Explicit Formulas for the Fourier Coefficients of Jacobi and Elliptic Modular Forms

by

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1. Introduction and Discussion. Fix a positive integer m and denote by $S_{2,m}$ and $S^*_{2,m}$ the space of holomorphic and skew-holomorphic Jacobi cusp forms of weight 2 and index m respectively. By definition these are the spaces of smooth functions $\phi(\tau,z)$ in two variables $\tau, z \in \mathbf{C}$, $\operatorname{Im}\tau > 0$, which are periodic in τ and z respectively with period 1, which satisfy $\phi(\frac{-1}{\tau},\frac{z}{\tau})e^{-2\pi i m\frac{z^2}{\tau}} = \tau^2\phi(\tau,z)$ if ϕ is a holomorphic Jacobi form, and $\phi(\frac{-1}{\tau},\frac{z}{\tau})e^{-2\pi i m\frac{z^2}{\tau}} = \overline{\tau}|\tau|\phi(\tau,z)$ if ϕ is skew-holomorphic, and the Fourier expansions of which have the form

$$\phi(\tau,z) = \sum_{\substack{\Delta,r\in\mathbf{Z}\\ \Delta\equiv r^2 \bmod 4m}} C_\phi(\Delta,r)\, e^{2\pi i\left(\frac{r^2-\Delta}{4m}u+\frac{r^2+|\Delta|}{4m}iv+rz\right)} \qquad (\tau = u+iv)$$

where the coefficients $C_\phi(\Delta,r)$ depend on r only modulo $2m$ and vanish for $\Delta \geq 0$ if ϕ is a holomorphic Jacobi form, and for $\Delta \leq 0$ if ϕ is skew-holomorphic.

Furthermore, we consider integral quadratic polynomials $[a,b,c](x) = ax^2+bx+c$. The group $SL_2(\mathbf{Z})$ acts on these by $[a,b,c]\circ\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)(x) = [a,b,c]\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)(\gamma x+\delta)^2$. For a pair of integers Δ, r with $\Delta \equiv r^2 \bmod 4m$ define

$$\mathcal{L}(\Delta,r) := \{\ [ma,b,c]\ | a,b,c\in\mathbf{Z},\ b^2-4mac=\Delta,\ b\equiv r \bmod 2m\ \}.$$

This set is invariant under $\Gamma_0(m) = \left(\begin{smallmatrix}\mathbf{Z}&\mathbf{Z}\\ m\mathbf{Z}&\mathbf{Z}\end{smallmatrix}\right)\cap SL_2(\mathbf{Z})$. For a fundamental discriminant Δ_0 which is a square modulo $4m$ we denote by $\chi_{\Delta_0} : \{[ma,b,c]|a,b,c\in\mathbf{Z}\}\to\{0,\pm1\}$ the generalized genus character introduced in [G-K-Z], i.e.

$$\chi_{\Delta_0}([ma,b,c]) = \begin{cases} \left(\dfrac{\Delta_0}{n}\right) & \text{if } \Delta_0 \text{ divides } b^2-4mac \text{ such that } (b^2-4mac)/\Delta_0 \\ & \quad\text{is a square modulo } 4m \text{ and } \gcd(a,b,c,\Delta_0)=1 \\ 0 & \text{otherwise.}\end{cases}$$

Here n is any integer relative prime to Δ_0 and represented by one of the quadratic forms $m_1ax^2+bxy+m_2cy^2$ with $m=m_1m_2$, $m_1,m_2>0$. (If $\gcd(a,b,c,\Delta_0)=1$ such n, m_1, m_2 exist, and if $\frac{(b^2-4mac)}{\Delta_0}$ is a square modulo $4m$ the value $\left(\frac{\Delta_0}{n}\right)$ is independent of the special choices of n or m_1, m_2, cf. [G-K-Z], Proposition 1.) Moreover, χ_{Δ_0} is $\Gamma_0(m)$-invariant. Finally, for any integral $[a,b,c]$ and any integer $N \neq 0$ set

$$\mathrm{sign}([a,b,c]) = \begin{cases} \mathrm{sign}(a) \text{ if } ac < 0 \\ 0 \qquad \text{otherwise,} \end{cases}$$

and

$$\varepsilon_N([a,b,c]) = \begin{cases} -\left(\frac{a}{N} - \frac{1}{2}\right) \text{ if } c = 0,\ 0 < a < N \\ +\left(\frac{c}{N} - \frac{1}{2}\right) \text{ if } a = 0,\ 0 < c < N \\ \quad 0 \qquad\qquad \text{otherwise.} \end{cases}$$

Note that for each discriminant Δ there are only finitely many integral polynomials $[a,b,c]$ with $b^2 - 4ac = \Delta$ such that $\mathrm{sign}([a,b,c])$ or $\varepsilon_N([a,b,c])$ is different from 0. Moreover, $\mathrm{sign}([a,b,c]) \neq 0$ implies $b^2 - 4ac > 0$, and $\varepsilon_N([a,b,c]) \neq 0$ implies that $b^2 - 4ac$ is a perfect square.

The aim of this paper is to state (and to prove) the following theorem.

Theorem. *For $A \in SL_2(\mathbf{Z})$ and integers Δ_0, r_0 with Δ_0 a fundamental discriminant, $\Delta_0 \equiv r_0^2 \bmod 4m$, define*

$$\phi_{A,\Delta_0,r_0}(\tau,z) = \sum_{\substack{\Delta, r \in \mathbf{Z} \\ \Delta \equiv r^2 \bmod 4m}} C_{A,\Delta_0,r_0}(\Delta,r)\, e^{2\pi i\left(\frac{r^2-\Delta}{4m}u + \frac{r^2+|\Delta|}{4m}iv + rz\right)}$$

where

$$C_{A,\Delta_0,r_0}(\Delta,r) = \sum_{Q \in \mathcal{L}(\Delta\Delta_0, rr_0)} \chi_{\Delta_0}(Q)\left[\mathrm{sign}(Q \circ A) + \varepsilon_{m|\Delta_0|}(Q \circ A)\right].$$

Then $\phi_{A,\Delta_0,r_0}(\tau,z)$ defines a holomorphic Jacobi form in $S_{2,m}$ if $\Delta_0 < 0$, and a skew-holomorphic Jacobi form in $S^*_{2,m}$ if $\Delta_0 > 0$. Moreover, any Jacobi form from $S_{2,m}$ or $S^*_{2,m}$ is obtained as a linear combination of the functions $\phi_{A,\Delta_0,r_0}(\tau,z)$.

Note that the sum defining $C_{A,\Delta_0,r_0}(\Delta,r)$ is actually finite since the expressions $\mathrm{sign}(Q \circ A)$ and $\varepsilon_{m|\Delta_0|}(Q \circ A)$ vanish for almost all Q with given discriminant $\Delta\Delta_0$ and that the term $\varepsilon_{m|\Delta_0|}(Q \circ A)$ always equals 0 unless $\Delta\Delta_0$ is a perfect square. Note also that $\phi_{A,\Delta_0,r_0}(\tau,z)$ for fixed Δ_0, r_0 depends only on the coset of A in $\Gamma_0(m)\backslash SL_2(\mathbf{Z})$, as is easily deduced from the $\Gamma_0(m)$-invariance of χ_{Δ_0} and $\mathcal{L}(\Delta\Delta_0, rr_0)$.

There are various intimate connections between Jacobi forms and elliptic modular forms so that, via these connections, the stated theorem can also be read as a theorem about elliptic modular forms. The deepest of such connections is given by the fact that the Fourier coefficients of an elliptic modular form, say a cusp form on $\Gamma_0(m)$ of weight two, and the values at the critical point of the twists of its L-series, or more generally its periods, are given by the Fourier coefficients of a certain Jacobi form from $S_{2,m}$ or $S^*_{2,m}$ which is naturally attached to it.

To state it more precisely, there is, for each pair of integers Δ_0, r_0 with $r_0^2 \equiv \Delta_0 \bmod 4m$ and Δ_0 being a fundamental discriminant, a lifting map $\mathcal{S}_{\Delta_0,r_0}$ which associates to a Jacobi form ϕ from $S_{2,m}$ or $S^*_{2,m}$ with Fourier coefficients $C_\phi(\Delta, r)$ an elliptic modular form with l-th Fourier coefficient $\sum_{a|l} \left(\frac{\Delta_0}{a}\right) C_\phi\left(\Delta_0 \left(\frac{l}{a}\right)^2, r_0 \frac{l}{a}\right)$. These maps are Hecke-equivariant, their images are all contained in a natural subspace $\mathcal{M}_2(m)$ of the space of all elliptic modular forms of weight 2 on $\Gamma_0(m)$, big enough to contain all newforms of level m, and, what is the essential part, there exist linear combinations of such maps which define isomorphisms of $S_{2,m} \oplus S^*_{2,m}$ with $\mathcal{M}_2(m)$ (cf. [S-Z],[S]). Thus for each $A \in SL_2(\mathbf{Z})$, each $\Delta_0, \Delta_1, r_0, r_1$ with Δ_0, Δ_1 fundamental and $r_0^2 \equiv \Delta_0 \bmod 4m$, $r_1^2 \equiv \Delta_1 \bmod 4m$ we may form the series

$$\sum_{l\geq 1} \left\{ \sum_{a|l} \left(\frac{\Delta_0}{a}\right) \sum_{Q \in \mathcal{L}\left(\Delta_0\Delta_1\left(\frac{l}{a}\right)^2, r_0 r_1 \frac{l}{a}\right)} \chi_{\Delta_1}(Q) \left[\operatorname{sign}(Q \circ A) + \varepsilon_{m|\Delta_1|}(Q \circ A)\right] \right\} e^{2\pi i l t}$$

(with t varying in the Poincaré upper half plane $\mathcal{H}$). This series then always defines an elliptic modular form on $\Gamma_0(m)$ of weight 2 (up to the addition of a constant if $\Delta_0 = 1$ which can be read off from the Corollary in section 2 and is due to the fact that certain cusp forms in $S^*_{2,m}$ lift to Eisenstein series). Moreover any new-form on $\Gamma_0(m)$ of weight 2 is a linear combination of such series. (The precise span of these series is the space of cusp forms in $\mathcal{M}_2(m)$ plus certain Eisenstein series coming from the (non-holomorphic) Eisenstein series E_2 of weight two on $SL_2(\mathbf{Z})$.)

