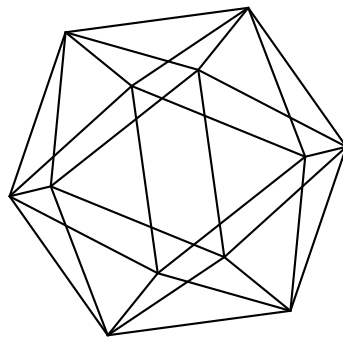# Max-Planck-Institut für Mathematik Bonn

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by

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# Spinorial Characterization of CR Structures, I

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### Abstract

We characterize CR structures of arbitrary type on Riemannian Spin$^c$ manifolds by the existence of a Spin$^c$ structure carrying a *partially pure* spinor field. Furthermore, we study the geometry and topology of Riemannian Spin$^c$ manifolds carrying a partially pure spinor which satisfies the generalized Killing equation in prescribed directions.

**Keywords:** Spin$^c$ structures, CR structures, complex structures, partially pure spinor fields, generalized Killing spinors, integrability condition, isometric immersions.

**Mathematics subject classifications (2010):** 32V05, 53C27, 53D10, 53B35.

## 1 Introduction

Spinors have played an important role in both physics and mathematics ever since they were discovered by É. Cartan in 1913. We refer the reader to Hitchin's seminal paper [42], as well as to [70, 54] for the more recent development of Seiberg-Witten theory and its notorious results on 4-manifold geometry and topology.

Cartan defined *pure spinors* [22, 23, 25] in order to characterize (almost) complex structures and, almost one hundred years later, they are still being used in related geometrical problems [16]. Furthermore, these spinor fields have been related to the notion of calibrations on a Spin manifold by Harvey and Lawson [32, 26], since distinguished differential forms are naturally associated to a spinor field and, in particular, give rise to special differential forms on immersed hypersurfaces. Pure spinors are also present in the Penrose formalism in General Relativity as they are implicit in Penrose's notion of "flag planes" [64, 65, 66].

There is the notion of abstract CR structure in odd dimensions which generalizes that of complex structure in even dimensions. This notion aims to describe intrinsically the property of being a hypersurface of a complex space form. This is done by distinguishing a distribution whose sections play the role of the holomorphic vector fields tangent to the hypersurface. It has been proved that every strictly pseudoconvex CR manifold has a canonical Spin$^c$ structure [67]. This fact, and the relation of pure spinors to complex structures, naturally led us to ask

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whether it is possible to characterize almost CR, CR and pseudoconvex CR structures by the existence of a Spin$^c$ structure carrying a special spinor field. In this paper, we define *partially pure spinors* on Riemannian Spin$^c$ manifolds, which will characterize almost CR structures of arbitrary type on such manifolds.

The existence of partially pure spinors on a Riemannian Spin$^c$ manifold $(M^n, g)$ implies the splitting of the tangent bundle $TM$ into two orthogonal distributions D and D$^\perp$. In this case, D can be endowed with an automorphism $J$ satisfying $J^2 = -1$, i.e. $M$ has an almost CR structure and the distribution D$^\perp$ can be endowed with a Spin$^c$ structure carrying a nonwhere zero spinor (see Theorem 3.4). The converse is also true, i.e. having an almost CR structure $(D, J)$ on a Riemannian manifold $(M^n, g)$ such that D$^\perp$ has a Spin$^c$ structure with a nowhere zero spinor field, implies that $M$ is a Spin$^c$ manifold carrying a partially pure spinor (see Theorem 3.4). After introducing the *integrability condition* for partially pure spinors (see Definition 3.3), this characterization can be extended to CR structures. For example, when D$^\perp$ has rank 0, we come to the notion of pure Spin$^c$ spinor and characterize almost complex, complex and Kähler structures (see Remarks 3.6 and 5.1). Notice that such characterization cannot be achieved using pure Spin spinors because not every almost complex manifold is Spin. However, every almost complex manifold is Spin$^c$. Now, if D$^\perp$ is the trivial line bundle, this characterization can be developed for pseudoconvex CR structures (see Theorem 5.5).

Note that, nowadays, the restriction of Spin$^c$ spinors is an effective tool for the study of hypersurface's geometry and topology , since these Spin$^c$ spinors contain more subtle geometric information than the classical Spin spinors [38, 59, 60, 61]. We prove that partially pure spinors appear naturally and implicitly in extrinsic Spin$^c$ geometry: consider a Kähler manifold endowed with a Spin$^c$ structure carrying a parallel spinor. It is known that the restriction $\psi$ of the parallel spinor to a real oriented hypersurface $M$ satisfies

$$\nabla_X \psi = -\frac{1}{2} II(X) \cdot \psi,$$

where $II$ denotes the second fundamental form of $M$, $\nabla$ is the Spin$^c$ covariant derivative on $M$ and "·" the Clifford multiplication on $M$ (see [52, 58]). Moreover, the spinor $\psi$ is partially pure and integrable (see Theorem 6.1).

The example above and the aforementioned distributions motivate us to consider the generalized Killing equation for partially pure spinors in prescribed directions:

$$\nabla_X \psi = E(X) \cdot \psi, \tag{1}$$

where $X \in \Gamma(\mathrm{D})$ or $\Gamma(\mathrm{D}^\perp)$ and $E$ is an endomorphism (not necessary symmetric) of D or D$^\perp$ .

Let us recall that generalized Killing spinors have played a key role in the study of intrinsic and extrinsic geometry. In the case of $M$ being a Spin manifold and $E = f\mathrm{Id}$, for some complex function $f$ on $M$, it is well known that the existence of such spinors (called parallel, Killing, imaginary Killing or generalized imaginary Killing spinors) imposes several restrictions on the geometry and topology of the manifold [49, 21, 69, 37, 30, 3, 8, 9, 10, 68]. In the case of $M$ being a Spin$^c$ manifold and $E = f\mathrm{Id}$ (see [56, 36, 62]), the geometry of the manifold is intertwined with the geometry of the auxiliary complex line bundle defining the Spin$^c$ structure. While the line bundle can be endowed, in principle, with an arbitrary connection and thus have an arbitrary curvature form $i\Omega$ (an imaginary 2-form on the manifold), Equation (1) determines a relationship between the geometries. For example, if the function $f$ is a real function, it must be a real constant if and only if $n \geq 4$ [36], and the manifold is not necessarily Einstein.

Parallel and Killing Spin$^c$ spinors have been useful in the study Kähler and Sasaki non-Einstein manifolds [56].

Now assume that $(M^n, g)$ is a Riemannian Spin$^c$ manifold carrying a generalized Killing spinor $\psi$. Then $-E$ must be the *energy-momentum* tensor $\ell^\psi$ defined on the complement of the zero set of the spinor $\psi$ by

$$g(\ell^\psi(X), Y) = \frac{1}{2}\text{Re} \ < X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2} > .$$

for any $X, Y \in \Gamma(TM)$. O. Hijazi [39] modified the spinorial Levi-Civita connection in the direction of the energy-momentum tensor, to get a lower bound for any eigenvalue $\lambda$ of the Dirac operator. This lower bound involves the scalar curvature Scal of the manifold and the energy-momentum tensor $\ell^\psi$, where $\psi$ is an eigenspinor associated with the eigenvalue $\lambda$. On Spin$^c$ manifolds, this lower bound involves additionally the curvature of the auxiliary line bundle [57]. Any eigenvalue $\lambda$ of the Dirac operator with eigenspinor $\psi$ satisfies

$$\lambda^2 \geq \inf_M (\frac{1}{4}\text{Scal} - \frac{c_n}{4}|\Omega| + |\ell^\psi|^2), \tag{2}$$

where $c_n = 2[\frac{n}{2}]^{\frac{1}{2}}$ and $|\Omega|$ is the norm of the curvature 2-form. The equality case in (2) is characterized by the existence of a spinor field $\psi$ satisfying (1) such that $\Omega \cdot \psi = i\frac{c_n}{2}|\Omega|\psi$. Since $\psi$ is an eigenspinor, the zero set is contained in a countable union of $(n-2)$-dimensional submanifolds and has locally finite $(n-2)$-dimensional Hausdroff density [4]. The trace of $\ell^\psi$ is equal to $\lambda$, so that (2) improves the well-known Friedrich Spin$^c$ inequality [30, 36].

Even though the energy-momentum tensor is not a geometric invariant since it depends on the spinor, the study of Equation (1) in extrinsic Spin or Spin$^c$ geometry is the key to a natural interpretation of this tensor. Indeed, on a Riemannian Spin$^c$ surface, the existence of a pair $(\psi, E)$ satisfying (1) is equivalent to the existence of a local immersion of the surface into $\mathbb{R}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$ with Weingarten tensor equal to $-2E$ [33, 60]. The energy-momentum tensor appears also naturally in the study of the variations of the spectrum of the Dirac operator [17, 58] and in the study of the Einstein-Dirac equation [34, 58]. As mentioned above, the energy-momentum tensor is, up to a constant, the second fundamental form of an isometric immersion into a Spin or Spin$^c$ manifold carrying a parallel spinor [52, 58]. Conversely, having a Riemannian Spin$^c$ manifold carrying a spinor field $\psi$ satisfying (1), the tensor $E$ can be realized as the Weingarten tensor of some isometric immersion of $M^n$ into a Spin$^c$ manifold $Z^{n+1}$ carrying parallel spinors [52, 5, 58, 53]. In [35], Equation (1) has been studied for an endomorphism $E$ which is not necessarily symmetric. The symmetric part of $E$ is $\ell^\psi$ and the skew-symmetric part of $E$ is $q^\psi$ defined on the complement set of zeroes of $\psi$ by

$$g(q^\psi(X), Y) = \frac{1}{2}\text{Re} \ < Y \cdot \nabla_X \psi - X \cdot \nabla_Y \psi, \frac{\psi}{|\psi|^2} >,$$

for all $X, Y \in \Gamma(TM)$. For Riemannian flows and if the normal bundle carries a parallel spinor, the tensor $q^\psi$ plays the role of the O'Neill tensor.

On a Riemannian Spin$^c$ manifold, the existence of a partially pure spinor satisfying the generalized Killing equation in prescribed directions restricts the geometry and the topology of the manifold. Indeed, if the partially pure spinor $\psi$ satisfies the generalized Killing equation in the horizontal directions (Theorem 4.1), the distribution D is involutive and hence the manifold cannot be pseudoconvex. If the spinor $\psi$ satisfies the generalized Killing equation in the vertical directions (Theorem 4.2), then the vertical distribution D$^\perp$ is totally geodesic and the manifold

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is foliated with Spin$^c$ leaves. Finally, if the partially pure spinor $\psi$ is parallel in the horizontal directions and generalized Killing in the vertical directions (Theorem 4.3), then the manifold is locally a Riemannian product of a Kähler manifold and a Spin$^c$ manifold carrying a generalized Killing spinor. Furthermore, if in the last situation the manifold is simply connected and the spinor is Killing in the vertical directions, then the manifold is isometric to the Riemannian product of a Kähler manifold and a Spin manifold carrying a Killing spinor (Corollary 4.10).

