+25q+27q^2+11q^3$	$6+21q+20q^2+17q^3$	$5+6q$
$\mathbf{E}_7$	$1+34q+60q^2+30q^3+3q^4$	$7+35q+40q^2+43q^3+3q^4$	$6+13q$
$\mathbf{E}_8$	$1+44q+118q^2+76q^3+17q^4$	$8+49q+87q^2+95q^3+17q^4$	$7+19q$
$\mathbf{F}_4$	$1+10q+5q^2$	$2+8q+6q^2$	1
$\mathbf{G}_2$	$1+3q$	$1+3q$	0

TABLE 3. The covering and deviation polynomials for $\mathfrak{Ab}(\mathfrak{g})$

Some observations related to Table 3

1. For $\mathbf{A}_{2n}$, we have $\deg\check{\mathcal{K}}_{\mathfrak{Ab}}-\deg\hat{\mathcal{K}}_{\mathfrak{Ab}}=1$. For all other cases the degrees are equal.

2. One may observe that there are several regularities in Table 3. For all classical series, both the covering polynomials satisfy the recurrence relation

$$\mathcal{K}_{\mathfrak{Ab}(\mathbf{X}_n)}(q)=2\mathcal{K}_{\mathfrak{Ab}(\mathbf{X}_{n-1})}(q)+(q-1)\mathcal{K}_{\mathfrak{Ab}(\mathbf{X}_{n-2})}(q)\ , \tag{5.5}$$

where $\mathbf{X}\in\{\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\}$ and $\mathcal{K}$ is either $\hat{\mathcal{K}}$ or $\check{\mathcal{K}}$. Furthermore, the sequence $\mathbf{E}_3=\mathbf{A}_2\times\mathbf{A}_1$, $\mathbf{E}_4=\mathbf{A}_4$, $\mathbf{E}_5=\mathbf{D}_5$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$ can be regarded as the 'exceptional' series, and for this series the same recurrence relation is satisfied for $\hat{\mathcal{K}}$ (but not for $\check{\mathcal{K}}$). It follows from Eq. 5.5 that $\mathcal{K}_{\mathfrak{Ab}(\mathbf{X}_n)}(1)=2\mathcal{K}_{\mathfrak{Ab}(\mathbf{X}_{n-1})}(1)$, which "explains" the equality $\#\mathfrak{Ab}(\mathbf{X}_n)=2^n$.

3. The upper covering polynomials for $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$ are the same. But the lower covering polynomial distinguishs these series. Furthermore, if the Dynkin diagram has no branching nodes, then $\hat{\mathcal{K}}_{\mathfrak{Ab}}(\mathfrak{g})$ depends only on n. That is, the upper covering polynomial for $\mathbf{F}_4$ (resp. $\mathbf{G}_2$) is equal to that for $\mathbf{A}_4$ (resp. $\mathbf{A}_2$).

On the other hand, the lower covering polynomials are equal for $\mathbf{A}_n$ and $\mathbf{D}_n$, and the deviation polynomial for $\mathbf{B}_n$ is equal to that for $\mathbf{A}_{n-1}$.

It would be interesting to find an explanation of these coincidences.

Remark. For the non-reduced root system $\mathbf{BC}_n$, one can also consider combinatorial Abelian ideals. However, these are exactly the same as in the symplectic case.

6. Some speculations

1^o. Is there a combinatorial interpretation of $\hat{\mathcal{K}}_{\mathcal{P}}(-1)$ and $\check{\mathcal{K}}_{\mathcal{P}}(-1)$? (At least, if $\mathcal{P}$ is a modular or distributive lattice.)