If $f(t) = \sum a(l)\, e^{2\pi i l t}$ is a new-cusp-Hecke eigen-form, $L(f,s) = \sum \frac{a(l)}{l^s}$ its L-series, ϕ the unique Jacobi cusp form in $S_{2,m}$ or $S^*_{2,m}$ corresponding to it via the lifting maps $\mathcal{S}_{\Delta_0,r_0}$ with coefficients $C_\phi(\Delta, r)$ then, by considering the adjoint maps of the $\mathcal{S}_{\Delta_0,r_0}$ (with respect to Petersson scalar products), it was proved in [G-K-Z] that

$$|C_\phi(\Delta, r)|^2 = \frac{\sqrt{|\Delta|}\langle\phi|\phi\rangle}{2\pi\langle f|f\rangle} L(f, \Delta, 1).$$

Here $L(f,\Delta,s) = \sum \left(\frac{\Delta}{l}\right) \frac{a(l)}{l^s}$ is the twisted L-series of f. (The symbols $\langle\cdot|\cdot\rangle$ denote suitably normalized Petersson scalar products the definition of which for Jacobi forms will be recalled in section 2, and Δ is assumed to be fundamental and relatively prime to m). This was proved for holomorphic Jacobi forms only but, without doubt, will be true for skew-holomorphic ones two. (One can copy the proof given in [G-K-Z], using the Proposition and its Corollary in section 2 of this paper to derive the corresponding result for skew-holomorphic forms.) Thus the stated Theorem can then be used to produce 'Tunnel-like' theorems in an algorithmic way, i.e. it can be used to describe explicitly a computable (recursive) function of fundamental discriminants (namely, the left hand side of the cited equation) which gives the values of all $L(f,\Delta,1)$ up to a non-zero constant independent of Δ.

As an illustration of the theorem we consider the simplest non-trivial case, i.e. the case m=11. The space of modular forms of weight 2 on $\Gamma_0(11)$ is spanned by the Eisenstein series $E(t) = E_2(t) - 11E_2(11t)$ with $E_2(t) = \frac{-1}{24} + \frac{1}{8\pi \operatorname{Im} t} + \sum_{l\geq 1}\left(\sum_{d|l} l\right) e^{2\pi i l t}$ and the cusp form $S(t) = \eta(t)^2\eta(11t)^2$ where $\eta(t) = e^{\frac{\pi i t}{12}}\prod_{n\geq 1}\left(1 - e^{2\pi i n t}\right)$. The space $S^*_{2,11}$ contains the 'trivial' cusp form

$$T(\tau,z) := \sum_{r,s\in\mathbf{Z}} (r-11s)e^{2\pi i\left(rs\tau + \frac{(r-11s)^2}{2m} iv + (r+11s)z\right)}$$

which satisfies $\mathcal{S}_{1,1}(T) = E$, as can easily be checked. The set $\Gamma_0(11)\backslash SL_2(\mathbf{Z})$ is indexed by elements of $\mathbf{P}^1(\mathbf{Z}/11\mathbf{Z})$ via $d \leftrightarrow \Gamma_0(11)\left(\begin{smallmatrix}0 & -1\\ 1 & d'\end{smallmatrix}\right)$for $d\in\mathbf{Z}/11\mathbf{Z}$ and with d' denoting any integer from the residue class d, and $\infty \leftrightarrow \Gamma_0(11)$. For $P\in\mathbf{P}^1(\mathbf{Z}/11\mathbf{Z})$ set $\phi_P = \phi_{A,1,1}$ (A any matrix corresponding to P). Now a short calculation shows $\phi_0 = \phi_\infty = 0$, $\phi_a = \phi_{-a}$ for all a, and

$$\phi_1 = \frac{9}{11}T,\ \phi_2 = \frac{3}{11}T,\ \phi_3 = \phi_4,\ \phi_5 = \frac{-3}{11}T,\ \phi_6 = \frac{3}{11}T.$$

Computing the first few coefficients of $\mathcal{S}_{1,1}(\phi_3)$ shows that they coincide with those of $\frac{1}{5}(S(t) + \frac{9}{11}E_2(t))$. Thus, by the theory quoted above, $\mathcal{S}_{1,1}(5\phi_3 + 3\phi_5)$ must be equal to $S(t)$, and this then yields funny formulas for the Fourier coefficients of $S(t)$ and the values $L(S,\Delta,1)$. The first few coefficients $C(\Delta,r)$ of $5\phi_3+3\phi_5$ are

Δ	1	4	5	9	12	16	20	25	33	36	37	44	45	48	49	53	56	60 ...
$C(\Delta, r_\Delta)$	1	-3	5	-2	5	4	5	0	0	6	5	0	0	10	-3	10	-10	-5 ...

(For each Δ the symbol r_Δ denotes that solution of $r^2 \equiv \Delta \bmod 44$ which satisfies $0\leq r\leq 11$).

The idea for the proof of the theorem is roughly as follows. Any elliptic cusp form on $\Gamma_0(m)$ of weight 2, considered as a holomorphic differential on the compactification of $\Gamma_0(m)\backslash\mathcal{H}$, is determined by its periods. As paths in the upper half plane for computing the periods one may restrict to hyperbolic lines which connect 0 and rational numbers equivalent to 0 modulo $\Gamma_0(m)$. By the so-called 'Manin trick' one can even restrict oneself to consider path integrals of the form $\int_{A0}^{Ai\infty}$ where A runs through a set of representatives of $\Gamma_0(m)\backslash SL_2(\mathbf{Z})$ (cf. [M]). On the other hand side there are the lifting maps $\mathcal{S}_{\Delta_0,r_0}$ from Jacobi forms to elliptic modular forms. Let $\mathcal{K}_{\Delta_0,r_0}(\tau,z;t)$ be the corresponding holomorphic kernel function with respect to the Petersson scalar product. Since a linear combination of the $\mathcal{S}_{\Delta_0,r_0}$ is an injection it is then clear that all the the Jacobi forms $\langle\mathcal{K}_{\Delta_0,r_0}(\tau,z;\cdot)|f\rangle$ together generate $S_{2,m}$ and $S^*_{2,m}$ (f runs through the set of cusp forms on $\Gamma_0(m)$). By the above any such scalar product can be written as a linear combination of path integrals $\int_{A0}^{Ai\infty}\mathcal{K}_{\Delta_0,r_0}(\tau,z;t)dt$. Hence all these path integrals together generate $S_{2,m}$ and $S^*_{2,m}$. Thus, if we have explicit formulas for the kernel functions and if we can carry out explicitly the integration along paths joining $A0$ and $Ai\infty$, then we can prove a theorem like the one stated here. (It must be added that these scetched arguments are not literally true since certain Jacobi forms in $S^*_{2,m}$ lift to Eisenstein series so that, for instance, some of the above path integrals are not a priori defined.)

For the case of holomorphic Jacobi forms the kernels $\mathcal{K}_{\Delta_0,r_0}$ have been constructed explicitly in [G-K-Z]. However, we do not take these kernels to deduce the Theorem but instead we take certain non-holomorphic kernel functions. These arise as special cases when one tries to construct Jacobi theta functions associated to indefinite quadratic forms. This construction can be done and one arrives at the exact pendant in the theory of Jacobi forms of the theta kernels which one uses in the theory of dual reductive pairs (cf.[S2]).

The reason for considering such theta kernels instead of the holomorphic $\mathcal{K}_{\Delta_0,r_0}$ is as follows. First of all, if one looks at the formula for $\mathcal{K}_{\Delta_0,r_0}$ as given in [G-K-Z] then it seems that the explicit computation of the integrals of $\mathcal{K}_{\Delta_0,r_0}$ along the paths from $A0$ and $Ai\infty$ involves some delicate convergence problems (which can perhaps be settled). Furthermore, we would like to treat the case of skew-holomorphic forms as well but corresponding results as for holomorphic Jacobi forms are not yet available in the literature. Now, apart from the fact that the above mentioned theta kernels add a new aspect to the theory of Jacobi forms (though not very surprising), they allow, and that is the main point, to give a reasonable short proof from scratch of the theorem, a proof which at the same time includes holomorphic as well as skew-holomorphic Jacobi forms. Finally, the investigation of these theta kernels as well as the computation of the corresponding path integrals exhibits, to our opinion, some amusing aspects (if one likes such computations).

There remain two points to be mentioned with respect to the proof. The minor one is that we shall not really consider the integrals $\int_{A0}^{Ai\infty}$ of the theta kernels but the symetrization $\int_{A0}^{Ai\infty} + \int_{-A0}^{-Ai\infty}$ (for $\Delta_0 < 0$) or the antisymetrization $\int_{A0}^{Ai\infty} - \int_{-A0}^{-Ai\infty}$ (for $\Delta_0 > 0$), both taken, so to speak, in the sense of the Cauchy Hauptwert (to be precise, we consider $\int_0^{i\infty}([A]^* \pm [gAg]^*)$ where $[A]^*$ is the pullback operation on differentials of the isomorphism $[A]$ on the upper half plane induced by A and g denotes the diagonal matrix with -1 and 1 in the diagonal). The reason for this is that the first integrals would not exist in general since the theta kernels involve contributions which come from elliptic Eisenstein series. However, to consider these symetrized or antisymetrized versions makes perfect sense. The mappings which associate to an elliptic cusp form its path integrals $\int_{A0}^{Ai\infty} + \int_{-A0}^{-Ai\infty}$ or $\int_{A0}^{Ai\infty} - \int_{-A0}^{-Ai\infty}$ respectively are both injective. It can be shown that they define interesting rational structures on the space of modular forms of weight 2 on $\Gamma_0(m)$. These rational structures are the natural generalizations to weight 2 forms on $\Gamma_0(m)$ of those rational structures considered in [K-Z], and they will be investigated (for arbitrary weight) in a forthcoming paper of J.Antoniadis.