The paper is organized as follows. In Section 2, we recall basic facts about Spin$^c$, complex, CR, and contact structures on manifolds. In Section 3, we define partially pure spinors and their integrability condition in order to characterize CR structures on Spin$^c$ manifolds (Theorem 3.4). The geometry of manifolds carrying partially pure spinors satisfying the generalized Killing equation in prescribed directions is then studied in Section 4. The special cases of Kähler and pseudoconvex CR structures is examined in Section 5. Finally, in Section 6, we address some extrinsic geometry questions including immersion theorems (Theorems 6.1 and 6.3).

## 2    Preliminaries

In this section, we recall basic facts about Spin$^c$ structures [48, 51, 29, 59, 18, 40], complex structures [55, 2, 11, 43], CR structures [28, 27] and contact structures [12, 20, 19, 13] on manifolds.

### 2.1    Spin$^c$ structures on manifolds

Let $(M^n, g)$ be an oriented Riemannian manifold of dimension $n \geqslant 2$ without boundary. We denote by $\mathrm{SO}M$ the $\mathrm{SO}(n)$-principal bundle over $M$ of positively oriented orthonormal frames. A Spin$^c$ structure on $M$ is given by a Spin$^c_n$-principal bundle $(\mathrm{Spin}^c M, \pi, M)$ and an $\mathbb{S}^1$-principal bundle $(\mathbb{S}^1 M, \pi, M)$ together with a double covering $\theta : \mathrm{Spin}^c M \longrightarrow \mathrm{SO}M \times_M \mathbb{S}^1 M$ such that $\theta(ua) = \theta(u)\xi(a)$, for every $u \in \mathrm{Spin}^c M$ and $a \in \mathrm{Spin}^c_n$, where $\xi$ is the 2-fold covering map of $\mathrm{Spin}^c_n$ over $\mathrm{SO}(n) \times \mathbb{S}^1$. Let $\Sigma M := \mathrm{Spin}^c M \times_{\rho_n} \Sigma_n$ be the associated spinor bundle where $\Sigma_n = \mathbb{C}^{2^{[\frac{n}{2}]}}$ and $\rho_n : \mathrm{Spin}^c_n \longrightarrow \mathrm{End}(\Sigma_n)$ denotes the complex spinor representation. A section of $\Sigma M$ will be called a spinor field. The Spin$^c$ bundle $\Sigma M$ is equipped with a natural Hermitian scalar product denoted by $< ., . >$ and with a Clifford multiplication denoted by "$\cdot$". We recall that if $n$ is even, the spinor bundle $\Sigma M$ splits into $\Sigma^+ M \oplus \Sigma^- M$ by the action of the complex volume element. Additionally, any connection 1-form $A : T(\mathbb{S}^1 M) \longrightarrow i\mathbb{R}$ on $\mathbb{S}^1 M$ together with the connection 1-form $\omega^M$ on $\mathrm{SO}M$ for the Levi-Civita connection $\nabla$, induce a connection on the principal bundle $\mathrm{SO}M \times_M \mathbb{S}^1 M$, and hence a covariant derivative $\nabla$ on $\Gamma(\Sigma M)$ [29, 58]. The curvature of $A$ is an imaginary valued 2-form denoted by $F_A = dA$, i.e., $F_A = i\Omega$, where $\Omega$ is a real valued 2-form on $\mathbb{S}^1 M$. We know that $\Omega$ can be viewed as a real valued 2-form on $M$ [29, 46]. In this case, $i\Omega$ is the curvature form of the auxiliary complex line bundle $L$ associated to the $\mathbb{S}^1$-principal bundle via the standard representation of the unit circle. Locally, a Spin bundle always exists, and so does the square root of the auxiliary line bundle $L$. We denote by $\Sigma' M$ the locally defined spinor bundle so that $\Sigma M = \Sigma' M \otimes L^{\frac{1}{2}}$, see [29, Appendix D], [51] and

[59]. This essentially means that, while the spinor bundle and $L^{\frac{1}{2}}$ may not exist globally, their tensor product (the Spin$^c$ bundle) does.

**Remark 2.1** *The analogous objects and properties exist for oriented Riemannian vector bundles [48, 29].*

## 2.2 Complex structures on manifolds

An almost complex structure on a differentiable manifold $M^n$ is given by a $(1,1)$-tensor $J$ satisfying $J^2 = -\text{Id}_{TM}$. The pair $(M, J)$ is then referred to as an almost complex manifold which must, therefore, have even real dimension, i.e. $n = 2m$. The integer $m$ is called the complex dimension of the manifold $M$. The endomorphism $J$ can be extended by $\mathbb{C}$-linearity to the complexified tangent bundle $T^{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ and $T^{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M$ where $T_{1,0}M$ (resp. $T_{0,1}M$) denotes the eigenbundle of $T^{\mathbb{C}}M$ corresponding to the eigenvalue $i$ (resp. $-i$) of $J$. The bundle $T_{1,0}M$ is given by

$$T_{1,0}M = \overline{T_{0,1}M} = \{X - iJX \mid X \in \Gamma(TM)\}.$$

An almost complex structure is a complex structure if and only if $T_{1,0}M$ is formally integrable, i.e., $[T_{1,0}M, T_{1,0}M] \subset T_{1,0}M$. This is equivalent to saying that the Nijenhuis tensor $N^J$ vanishes. The Nijenhuis tensor $N^J$ is the $(2,1)$-tensor defined, for any $X, Y \in \Gamma(TM)$, by

$$N^J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

Now, fix a Hermitian metric $g$ compatible with the almost complex structure, i.e., a Riemannian metric $g$ with the property

$$g(JX, JY) = g(X, Y),$$

for any $X, Y \in \Gamma(TM)$ so that $J$ is orthogonal. Then $\alpha(X, Y) = g(X, JY)$ is a 2-form on $M$. We will call $(M, J, g)$ an almost Hermitian manifold. A Kähler manifold is an almost Hermitian manifold $(M, J, g)$ such that $J$ is a parallel complex structure, $\nabla J = 0$, where $\nabla$ is the Levi-Civita connection on $M$.

Every almost Hermitian manifold $(M^n, J, g)$ has a *canonical Spin$^c$ structure* whose complex spinorial bundle is given by $\Sigma M = \Lambda^{0,*}M = \oplus_{r=0}^{m} \Lambda^{0,r}M$, where $\Lambda^{0,r}M = \Lambda^r(T_{0,1}^*M)$ is the bundle of complex forms of type $(0, r)$. The auxiliary bundle of this canonical Spin$^c$ structure is given by $K_M^{-1}$, where $K_M$ is the canonical bundle of $M$ given by $K_M = \Lambda^m(T_{1,0}^*M)$ [29, 50]. The auxiliary line bundle $K_M^{-1}$ has a canonical holomorphic connection induced from the Levi-Civita connection whose curvature form is given by $i\Omega = i\rho$ where $\rho$ is the Ricci 2-form given by $\rho(X, Y) = \text{Ric}(X, JY)$. Here Ric denotes the Ricci tensor of $M$. For any other Spin$^c$ structure the spinorial bundle can be written as [29, 38]: $\Sigma M = \Lambda^{0,*}M \otimes \mathcal{L}$, where $\mathcal{L}^2 = K_M \otimes L$ and $L$ is the auxiliary bundle associated with this Spin$^c$ structure. In this case, the 2-form $\alpha$ can be considered as an endomorphism of $\Sigma M$ via Clifford multiplication and it acts on a spinor $\psi$ locally by [29, 45, 41]:

$$\alpha \cdot \psi = -\frac{1}{2} \sum_{j=1}^{m} e_j \cdot Je_j \cdot \psi,$$

where $\{e_1, e_2, ..., e_n\}$ is any local oriented orthonormal frame of $TM$. Moreover, we have the well-known orthogonal splitting

$$\Sigma M = \oplus_{r=0}^{m} \Sigma_r M,$$

where $\Sigma_r M$ denotes the eigensubbundle corresponding to the eigenvalue $i(m - 2r)$ of $\alpha$, with complex rank $\binom{m}{k}$. For any $Z \in \Gamma(T_{1,0}M)$ and for any $\psi \in \Gamma(\Sigma_r M)$, we have $Z \cdot \psi \in \Gamma(\Sigma_{r+1}M)$ and $\overline{Z} \cdot \psi \in \Gamma(\Sigma_{r-1}M)$.

With respect to the isomorphism $\Sigma M = \Lambda^{0,*}M \otimes \mathcal{L}$, the bundle $\Sigma_r M$ corresponds to $\Lambda^{0,r}M \otimes \mathcal{L}$. In fact, there is an isomorphism between $\Lambda^{0,r}M \otimes \Sigma_0 M$ and $\Sigma_r M$ for any $r = 0, \cdots, m$ [29, 45]. Hence, $\Sigma M = \Lambda^{0,*}M \otimes \Sigma_0 M$. But, $(\Sigma_0 M)^2 = K_M \otimes L$ so that $\mathcal{L} = \Sigma_0 M$, which gives $\Sigma_r M \simeq \Lambda^{0,r}M \otimes \mathcal{L}$. For the canonical Spin$^c$ structure on an almost Hermitian manifold, the subbundle $\Sigma_0 M$ is trivial [29] ($\Sigma_0 M = \Lambda^{0,0}M$). Hence, if the manifold is Kähler, this Spin$^c$ structure admits parallel spinors (complex constant functions) lying in $\Sigma_0 M$ [56].

