HYPERBOLICITY OF GENERAL DEFORMATIONS

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Abstract. We modify the deformation method from [9] in order to construct further examples of Kobayashi hyperbolic surfaces in $\mathbb{P}^3$ of any even degree $d \ge 8$.

Given a hypersurface $X_d = f_d^*(0)$ in $\mathbb{P}^n$ of degree d, we say that a (very) general small deformation of X_d is hyperbolic if for any (very) general degree d hypersurface $X_\infty = g_d^*(0)$ and for all sufficiently small $\varepsilon \in \mathbb{C} \setminus \{0\}$ (depending on X_∞) the hypersurface $X_{d,\varepsilon} = (f_d + \varepsilon g_d)^*(0)$ is Kobayashi hyperbolic. With this definition let us formulate the following weakened version of the Kobayashi Conjecture.

Conjecture. For every hypersurface X_d in $\mathbb{P}^n$ of degree $d \ge 2n-1$, a (very) general small deformation of X_d is Kobayashi hyperbolic.

According to the Kobayashi Conjecture, a (very) general surface X_d of degree $d \ge 5$ in $\mathbb{P}^3$ is Kobayashi hyperbolic. This is known to hold indeed for a very general surface of degree at least 21 [2, 7].

By Brody's Theorem, a compact complex space X is hyperbolic if and only if any holomorphic map $\mathbb{C} \to X$ is constant. Hence the proof of hyperbolicity reduces to a certain degeneration principle for entire curves in X. The Green-Griffiths' proof of Bloch's Conjecture [6] provides a kind of such degeneration principle. It was shown by McQuillen [7] and, independently, by Demailly-El Goul [2] (according with this principle) that every entire curve $\varphi : \mathbb{C} \to X$ in a very general surface $X \subseteq \mathbb{P}^3$ of degree $d \ge 36$ ($d \ge 21$, respectively) satisfies a certain algebraic differential equation. See also [8, 12] for recent advances in higher dimensions.

In [9] we exhibited examples of some special surfaces X_d in $\mathbb{P}^3$ of any given degree $d \ge 8$ such that a general small deformation of X_d is Kobayashi hyperbolic. In these examples $X_d = X'_{d'} \cup X''_{d''}$, where $d = d' + d''$, is a union of two cones in $\mathbb{P}^3$ with distinct vertices over plane hyperbolic curves in general position.

Let us indicate briefly the deformation method used in [9] for constructing examples of small degree hyperbolic hypersurfaces (see also the references in [9, 10]). Given two hypersurfaces $X_{d,0}$ and $X_{d,\infty}$ in $\mathbb{P}^n$ of the same degree d, we consider the pencil of hypersurfaces $\{X_{d,\varepsilon}\}_{\varepsilon\in\mathbb{C}}$ generated by $X_{d,0}$ and $X_{d,\infty}$. Assuming that for a sequence $\varepsilon_n \to 0$, the hypersurfaces X_{d,ε_n} are not hyperbolic, there exists a sequence of Brody entire curves $\varphi_n : \mathbb{C} \to X_{d,\varepsilon_n}$ which converges to a (non-constant) Brody curve $\varphi : \mathbb{C} \to X_{d,0}$. Suppose in addition that the hypersurface $X_{d,0}$ admits a rational map to a hyperbolic variety $\pi : X_{d,0} \dashrightarrow Y_0$ (to a curve Y_0 of genus ≥ 2 in case where $\dim X_{d,0} = 2$). Then necessarily $\pi \circ \varphi = \mathrm{cst}$, provided

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that the composition $\pi \circ \varphi$ is well defined. Anyhow, the limiting Brody curve $\varphi : \mathbb{C} \to X_{d,0}$ degenerates.

For a union $X_{d,0} = X'_{d',0} \cup X''_{d'',0}$ of two cones in general position in $\mathbb{P}^3$ as in [9], there is a further degeneration principle. It prohibits to the image $\varphi(\mathbb{C})$ to meet the double curve $D = X'_{d',0} \cap X''_{d'',0}$ outside the points of $D \cap X_{d,\infty}$. Using the assumptions that d', $d'' \geq 4$ and $X_{d,\infty}$ is general this forces φ to be constant, contrary to our construction.

This applies in particular to the union of two quartic cones $X'_{4,0} \cup X''_{4,0}$ in $\mathbb{P}^3$ in general position. Modifying the construction in [9], in the present note we establish, in particular, hyperbolicity of a general deformation of a double quartic cone in $\mathbb{P}^3$, see Example 2.3 below.

1. Some technical lemmas

Here we expose some preliminary facts that will be used in the next section. We let Δ denote the unit disc in $\mathbb{C}$, B^n the unit ball in $\mathbb{C}^n$ and $\mathrm{Hol}(B^n)$ the space of all holomorphic functions on B^n. For two complex spaces X and Y, $\mathrm{Hol}(X, Y)$ stands for the space of all holomorphic maps $X \to Y$ with the usual topology.