A more severe point is that the proof of the theorem is not completely independent of the literature. We have to cite the fact that the intersection of the kernels of all the $\mathcal{S}_{\Delta_0,r_0}$ equals 0, or, what sounds less complicated, that for any nonzero Jacobi form there is at least one non-zero Fourier coefficient $C(\Delta_0 l^2, r_0 l)$ with some integer l and fundamental Δ_0 such that $r_0^2 \equiv \Delta_0 \bmod 4m$. This seems to be a fairly deep fact. Its proof depends on a trace formula for Jacobi forms and was given in [S-Z] for holomorphic Jacobi forms. A corresponding proof for the case of skew-holomorphic Jacobi forms is not yet available in the published literature and will be given in [S]. The suspicious reader may thus divide the stated theorem into two, one, unchanged but valid only for holomorphic Jacobi forms, and a second one for skew-holomorphic Jacobi forms, but which must then be stated in the weaker form that the span of the $\phi_{A,\Delta_0,r_0}(\tau, z)$ is the orthogonal complement in $S_{2,m}^*$ with respect to the Petersson scalar product of the intersection of the kernels of the $\mathcal{S}_{\Delta_0,r_0}$. This is what we shall actually prove (cf. the end of section 2). It is perhaps worthwhile to mention that one could possibly circumvent this problem by considering more general theta kernels which would yield lifting maps $\mathcal{S}_{\Delta_0,r_0}$ associated to arbitrary (i.e. not necessarily fundamental) discriminants Δ_0. This would mean to generalize the equation (3) in section 3, or, to say it in other terms, to study how many and what $SL_2(\mathbf{Z})$-invariant vectors are contained in the Weil-representation associated to a certain finite quadratic module the definition of which can be read off from the cited identity. However, we did not pursue this here.

A theorem of similar type as the above was proved by Kohnen and Zagier in [K-Z] (in the quoted article it appears more as an incidental remark than as a theorem; loc.cit.,p.236). They show that for each k any modular form on $\Gamma_0(4)$ of weight $k+\frac{1}{2}$ from the Kohnen-plus space can be written as a linear combination of certain explicitly given functions which themselves are not modular functions but look very much like theta series with spherical polynomials associated to the indefinite ternary form b^2-4ac. Their proof is based on the fact that the coefficients of a Kohnen-plus space Hecke-eigenform are essentially given by the periods of the (via Shimura lift) associated modular form of level 1 and even weight around closed geodesics, and on an identity expressing such cycle integrals explicitly as linear combination of what are usually called the periods of this associated form (i.e. the values at the integer points in the critical strip of the L-series of this form; loc.cit.Theorem 7). Although it is not this procedure that we try to carry over to Jacobi forms to derive the stated theorem it probably could be done (and, as should be added, the quoted article stimulated us very much to meditate on such a theorem). The formulas given above for the Fourier coefficients of weight 2 modular forms on $\Gamma_0(m)$ should also be very closely related to the formulas given in [M] for the Fourier coefficients of Hecke eigenforms. It may be worthwhile to make this connection explicit.

Finally, it should be possible to generalize the theorem to the case of higher weight. The corresponding theta kernels are easily constructed and on the other hand side the Eichler-Shimura isomorphism will play a rôle. Since we did not want to gain overweight by too many technical considerations, we did not try to fulfill this program here, and we shall publish the corresponding computations elsewhere.

2. Proof. As in the theorem, fix a fundamental discriminant Δ_0 and an integer r_0 such that $\Delta_0 \equiv r_0^2 \bmod 4m$. Set

$$(1) \qquad \vartheta_{\Delta_0,r_0}(\tau,z;t) := \sum_{\substack{\Delta,r\in\mathbf{Z}\\ \Delta\equiv r^2 \bmod 4m}} C_v(\Delta,r;t)\, e^{2\pi i\left(\frac{r^2-\Delta}{4m}u+\frac{r^2+\sigma\Delta}{4m}iv+rz\right)}$$

where

$$C_v(\Delta, r; t) = v^{\frac{1}{2}} \sum_{Q \in \mathcal{L}(\Delta\Delta_0, rr_0)} \chi_{\Delta_0}(Q) \, \frac{Q(t)}{\eta^2} \exp\left(-\frac{\pi v \widehat{Q}(t)^2}{m|\Delta_0|\eta^2} \right)$$

and $\sigma = \operatorname{sign}(\Delta_0)$. Here

$$\tau = u + iv, \ t = \xi + i\eta \in \mathcal{H}, \ z \in \mathbf{C}$$

($\mathcal{H}$=Poincaré-upper half plane of complex numbers with positive imaginary part) and we use the notation

$$\widehat{[a, b, c]}(t) := a|t|^2 + b\xi + c \ .$$

Taking absolute values and writing $[a, b, c]$ for Q and $(b^2 - 4ac)/\Delta_0$ for Δ we see that the series defining $\vartheta_{\Delta_0, r_0}(\tau, z; t)$ is dominated by

$$\text{(2)} \qquad v^{\frac{1}{2}} \sum_{\substack{r, a, b, c \in \mathbf{Z}, \ m | a \\ b \equiv r r_0 \bmod 4m}} \mathrm{e}^{-2\pi i \left(\frac{r^2}{4m} v + r \operatorname{Im}(z) \right)} |\chi_{\Delta_0}(Q)| \, \frac{|at^2 + bt + c|}{\eta^2} \, \mathrm{e}^{\frac{-\pi v}{2m|\Delta_0|} F_\eta(a,b,c)}$$

where $F_\eta(a, b, c) = (b^2 - 4ac) + \frac{2}{\eta^2}(a|t|^2 + b\xi + c)^2$. Since it is easily checked that, for fixed t, the quadratic form $F_\eta(a, b, c)$ is positive definite, we deduce that the series in (1) is normally convergent, i.e. uniformly convergent on compact subsets in the $(\tau, z; t)$-domain, and defines a smooth function in τ, z, t . Moreover $\vartheta_{\Delta_0, r_0}(\tau, z; t)$ is obviously periodic in τ and z with period 1, and it can be checked by a tedious but routine application of the Poisson summation formula that $\vartheta_{\Delta_0, r_0}(\tau, z; t)$ transforms like a Jacobi form of weight 2 and index m under $\tau \mapsto \frac{-1}{\tau}$, $z \mapsto \frac{z}{\tau}$, like a holomorphic one for negative Δ_0, and like a skew-holomorphic one for positive Δ_0. However, this also follows from the Proposition below and we skip this computation.

Fix a matrix $A \in SL_2(\mathbf{Z})$ and define

$$\text{(3)} \qquad \phi(\tau, z) := \int_0^{i\infty} \left(\vartheta_{\Delta_0, r_0}(\tau, z; At) \overline{d(At)} - \operatorname{sign}(\Delta_0) \vartheta_{\Delta_0, r_0}(\tau, z; gAgt) \overline{d(gAgt)} \right) .$$

Here $At = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$ for $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix} \right)$, thus $d(At) = \frac{dt}{(\gamma\tau + \delta)^2}$, and g denotes the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$, thus $gAg = \left(\begin{smallmatrix} \alpha & -\beta \\ -\gamma & \delta \end{smallmatrix} \right)$. The integral has to be taken along the line $t = i\eta$ with η ranging from 0 to ∞. We shall see below that the integral converges absolutely. Thus $\phi(\tau, z)$ is well defined and, of course, it inherits from $\vartheta_{\Delta_0, r_0}(\tau, z; t)$ the periodicity in τ, z and the invariance property with respect to $\tau \mapsto \frac{-1}{\tau}$, $z \mapsto \frac{z}{\tau}$. Thus, it will be a Jacobi form if it has the correct Fourier development. Indeed, we shall show next that $\phi(\tau,z)$ has a Fourier development which coincides, up to a non-zero constant, with that of $\phi_{A,\Delta_0,r_0}(\tau,z)$ as given in the theorem. This will then prove part of the theorem, namely that $\phi_{A,\Delta_0,r_0}(\tau,z)$ is a Jacobi form.

To compute $\phi(\tau,z)$ we first of all rewrite the integrand. Using the easily checked identities

$$\text{(4)}\qquad \frac{Q(At)}{(\operatorname{Im} At)^2}(\gamma\bar{t}+\delta)^{-2} = \frac{Q\circ A(t)}{(\operatorname{Im} t)^2}, \qquad \frac{\widehat{Q}(At)}{\operatorname{Im} At} = \frac{\widehat{Q\circ A}(t)}{\operatorname{Im} t}$$

we can write

$$\begin{aligned}\widetilde{C}_v(\Delta,r;t) &:= C_v(\Delta,r;At)(\gamma\bar{t}+\delta)^{-2} + C_v(\Delta,r;gAgt)(-\gamma\bar{t}+\delta)^{-2} \\ &= v^{\frac12}\Bigg\{\sum_{Q\in\mathcal{L}(\Delta\Delta_0,rr_0)} \chi_{\Delta_0}(Q)\,\frac{Q\circ A(t)}{\eta^2}\,\mathrm{e}^{-\lambda\frac{\widehat{Q\circ A}(t)^2}{\eta^2}} \\ &\qquad -\operatorname{sign}(\Delta_0)\sum_{Q\in\mathcal{L}(\Delta\Delta_0,rr_0)} \chi_{\Delta_0}(Q)\,\frac{Q\circ gAg(t)}{\eta^2}\,\mathrm{e}^{-\lambda\frac{\widehat{Q\circ gAg}(t)^2}{\eta^2}}\Bigg\},\end{aligned}$$

where we put $\lambda = \frac{\pi v}{m|\Delta_0|}$. Replace Q by $-Q\circ g$ in the second sum. Since $[a,b,c]\circ g = [a,-b,c]$ we find that $\chi_{\Delta_0}(-Q\circ g) = \operatorname{sign}(\Delta_0)\chi_{\Delta_0}(Q)$ and that, for $t = i\eta$ and $Q\circ A =: [a,b,c]$, one has $Q\circ A(t) + Q\circ g^2Ag(t) = 2\left(-a\eta^2+c\right)$ and $\widehat{Q\circ g^2Ag}(t) = \widehat{Q\circ A}(t) = a\eta^2+c$. Thus

$$\widetilde{C}_v(\Delta,r;t) = \frac{2v^{\frac12}}{\eta}\sum_{[a,b,c]\in\mathcal{L}(\Delta\Delta_0,rr_0)\circ A} \chi_{\Delta_0}([a,b,c]\circ A^{-1})\left(-a\eta+\frac{c}{\eta}\right)\mathrm{e}^{-\lambda\left(a\eta+\frac{c}{\eta}\right)^2}.$$

Take here now a typical term with $ac\neq 0$ and set $\eta = \sqrt{|\frac{c}{a}|}e^{\theta}$. Then

$$\int_0^\infty \left(-a\eta+\frac{c}{\eta}\right)\mathrm{e}^{-\lambda\left(a\eta+\frac{c}{\eta}\right)^2}\frac{d\eta}{\eta} = -\operatorname{sign(a)}\sqrt{|ac|}\int_{-\infty}^{+\infty}\mathrm{e}^{-\lambda|ac|c(\theta)^2}dc(\theta)$$

where $c(\theta) = 2\cosh(\theta)$ if $ac>0$ and $c(\theta) = 2\sinh(\theta)$ if $ac<0$. Thus the latter integral vanishes for $ac>0$ (since then the integrand is odd) and otherwise equals $-\operatorname{sign}(a)\sqrt{|ac|}\int_{-\infty}^{+\infty}\mathrm{e}^{-\lambda|ac|x^2}dx = -\operatorname{sign}(a)\sqrt{\frac{m|\Delta_0|}{v}}$.