**Remark 2.2** *In a similar way, we can define* Spin$^c$ *structures on vector bundles and* SO$(k)$*-principal fiber bundles [48, 29]. For instance, every complex vector bundle $E$ over $M$ has a canonical* Spin$^c$ *structure carrying a nowhere zero spinor $\psi \in \Gamma(\Sigma_0 E)$. Any* SO$(k)$*-principal fiber bundle $Q$ over $M$ ($k = 2l$) which has a* U$_l$*-reduction has a canonical* Spin$^c$ *structure.*

## 2.3   CR structures on manifolds

Let $M^n$ be an oriented smooth manifold of dimension $n$. Let $l \in \mathbb{N}$ be an integer such that $1 \leq l \leq [\frac{n}{2}]$. An almost CR structure on $M^n$ is a complex subbundle $T_{1,0}M$ of $TM \otimes \mathbb{C}$ of complex rank $l$ such that $T_{1,0}M \cap \overline{T_{1,0}M} = \{0\}$. The integers $l$ and $k := n - 2l$ are respectively the CR dimension and the CR codimension of the almost CR structure and $(l, k)$ is its type. A CR structure on $M^n$ of type $(l, k)$ is an almost CR structure of type $(l, k)$ such that $T_{1,0}M$ is formally integrable. An almost CR manifold (resp. a CR manifold) $M^n$ of type $(m, 0)$ is an almost complex manifold (resp. a complex manifold) [47]. Having a CR structure is equivalent to having a real subbundle $H(M)$ of $TM$ of real rank $2l$ and a bundle automorphism $J$ of $H(M)$ such that $J^2 = -\mathrm{Id}$ such that for every $X, Y \in \Gamma(H(M))$, we have

$$[X, Y] - [JX, JY] \in \Gamma(H(M)) \quad \text{and} \quad J([X, Y] - [JX, JY]) = [JX, Y] + [X, JY].$$

The equivalence is given as follows. Given $H(M)$ and $J$ as above, extend $J$ by complex linearity to $H(M) \otimes \mathbb{C}$ and let $T_{1,0}M = \{Z \in H(M) \otimes \mathbb{C}, JZ = iZ\}$. Conversely, given $T_{1,0}M$, let $J_1$ be the automorphism of $T_{1,0}M \oplus \overline{T_{1,0}M}$ which acts as multiplication by $i$ (resp. $-i$) on $T_{1,0}M$ (resp. $\overline{T_{1,0}M}$). Take $H(M) = \mathrm{Re}\,(T_{1,0}M \oplus \overline{T_{1,0}M})$ and let $J$ to be the restriction of $J_1$ to $H(M)$.

Now, we will consider a CR manifold $M^n$ of hypersurface type, i.e., a CR manifold of type $(m, 1)$. Assume $M$ to be orientable. In this case, the dimension of $M$ is odd, $n = 2m + 1$ and there exists a global 1-form $\theta$, called a pseudohermitian structure on $M$ such that $H(M) = \ker \theta$. The Levi form is given by

$$G_\theta(X, Y) = d\theta(JX, Y) \quad \text{for} \quad X, Y \in \Gamma(H(M)).$$

The given CR manifold is nondegenerate (resp. strictly pseudoconvex) if $G_\theta$ is nondegenerate (resp. positive definite). If $M$ is nondegenerate, we consider $T$ the characteristic direction of $d\theta$, i.e., the unique global nowhere zero tangent vector field $T$ on $M$ determined by $\theta(T) = 1$ and $T \lrcorner d\theta = 0$. If $M$ is a strictly pseudoconvex CR manifold, we define a Riemannian metric $g_\theta$, called the Tanaka-Webster metric, by

$$g_\theta(X, Y) = G_\theta(X, Y), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1,$$

for any $X, Y \in \Gamma(H(M))$. On a strictly pseudoconvex CR manifold $(M, g_\theta)$, there exists a unique linear connection $\nabla$ with torsion $T^\nabla$ such that $H(M)$ is parallel with respect to $\nabla$ and $\nabla J = \nabla g_\theta = 0$. The pseudo-Hermitian Tanaka torsion is then defined by $\tau(X) = T^\nabla(T, X)$ for any $X \in \Gamma(TM)$.

Every strictly pseudoconvex CR manifold $(M, g_\theta)$ has a canonical Spin$^c$ structure whose spinor bundle can be identified with the bundle $\Lambda_H^{0,*}M$ and whose auxiliary line bundle is $(K_M)^{-1}$ where $K_M = \Lambda_H^{m,0}M$ [67]. Here,

$$\Lambda_H^{0,*}M = \oplus_{r=0}^m \Lambda_H^{0,r}M = \oplus_{r=0}^m \Lambda^r(T_{0,1}^*M).$$

For any other Spin$^c$ structure, with auxiliary line bundle $L$, the spinorial bundle can be written as follows [67]: $\Sigma M = \Lambda_H^{0,*}M \otimes \mathcal{L}$, where $\mathcal{L}^2 = K_M \otimes L$. Moreover, the action of the 2-form $\alpha = d\theta$ via Clifford multiplication gives the orthogonal splitting [6, 7, 67]: $\Sigma M = \oplus_{r=0}^m \Sigma_r M$, where $\Sigma_r M$ is the eigenbundle corresponding to the eigenvalue $i(m - 2r)$ of $\alpha$ [6, 7]. As in the complex case, we have $Z \cdot \psi \in \Gamma(\Sigma_{r+1}M)$ and $\overline{Z} \cdot \psi \in \Gamma(\Sigma_{r-1}M)$ for any $Z \in \Gamma(T_{1,0}M)$ and $\psi \in \Gamma(\Sigma_r M)$. Note that for the canonical Spin$^c$ structure, the subbundle $\Sigma_0 M$ is trivial, i.e., $\Sigma_0 M = \Lambda_H^{0,0}M$.

## 2.4 Contact structures on manifolds

Let $M^{2m+1}$ be an oriented smooth manifold and $(\mathfrak{X}, \xi, \eta)$ a synthetic object consisting of a $(1,1)$-tensor field $\mathfrak{X} : TM \longrightarrow TM$, a tangent vector field $\xi$, and a differential 1-form $\eta$ on $M$. $(\mathfrak{X}, \xi, \eta)$ is an almost contact struture if

$$\mathfrak{X}^2 = -\mathrm{Id} + \eta \otimes \xi, \quad \mathfrak{X}\xi = 0, \quad \eta(\xi) = 1, \quad \eta \circ \mathfrak{X} = 0.$$

An almost contact structure is said to be normal if, for all $X, Y \in \Gamma(TM)$, $N^{\mathfrak{X}} = d\eta \otimes \xi$, where $N^{\mathfrak{X}}$ is the Nijenhuis contact tensor of $\mathfrak{X}$ defined by

$$N^{\mathfrak{X}}(X, Y) = -\mathfrak{X}^2[X, Y] + \mathfrak{X}[\mathfrak{X}X, Y] + \mathfrak{X}[X, \mathfrak{X}Y] - [\mathfrak{X}X, \mathfrak{X}Y],$$

for all $X, Y \in \Gamma(TM)$. Since $\mathfrak{X}^2 \neq -\mathrm{Id}$, this tensor differs slightly from the one defined for almost-complex structures. We recall that the Nijenhuis tensor of an almost-complex manifold vanishes if and only if the almost-complex structure is integrable. No such interpretation can be given in the case of a contact manifold. A Riemannian metric $g$ is said to be compatible with the almost contact structure if

$$g(\mathfrak{X}X, \mathfrak{X}Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all $X, Y \in \Gamma(TM)$. An almost contact structure $(\mathfrak{X}, \xi, \eta)$ together with a compatible Riemannian metric $g$ is called an almost contact metric structure. Given an an almost contact metric structure $(\mathfrak{X}, \xi, \eta, g)$ one defines a 2-form $\alpha$ by $\alpha(X, Y) = g(X, \mathfrak{X}Y)$ for all $X, Y \in \Gamma(TM)$. Now, $(\mathfrak{X}, \xi, \eta, g)$ is said to satisfy the contact condition if $\alpha = d\eta$ and if it is the case, $(\mathfrak{X}, \xi, \eta, g)$ is called a contact metric structure on $M$. A contact metric structure $(\mathfrak{X}, \xi, \eta, g)$ that is also normal is called a Sasakian structure (and $M$ a Sasaki manifold).

Every strictly pseudoconvex CR manifold is also a contact metric manifold [28, 12]. Moreover, this contact metric structure is a Sasaki structure if and only if the Tanaka torsion vanishes, i.e., $\tau = 0$ [28, 12]. Conversely, a metric contact manifold has a natural almost CR structure [28, 12]. This almost CR structure is a CR structure (and then automatically strictly pseudoconvex) if and only if $\mathfrak{X} \circ N^{\mathfrak{X}}(X, Y) = 0$ for all $X, Y \in \Gamma(\ker \eta)$ [63].

# 3 Partially pure Spin$^c$ spinors and almost CR structures

In this section, we characterize the existence and integrability of an almost CR structure on a Riemannian manifold by the existence of a Spin$^c$ structure carrying an integrable partially pure spinor.

**Definition 3.1** *Let $(M^n, g)$ be an oriented Riemannian* Spin$^c$ *manifold. A nowhere zero spinor field $\psi$ is called a* partially pure *spinor if there exists a distribution* $D \subset TM$ *of constant rank whose fiber at each point $x \in M$ is given by*

$$D_x = \{X \in T_x M \mid X \cdot \psi = i\, Y \cdot \psi,\ for\ some\ \ Y \in T_x M \smallsetminus \{0\}\}.$$

Multiplying the defining equation $X \cdot \psi = iY \cdot \psi$ by $i$ gives $Y \cdot \psi = -iX \cdot \psi$ so that by setting

$$JX := -Y$$

we get a well defined endomorphism $J$ of D such that $J^2 = -\mathrm{Id}$. This implies that the rank of D is even, say $2m$. Moreover, this almost complex structure is orthogonal. Indeed, for every $X \in \Gamma(D)$, it follows

$$X \cdot JX \cdot \psi = -i|X|^2 \psi \quad \text{and} \quad JX \cdot X \cdot \psi = i|JX|^2 \psi.$$

Hence,

$$-2g(X, JX)\psi = X \cdot JX \cdot \psi + JX \cdot X \cdot \psi = i(|JX|^2 - |X|^2)\psi,$$

so that $g(X, JX) = 0$ and $|X| = |JX|$.

A partially pure spinor is called *m-partially pure* or *of rank m* if the real rank of D is $2m$. If $n = 2m$, the $m = \frac{n}{2}$-partially pure spinor is called a *pure spinor*. Let $D^\perp$ denote the orthogonal complement of D with respect to the metric $g$. Next, we prove a lemma that will be used throughout the paper.