2^o. Various examples considered in the paper show that in many cases the deviation polynomial of a poset has the nonzero coefficients of the same sign. It would interesting to find a general pattern for this phenomenon. Of course, it is not always the case. For instance, if $\mathcal{D}_{\mathcal{P}_1}$ (resp. $\mathcal{D}_{\mathcal{P}_2}$) has positive (resp. negative) coefficients, then Lemma 1.2(ii) shows that the deviation polynomial of $\mathcal{P}_1 \times \mathcal{P}_2$ may have coefficients of both signs. It is not hard to produce a concrete example. However, I conjecture that the following is true:

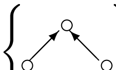
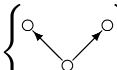
Suppose $\mathcal{P} = J(\mathcal{L})$ and $\mathcal{P}(\leqslant m)$ is the subposet of ideals of cardinality at most m. Then $\mathcal{D}_{\mathcal{P}(\leqslant m)}$ has non-positive coefficients.

In the special case of $J(\mathcal{L}) \setminus \{\hat{1}\}$ this is verified in Section 1.

3^o. From the definition of $\mathcal{D}_{\mathcal{P}}$, it follows that $\mathcal{D}_{\mathcal{P}}(1) = \frac{1}{2}(\hat{\mathcal{K}}''_{\mathcal{P}}(1) - \check{\mathcal{K}}''_{\mathcal{P}}(1))$. Therefore

$$\mathcal{D}_{\mathcal{P}}(1) = \frac{\#\{(x, y_1, y_2) \in \mathcal{P}^3 \mid y_1 \to x,\ y_2 \to x\} - \#\{(x_1, x_2, y) \in \mathcal{P}^3 \mid y \to x_1,\ y \to x_2\}}{2},$$

where $y_1 \neq y_2$ and $x_1 \neq x_2$. In other words,

$2\mathcal{D}_{\mathcal{P}}(1) = \#\Big\{$  $\Big\} - \#\Big\{$  $\Big\}$,

the difference between the number of two types of configurations in $\mathcal{H}(\mathcal{P})$. These configurations are said to be $\wedge$-triples and $\vee$-triples, respectively. Using this interpretation, one obtains the following result.

6.1 Proposition. *Suppose $\tilde{\mathcal{P}}$ is a distributive lattice and $\mathcal{P} \subset \tilde{\mathcal{P}}$ a subposet such that if $I \in \mathcal{P}$ and $I' \preccurlyeq I$ ($I' \in \tilde{\mathcal{P}}$), then $I' \in \mathcal{P}$. Then $\mathcal{D}_{\mathcal{P}}(1) \leqslant 0$. Furthermore, $\mathcal{D}_{\mathcal{P}}(1) = 0$ if and only if $\mathcal{P}$ is a distributive lattice if and only if $\mathcal{D}_{\mathcal{P}} \equiv 0$.*

Proof. Here each $\wedge$-triple can be completed to a *diamond* inside $\mathcal{P}$, i.e., the configuration of the form '$\diamond$'. This provides an injection of the set of $\wedge$-triples to the set of $\vee$-triples. If this is a bijection, i.e., each $\vee$-triple can be included in a diamond, then $\mathcal{P}$ has a unique maximal element. Hence $\mathcal{P}$ is a distributive lattice and $\mathcal{D}_{\mathcal{P}} = 0$. $\square$

4^o. Using Table 3, one can compute the values $-\mathcal{D}_{\mathfrak{Ab}}(1)$. In the serial cases, these values are quite simple:

2^{n-2} for $\mathbf{A}_n$, $n \geqslant 2$; 2^{n-3} for $\mathbf{B}_n$, $n \geqslant 3$; 0 for $\mathbf{C}_n$, $n \geqslant 1$; $2^{n-3} + 2^{n-4}$ for $\mathbf{D}_n$, $n \geqslant 4$.
Hopefully, there could be a uniform general description for them. One may notice that if the Dynkin diagram has no branching nodes, then this value equals 2^m with $m = \#(\Pi_l) - 2$; or 0, if $\#(\Pi_l) = 1$. (The case of $\mathbf{G}_2$ and $\mathbf{F}_4$ is included here.) But I have no idea how to explain the values for series $\mathbf{D}$ and $\mathbf{E}$.

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