To handle terms with $ac=0$ we rewrite the contribution to $\widetilde{C}_v(\Delta,r;t)$ resulting from these terms as

$$-2v^{\frac12}\Bigg\{\sum_{\substack{a \bmod m|\Delta_0|,\ b\in\mathbb{Z}\\ [a,b,0]\in\mathcal{L}(\Delta\Delta_0,rr_0)\circ A}}\chi_{\Delta_0}([a,b,0]\circ A^{-1})\,\theta_a(\eta) - \sum_{\substack{c \bmod m|\Delta_0|,\ b\in\mathbb{Z}\\ [0,b,c]\in\mathcal{L}(\Delta\Delta_0,rr_0)\circ A}}\chi_{\Delta_0}([0,b,c]\circ A^{-1})\,\theta_c(\frac{1}{\eta})\frac{1}{\eta^2}\Bigg\}$$

where

$$\theta_a(\eta) = \sum_{\substack{x \in \mathbf{Z} \\ x \equiv a \bmod m|\Delta_0|}} x\, e^{-\lambda x^2 \eta^2}$$

and where we used that $\chi_{\Delta_0}([a,b,c] \circ A^{-1})$ depends on a only modulo $m|\Delta_0|$. Now by a standard computation

$$\int_0^\infty \theta_a(\eta) d\eta = \frac{1}{2}\sqrt{\frac{m|\Delta_0|}{v}} \left(\zeta\left(0, \frac{a}{m|\Delta_0|}\right) - \zeta\left(0, \frac{-a}{m|\Delta_0|}\right)\right)$$

where

$$\zeta(s,u) = \sum_{\substack{x>0 \\ x \equiv u \bmod \mathbf{Z}}} \frac{1}{x^s}$$

is the Hurwitz zeta-function, i.e. $\zeta(0,u) = \frac{1}{2} - u$ for $0 < u \leq 1$. The integral $\int_0^\infty \theta_c(\frac{1}{\eta})\frac{1}{\eta^2} d\eta$ is treated in exactly the same way after substituting $\eta \mapsto \frac{1}{\eta}$.

Thus using the notation introduced for the statement of the theorem and disregarding for the moment the question of whether interchanging of integration and summation for the computation of $\phi(\tau, z)$ is allowed or not (in fact, it is) we may summarize as:

$$\int_0^\infty \widetilde{C}_v(\Delta, r; \eta)\overline{d(i\eta)} = 2i\sqrt{m|\Delta_0|} \times \left(\Delta, r\text{-th coefficient of } \phi_{A,\Delta_0,r_0}(\tau, z)\right).$$

But then, replacing the coefficients $C_v(\Delta, r; t)$ in the defining equation (1) of $\vartheta_{\Delta_0,r_0}(\tau, z; t)$ by the integrals $\int_0^\infty \widetilde{C}_v(\Delta, r; \eta)\overline{d(i\eta)}$ in order to obtain $\phi(\tau, z)$, we deduce $\phi(\tau, z) = 2i\sqrt{m|\Delta_0|}\phi_{A,\Delta_0,r_0}(\tau, z)$. (When doing this replacement note that one can at the same time replace the $\sigma\Delta$ in (1) by $|\Delta|$ since $\int_0^\infty \widetilde{C}_v(\Delta, r; \eta)\overline{d(i\eta)}$, i.e. the Δ, r-th coefficient of $\phi_{A,\Delta_0,r_0}(\tau, z)$, vanishes for Δ negative or positive accordingliy as Δ_0 is positive or negative.)

To complete the proof of the first part of the theorem, i.e. that $\phi_{A,\Delta_0,r_0}(\tau, z)$ is a cusp form, it remains to show that for any r with $r^2 \equiv 0 \bmod 4m$ the $0, r$-th coefficient of $\phi_{A,\Delta_0,r_0}(\tau, z)$ vanishes, i.e. that

$$\sum_{\substack{0<a<m|\Delta_0| \\ [a,0,0] \in \mathcal{L}(0, rr_0) \circ A}} \chi_{\Delta_0}([a,0,0] \circ A^{-1}) \left(\frac{a}{m|\Delta_0|} - \frac{1}{2}\right)$$

$$= \sum_{\substack{0<c<m|\Delta_0| \\ [0,0,c] \in \mathcal{L}(0, rr_0) \circ A}} \chi_{\Delta_0}([0,0,c] \circ A^{-1}) \left(\frac{c}{m|\Delta_0|} - \frac{1}{2}\right).$$

But this can be checked by a standard computation. (In fact, by the Dirichlet class number formula each side equals $-h'(\Delta_0)$ if $\Delta_0 < 0$ and $rr_0 \equiv 0 \bmod 2m$, and 0 otherwise, where $h'(\Delta_0)$ denotes the class number of $\mathbf{Q}(\sqrt{\Delta_0})$ for $\Delta_0 < -4$, and $h'(-3) = \frac{1}{3}, h'(-4) = \frac{1}{2}$.)

This completes the proof that $\phi_{A,\Delta_0,r_0}(\tau, z)$ is a Jacobi cusp form apart from some estimates which seem to be indispensable now. So assume first of all that $ac \neq 0$. Then one has

$$\begin{aligned} e^{2\lambda ac} \int_0^\infty \left|-a\eta + \frac{c}{\eta}\right| e^{-\lambda\left(a\eta+\frac{c}{\eta}\right)^2} \frac{d\eta}{\eta} &= \sqrt{|ac|} \int_0^\infty \left|\eta \pm \frac{1}{\eta}\right| e^{-\lambda|ac|\left(\eta^2+\frac{1}{\eta^2}\right)} \frac{d\eta}{\eta} \\ = 2\sqrt{|ac|} \int_1^\infty \left|\eta \pm \frac{1}{\eta}\right| e^{-\lambda|ac|\left(\eta^2+\frac{1}{\eta^2}\right)} \frac{d\eta}{\eta} &\le 2\sqrt{|ac|} \int_1^\infty 2\eta\, e^{-\lambda|ac|\eta^2}\, d\eta \\ &= \frac{2}{\lambda\sqrt{|ac|}} e^{-\lambda|ac|} \end{aligned}$$

(the first equality is obtained by setting $\eta \mapsto \sqrt{|ac|}\eta$, the second one by observing that the differential under the integral is invariant by $\eta \mapsto \frac{1}{\eta}$, the inequality by a crude estimate.) Furthermore

$$\int_0^\infty |\theta_a(\eta)| d\eta = \lambda^{-\frac{1}{2}} \int_0^\infty \left| \sum_{\substack{x \in \mathbf{Z} \\ x \equiv a \bmod m|\Delta_0|}} x\, e^{-\eta^2 x^2} \right| d\eta$$

(set $\eta \mapsto \lambda^{\frac{1}{2}}\eta$ in the first integral), and the latter integral is bounded by a constant γ independent of a. Now

$$\int_0^{i\infty} \left|\vartheta_{\Delta_0,r_0}(\tau, z; At)\overline{d(At)} - \operatorname{sign}(\Delta_0)\vartheta_{\Delta_0,r_0}(\tau, z; gAgt)\overline{d(gAgt)}\right|$$

$$\begin{aligned} \le 2v^{\frac{1}{2}} \sum_{r\in\mathbf{Z}} e^{-2\pi\left(\frac{r^2}{4m}v + r\operatorname{Im}(z)\right)} \Bigg\{ &\sum_{\substack{a,b,c\in\mathbf{Z} \\ ac\neq 0}} e^{-\pi\frac{(b^2-4ac)}{2m|\Delta_0|}v} \int_0^\infty \left|-a\eta + \frac{c}{\eta}\right| e^{-\frac{\pi v}{m|\Delta_0|}\left(a\eta+\frac{c}{\eta}\right)^2} \frac{d\eta}{\eta} \\ &+ \sum_{a,c \bmod m|\Delta_0|} \left(\int_0^\infty |\theta_a(\eta)| d\eta + \int_0^\infty |\theta_c(\eta)| d\eta\right) \Bigg\} \end{aligned}$$

where we wrote $(b^2 - 4ac)/\Delta_0$ for Δ in the definition of $\vartheta_{\Delta_0,r_0}(\tau, z; t)$ and where we have eventually enlarged by summing over all $a, b, c \in \mathbf{Z}$. Thus, since by the given estimates for the integrals the right hand side is clearly convergent, we have justified the above computation of $\phi_{A,\Delta_0,r_0}(\tau, z)$. Note that we have also proved

(5)
$$\int_0^{i\infty} \left|\vartheta_{\Delta_0,r_0}(\tau, z; At)\overline{d(At)} - \operatorname{sign}(\Delta_0)\vartheta_{\Delta_0,r_0}(\tau, z; gAgt)\overline{d(gAgt)}\right| e^{-2\pi m \operatorname{Im}(z)^2/v} = \mathcal{O}(1)$$

for $v \to \infty$ where the $\mathcal{O}$-constant is independent of u and z.