**Lemma 3.2** *Let $(M^n, g)$ be an oriented Riemannian* Spin$^c$ *manifold carrying an m-partially pure spinor $\psi$.*

1. *For all $X, Y \in \Gamma(D)$,*

$$\begin{aligned}([X, Y] - [JX, JY]) \cdot \psi &= -i[X, JY]\psi - i[JX, Y] \cdot \psi \\ &\quad + (X + iJX) \cdot \nabla_{Y+iJY}\psi - (Y + iJY) \cdot \nabla_{X+iJX}\psi. \quad (3)\end{aligned}$$

2. *For every non-zero vector field $u \in \Gamma(D^\perp)$, $u \cdot \psi$ is a local partially pure spinor field with respect to $(D, J)$ of rank at least $m$.*

3. *Let $u \in \Gamma(TM)$ be such that for every $X \in \Gamma(D)$, we have*

$$X \cdot u \cdot \psi = -iJX \cdot u \cdot \psi.$$

*Then $u \in \Gamma(D^\perp)$.*

**Proof.** For any $X \in \Gamma(D)$, we have $(X + iJX) \cdot \psi = 0$.

8

1. Differentiating this identity, we get

$$\nabla_Y X \cdot \psi + X \cdot \nabla_Y \psi + i\nabla_Y(JX) \cdot \psi + iJX \cdot \nabla_Y \psi = 0, \qquad (4)$$

where $Y \in \Gamma(\mathrm{D})$. Exchanging $X$ and $Y$ gives

$$\nabla_X Y \cdot \psi + Y \cdot \nabla_X \psi + iJY \cdot \nabla_X \psi + i\nabla_X(JY) \cdot \psi = 0. \qquad (5)$$

Subtract (4) from (5)

$$\begin{aligned}
[X,Y] \cdot \psi &= -i\nabla_X(JY) \cdot \psi + i\nabla_Y(JX) \cdot \psi - iJY \cdot \nabla_X \psi \\
&\quad + iJX \cdot \nabla_Y \psi + X \cdot \nabla_Y \psi - Y \cdot \nabla_X \psi.
\end{aligned} \qquad (6)$$

Substituting $X$ with $JX$, and $Y$ with $JY$ in (6)

$$\begin{aligned}
[JX, JY] \cdot \psi &= i\nabla_{JX} Y \cdot \psi - i\nabla_{JY}(X) \cdot \psi + iY \cdot \nabla_{JX} \psi \\
&\quad - iX \cdot \nabla_{JY} \psi + JX \cdot \nabla_{JY} \psi - JY \cdot \nabla_{JX} \psi.
\end{aligned} \qquad (7)$$

Finally, subtracting (7) from (6), we get (3).

2. It is straightforward since for any $X \in \Gamma(\mathrm{D})$ and $u \in \Gamma(\mathrm{D}^\perp)$, we have $X \cdot u = -u \cdot X$ and $JX \cdot u = -u \cdot JX$.

3. Assume that $X \cdot u \cdot \psi = -iJX \cdot u \cdot \psi$ for every $X \in \Gamma(\mathrm{D})$. Since $X \in \Gamma(\mathrm{D})$, we have $X \cdot \psi = -iJX \cdot \psi$. Thus,
$$-2g(u, X)\psi = 2ig(u, JX)\psi,$$
so that $g(u, X) = -ig(u, JX)$, i.e., $g(u, X) = g(u, JX) = 0$. $\qquad \square$

**Definition 3.3** *An $m$-partially pure spinor $\psi$ is called* integrable *if*

$$(X + iJX) \cdot \nabla_{(Y+iJY)} \psi = (Y + iJY) \cdot \nabla_{(X+iJX)} \psi,$$

*for every $X, Y \in \Gamma(\mathrm{D})$.*

**Theorem 3.4** *Let $M$ be an oriented $n$-dimensional Riemannian manifold. Then, the following two statements are equivalent:*

(a) *$M$ admits a $\mathrm{Spin}^c$ structure carrying a partially pure spinor (resp. an integrable partially pure spinor) of rank $m \leq [n/2]$.*

(b) *$M$ admits an orthogonal almost CR structure $(\mathrm{D}, J)$ (resp. a CR structure) of type $(m, n - 2m)$, $m \leq [n/2]$, and the orthogonal distribution $\mathrm{D}^\perp$ has a $\mathrm{Spin}^c$ structure carrying a nowhere zero spinor field.*

**Proof**. (a)$\Rightarrow$(b). Let $\psi$ be a partially pure spinor on $M$. By definition, the manifold $M$ admits an orthogonal almost CR structure $(H(M) := \mathrm{D}, J)$ of type $(m, k)$, where $k = n - 2m$, The bundle $H(M)$ has a canonical $\mathrm{Spin}^c$ structure induced by its orthogonal almost complex structure. Since $M$ is also $\mathrm{Spin}^c$, $\mathrm{D}^\perp$ must also be $\mathrm{Spin}^c$. Moreover [24, 15]

$$\Sigma M \;\cong\; \Sigma \mathrm{D} \otimes \Sigma \mathrm{D}^\perp.$$

Note that $\psi$ must be of form $\psi = \tau_0 \otimes \varphi$, where $\tau_0$ is the nowhere zero spinor in $\Sigma_0 D$ (the only spinor satisfying $(X + iJX) \cdot \tau_0 = 0$ for $X \in \Gamma(D)$), and $\varphi$ is a spinor in $\Sigma D^{\perp}$. Since $\psi$ is nowhere zero, $\varphi$ has no zeroes. If moreover $\psi$ is integrable, (3) implies that the almost CR structure $(D, J)$ is a CR structure.

(b)$\Rightarrow$(a). We denote by $L$ the auxiliary complex line bundle of the $\mathrm{Spin}^c$ structure of $D^{\perp}$. Since $M$ has an almost CR structure, D has a canonical $\mathrm{Spin}^c$ structure carrying a nonwhere zero spinor $\tau_0 \in \Sigma_0 D$ (again, the only spinor satisfying $(X + iJX) \cdot \tau_0 = 0$ for $X \in \Gamma(D)$). Now, we have

$$
\begin{aligned}
\Sigma D \otimes \Sigma D^{\perp} &= \Sigma' D \otimes \kappa_D^{-1/2} \otimes \Sigma' D^{\perp} \otimes L^{1/2} \\
&\cong \Sigma' D \otimes \Sigma' D^{\perp} \otimes (\kappa_D^{-1} \otimes L)^{1/2} \\
&\cong \Sigma' M \otimes (\kappa_D^{-1} \otimes L)^{1/2},
\end{aligned}
$$

Since $\Sigma' M \otimes (\kappa_D^{-1} \otimes L)^{1/2}$ is globally defined, $M$ admits a $\mathrm{Spin}^c$ structure with auxiliary line bundle $\kappa_D^{-1} \otimes L$. Furthermore, it carries the nowhere zero spinor $\psi = \tau_0 \otimes \varphi$, where $\varphi$ is the nowhere zero spinor for $D^{\perp}$. Note that due to its factor $\tau_0$, $\psi$ is partially pure of rank $m$ on $M$. If moreover, the almost CR structure is a CR structure, by (3) the spinor $\psi$ is integrable. $\qquad\square$

## Remark 3.5

1. *Note that our definitions of pure and partially pure spinors make no emphasis on isotropic subspaces (see [48, 22, 23]). Indeed, we do not impose the rank of D to be maximal. For instance, a pure spinor on a complex manifold could be a partially pure spinor for a distribution of positive codimension invariant by the complex structure, if it exists.*

2. *Note that the partially pure spinor $\psi$ in Theorem 3.4 may not be unique, since the spinor $\varphi$ on $D^{\perp}$ may not be unique. For instance, this will be the case when $D^{\perp}$ is parallelizable.*

3. *For partially pure spinors, we should point out that if we replace $-J$ by $J$, we have to consider the anti-canonical $\mathrm{Spin}^c$ structure in all the proofs. In this case, the integrability condition of a partially pure spinor $\psi$ is given by*

$$
(X - iJX) \cdot \nabla_{(Y-iJY)} \psi = (Y - iJY) \cdot \nabla_{(X-iJX)} \psi,
$$

*for every $X, Y \in \Gamma(D)$.*

**Remark 3.6** *Let $M$ be an oriented $n$-dimensional Riemannian manifold. In the following cases it is note necessary to assume that $D^{\perp}$ is $\mathrm{Spin}^c$ carrying a nowhere zero spinor:*

- *If $n = 2m$, $D^{\perp} = \{0\}$. Then the manifold $(M^n, g)$ is a Riemannian $\mathrm{Spin}^c$ manifold carrying a pure spinor (resp. integrable pure spinor) if and only if it has an almost complex structure (resp. complex structure) $J$ such that $(M, g, J)$ is an almost Hermitian (resp. Hermitian) manifold.*

- *If $n = 2m + 1$, $D^{\perp}$ is a trivial real line bundle. Then the manifold admits an almost CR structure (resp. a CR structure) of hypersurface type if and only if it is a Riemannian $\mathrm{Spin}^c$ manifold carrying an $m$-partially pure spinor (resp. an integrable $m$-partially pure spinor).*

- If $n = 2m + 2$, $D^\perp$ is an oriented rank 2 Riemannian vector bundle, i.e. it is a complex line bundle and it has a $\mathrm{Spin}^c$ structure with a nowhere zero spinor.

- If $D^\perp$ is a parallelizable vector bundle. □

**Examples 3.7** *Consider de following manifolds.*

- *The Heisenberg group $H^{2m+1}$ of dimension $2m + 1$ is a strictly pseudoconvex CR manifold of type $(m, 1)$ (see [28]) so that it carries $m$-partially pure integrable spinor fields.*

- *The odd-dimenisonal spheres are strictly pseudoconvex CR manifold of type $(m, 1)$ (see [28]) so that it carries $m$-partially pure integrable spinor fields. This can be checked using the homegeneous description of the sphere.*

- *The $(2m - 3)$-dimensional Stiefel manifold*

$$V_{m,2} = \frac{\mathrm{SO}(m)}{\mathrm{SO}(m - 2)}$$

*carries two non-zero Killing spinors for an appropriate metric [44], one of which turns out to be also $m$-partially pure.*

- *Similar to the space above, it can be shown that the $(4n - 4)$-dimensional homogeneous space*

$$\frac{\mathrm{U}(n)}{\mathrm{U}(n - 2)}$$

*carries an integrable $(2n - 4)$-partially pure spinor, i.e. it has a CR structure of codimension 4. Note that, in this example, the spinor is also pure, but there is a codimension 4 distribution invariant by the global almost complex structure. This invariant distribution constitutes the CR distribution of codimenison 4.*

In low dimensions, we have the following.

**Proposition 3.8** *Let $(M, g)$ be an oriented Riemannian $\mathrm{Spin}^c$ manifold of dimension 4 or 6, every non-zero positive (resp. negative) spinor field $\psi \in \Gamma(\Sigma^+ M)$ (resp. $\psi \in \Gamma(\Sigma^- M)$) is pure, i.e., every nowhere-vanishing positive or negative spinor field uniquely determines an almost complex structure.*

**Proof.** In dimensions $2m \leq 6$, every nonwhere zero positive or negative spinor is pure since the group $\mathrm{Spin}_{2m}^c$ acts transitively on the unit sphere of $\Sigma^\pm M$. □

**Proposition 3.9** *Let $(M, g)$ be an oriented Riemannian $\mathrm{Spin}^c$ manifold of dimension 3 (resp. 5), every nowhere vanishing spinor field $\psi \in \Gamma(\Sigma M)$ is 1-partially pure (resp. 2-partially pure), i.e. every nowhere-vanishing spinor field uniquely determines an orthogonal almost CR structure of codimension 1.*

**Proof.** Using the concrete realization of the $\mathrm{Spin}^c$ representation [36, 44], one immediately proves that in dimension 3 and 5, for every nowhere zero spinor $\psi$, there exists a unique unit vector field satisfying $\xi \cdot \psi = -i\psi$. Hence,$-\xi^2 = -\xi \cdot \xi = \mathrm{Id}$ acts as identity on the $\mathrm{Spin}^c$

bundle. Thus, $\Sigma M = \Sigma_1 M \oplus \Sigma_{-1} M$, where $\Sigma_1 M$ (resp. $\Sigma_{-1} M$) denotes the eigensubbundle corresponding to the eigenvalue 1 (resp. $-1$). It is clear that $\psi \in \Gamma(\Sigma_1 M)$. For the dimension 3, we consider $\{e_1, e_2, \xi\}$ an orthonormal frame of $TM$. It is known that the complex volume element acts on $\Sigma M$ as the identity. So, $-e_1 \cdot e_2 \cdot \xi \cdot \psi = \psi$. Finally, we get $e_1 \cdot \psi = -i e_2 \cdot \psi$, i.e. $\psi$ is a partially pure spinor of rank 1. In dimension 5, the bundle $\Sigma_1 M$ is of complex dimension 2. A similar proof as in dimension 4 (where $\Sigma^+ M$ is also of complex dimension 2) gives that $\psi$ is a partially pure spinor of rank 2. $\qquad\square$

# 4   Generalized Killing conditions in prescribed directions

In this section, we will study Riemannian Spin$^c$ manifolds carrying partially pure spinors which are generalized Killing spinors in prescribed directions.