Lemma 1.1. *Let $f_0, f_\infty \in \mathrm{Hol}(B^n)$ be such that $f_0(0) = f_\infty(0) = 0$ and the divisors $X_0 = f_0^*(0)$ and $X_\infty = f_\infty^*(0)$ have no common component passing through 0. Let $\Gamma = X_0 \cap X_\infty$ and $X_\varepsilon = f_\varepsilon^{-1}(0)$, where $f_\varepsilon = f_0 + \varepsilon f_\infty$. We assume that $\bigtriangledown f_0|_\Gamma = 0$. Let further $\varphi_n \in \mathrm{Hol}(\Delta, X_{\varepsilon_n})$, where $\varepsilon_n \longrightarrow 0$, be a sequence of holomorphic discs which converges to $\varphi \in \mathrm{Hol}(\Delta, X_0)$ with $\varphi(0) = 0$. Then necessarily $d\varphi(0) \in T_0 X_\infty$.*

Proof. The assertion is clearly true in the case where $\varphi(\Delta) \subseteq \Gamma$. So we will assume further that $\varphi(\Delta) \nsubseteq \Gamma$.

Claim 1. Under the assumptions as above $\varphi_n(t_n) \in \Gamma$ for some sequence $t_n \longrightarrow 0$.

Proof of Claim 1. Let us consider the holomorphic map $F : B^n \to \mathbb{C}^2$, $z \longmapsto (f_0(z), f_\infty(z))$. It is easily seen that F possesses the following properties:

$F(0) = 0$;

$F^{-1}(0) = \Gamma$;

$F(X_{\varepsilon_n}) \subseteq l_n$, where $l_n := \{x + \varepsilon_n y = 0\} \subseteq \mathbb{C}^2$;

$F(X_0) \subseteq l_0 := \{x = 0\}$;

$F \circ \varphi_n(\Delta) \subseteq l_n$;

$F \circ \varphi(\Delta) \subseteq l_0$, $F \circ \varphi(0) = 0$, $F \circ \varphi \not\equiv 0$.

We let $F \circ \varphi_n = (x_n(t), y_n(t))$ and $F \circ \varphi = (0, y(t))$. Thus $x_n \longrightarrow 0$ and $y_n \longrightarrow y$ as $n \longrightarrow \infty$. Since $y(0) = 0$ and $y \not\equiv 0$, we have $y_n \not\equiv 0$. By Rouché's Theorem there exists a sequence $t_n \longrightarrow 0$ such that $y_n(t_n) = 0$, so also $x_n(t_n) = -\varepsilon_n y_n(t_n) = 0$. Hence $\varphi_n(t_n) \in \Gamma = X_0 \cap X_\infty$, as claimed. □

It will be convenient for the rest of the proof to replace the given sequence (φ_n) by a new one (ψ_n). We let $\psi_n(t) = \varphi_n(a_n(t - t_n))$ with (t_n) as in Claim 1 and $a_n \longrightarrow 1$ chosen appropriately so that $\psi_n \in \mathrm{Hol}(\Delta, X_{\varepsilon_n})$ and $\psi_n \longrightarrow \varphi$ as $n \longrightarrow \infty$. Moreover $p_n := \psi_n(0) \in \Gamma$ $\forall n \geq 1$ and $v_n := d\psi_n(0) \longrightarrow v := d\varphi(0)$ when $n \longrightarrow \infty$. Now the assertion follows immediately from the next claim.

Claim 2. $v_n \in T_{p_n} X_\infty$ $\forall n \geq 1$.

Proof of Claim 2. We have:

$$\psi_n(t) = p_n + tv_n + \mathrm{HOT}(t) \qquad \text{and} \qquad f_{\varepsilon_n}(x) = \langle \bigtriangledown f_{\varepsilon_n}(p_n), x - p_n\rangle + \mathrm{HOT}(x-p_n)\,,$$

where HOT stands as usual for the higher order terms. Hence

$$f_{\varepsilon_n} \circ \psi_n(t) = \langle \bigtriangledown f_{\varepsilon_n}(p_n), v_n\rangle \cdot t + \mathrm{HOT}(t)\,. \tag{1}$$

From (1) and the identity $f_{\varepsilon_n} \circ \psi_n \equiv 0$ we obtain

$$0 = \langle \bigtriangledown f_{\varepsilon_n}(p_n), v_n\rangle = \langle \bigtriangledown f_0(p_n), v_n\rangle + \varepsilon_n \langle \bigtriangledown f_\infty(p_n), v_n\rangle = \varepsilon_n \langle \bigtriangledown f_\infty(p_n), v_n\rangle\,.$$

Indeed, by our assumption $\bigtriangledown f_0|_\Gamma = 0$, in particular $\bigtriangledown f_0(p_n) = 0$ $\forall n \geq 1$. This proves the claim. □

In the following corollary we adjust Lemma 1.1 to the situation of the Hurwitz type lemma from [9]. The proof is easy and so we leave it to the reader.