To prove the converse, i.e. that the $\phi_{A,\Delta_0,r_0}(\tau,z)$ span $S_{2,m}$ and $S^*_{2,m}$ we have to introduce the Petersson scalar product. Let $\phi(\tau,z)$ be any Jacobi cusp form from $S_{2,m}$ or $S^*_{2,m}$, and let $\psi(\tau,z)$ be any, say smooth, function on $\mathcal{H}\times\mathbf{C}$ such that $\psi(\tau,z)\mathrm{e}^{-2\pi m \mathrm{Im}\,(z)^2/v} = \mathcal{O}(v^k)$ for $v\to\infty$ with some k and an $\mathcal{O}$-constant which is independent of u and z. Then the Petersson scalar product $\langle\phi|\psi\rangle$ of $\phi(\tau,z)$ and $\psi(\tau,z)$ is defined by

$$\langle\phi|\psi\rangle = \frac{1}{2}\int\int_{\mathcal{F}}\int_0^1\int_0^1 \phi(\tau,\lambda\tau+\mu)\overline{\psi(\tau,\lambda\tau+\mu)}\mathrm{e}^{-4\pi m\lambda^2 v}d\lambda d\mu du dv.$$

Here, with respect to u and v, the integral has to be taken over the standard fundamental domain $\mathcal{F}$ for $\mathcal{H}$ modulo $SL_2(\mathbf{Z})$, i.e. $\mathcal{F}=\{\tau\in\mathcal{H}|-\frac{1}{2}\le u\le\frac{1}{2},|\tau|\ge 1\}$, and the integral is absolutely convergent since for a cusp form $\phi(\tau,z)$ the expression $\phi(\tau,z)\mathrm{e}^{-2\pi m\mathrm{Im}\,(z)^2/v}$ is exponentially decreasing uniformly in u and z for $v\to\infty$. In particular, this defines a non-degenerate scalar product on the finite dimensional spaces $S_{2,m}$ and $S^*_{2,m}$. In the next section we shall need a more conceptional way to look at the Petersson scalar product.

To explain this denote by $\mathcal{J}(\mathbf{Z}) = SL_2(\mathbf{Z}) \propto \mathbf{Z}^2$ the Jacobi-group over $\mathbf{Z}$, i.e. the set of all pairs $A[\lambda,\mu]$ ($A\in SL_2(\mathbf{Z})$ and $\lambda,\mu\in\mathbf{Z}$) equipped with the multiplication $A[\lambda,\mu]\bullet A'[\lambda',\mu'] = AA'[(\lambda,\mu)A'+(\lambda,\mu)]$. The Jacobi group acts on $\mathcal{H}\times\mathbf{C}$ by $\Upsilon\cdot(\tau,z) = \left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta},\frac{z+\lambda\tau+\mu}{\gamma\tau+\delta}\right)$ (for $\Upsilon=\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)[\lambda,\mu]$) and on functions $\phi(\tau,z)$ by

$$\left(\phi|\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\right)(\tau,z) = \frac{\mathrm{e}^{-2\pi i m\frac{\gamma z^2}{\gamma\tau+\delta}}}{(\gamma\tau+\delta)^2}\,\phi\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta},\frac{z}{\gamma\tau+\delta}\right)$$

and

$$\phi|[\lambda,\mu](\tau,z) = \mathrm{e}^{2\pi i m(\lambda^2\tau+2\lambda z)}\,\phi(\tau,z+\lambda\tau+\mu),$$

and also by

$$\left(\phi|^*\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\right)(\tau,z) = \frac{\mathrm{e}^{-2\pi i m\frac{\gamma z^2}{\gamma\tau+\delta}}}{(\gamma\overline{\tau}+\delta)|\gamma\tau+\delta|}\,\phi\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta},\frac{z}{\gamma\tau+\delta}\right)$$

and $\phi|^*[\lambda,\mu]=\phi|[\lambda,\mu]$. It is easily checked that a holomorpic or skew-holomorphic Jacobi form ϕ (of weight 2 and index m) satisfies $\phi|\Upsilon=\phi$ or $\phi|^*\Upsilon=\phi$ respectively for all $\Upsilon\in\mathcal{J}(\mathbf{Z})$. If $\phi(\tau,z)$ and $\psi(\tau,z)$ are as above and if $\psi(\tau,z)$ additionally satisfies the same transformation law with respect to $\mathcal{J}(\mathbf{Z})$ as $\phi(\tau,z)$ then the expression

$\phi(\tau,z)\overline{\psi(\tau,z)}\mathrm{e}^{-4\pi m y^2/v}v^2$ $(z = x+iy)$ is invariant by replacing (τ, z) by $\Upsilon \cdot (\tau, z)$ for all $\Upsilon \in \mathcal{J}(\mathbf{Z})$. Moreover

$$\langle \phi | \psi \rangle = \int_{\mathcal{J}(\mathbf{Z})\backslash \mathcal{H}\times\mathbf{C}} \phi(\tau,z)\overline{\psi(\tau,z)}\mathrm{e}^{-4\pi m y^2/v}v^2 \, dV,$$

where $dV = \frac{du dv dx dy}{v^3}$ is the $\mathcal{J}(\mathbf{Z})$-invariant volume element on $\mathcal{H}\times\mathbf{C}$ and the integral has to be taken over any fundamental domain of $\mathcal{H}\times\mathbf{C}$ modulo $\mathcal{J}(\mathbf{Z})$.

Take now $\phi(\tau,z)$ from the orthogonal complement of the span of the $\phi_{A,\Delta_0,r_0}(\tau,z)$ in $S_{2,m}$ and $S^*_{2,m}$. Thus

$$\langle \phi | \phi_{A,\Delta_0,r_0} \rangle = 0 \tag{6}$$

for all A, Δ_0, r_0. We have to prove that (6) implies $\phi \equiv 0$. To interpret (6) note that $\overline{\vartheta_{\Delta_0,r_0}(\tau,z;t)}$ as a function of t, for fixed τ and z behaves like an element from $M_2(\Gamma_0(m))$, the space of elliptic modular forms on $\Gamma_0(m)$ of weight 2. Indeed, the transformation law $\vartheta_{\Delta_0,r_0}\left(\tau,z;\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)(\gamma\tau+\delta)^{-2} = \vartheta_{\Delta_0,r_0}(\tau,z;t)$ for all $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma_0(m)$ is immediately clear from the definition of $\vartheta_{\Delta_0,r_0}(\tau,z;t)$ using (4) and the invariance of the $\mathcal{L}(\Delta,r)$ and χ_{Δ_0} under $\Gamma_0(m)$. This transformation law with respect to $\Gamma_0(m)$ is then also fulfilled by $f(t) := \langle \phi | \vartheta_{\Delta_0,r_0}(\cdot,\cdot;t)\rangle$. Here, of course, we have to check that the Petersson scalar product is defined, i.e. that $\vartheta_{\Delta_0,r_0}(\tau,z;t)$ as a function in τ, z satisfies the correct boundedness condition. But this follows easily from (2) which even shows that $\vartheta_{\Delta_0,r_0}(\tau,z;t)\,\mathrm{e}^{-2\pi m y^2/v}$ can be bounded by a polynomial in v and η which does not depend on u, z, ξ. Thus, for any cusp form ϕ, the function $f(t)$ is smooth on $\mathcal{H}$ and can be bounded by a polynomial in η independently of ξ. We shall even show below that $f(t)$ is an element of $M_2(\Gamma_0(m))$. Using (5) to justify interchanging of integrals we may rewrite (6) as

$$\int_0^{i\infty} \left(\langle \phi|\vartheta_{\Delta_0,r_0}(\cdot,\cdot;At)\rangle d(At) - \mathrm{sign}(\Delta_0)\langle \phi|\vartheta_{\Delta_0,r_0}(\cdot,\cdot;gAgt)\rangle d(gAgt)\right) = 0 \tag{7}$$

and this is then a statement about modular forms. We apply the following Lemma.

Lemma. *Let f(t) be a modular form from $M_2(\Gamma_0(m))$, let $\varepsilon \in \{\pm 1\}$. Assume that for each $A \in SL_2(\mathbf{Z})$ the integral*

$$\int_0^{i\infty} \left(f(At)d(At) + \varepsilon f(gAgt)d(gAgt)\right),$$

taken along the path $t = i\eta$, $\eta \in \mathbf{R}$, is absolutely convergent and equal to 0. Then $f(t)$ is identically 0.

This Lemma is probably known to the specialists, but for the sake of completeness we give the short proof in Section 4.

Applying this Lemma we deduce from (7) that $\langle\phi|\vartheta_{\Delta_0,r_0}(\cdot,\cdot;t)\rangle = 0$ for all A, Δ_0, r_0. To investigate this we need the following Proposition.

Proposition. *For $\tau, t \in \mathcal{H}, z \in \mathbf{C}$, $\tau = u + iv$, $t = \xi + i\eta$ define*

$$\theta(\tau,z,t) = \frac{\partial}{\partial t} \sum_{\substack{r,s\in\mathbf{Z} \\ r\equiv sr_0 \bmod 2m}} e^{2\pi i\left(\frac{r^2-s^2\Delta_0}{4m}u + \frac{r^2+s^2|\Delta_0|}{4m}iv + rz + s|\Delta_0|\xi\right)} e^{-\pi m|\Delta_0|\eta^2/v}.$$

Then one has

$$\vartheta_{\Delta_0,r_0}(\tau,z;t) = \frac{\sqrt{m}}{\pi\varepsilon}\left\{ \left(\tfrac{\Delta_0}{0}\right)\theta(\tau,z,0) + \sum_A \sum_{l\geq 1} \left(\tfrac{\Delta_0}{l}\right) \frac{e^{-2\pi i m\frac{\gamma z^2}{\gamma\tau+\delta}}}{(\gamma\tau+\delta)^\times} \theta\left(\tfrac{\alpha\tau+\beta}{\gamma\tau+\delta}, \tfrac{z}{\gamma\tau+\delta}, \tfrac{lt}{\Delta_0}\right)\right\}.$$

Here A runs through a set of representatives $A = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ for $\left(\begin{smallmatrix}1 & \mathbf{Z} \\ 0 & 1\end{smallmatrix}\right)\backslash SL_2(\mathbf{Z})$ and for each such A the expression $(\gamma\tau+\delta)^\times$ equals $(\gamma\tau+\delta)^2$ for negative Δ_0, and it equals $(\gamma\bar{\tau}+\delta)|\gamma\tau+\delta|$ for positive Δ_0. Moreover, $\varepsilon = i$ for $\Delta_0 < 0$, $\varepsilon = 1$ for $\Delta_0 > 0$, and $\left(\frac{\Delta_0}{0}\right) = 1$ if $\Delta_0 = 1$ and $\left(\frac{\Delta_0}{0}\right) = 0$ otherwise.