## 4.1   Generalized Killing in $D$ directions

**Theorem 4.1** *Let $(M^n, g)$ be an oriented Riemannian* Spin$^c$ *manifold carrying an $m$-partially pure spinor $\psi$ such that*
$$\nabla_Y \psi = E(Y) \cdot \psi, \quad for \quad Y \in \Gamma(D),$$
*where $E \in \Gamma(\mathrm{End}(D))$. Then,*

1. *The spinor field $\psi$ is integrable if and only if $J \circ A = A \circ J$, where $A$ is the antisymmetric part of $E$. In particular, $\psi$ is integrable if $E$ is symmetric.*

2. *If $A = 0$ ($E$ is symmetric), D is involutive if and only if $E \circ J = -J \circ E$. In this case $\nabla J$ is symmetric, i.e. $\nabla J(X, Y) = \nabla J(Y, X)$ for any $X, Y \in \Gamma(D)$.*

3. *D is D-parallel if and only if $E = 0$ if and only if $\nabla J(X, Y) = 0$ for any $X, Y \in \Gamma(D)$. In particular, D is involutive and totally geodesic.*

**Proof.** 1. For all $X, Y \in \Gamma(D)$, we have
$$
\begin{aligned}
(X + iJX) \cdot (EY + iE(JY)) \cdot \psi &= X \cdot EY \cdot \psi + iX \cdot EJY \cdot \psi \\
&\quad + iJX \cdot EY \cdot \psi - JX \cdot EJY \cdot \psi \\
&= -EY \cdot X \cdot \psi - 2g(X, EY)\psi \\
&\quad - iEJY \cdot X\psi - 2ig(X, EJY)\psi \\
&\quad - iEY \cdot JX \cdot \psi - 2ig(JX, EY)\psi \\
&\quad + EJY \cdot JX \cdot \psi + 2g(JX, EJY)\psi.
\end{aligned}
$$

Using that $X \cdot \psi = -iJX \cdot \psi$, we get
$$
\begin{aligned}
(X + iJX) \cdot (EY + iEJY) \cdot \psi &= -2g(X, EY)\psi + 2g(JX, EJY)\psi \\
&\quad - 2ig(X, EJY)\psi - 2ig(JX, EY)\psi. \tag{8}
\end{aligned}
$$

Substituting $X$ with $Y$, and $Y$ with $X$ in (8) gives
$$
\begin{aligned}
(Y + iJY) \cdot (EX + iEJX) \cdot \psi &= -2g(Y, EX)\psi + 2g(JY, EJX)\psi \\
&\quad - 2ig(Y, EJX)\psi - 2ig(JY, EX)\psi. \tag{9}
\end{aligned}
$$

Now, we write $E = S + A$, where $S$ and $A$ are respectively the symmetric and the antisymmetric parts of $E$. Equations (8) and (9) give that $\psi$ is integrable if and only if

$$-4g(AY, X)\psi + 4g(JX, AJY)\psi = 4ig(JAX, Y)\psi - 4ig(AJX, Y)\psi. \tag{10}$$

Since $\psi$ is a nowhere zero spinor field, we get

$$g(AY, X) - g(JX, AJY) = g(JAX, Y) - g(AJX, Y) = 0.$$

Thus, $A \circ J = J \circ A$. Conversely, if $A \circ J = J \circ A$, then (10) holds and $\psi$ is integrable.

2. Assuming that $E$ is symmetric ($A = 0$), we get from Lemma 3.2, for all $X, Y \in \Gamma(\mathrm{D})$

$$
\begin{aligned}
[X, Y] \cdot \psi &= -i\nabla_X(JY) \cdot \psi + i\nabla_Y(JX) \cdot \psi - iJY \cdot E(X) \cdot \psi \\
&\quad + iJX \cdot E(Y) \cdot \psi + X \cdot E(Y) \cdot \psi - Y \cdot E(X) \cdot \psi.
\end{aligned} \tag{11}
$$

We also know

$$-iJY \cdot EX \cdot \psi = Y \cdot EX \cdot \psi + 2g(EX, Y)\psi + 2ig(JY, EX)\varphi,$$

$$iJX \cdot EY \cdot \psi = -X \cdot EY \cdot \psi - 2g(EY, X)\psi - 2ig(JX, EY)\varphi.$$

Inserting these two equations in (11), we get

$$
\begin{aligned}
[X, Y] \cdot \psi &= -i\nabla_X(JY) \cdot \psi + i\nabla_Y(JX) \cdot \psi \\
&\quad + 2g(EX, Y)\psi + 2ig(JY, EX)\psi \\
&\quad - 2g(EY, X)\psi - 2ig(JX, EY)\psi \\
&= -i\nabla_X(JY) \cdot \psi + i\nabla_Y(JX) \cdot \psi \\
&\quad + 2ig(JY, EX)\psi - 2ig(JX, EY)\psi.
\end{aligned} \tag{12}
$$

Hence $E \circ J = -J \circ E$ implies that $[X, Y] \in \Gamma(\mathrm{D})$ and $J([X, Y]) = \nabla_X(JY) - \nabla_Y(JX)$. Conversely, if $[X, Y] \in \Gamma(\mathrm{D})$, then taking the real scalar product of (12) with $\psi$ gives

$$
\begin{aligned}
< [X, Y] \cdot \psi, \psi > &= 2i(g(JY, EX) - g(JX, EY))|\psi|^2 \\
&\quad - i < (\nabla_X(JY) - \nabla_Y(JX)) \cdot \psi, \psi >.
\end{aligned} \tag{13}
$$

Since $[X, Y] \in \Gamma(\mathrm{D})$, we have $< [X, Y] \cdot \psi, \psi >= 0$. Hence, taking the imaginary part of (13), we obtain

$$(g(JY, EX) - g(JX, EY))|\psi|^2 = 0,$$

which gives that $E \circ J = -J \circ E$. Moreover, $J([X, Y]) = \nabla_X(JY) - \nabla_Y(JX)$ implies

$$J(\nabla_X Y) - J(\nabla_Y X) = \nabla_X(JY) - \nabla_Y(JX).$$

Hence, $\nabla J(X, Y) = \nabla J(Y, X)$ for any $X, Y \in \Gamma(\mathrm{D})$.

3. Assume that D is D parallel. For any $X \in \Gamma(\mathrm{D})$, taking the covariant derivative of $X \cdot \psi = -iJX \cdot \psi$ in direction of $Y \in \Gamma(\mathrm{D})$ implies

$$\nabla_Y X \cdot \psi + X \cdot EY \cdot \psi = -i\nabla_Y(JX) \cdot \psi - iJX \cdot EY \cdot \psi. \tag{14}$$

Hence,

$$\nabla_Y X \cdot \psi - 2g(X, EY)\psi = -i\nabla_Y(JX) \cdot \psi + 2ig(JX, EY)\psi. \tag{15}$$

Take the scalar product of the last identity with $\psi$ and using that $< \nabla_Y X \cdot \psi, \psi >=< \nabla_Y(JX) \cdot \psi, \psi >= 0$, we get $g(EY, X) = ig(EY, JX) = 0$ for any $X, Y \in \Gamma(\mathrm{D})$. Hence $E = 0$. Now, if $E = 0$, (14) implies that $\nabla_Y X \cdot \psi = -i\nabla_Y(JX) \cdot \psi$. This implies that D is D-parallel. Moreover, $J(\nabla_Y X) = \nabla_Y(JX)$, i.e., $\nabla J = 0$. Now, assume that $\nabla J = 0$, then by (15), we get that $g(EY, X) = 0$ and for all $X, Y \in \Gamma(\mathrm{D})$, i.e., $E = 0$. $\qquad\square$

**Remark 4.2** *The statement 2 in Theorem 4.1 implies that $\nabla J$ is symmetric on $\mathrm{D}$ when $E$ is symmetric and $\mathrm{D}$ is involutive. It may be interesting to study the geometry of manifolds satisfying this condition. Indeed, D. V. Alekseevsky, V. Cortés and C. Devchand introduced in [1] the notion of a special complex manifold: a complex manifold (M,J) with a flat torsionfree connection $\nabla$ such that $\nabla J$ is symmetric. This generalises Freed's definition of (affine) special Kähler manifolds.*

**Corollary 4.3** *Let $(M^{2m+1}, g)$ be an oriented Riemannian $\mathrm{Spin^c}$ manifold carrying an m-partially pure spinor $\psi$ such that*

$$\nabla_Y \psi = E(Y) \cdot \psi, \quad for \quad Y \in \Gamma(\mathrm{D}),$$

*where $E \in \Gamma(\mathrm{End D})$ is symmetric. Then, $\psi$ is integrable. Moreover, if $E \circ J = -J \circ E$, then the CR manifold $M$ cannot be a pseudoconvex CR manifold.*

**Remark 4.4** *The obstruction in Corollary 4.3 can be related to a famous extrinsic problem in CR geometry: let $M$ be a real hypersurface of a Kähler manifold. When do the induced metric and the Webster metric of $M$ coincide?. For instance, none of the Webster metrics of $\partial V$ (the boundary of the Siegel domain $V$ in $\mathbb{C}^2$) coincides with the metric induced on $\partial V$ from the standard (flat) Kähler metric of $\mathbb{C}^2$.*

## 4.2 Generalized Killing in $\mathrm{D}^\perp$ directions

**Theorem 4.5** *Let $M$ be an oriented Riemannian $\mathrm{Spin^c}$ manifold carrying an m-partially pure spinor field $\psi$ such that*

$$\nabla_u \psi = E(u) \cdot \psi, \quad for \; all \quad u \in \Gamma(\mathrm{D}^\perp),$$