We shall prove the proposition in the next section. Note that the given formula for $\vartheta_{\Delta_0,r_0}(\tau,z;t)$ may also be written as

$$\vartheta_{\Delta_0,r_0}(\tau,z;t)$$

$$= \frac{\sqrt{m}|\Delta_0|}{\pi\varepsilon}\left\{ \left(\tfrac{\Delta_0}{0}\right)\pi i T(\tau,z) + \sum_{\Upsilon}\sum_{l\geq 1}\sum_{s\in\mathbf{Z}} \left(\tfrac{\Delta_0}{l}\right) \frac{1}{l}\frac{\partial}{\partial t}\left(\kappa_{\Delta_0 s^2, r_0 s}|^\times \alpha\right)(\tau,z;lst)\right\}$$

with

$$\kappa_{\Delta,r}(\tau,z;t) = e^{2\pi i\left(\frac{r^2-\Delta}{4m}u + \frac{r^2+|\Delta|}{4m}iv + rz\right)} e^{2\pi i \operatorname{sign}(\Delta)\,\xi} e^{-\frac{\pi m\eta^2}{|\Delta|v}}$$

and

$$T(\tau,z) = \sum_{r,s\in\mathbf{Z}} (r - ms)\, e^{2\pi i\left(rs\tau + \frac{(r-ms)^2}{2m}iv + (r+ms)z\right)}.$$

Here Υ runs through a complete set of representatives for $\mathcal{J}(\mathbf{Z})_\infty\backslash\mathcal{J}(\mathbf{Z})$ (with $\mathcal{J}(\mathbf{Z})_\infty = [0,\mathbf{Z}]\left(\begin{smallmatrix}1 & \mathbf{Z} \\ 0 & 1\end{smallmatrix}\right)$) and '$|^\times$' stands for the action '$|$' or '$|^*$' accordingly as Δ_0 is negative or positive. This action refers to the first pair of variables, of course. A simple consequence of this formula is then

Corollary. *Let $\phi(\tau, z)$ be a Jacobi cusp form from $S_{2,m}$ or $S^{*}_{2,m}$ with Fourier coefficients $C_\phi(\Delta, r)$. Then $\langle \phi | \vartheta_{\Delta_0, r_0}(\cdot, \cdot; t)\rangle$ is a modular form of weight 2 on $\Gamma_0(m)$ with Fourier expansion*

$$\frac{\langle \phi | \vartheta_{\Delta_0, r_0}(\cdot, \cdot; t)\rangle}{(-i\varepsilon)\sqrt{m|\Delta_0|}} = (\tfrac{\Delta_0}{0})\,\langle \phi | T\rangle + \sum_{l \geq 1} \left(\sum_{a|l} (\tfrac{\Delta_0}{l}) C_\phi(\Delta_0(\tfrac{l}{a})^2, r_0 \tfrac{l}{a}) \right) \mathrm{e}^{2\pi i l t}$$

From the Corollary (proof in the next section) we now obtain that our $\phi(\tau, z)$, i.e. any $\phi(\tau, z)$ which is orthogonal to the span of the $\vartheta_{\Delta_0, r_0}(\tau, z; t)$, must necessarily satisfy $C_\phi(\Delta_0 l^2, r_0 l) = 0$ for all $l \geq 1$ and all fundamental discriminants Δ_0 and all r_0 such that $\Delta_0 \equiv r_0^2 \bmod 4m$, or, eqivalently, that it must necessarily lie in the intersection of the kernels of all the lifting maps $\phi(\tau, z) \mapsto \langle \phi | \vartheta_{\Delta_0, r_0}(\cdot, \cdot; t)\rangle$. But this implies $\phi \equiv 0$. For holomorphic Jacobi forms this was proved in [S-Z](Theorem 3), for skew-holomorphic ones this will be proved in [S]. This completes the proof of the theorem.

3. Proof of the Proposition and its Corollary. Summing over $Q = [a, b, c]$ and replacing the discriminants Δ by $(b^2 - 4mac)/\Delta_0$ in the definition of $\vartheta_{\Delta_0, r_0}(\tau, z; t)$ we can write

$$\vartheta_{\Delta_0, r_0}(\tau, z; t) = \frac{v^{\frac{1}{2}}}{\eta^2} \sum_{\substack{r, a, b, c \in \mathbf{Z} \\ b \equiv r r_0 \bmod 2m}} \mathrm{e}^{2\pi i \left(\frac{r^2 \Delta_0 - b^2}{4m\Delta_0} u + \frac{r^2|\Delta_0| + b^2}{4m|\Delta_0|} iv + rz \right)} f_{\tau,t}(r, a, b)$$

where

$$f_{\tau,t}(r, a, b) = \sum_{\substack{c \in \mathbf{Z} \\ r^2\Delta_0 \equiv b^2 - 4mac \bmod 4m|\Delta_0|}} \mathrm{e}^{2\pi i \left(\frac{ac}{\Delta_0} u - \frac{ac}{|\Delta_0|} iv \right)} \times$$
$$\times \chi_{\Delta_0}([ma, b, c])\,(mat^2 + bt + c)\,\mathrm{e}^{-\frac{\pi v}{m|\Delta_0|\eta^2}(am|t|^2 + b\xi + c)^2}.$$

To $f_{\tau,t}(r, a, b)$, the sum over c, we now apply the Poisson summation formula. Thus we write

$$f_{\tau,t}(r, a, b) = \sum_{d \in \mathbf{Z}} \psi_{r,a,b}(d)\, g\left(\frac{d}{|\Delta_0|} \right)$$

where

$$\psi_{r,a,b}(d) = \frac{1}{|\Delta_0|} \sum_{\substack{c \bmod |\Delta_0| \\ r^2\Delta_0 \equiv b^2 - 4mac \bmod 4m|\Delta_0|}} \chi_{\Delta_0}([ma, b, c])\, \mathrm{e}^{2\pi i \frac{cd}{|\Delta_0|}}$$

and

$$g(d) = \int_{-\infty}^{+\infty} \mathrm{e}^{-2\pi i \frac{a\bar{\tau}}{|\Delta_0|} c}\,(mat^2 + bt + c)\,\mathrm{e}^{-\frac{\pi v}{m|\Delta_0|\eta^2}(ma|t|^2 + b\xi + c)^2}\, \mathrm{e}^{-2\pi i c d} dc.$$

Here we used that $\chi_{\Delta_0}([ma,b,c])$ depends on c only modulo $|\Delta_0|$. Furthermore we use

$$\tilde{\tau} = \begin{cases} \tau & \text{if } \Delta_0 < 0 \\ -\bar{\tau} & \text{if } \Delta_0 > 0 \end{cases}.$$

Now, by a simple computation

$$g(d) = \frac{\sqrt{m|\Delta_0|}\eta^2}{\pi v^{\frac{1}{2}}} \frac{1}{\frac{a}{|\Delta_0|}\tilde{\tau} + d} \frac{\partial}{\partial \bar{t}} \,\mathrm{e}^{2\pi i (ma\xi^2 + b\xi)(\frac{a}{|\Delta_0|}\tilde{\tau}+d)} \,\mathrm{e}^{-\pi m |\Delta_0| \eta^2 \frac{|\frac{a}{|\Delta_0|}\tilde{\tau}+d|^2}{v}}$$

for $(a,d) \neq 0$, and

$$g(d) = i\left(\frac{m|\Delta_0|}{v}\right)^{\frac{1}{2}} \eta^2 b$$

for $(a,d) = 0$. For $(a,d) \neq 0$ let l be the greatest common divisor of a,d, set $\gamma = a/l$, $\delta = d/l$ and choose any matrix A in $SL_2(\mathbf{Z})$ such that $A = \left(\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right)$. Then we can write

$$g\left(\frac{ld}{|\Delta_0|}\right) = \frac{\sqrt{m}|\Delta_0|^{\frac{3}{2}}\eta^2}{\pi v^{\frac{1}{2}}} \frac{1}{l(\gamma\tilde{\tau}+\delta)} \frac{\partial}{\partial \bar{t}} \,\mathrm{e}^{2\pi i (ml\gamma\xi^2 + b\xi)\frac{l(\gamma\tilde{\tau}+\delta)}{|\Delta_0|}} \,\mathrm{e}^{-\pi \frac{ml^2\eta^2}{|\Delta_0| \operatorname{Im} A\tilde{\tau}}}.$$

Inserting this into the formula for $f_{\tau,t}(r,a,b)$ and then summing in the resulting formula for $\vartheta_{\Delta_0,r_0}(\tau,z;t)$ over $l \geq 1$ and a complete set of representatives A for $\left(\begin{smallmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{smallmatrix}\right)\backslash SL_2(\mathbf{Z})$ instead of $a,d \in \mathbf{Z}$ we obtain

$$\vartheta_{\Delta_0,r_0}(\tau,z;t) = \frac{\sqrt{m|\Delta_0|}}{2\pi} \frac{\partial}{\partial z'} \varphi(\tau,z,0;0,0)$$

$$+ \frac{\sqrt{m}|\Delta_0|^{\frac{3}{2}}}{\pi} \sum_A \sum_{l \geq 1} \frac{1}{l(\gamma\tilde{\tau}+\delta)} \frac{\partial}{\partial \bar{t}} \,\mathrm{e}^{2\pi i m l \gamma \xi^2 \frac{l(\gamma\tilde{\tau}+\delta)}{|\Delta_0|}} \varphi\left(\tau, z, \xi\tfrac{l(\gamma\tilde{\tau}+\delta)}{|\Delta_0|}; l\gamma, l\delta\right) \mathrm{e}^{-\pi \frac{ml^2\eta^2}{|\Delta_0| \operatorname{Im} A\tilde{\tau}}}.$$

Here for any $a, d \in \mathbf{Z}$ and any $z' \in \mathbf{C}$

$$\varphi(\tau,z,z';a,d) = \sum_{\substack{r,b \in \mathbf{Z} \\ b \equiv r r_0 \bmod 2m}} \psi_{r,a,b}(d)\, \mathrm{e}^{2\pi i \left(\frac{r^2\Delta_0 - b^2}{4m\Delta_0} u + \frac{r^2|\Delta_0| + b^2}{4m|\Delta_0|} iv + rz + bz'\right)}.$$

We shall prove in a moment that

$$\mathrm{e}^{2\pi i m l \gamma \xi^2 \frac{l(\gamma\tilde{\tau}+\delta)}{|\Delta_0|}} \varphi\left(\tau, z, \xi\tfrac{l(\gamma\tilde{\tau}+\delta)}{|\Delta_0|}; l\gamma, l\delta\right)$$

$$= \frac{\mathrm{e}^{-2\pi i m \frac{\gamma' z^2}{\gamma'\tau+\delta'}}}{[(\gamma'\tau+\delta')(\gamma\tilde{\tau}+\delta)]^{\frac{1}{2}}} \varphi\left(A'\tau, \frac{z}{(\gamma'\tau+\delta')}, \frac{l\xi}{|\Delta_0|}; 0, l\right). \tag{1}$$