*where $E \in \Gamma(\mathrm{End}(\mathrm{D}^\perp))$. Then,*

1. *The distribution $\mathrm{D}$ is $\mathrm{D}^\perp$-parallel and the orthogonal almost complex structure $J$ on $\mathrm{D}$ is $\mathrm{D}^\perp$-parallel*

$$\nabla J(X, u) = 0, \quad for \quad X \in \Gamma(\mathrm{D}), u \in \Gamma(\mathrm{D}^\perp).$$

2. *The orthogonal distribution $\mathrm{D}^\perp$ is $\mathrm{D}^\perp$-parallel, i.e. totally geodesic. In particular, $\mathrm{D}^\perp$ is involutive and the manifold $M$ is foliated with $\mathrm{Spin^c}$ leaves.*

**Proof.** 1. For every $X \in \Gamma(\mathrm{D})$, take the covariant derivative of $X \cdot \psi = -iJX \cdot \psi$,

$$\nabla_u X \cdot \psi + X \cdot \nabla_u \psi = -i\nabla_u(JX) \cdot \psi - iJX \cdot \nabla_u \psi,$$

for every $u \in \Gamma(\mathrm{D}^\perp)$. Since $\nabla_u \psi = E(u) \cdot \psi$ for every $u \in \Gamma(\mathrm{D}^\perp)$, we have

$$\nabla_u X \cdot \psi + X \cdot E(u) \cdot \psi = -i\nabla_u(JX) \cdot \psi - iJX \cdot E(u) \cdot \psi.$$

By Lemma 3.2, we know that $X \cdot E(u) \cdot \psi = -iJX \cdot E(u) \cdot \psi$, so that

$$\nabla_u X \cdot \psi = -i\nabla_u(JX) \cdot \psi.$$

This means that D is parallel in the $D^\perp$-direction and $J(\nabla_u X) = \nabla_u(JX)$, i.e., $\nabla J(X, u) = 0$.

2. From the previous paragraph, we have

$$\nabla_v X \cdot u \cdot \psi = -i\nabla_v(JX) \cdot u \cdot \psi$$

for any $X \in \Gamma(D)$ and $v \in \Gamma(D^\perp)$. By Lemma 3.2, since $\psi$ is an $m$-partially pure spinor, $u \cdot \psi$ satisfies $X \cdot u \cdot \psi = -iJX \cdot u \cdot \psi$ for all $X \in \Gamma(D)$. Take the covariant derivative in the direction of $v \in \Gamma(D^\perp)$ to get

$$
\begin{aligned}
\nabla_v X \cdot u \cdot \psi + X \cdot \nabla_v u \cdot \psi + X \cdot u \cdot \nabla_v \psi &= -i\nabla_v(JX) \cdot u \cdot \psi - iJX \cdot \nabla_v u \cdot \psi \\
&\quad -iJX \cdot u \cdot \nabla_v \psi.
\end{aligned}
\tag{16}
$$

Since $\psi$ satisfies $\nabla_v \psi = E(v) \cdot \psi$, we have

$$
\begin{aligned}
X \cdot u \cdot \nabla_v \psi &= X \cdot u \cdot E(v) \cdot \psi \\
&= -iJX \cdot u \cdot E(v) \cdot \psi \\
&= -iJX \cdot u \cdot \nabla_v \psi.
\end{aligned}
$$

Hence

$$X \cdot \nabla_v u \cdot \psi = -iJX \cdot \nabla_v u \cdot \psi. \tag{17}$$

This means, by Lemma 3.2, that $D^\perp$ is $D^\perp$-parallel, totally geodesic and involutive. $\qquad\square$

**Examples 4.6**    *1. Consider $\mathbb{C}^2$ as a flat Kähler manifold with the euclidean metric and its standard complex structure given by multiplication by $i$. We consider the corresponding $\mathrm{Spin}^c$ structure with flat auxiliary line bundle and the corresponding parallel pure spinor $\psi \in \Sigma^+\mathbb{C}^2$. The unit sphere $\mathbb{S}^3 \hookrightarrow \mathbb{C}^2$ inherits a CR-structure of type $(1,1)$ from the complex structure on $\mathbb{C}^2$ and a $\mathrm{Spin}^c$ structure whose auxiliary line bundle is also flat (then it is the unique $\mathrm{Spin}$ structure on $\mathbb{S}^3$). In Section 6, we will prove that the restriction of $\psi$ to $\mathbb{S}^3$ gives an an integrable 1-partially pure spinor $\varphi \in \Sigma\mathbb{S}^3 \cong \Sigma^+\mathbb{C}^2|_{\mathbb{S}^3}$ on $\mathbb{S}^3$ (see Proposition 6.1). Moreover, the spinor $\varphi$ satisfies (see Section 6)*

$$\nabla_X^{\mathbb{S}^3} \varphi = -\frac{1}{2}II(X) \bullet \varphi = \frac{1}{2}X \bullet \varphi,$$

*where $II$ denotes the second fundamental form of $\mathbb{S}^3$ as an endomorphism of its tangent bundle and "$\bullet$" denotes the Clifford multiplication on $\mathbb{S}^3$. Thus, $\varphi$ is a real Killing spinor. Now consider the product $\underline{\mathbb{S}^3} \times \mathbb{S}^3 \hookrightarrow \mathbb{C}^2 \times \mathbb{C}^2$. The spinor $\psi \otimes \psi \in \Gamma(\Sigma^+\mathbb{C}^2 \otimes \Sigma^+\mathbb{C}^2)$ is parallel and pure, so that its restriction $\varphi \otimes \varphi \in \Sigma(\underline{\mathbb{S}^3} \times \mathbb{S}^3) \cong (\Sigma^+\mathbb{C}^2 \otimes \Sigma^+\mathbb{C}^2)|_{\mathbb{S}^3 \times \mathbb{S}^3}$ [24, 15] is an integrable partially pure spinor (see Proposition 6.1), which is also Killing of Killing constant $\frac{1}{2}$ because $\underline{\mathbb{S}^3} \times \mathbb{S}^3$ is totally umbilic in $\mathbb{C}^2 \times \mathbb{C}^2$. Hence,*

$$\nabla_{X_1+X_2} \varphi \otimes \varphi = \frac{1}{2}(X_1 + X_2) \cdot (\varphi \otimes \varphi).$$

*Here $X_1 \in \Gamma(\Sigma\underline{\mathbb{S}^3})$, $X_2 \in \Gamma(\Sigma\mathbb{S}^3)$, $\nabla$ is the twisted connection of $\nabla^{\mathbb{S}^3}$ and $\nabla^{\underline{\mathbb{S}^3}}$ and "$\cdot$" denotes the Clifford multiplication on the product $\underline{\mathbb{S}^3} \times \mathbb{S}^3$ [24, 15]. Thus, the product of spheres satisfies the hypotheses of Proposition 4.5 and it is known that it is foliated by the 2-tori.*

2. *A Riemannian manifold is said to be homogeneous if its isometry group acts transitively on it, i.e., for any two points $p$ and $q$, there exists an isometry that maps $p$ to $q$. A homogeneous manifold is necessarily complete. It is a classical result of Riemannian geometry that a homogeneous 2-manifold has constant curvature. Consequently, up to homotheties there are only three simply connected homogeneous 2-manifolds: the Euclidean plane $\mathbb{R}^2$, the sphere $\mathbb{S}^2$ and the hyperbolic plane $\mathbb{H}^2$. In dimension 3, the classifcation of simply connected homogeneous manifolds is also well-known but more examples arise. Such a manifold has an isometry group of dimension 3, 4 or 6. When the dimension of the isometry group is 6, then we have a space form ($\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$). When the isometry group has dimension 3, then we have the solvable group $\mathrm{Sol}_3$. The ones with a 4-dimensional isometry group are denoted by $\mathbb{E}(\kappa, \tau)$. All manifolds $\mathbb{E}(\kappa, \tau)$ have the property that there exists a Riemannian fibration*

$$\mathbb{E}(\kappa, \tau) \longrightarrow \mathbb{M}^2(\kappa),$$

*over the simply connected surface $\mathbb{M}^2(\kappa)$ of curvature $\kappa$ with bundle curvature $\tau$. The bundle curvature $\tau$ measures the defect to be a product. When $\tau = 0$, the fibration is trivial, i.e., $\mathbb{E}(\kappa, \tau)$ is nothing but the product space $\mathbb{M}^2(\kappa) \times \mathbb{R}$. There exist five different kinds of manifolds according to the parameters $\tau$ and $\kappa$ : the product spaces $\mathbb{S}^2(\kappa) \times \mathbb{R}$ and $\mathbb{H}^2(\kappa) \times \mathbb{R}$, Berger spheres, the Heisenberg group $\mathrm{Nil}_3$ and the universal cover of the Lie group $\mathrm{PSL}_2(\mathbb{R})$. Homogeneous 3-manifolds are also related to "Thurston geometries". In fact, all homogeneous manifolds of dimension 3, i.e., $\mathbb{R}^3, \mathbb{H}^3, \mathbb{S}^3, \mathrm{Sol}_3$ and all $\mathbb{E}(\kappa, \tau)$, except Berger spheres are the eight geometries of Thurston.*

*Berger spheres $\mathbb{E}(\kappa, \tau)$ can be isometrically immersed into the complex space form $\mathbb{M}^4(\frac{\kappa}{4} - \tau^2)$ of constant holomorphic sectional curvature $\kappa - 4\tau^2$. Moreover, the second fundamental form of the immersion is given by*

$$II(X) = -\tau X - \frac{4\tau^2 - \kappa}{\tau} g_{\mathbb{M}^4}(X, \xi)\xi,$$

*for any $X \in \Gamma(T\mathbb{E}(\kappa, \tau))$. Here we recall that $\xi := -J\nu$ is the vector field defining the Sasaki structure on Berger spheres and $\nu$ is the normal vector of the immersion. The restriction of the parallel pure spinor on $\mathbb{M}^4$ induces an 1-partially pure spinor field $\varphi$ on $\mathbb{E}(\kappa, \tau)$ satisfying (see Proposition 6.1)*

$$\nabla_X \varphi = \frac{\tau}{2} X \bullet \varphi + \frac{4\tau^2 - \kappa}{8\tau} \eta(X)\xi \bullet \varphi,$$

*where $\eta$ is the 1-form given by $\eta(X) = g(X, \eta)$ for any $X \in \Gamma(T\mathbb{E}(\kappa, \tau))$. As in the first example, the product $\mathbb{E}(\kappa, \tau) \times \mathbb{E}(\kappa, \tau)$ carries a spinor field $\varphi \otimes \varphi$ which is a partially pure spinor and it satisfies*

$$\nabla_{X_1 + X_2} \varphi \otimes \varphi = \frac{\tau}{2}(X_1 + X_2) \cdot (\varphi \otimes \varphi) \quad + \quad \frac{4\tau^2 - \kappa}{8\tau} \eta_1(X_1)\xi_1 \cdot (\varphi \otimes \varphi)$$
$$+ \frac{4\tau^2 - \kappa}{8\tau} \eta_2(X_2)\xi_2 \cdot (\varphi \otimes \varphi),$$