Here on the right hand side that squareroot has to be taken which is positive or has positive imaginary part. Moreover $A' = \left(\begin{smallmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{smallmatrix}\right)$ equals A if Δ_0 is negative and it equals $\left(\begin{smallmatrix} \alpha & -\beta \\ -\gamma & \delta \end{smallmatrix}\right)$ if Δ_0 is positive. Now

$$\psi_{r,0,b}(l) = \frac{1}{|\Delta_0|} \sum_{c \bmod |\Delta_0|} \chi_{\Delta_0}([0,b,c])\, \mathrm{e}^{2\pi i \frac{cl}{|\Delta_0|}}$$

if $r^2\Delta_0 \equiv b^2 \bmod 4m|\Delta_0|$ and $= 0$ otherwise. So assume $r^2\Delta_0 \equiv b^2 \bmod 4m|\Delta_0|$. Then $\Delta_0 | b^2$, thus $\chi_{\Delta_0}([0,b,c]) = \left(\frac{\Delta_0}{c}\right)$ and hence $\psi_{r,0,b}(l) = \left(\frac{\Delta_0}{l}\right) \varepsilon |\Delta_0|^{-\frac{1}{2}}$ since Δ_0 is fundamental (recall $\varepsilon = i, 1$ for $\Delta_0 < 0, > 0$, respectively). For the same reason we find that $r^2 \equiv \frac{b^2}{|\Delta_0|} \bmod 4m$ and $b \equiv r r_0 \bmod 2m$ imply $\Delta_0 | b$ and $r \equiv \frac{b}{\Delta_0} r_0 \bmod 2m$, and vice versa. Thus, summing in the sum defining $\varphi(\tau, z, \xi; 0, l)$ over $\Delta_0 s$ instead of b we can write in the notation of the proposition

$$\frac{\partial}{\partial \bar{t}} \varphi(\tau, z, \mathrm{sign}(\Delta_0)\xi; 0, l)\, \mathrm{e}^{-\pi m |\Delta_0| \eta^2 / v} = \left(\tfrac{\Delta_0}{l}\right) \varepsilon |\Delta_0|^{-\frac{1}{2}} \theta(\tau, z; t).$$

Inserting this in the last formula for $\vartheta_{\Delta_0, r_0}(\tau, z; t)$ and summing over A' instead of A if $\Delta_0 > 0$, thereby noticing that $A(\tilde{\tau}) = A(-\overline{\tau}) = -A'\overline{\tau}$, we now easily recognize the asserted formula.

To prove (1) we write first of all for $a, d \in \mathbf{Z}$

$$\varphi(\tau, z, z'; a, d) = \sum_{\substack{r \bmod 2m,\ b \bmod 2m|\Delta_0| \\ b \equiv r r_0 \bmod 2m}} \psi_{r,a,b}(d)\, \vartheta_{m,r}(\tau, z)\, \vartheta_{m|\Delta_0|, b}(\tilde{\tau}, z'),$$

where $\vartheta_{N,\rho}$ for any N, ρ is the basic function

$$\vartheta_{N,\rho}(\tau, z) = \sum_{\substack{s \in \mathbf{Z} \\ s \equiv \rho \bmod 2N}} \mathrm{e}^{2\pi i \left(\frac{s^2}{4N}\tau + sz\right)}.$$

We shall show that for all $a, d \in \mathbf{Z}$ and for all $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SL_2(\mathbf{Z})$

$$(2) \quad \frac{\mathrm{e}^{-2\pi i m \left(\frac{\gamma' z^2}{\gamma'\tau + \delta'} + \frac{\gamma z'^2}{\gamma\tilde{\tau} + \delta}\right)}}{[(\gamma'\tau + \delta')(\gamma\tilde{\tau} + \delta)]^{\frac{1}{2}}}\, \varphi\left(\frac{\alpha'\tau + \beta'}{\gamma'\tau + \delta'}, \frac{z}{\gamma'\tau + \delta'}, \frac{z'}{\gamma\tilde{\tau} + \delta}; (a,d)M^{-1}\right) = \varphi(\tau, z, z'; a, d)$$

where $\left(\begin{smallmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{smallmatrix}\right) = A$ if $\Delta_0 < 0$ and $= \left(\begin{smallmatrix} \alpha & -\beta \\ -\gamma & \delta \end{smallmatrix}\right)$ otherwise. Replacing z' by $\xi \frac{l(c\tilde{\tau} + d)}{|\Delta_0|}$ and a, d by $l\gamma, l\delta$ this then clearly implies (1).

Now the left hand side of (2) defines an action of $SL_2(\mathbf{Z})$ on functions in τ, z, z' as is easily proved. Thus to verify (2) it suffices to check it for some generators A of $SL_2(\mathbf{Z})$, say $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The first case is trivially verified. To treat the second one recall (or prove by Poisson summation)

$$\frac{\mathrm{e}^{-2\pi i N\frac{z^2}{\tau}}}{\tau^{\frac{1}{2}}} \vartheta_{N,\rho}\left(\frac{-1}{\tau}, \frac{z}{\tau}\right) = \frac{\mathrm{e}^{-\frac{\pi i}{4}}}{\sqrt{2N}} \sum_{\sigma \bmod 2N} \mathrm{e}^{\frac{-2\pi i \rho\sigma}{2N}} \vartheta_{N,\sigma}(\tau, z).$$

Thus, for $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ the left hand side of (2) becomes

$$\frac{1}{2\varepsilon m|\Delta_0|^{\frac{1}{2}}} \sum_{\substack{r \bmod 2m \\ b \bmod 2m|\Delta_0| \\ b \equiv r r_0 \bmod 2m}} \psi_{r,-d,b}(a) \sum_{\substack{r' \bmod 2m \\ b' \bmod 2m|\Delta_0|}} \mathrm{e}^{2\pi i\left(\frac{rr'\Delta_0 - bb'}{2m|\Delta_0|}\right)} \vartheta_{2m,r'}(\tau, z)\,\vartheta_{2m|\Delta_0|,b'}(\tilde{\tau}, z')$$

with $\varepsilon = i$ for $\Delta_0 < 0$ and $= 1$ otherwise. We thus have to show that

$$\frac{1}{2\varepsilon m|\Delta_0|^{\frac{1}{2}}} \sum_{\substack{r \bmod 2m,\ b \bmod 2m|\Delta_0| \\ b \equiv r r_0 \bmod 2m}} \psi_{r,-d,b}(a) \mathrm{e}^{2\pi i\left(\frac{rr'\Delta_0 - bb'}{2m|\Delta_0|}\right)} = \psi_{r',a,b'}(d)$$

if $b' \equiv r' r_0 \bmod 2m$ and $= 0$ otherwise. Inserting the defining formula for $\psi_{\cdot,\cdot,\cdot}(\cdot)$ and taking on both sides of the last equation the finite Fourier transform with respect to a modulo $|\Delta_0|$ all culminates in the identity

$$\frac{1}{2\varepsilon m|\Delta_0|^{\frac{1}{2}}} \sum_{\substack{r \bmod 2m,\ b \bmod 2m|\Delta_0| \\ a \bmod |\Delta_0|,\ c \bmod |\Delta_0| \\ b \equiv r r_0 \bmod 2m \\ r^2\Delta_0 \equiv b^2 - 4mac \bmod 4m|\Delta_0|}} \chi_{\Delta_0}([ma, b, c])\, \mathrm{e}^{2\pi i\left(\frac{rr'\Delta_0 - bb' + 2m(ac' + a'c)}{2m|\Delta_0|}\right)}$$

(3)

$$= \begin{cases} \chi_{\Delta_0}([ma', b', c']) & \text{if } \left[\begin{array}{c} b \equiv r r_0 \bmod 2m \quad \text{and} \\ r^2\Delta_0 \equiv b^2 - 4mac \bmod 4m|\Delta_0| \end{array}\right] \\ 0 & \text{otherwise,} \end{cases}$$

This can now be proved using standard Gauss sum identities and we leave this to the reader.

The proof of the corollary is the usual exersise in unfolding an integral. First of all write, according to the formula immediately before the Corollary,

$$\left[\left(\tfrac{\sqrt{m}|\Delta_0|}{\pi\epsilon}\right)^{-1} \phi(\tau, z)\overline{\vartheta_{\Delta_0,r_0}(\tau, z; t)} + \pi i\left(\tfrac{\Delta_0}{0}\right)\phi(\tau, z)\overline{T(\tau, z)}\right] \mathrm{e}^{-4\pi m y^2/v} v^2$$

$$= \sum_{\Upsilon} \sum_{l\geq 1} \sum_{s\in\mathbf{Z}} \left(\tfrac{\Delta_0}{l}\right) \frac{1}{l} \frac{\partial}{\partial t} \phi(\tau', z') \overline{\kappa_{\Delta_0 s^2, r_0 s}(\tau', z'; lst)} \mathrm{e}^{-4\pi m y'^2/v'} v'^2.$$

Here we still use y and v for the imaginary parts of z and τ respectively. Furthermore Υ runs through a complete set of representatives for $\mathcal{J}(\mathbf{Z})_\infty \backslash \mathcal{J}(\mathbf{Z})$ and for each such Υ we use $(\tau', z') = \Upsilon \bullet (\tau, z)$, y' and v' denoting the imaginary parts of z' and τ' respectively. Thus unfolding the integral of the right hand side taken over a fundamental domain of $\mathcal{H} \times \mathbf{C}$ modulo $\mathcal{J}(\mathbf{Z})$ with respect to the $\mathcal{J}(\mathbf{Z})$-invariant measure $dV = \frac{du dv dx dy}{v^3}$ we obtain the expression

$$\int_{\mathcal{J}(\mathbf{Z})_\infty \backslash \mathcal{H}\times\mathbf{C}} \sum_{\alpha} \sum_{l\geq 1} \sum_{s\in\mathbf{Z}} \left(\frac{\Delta_0}{l}\right) \frac{1}{l} \frac{\partial}{\partial t} \phi(\tau, z) \overline{\kappa_{\Delta_0 s^2, r_0 s}(\tau, z; lst)} \mathrm{e}^{-4\pi m y^2/v} v^2 \, dV.$$

But

$$\kappa_{\Delta, r}(\tau, z; t) = \mathrm{e}^{2\pi i\left(\frac{r^2-\Delta}{4m} u + \frac{r^2+|\Delta|}{4m} iv + rz\right)} \mathrm{e}^{2\pi i \sigma \xi} \, \mathrm{e}^{-\frac{\pi m \eta^2}{|\Delta| v}}$$

(with $\sigma = \operatorname{sign}(\Delta_0)$), a fundamental domain for $\mathcal{H} \times \mathbf{C}$ modulo $\mathcal{J}(\mathbf{Z})_\infty$ is given by the set $\{(\tau, \lambda\tau + \mu) | 0 \leq u, \lambda, \mu \leq 1,\ 0 < v\}$, and for $z = \lambda\tau + \mu$ one has $dV = \frac{du dv d\lambda d\mu}{v^2}$. Thus, carrying out the integration with respect to u and μ we obtain

$$\sum_{l\geq 1} \sum_{s\in\mathbf{Z}} \left(\frac{\Delta_0}{l}\right) \frac{1}{l} \frac{\partial}{\partial t} C_\phi(\Delta_0 s^2, r_0 s) \, \mathrm{e}^{-2\pi i \sigma s l \xi} I_{s,l}(\eta)$$

where

$$\begin{aligned} I_{s,l}(\eta) &= \int_0^\infty \int_{-\infty}^{+\infty} \mathrm{e}^{-4\pi\left(\frac{r_0^2+|\Delta_0|}{4m} s^2 + r_0 s\lambda + m\lambda^2\right) v} \, \mathrm{e}^{\frac{-\pi m l^2 \eta^2}{|\Delta_0| v}} \, d\lambda dv \\ &= \frac{1}{2\sqrt{m}} \int_0^\infty \mathrm{e}^{-\pi\left(\frac{|\Delta_0| s^2 v}{m} + \frac{m l^2 \eta^2}{|\Delta_0| v}\right)} v^{-\frac{1}{2}} dv \\ &= \frac{1}{2} \left(\frac{l\eta}{|\Delta_0||s|}\right)^{\frac{1}{2}} \int_0^\infty \mathrm{e}^{-\pi l \eta |s| (w + \frac{1}{w})} w^{-\frac{1}{2}} dw \\ &= \left(\frac{l\eta}{|\Delta_0||s|}\right)^{\frac{1}{2}} \mathrm{e}^{-2\pi l |s| \eta} \int_0^\infty \mathrm{e}^{-\pi l |s| \eta \left(\sqrt{w} - \frac{1}{\sqrt{w}}\right)^2} d\sqrt{w} \\ &= \left(\frac{l\eta}{|\Delta_0||s|}\right)^{\frac{1}{2}} \mathrm{e}^{-2\pi l |s| \eta} \int_{-\infty}^{+\infty} \mathrm{e}^{-4\pi l |s| \eta \sinh^2\theta} \mathrm{e}^\theta d\theta \\ &= \left(\frac{l\eta}{|\Delta_0||s|}\right)^{\frac{1}{2}} \mathrm{e}^{-2\pi l |s| \eta} \int_{-\infty}^{+\infty} \mathrm{e}^{-4\pi l |s| \eta \sinh^2\theta} \cosh\theta d\theta = \frac{1}{2|\Delta_0|^{\frac{1}{2}} |s|} \mathrm{e}^{-2\pi l |s| \eta}. \end{aligned}$$

Observing that $\frac{\partial}{\partial t}e^{2\pi i(-\sigma sl\xi+l|s|i\eta)}=0$ for $-\sigma s\le 0$, and that $C_\phi(\Delta,-r)=-C_\phi(\Delta,r)$ for skew-holomorphic ϕ (as it is easily deduced from $\phi(\tau,z)=\left(\phi|^*\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)\right)(\tau,z)=-\phi(\tau,-z)$) we obtain the formula for the Fourier development of $\langle\phi|\vartheta_{\Delta_0,r_0}(\cdot,\cdot;t)\rangle$ as given in the Corollary. Note that this formula shows in particular that $\langle\phi|\vartheta_{\Delta_0,r_0}(\cdot,\cdot;t)\rangle$ is holomorphic. Since we saw in section 2 that it is bounded by a polynomial in η independently of ξ we deduce that it must even be regular at the cusps. Thus it is a modular form.

4. Proof of the Lemma. For $t=i\eta$ one has $gAgt=-A\bar{t}$ and thus $f(gAgt)d(gAgt)=-f(-A\bar{t})d(A\bar{t})$. Decompose $f(t)$ as $f_+(t)+if_-(t)$ with $f_\pm(t)=\frac{1}{2\sqrt{\pm 1}}\left(f(t)\pm\overline{f(-\bar{t})}\right)$. The modular forms $f_+(t)$ and $f_-(t)$ have real Fourier coefficients, i.e satisfy $f_\pm(-\bar{t})=\overline{f_\pm(t)}$, and hence $f_\pm(-A\bar{t})d(A\bar{t})=\overline{f_\pm(At)d(At)}$. Thus, for $t=i\eta$, the differential $f(At)d(At)+\varepsilon f(gAgt)d(gAgt)$ equals $2\mathrm{Re}\,[f_+(At)d(At)]+2i\mathrm{Re}\,[f_-(At)d(At)]$ if $\varepsilon=-1$ and equals the same expression but with Re replaced by Im and multiplied by i if $\varepsilon=+1$. We assume the first case, the other one can be treated similarily. The assumption about f then implies that for all $A\in SL_2(\mathbf{Z})$ both integrals $\int_0^{i\infty}\mathrm{Re}\,[f_\pm(At)d(At)]$ are absolutely convergent and equal to zero. Thus we may assume that f equals f_+ or f_-, or, more generally, that f itself has the property that all the integrals $\int_0^{i\infty}\mathrm{Re}\,[f(At)d(At)]$ converge absolutely and equal 0, and we have to show that $f(t)$ vanishes identically.

To prove this, consider $\varphi(B):=\int_{t_0}^{Bt_0}\mathrm{Re}\,[f(t)d(t)]$ for $B\in\Gamma_0(m)$ and $t_0\in\mathcal{H}$. Note that $\varphi(B)$ does not depend on the choice of t_0. Now let $A,B\in SL_2(\mathbf{Z})$ such that $ABA^{-1}\in\Gamma_0(m)$, let $B=\pm T^{n_1}ST^{n_2}S\dots T^{n_r}S$ with $n_j\in\mathbf{Z}$ and T, S denoting the generators $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ of $SL_2(\mathbf{Z})$ respectively, and set $B_j:=\pm T^{n_1}ST^{n_2}S\dots T^{n_j}S$, $B_0:=1$. Write $\varphi(ABA^{-1})=\int_{t_1}^{Bt_1}\mathrm{Re}\,[f(At)d(At)]$ $(t_1=A^{-1}t_0)$, $\int_{t_1}^{Bt_1}=\int_{t_1}^{B_1t_1}+\int_{B_1t_1}^{B_2t_1}+\dots+\int_{B_{r-1}t_1}^{B_rt_1}$, and $\int_{B_jt_1}^{B_{j+1}t_1}\mathrm{Re}\,[f(At)d(At)]=\int_{t_1}^{T^{n_{j+1}}St_1}\mathrm{Re}\,[f(AB_jt)d(AB_jt)]$. Note that one has $T^{n_{j+1}}St_1=-\frac{1}{t_1}+n_{j+1}$. Thus, setting $t_1=i\eta$ and letting η tend to 0, it is easily deduced from the assumptions about f, that $\int_{t_1}^{T^{n_{j+1}}St_1}\mathrm{Re}\,[f(AB_jt)d(AB_jt)]\to n_{j+1}f(AB_ji\infty)$. Here $f(s)$, for any rational number s or $s=i\infty$, denotes the constant term in the Fourier expansions of $f(t)$ at the cusp s. Summarizing, we have $\varphi(ABA^{-1})=\sum_{j=0}^{r-1}n_{j+1}f(AB_ji\infty)$.

In particular, choosing $B=1=-TSTSTS$ in this identity and observing $\varphi(1)=0$, we obtain $f(Ai\infty)+f(A1)+f(A0)=0$ for all $A\in SL_2(\mathbf{Z})$. But this implies $f(s)=0$ for any cusps s. Namely, write $s=\frac{\alpha}{\gamma}$ with relative prime integers α,γ, and choose integers β,δ such that $\alpha\delta-\beta\gamma=1$. We can even choose β,δ such that δ and $\gamma+\delta$ are prime to m, except in the case m even and γ odd, where we choose β,δ such that $\gamma+\delta$ is prime to m and $\gcd(m,\delta)=2$ (If a given solution δ of $\alpha\delta-\beta\gamma=1$ has not these properties then choose an integer ν such that $\delta + \gamma\nu \equiv -2\gamma \bmod m'$ (resp. $\bmod 2m'$ if m is even and γ is odd), where m' is the product of all primes of m which do not divide γ, and replace δ, β by $\delta + \gamma\nu, \beta - \alpha\nu$). Setting $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, we find $Ai\infty = s$, $A1 = \frac{\alpha+\beta}{\gamma+\delta}$, $A0 = \frac{\beta}{\delta}$, and, since $\gamma + \delta$, δ are prime to m (except for the case ...) the cusps $A1$, $A0$ are equivalent modulo $\Gamma_0(m)$ to the cusp 0 (except for the case ... where $A0$ is equivalent to $\frac{1}{2}$). Thus, we have $f(s) + 2f(0) = 0$ (or $f(s) + f(0) + f(\frac{1}{2}) = 0$ if m is even and γ is odd). Since this equation holds for any s, we now deduce that $f(t)$ vanishes at the cusps.

But then we conclude that $\mathrm{Re}\,[f(t)dt]$ induces a harmonic differential on the compactification of $\Gamma_0(m)\backslash\mathcal{H}$, which, by the above, satisfies $\int_{t_0}^{Bt_0} \mathrm{Re}\,[f(t)dt] = 0$ for all $B \in SL_2(\mathbf{Z})$ and all t_0. Hence the function $F(t) := \int_{t_0}^{t} \mathrm{Re}\,[f(t')dt']$ induces a harmonic function on this compact Riemann surface (to prove the invariance under $\Gamma_0(m)$, use $F(Bt) = F(t) + \int_{t_0}^{Bt_0} \mathrm{Re}\,[f(t')dt']$ for $B \in \Gamma_0(m)$), hence $F(t)$ is constant, hence $\mathrm{Re}\,[f(t)dt] \equiv 0$, and since $f(t)$ is holomorphic, this finally implies that $f(t)$ vanishes identically.

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