*i.e., $\nabla_{X_1 + X_2} \varphi \otimes \varphi = E(X_1 + X_2) \cdot \varphi \otimes \varphi$ for*

$$E(X_1 + X_2) = \frac{\tau}{2}(X_1 + X_2) + \frac{4\tau^2 - \kappa}{8\tau} \eta_1(X_1)\xi_1 + \frac{4\tau^2 - \kappa}{8\tau} \eta_2(X_2)\xi_2.$$

*Thus, the product of Berger spheres satisfies the hypotheses of Proposition 4.5 and it is known that it is foliated by the 2-tori.*

3. The $(2m - 3)$-dimensional Stiefel manifold

$$V_{m,2} = \frac{\mathrm{SO}(m)}{\mathrm{SO}(m - 2)}$$

carries an $m$-partially pure Killing spinor for an appropriate metric [44]. This is consistent with it being a circle fibration over the real Grassmannian

$$\frac{\mathrm{SO}(m)}{\mathrm{SO}(m - 2) \times \mathrm{SO}(2)}.$$

4. It can be shown that the $(4n - 4)$-dimensional homogeneous space

$$\frac{\mathrm{U}(n)}{\mathrm{U}(n - 2)}$$

carries an integrable $(2n-4)$-partially pure spinor which is Killing in the vertical directions. This is consistent with it being an $\mathrm{U}(2)$-fibration over the complex Grassmannian

$$\frac{\mathrm{U}(n)}{\mathrm{U}(n - 2) \times \mathrm{U}(2)}.$$

## 4.3   Generalized Killing preserving the split $\mathrm{D} \oplus \mathrm{D}^\perp$

**Theorem 4.7** *Let $(M^n, g)$ be an oriented Riemannian* Spin$^c$ *manifold carrying an $m$-partially pure spinor $\psi$ such that*

$$\nabla_X \psi = E(X) \cdot \psi,$$

*for all $X \in \Gamma(TM)$ and $E \in \Gamma(\mathrm{End}(TM))$. Assume that $E = 0$ on $\mathrm{D}$ and $\mathrm{D}^\perp$ is $E$-invariant. Then, $M$ is locally the Riemannian product of a Kähler manifold and a* Spin$^c$ *manifold carrying a generalized Killing spinor.*

**Proof.** By Theorems 4.1 and 4.5, the distributions $\mathrm{D}$ and $\mathrm{D}^\perp$ are parallel in all directions. Hence, $M$ is a local Riemannian product, where the integral manifold of $\mathrm{D}$ must be Kähler. $\square$

**Remark 4.8**    *1. The last theorem is related to Moroianu's result on parallel* Spin$^c$ *spinors. In [56], the existence of $\mathrm{D}$ is derived from the parallelnes hypothesis, while here, it is assumed. Here, however, we relax the parallelness condition in the $\mathrm{D}^\perp$ direction.*

2. *We can see that Theorem 4.5 holds also if $E$ is a complex endomorphism of the complexified vector bundle of $\mathrm{D}^\perp$. Thus, Theorem 4.7 is also true for complex endomorphism $E$. For example, $E$ could be an imaginary endomorphism of $TM$.*

**Example 4.9** [14] *Let $N$ be a totally geodesic CR-submanifold of a Kähler manifold $M$. Then $N$ is locally the Riemannian product of a Kähler submanifold and a totally real submanifold.* $\square$

**Corollary 4.10** *Let $(M^n, g)$ be a simply connected Riemannian* Spin$^c$ *manifold carrying an $m$-partially pure spinor $\psi$ such that*

$$\nabla_X \psi = E(X) \cdot \psi,$$

*for all $X \in \Gamma(TM)$, where $E \in \Gamma(\mathrm{End}(TM))$ with $E = 0$ on $\mathrm{D}$ and $E = \lambda Id$ on $\mathrm{D}^\perp$, $\lambda \in \mathbb{R}$. Then, $M$ is isometric to the Riemannian product of a Kähler manifold and a* Spin *manifold carrying a Killing spinor.*

**Proof.** By Theorem 4.7, $(M^n, g)$ is the Riemannian product of a Kähler manifold $M_1$ and a Spin$^c$ manifold $M_2$ with a Killing spinor of Killing constant $\lambda$. By the Spin$^c$ Ricci identity [36, 29, 59], we have

$$\mathrm{Ric}(u) \cdot \psi - i(u \lrcorner \Omega) \cdot \psi = 4(n-1)\lambda^2 u \cdot \psi, \tag{18}$$

for any $u \in \Gamma(\mathrm{D}^\perp) = \Gamma(TM_2)$ and

$$\mathrm{Ric}(X) \cdot \psi = i(X \lrcorner \Omega) \cdot \psi. \tag{19}$$

for any $X \in \Gamma(\mathrm{D}) = \Gamma(TM_1)$, where Ric denotes the Ricci tensor as symmetric endomorphism of $M$. We want to prove that for every element in $u \in \mathrm{D}^\perp$, we have $\mathrm{Ric}(u) = 4(n-1)\lambda^2 u$. Let $Y = \mathrm{Ric}(u) - 4(n-1)\lambda^2 u$. By Equation (18), we have that $Y \in \Gamma(\mathrm{D})$, so that $g(v, Y) = 0$ for all $v \in \Gamma(\mathrm{D}^\perp)$. Now, let $Z \in \Gamma(\mathrm{D})$. Then

$$
\begin{aligned}
g(Y, Z) &= g(\mathrm{Ric}(u) - 4(n-1)\lambda^2 u, Z) \\
&= g(\mathrm{Ric}(u), Z) \\
&= g(u, \mathrm{Ric}(Z)) \\
&= 0,
\end{aligned}
$$

since, by (19), $\mathrm{Ric}(Z) \in \Gamma(\mathrm{D})$. $\qquad\square$

# 5 Special cases : Kähler and pseudonconvex CR structures

**Corollary 5.1** *Let $(M^n, g)$ be an oriented Riemannian manifold. The manifold $M$ has a* Spin$^c$ *structure carrying a parallel pure spinor $\psi$ if and only if $M$ is a Kähler manifold.* $\qquad\square$

Corollary 5.1 was proved implicitly by A. Moroianu in [56] to classify simply connected Spin$^c$ manifolds.

**Corollary 5.2** *Let $M$ be a simply connected irreducible Kähler manifold. The only* Spin$^c$ *structures on $M$ carrying a parallel pure spinor are the canonical and the anti-canonical ones. Moreover, the space of parallel pure spinors is $1$-dimensional for the canonical and the anti-canonical* Spin$^c$ *structures.*

**Proof.** On a simply connected irreducible Kähler manifold, the only Spin$^c$ structures carrying a parallel spinor are the canonical and the anti-canonical ones [56]. Thus, Corollary 5.1 gives the result. $\qquad\square$

**Corollary 5.3 ([48])** *Let $(M^n, g)$ be a simply connected Riemannian manifold. $M$ is a Kähler Ricci-flat manifold if and only if $M$ has a* Spin *structure carrying a parallel pure spinor field.*

**Proof.** Assume having a Kähler Ricci-flat manifold. Thus, by Corollary 5.1, it has a Spin$^c$ structure with a parallel pure spinor. Since $(M^n, g)$ is a Ricci flat and a simply connected manifold, the auxiliary line bundle is trivial and it is endowed with the trivial connection [56]. Thus, the Spin$^c$ structure is a Spin structure [56]. Conversely, having a Spin structure with a parallel pure spinor implies, by Corollary 5.1, that the manifold is Kähler. Because $M$ is Spin, the existence of a parallel spinor implies that $M$ is Ricci flat. $\qquad\square$

Proposition 3.9 and Corollary 5.1 imply the following.

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**Corollary 5.4** *In dimension 4 or 6, a manifold carries a* Spin$^c$ *structure with a parallel spinor if and only if it is a Kähler manifold.* □

Next, we will focus our attention on oriented Riemannian manifolds of dimension $n = 2m + 1$ carrying an integrable $m$-partially pure spinor field $\psi$.

Let $(M^{2m+1}, g)$ be a Riemannian manifold carrying an $m$-partially pure spinor field $\psi$. We have $TM = D \oplus D^\perp$, where $D^\perp$ is a trivial real line bundle over $M$. Consider the following vector field $\xi^\psi$ defined by

$$g(X, \xi^\psi) = i < X \cdot \psi, \psi >,$$

for all $X \in \Gamma(TM)$. By definition, if $X \in \Gamma(D)$, then $(X + iJX) \cdot \psi = 0$. Taking the scalar product of the last equality with $\psi$ gives

$$< X \cdot \psi, \psi >= i < JX \cdot \psi, \psi >= 0.$$

Hence $g(X, \xi^\psi) = 0$. We denote by $\theta^\psi$ the 1-form associated with $\xi^\psi$ with respect to the metric $g$. and by $G_{\theta^\psi}$ the symetric 2-form defined by

$$G_{\theta^\psi}(X, Y) = d\theta^\psi(X, JY),$$

for any $X, Y \in \Gamma(D)$.

**Theorem 5.5** *Let $M^{2m+1}$ be an oriented smooth manifold. Then, $M^{2m+1}$ is a pseudoconvex CR manifold if and only if it has a* Spin$^c$ *structure carrying an $m$-partially pure integrable spinor such that $G_{\theta^\psi}$ is positive definite.*

**Proof.** Assume that $M$ is a pseudoconvex CR manifold. Then $H(M) = \ker \theta$ for some hermitian structure $\theta$. We consider the Tanaka-Webster metric $g_\theta$, which is Riemannian. It is known that $(M^{2m+1}, g_\theta)$ has a canonical Spin$^c$ structure carrying an integrable $m$-partially pure spinor field $\psi$ for which $D = H(M)$. Indeed, $\psi \in \Gamma(\Sigma_0 M)$ is the nowhere zero spinor field trivializing $\Sigma_0 M$ so that we can choose it to be a nonzero constant function and $|\psi| = 1$. Moreover, we recall that there exists a unique vector field $T$ such that $\theta(T) = 1$ and $T \lrcorner d\theta = 0$. If we prove that $T = \xi^\psi$ (and hence $\theta = \theta^\psi$) then $G_{\theta^\psi}$ is positive definite. First, we claim that $\xi^\psi$ cannot be zero. Indeed, if $\xi^\psi = 0$, then $g(X, \xi^\psi) = 0$ for any $X \in \Gamma(TM)$. In particular, $g(T, \xi^\psi) = 0$. Since $T \cdot \psi = -i\psi$, we get $|\psi| = 0$, a contradiction. Thus, one can assume that $\xi^\psi$ is of unit length. For any $X \in \Gamma(D = H(M))$, we have $g(X, \xi^\psi) = 0$ and $\xi^\psi$ is colinear to $T$. Since $\xi^\psi$ has length 1, $T = \xi^\psi$.

Now assume that a Riemannian manifold $(M^{2m+1}, g)$ carries an $m$-partially pure spinor which is integrable and such that $G_{\theta^\psi}$ is positive definite. Then, we have a CR structure such that $TM = D \oplus D^\perp$. It remains to prove that it is a pseudoconvex structure. For this, it is sufficient to prove that $\ker \theta^\psi = D$. By definition, for any $X \in \Gamma(D)$, we have $< X \cdot \psi, \psi >= 0$. Hence $g(X, \xi^\psi) = 0$ and $\theta^\psi(X) = 0$. □

**Remark 5.6** *From the proof of Theorem 5.5, a $(2m+1)$-dimensional Riemannian* Spin$^c$ *manifold carrying an $m$-partially pure integrable spinor such that $G_{\theta^\psi}$ is positive definite, is a pseudoconvex manifold. Moreover, in this case, $T = \xi^\psi$ and the Tanaka Webster metric is given by $g_{\theta^\psi}$, i.e.,*

$$g_{\theta^\psi}(X, Y) = G_{\theta^\psi}(X, Y), \quad for \ any \quad X, Y \in H(M).$$

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**Definition 5.7** *On an oriented Riemannian* Spin$^c$ *manifold* $(M^{2m+1}, g)$, *an* $m$-*partially integrable pure spinor field is called pseudoconvex if* $G_{\theta^\psi}$ *is positive definite.*

**Corollary 5.8** *Every oriented contact Riemannian manifold has a* Spin$^c$ *structure carrying an* $m$-*partially pure spinor. Moreover, this spinor is integrable if the contact structure is normal, i.e., if $M$ is a Sasaki manifold.*

**Proof.** We know that any contact Riemannian manifold has a Spin$^c$ structure. Moreover, for this Spin$^c$ structure, we have $\Sigma_0 M$ is trivial. Then, $M$ carries an $m$-partially pure spinor. Similar to Lemma 3.2, we have

$$(N^{\mathfrak{X}}(X,Y) + \eta([X,Y])\xi) \cdot \psi = (X + i\mathfrak{X}X)\nabla_{Y+i\mathfrak{X}Y}\psi - (Y + i\mathfrak{X}Y)\nabla_{X+i\mathfrak{X}X}\psi,$$

for all $X, Y \in \Gamma(\mathrm{D})$. But $d\eta(X,Y) = -\eta([X,Y])$ for all $X, Y \in \Gamma(\ker\eta)$. Hence, if the contact metric is normal then $\psi$ is integrable. $\qquad\square$

**Corollary 5.9** *Let $M$ be an oriented contact Riemannian manifold. Then, it is a* Spin$^c$ *manifold carrying an integrable pseudoconvex* $m$-*partially pure spinor if and only if* $\mathfrak{X} \circ N^{\mathfrak{X}} = 0$.
$\square$

**Corollary 5.10** *$M$ be a Sasaki manifold satisfying $N^{\mathfrak{X}} \circ \mathfrak{X} = 0$. Then, it is a* Spin$^c$ *manifold carrying a pseudoconvex integrable* $m$-*partially pure spinor field. Conversely, if $M^{2m+1}$ is a Riemannian* Spin$^c$ *manifold carrying a pseudoconvex integrable* $m$-*partially pure spinor field such that $\tau = 0$, then $M$ is a Sasaki manifold.* $\qquad\square$

# 6  Isometric immersions via partially pure Spin$^c$ spinors

Let $N^{2m-1}$ be an oriented real hypersurface of a Kähler manifold $(M^{2m}, \overline{g}, J)$ endowed with the metric $g$ induced by $\overline{g}$. We denote by $\nu$ the unit normal inner vector globally defined on $M$ and by $II$ the second fundamental form of the immersion. Moreover, the complex structure $J$ induces on $N$ an almost contact metric structure $(\mathfrak{X}, \xi, \eta, g)$, where $\mathfrak{X}$ is the $(1,1)$-tensor defined by $g(\mathfrak{X}X, Y) = \overline{g}(JX, Y)$ for all $X, Y \in \Gamma(TN)$, $\xi = -J\nu$ is a tangent vector field and $\eta$ the 1-form associated with $\xi$, that is $\eta(X) = g(\xi, X)$ for all $X \in \Gamma(TN)$. Then, for every $X \in \Gamma(TN)$

$$\mathfrak{X}^2 X = -X + \eta(X)\xi, \quad g(\xi, \xi) = 1, \quad \text{and} \quad \mathfrak{X}\xi = 0. \tag{20}$$

Moreover, from the relation between the Riemannian connections $\overline{\nabla}$ of $M$ and $\nabla$ of $N$, $\overline{\nabla}_X Y = \nabla_X Y + g(IIX, Y)\nu$, we deduce the two following identities:

$$(\nabla_X \mathfrak{X})Y = \eta(Y)IIX - g(IIX, Y)\xi \quad \text{and} \quad \nabla_X \xi = \mathfrak{X}IIX,$$

for every $X, Y \in \Gamma(TN)$. It is not difficult to see that we can choose $\{e_1, e_2 = \mathfrak{X}e_1, \dots, e_{2m-3}, e_{2m-2} = \mathfrak{X}e_{2m-3}, \xi\}$ an orthonormal frame of $N$ such that $\{e_1, e_2 = \mathfrak{X}e_1, \dots, e_{2m-3}, e_{2m-2} = \mathfrak{X}e_{2m-3}, \xi, \nu = J\xi\}$ is an orthonormal frame of $M$.

**Theorem 6.1** *Let $(M^{2m}, g, J)$ be a complex manifold. Then, any real oriented hypersurface $N$ of $M$ has a* Spin$^c$ *structure carrying an* $(m-1)$-*partially pure spinor which is integrable.*

**Proof.** Since $M$ is a complex manifold, it has a canonical $\mathrm{Spin}^c$ structure carrying a pure integrable spinor field $\psi \in \Gamma(\Sigma_0 M)$. The restriction of this $\mathrm{Spin}^c$ structure to an oriented real hypersurface $N^{2m-1}$ gives a $\mathrm{Spin}^c$ structure carrying a spinor field $\varphi = \psi|_N$ satisfying, for all $X \in \Gamma(TN)$,

$$\nabla_X^N \varphi = \nabla_X^M \psi|_M - \frac{1}{2} II(X) \bullet \varphi.$$

We will prove that the spinor field $\varphi$ is an integrable $(m-1)$-partially pure spinor field. For all $j = 1, \cdots, 2m-2$, we have

$$(e_j + i\mathfrak{X} e_j) \bullet \varphi = (e_j + i\mathfrak{X} e_j) \cdot \nu \cdot \psi|_M = -\nu \cdot (e_j + i\mathfrak{X} e_j) \cdot \psi|_M = 0,$$

since $(e_j + i\mathfrak{X} e_j) \cdot \psi = \underbrace{(e_j + iJe_j) \cdot \psi = 0}_{\psi \in \Gamma(\Sigma_0 M)}$. Then, the distribution

$$\mathrm{D} = \{X \in \Gamma(TN), \quad X \bullet \varphi = -i\mathfrak{X} X \bullet \varphi\},$$

is of constant rank $(m-1)$, i.e. $\varphi$ is a $(m-1)$-partially pure spinor. Moreover, we can see that the distribution D is nothing more than the natural almost CR structure on a real hypersurface of a complex manifold, i.e., $\mathrm{D} = H(N)$, where $H(N) = \mathrm{Re}\,(T_{1,0} N \oplus \overline{T_{1,0} N})$ and

$$T_{1,0} N = \{Z \in H(N) \otimes \mathbb{C}, \mathfrak{X} Z = iZ\} \subset T_{1,0} M.$$

By the $\mathrm{Spin}^c$ Gauss formula, the spinor $\varphi$ is integrable if and only if $\psi$ is integrable because

$$(X + iJX) \bullet II(Y + iJY) \bullet \varphi = (Y + iJY) \bullet II(X + iJX) \bullet \varphi,$$

for all $X, Y \in \Gamma(\mathrm{D})$. Since $\psi$ is integrable, $\varphi$ is also integrable. $\qquad\square$

**Remark 6.2** *From Proposition 6.1, any real oriented hypersurface $N$ of a Kähler manifold $M$ has a $\mathrm{Spin}^c$ structure carrying an $(m-1)$-partially integrable pure spinor $\varphi$ satisfying*

$$\nabla_X^N \varphi = -\frac{1}{2} II(X) \bullet \varphi,$$

*for all $X \in \Gamma(TN)$.*

**Theorem 6.3** *Let $(N^{2m+1}, g)$ be an oriented almost contact metric $\mathrm{Spin}^c$ manifold carrying an parallel $m$-partially pure spinor $\psi$, and $I = [0,1]$. Then the product $\mathcal{Z} := M \times I$ endowed with the metric $dt^2 + g$ and the $\mathrm{Spin}^c$ structure arising from the given one on $M$ is a Kähler manifold having a parallel spinor $\varphi$ whose restriction to $M$ is just $\varphi$.*

**Proof.** First, the pull back of the $\mathrm{Spin}^c$ structure on $M$ defines a $\mathrm{Spin}^c$ structure on $M \times I$. Moreover, from the spinor field $\psi$, we can construct on $M \times I$ a parallel spinor $\psi$. It remains to show that $M \times I$ is Kähler. We define the endomorphism $\overline{J}$ by

$$\overline{J}(X) = J(X) \text{ for any } X \in \Gamma(\mathrm{D}), \quad \overline{J}(T) = \nu \quad \overline{J}(\nu) = -T.$$

It is easy to prove that $(M \times I, \overline{J}, g + dt^2)$ is an almost Hermitian manifold. Moreover, since $T \bullet \varphi = -i\varphi$, then

$$\nabla_T T = \nabla_X T = \nabla_{JX} T = 0.$$

because the immersion is totally geodesic, we get

$$\overline{\nabla}_T T = \overline{\nabla}_X T = \overline{\nabla}_{JX} T = 0. \tag{21}$$

Now, since $\varphi$ is parallel on $M$, we get for any $X \in \Gamma(\mathrm{D})$,

$$\nabla J = 0 \text{ on } \mathrm{D} \text{ and } \nabla_T X \in \Gamma(\mathrm{D}) \text{ with } \nabla_T JX = J(\nabla_T X). \tag{22}$$

Finally, using (21) and (22), we conclude that $\overline{\nabla J} = 0$ on $M \times I$, which gives that $M \times I$ is Kähler. $\qquad\square$

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