Corollary 1.2. *Let us consider a pencil of degree d hypersurfaces*

$$X_\varepsilon = (f_0 + \varepsilon f_\infty)^*(0) \qquad in \quad \mathbb{P}^{n+1}$$

generated by

$$X_0 = X_0' \cup X_0'' = f_0^*(0) \quad and \quad X_\infty = f_\infty^*(0)\,.$$

We let $D = X_0' \cap X_0''$. Then for any sequence of entire curves $\varphi_n : \mathbb{C} \to X_{\varepsilon_n}$ which converges to $\varphi : \mathbb{C} \to X_0'$ the following alternative holds:

- *either $\varphi(\mathbb{C}) \subseteq D$, or*
- *$\varphi(\mathbb{C}) \cap D \subseteq D \cap X_\infty$ and $d\varphi(t) \in T_P X_0' \cap T_P X_\infty \quad \forall P = \varphi(t) \in D \cap X_\infty$.*

To reformulate this corollary, let us choose an affine chart U in $\mathbb{P}^{n+1}$. Letting $\tilde{f}_\infty = 0$ ($f_{01} = 0,\ f_{02} = 0$, respectively) be a polynomial equation of $X_\infty \cap U$ (of $X_0' \cap U,\ X_0'' \cap U$, respectively), by Corollary 1.2 we have $(\tilde{f}_\infty \circ \varphi)'(t) = 0$ every time when $(f_{01} \circ \varphi)(t) = 0 = (f_{02} \circ \varphi)(t)$, provided that $f_{0i} \circ \varphi$, $i = 1, 2$, do not vanish identically and simultaneously.

Next we study an enumeration problem for the intersection of a general hypersurface with the generators of a given cone in $\mathbb{P}^{n+1}$.

Proposition 1.3. *We let $X \subseteq \mathbb{P}^{n+1}$ be a cone over a variety $Y \subseteq \mathbb{P}^n$. We consider also a general hypersurface $X' \subseteq \mathbb{P}^{n+1}$ of degree $e \geq 2\dim Y$. Then X' meets every generator $l = (PQ)$ of X, where Q runs over Y, in at least $k = e - 2\dim Y$ points transversally.*

Before giving the proof let us introduce some notation. For a pair $n, e \in \mathbb{N}$ we let $\mathbb{F}(n+1, e)$ denote the vector space of all homogeneous forms in $n+2$ variables of degree $e \geq 1$ and $\mathbb{P}(n+1, e)$ its projectivization. Given a projective variety $Y \subseteq \mathbb{P}^n$ and a cone $X \subseteq \mathbb{P}^{n+1}$ over Y with vertex P, for every $k \geq 1$ we consider the subset $\mathbb{F}(Y, e, k) \subseteq \mathbb{F}(n+1, e)$ of all forms $f \in \mathbb{F}(n+1, e)$ such that the intersection divisor $f^*(0) \cdot (PQ)$ has at most $k-1$ reduced points on at least one generator $l = (PQ)$ $(Q \in Y)$ of X. We let $\mathbb{P}(Y, e, k)$ denote the projectivization of $\mathbb{F}(Y, e, k)$.

Lemma 1.4. *$\mathbb{P}(Y, e, k)$ is a Zariski closed subset of $\mathbb{P}(n+1, e)$.*

Proof. Blowing up $\mathbb{P}^{n+1}$ with center at P yields a fiber bundle $\xi : \widehat{\mathbb{P}}^{n+1} \to \mathbb{P}^n$ with fiber $\mathbb{P}^1$. We let $\mathrm{Symm}_e(\xi)$ denote the eth symmetric power[1] of ξ over $\mathbb{P}^n$. Its fiber over a point $Q \in \mathbb{P}^n$ consists of all effective divisors on $\xi^{-1}(Q) \cong \mathbb{P}^1$ of degree e. Given a partition

$$e = \sum_{i=1}^{k} n_i \qquad \text{with} \qquad 1 \le n_1 \le n_2 \le \ldots \le n_s$$

we let $\Sigma_{\bar{n}}$, where $\bar{n} = (n_1, \ldots, n_s)$, denote the closed subbundle of $\mathrm{Symm}_e(\xi)$ whose fiber over Q consists of all effective divisors on $\xi^{-1}(Q)$ of the form

$$\sum_{i=1}^{s} n_i [p_i], \qquad \text{where} \quad p_i \in \xi^{-1}(Q)\,.$$

We also let

$$\Sigma_k = \bigcup_{\bar{n}: n_k \ge 2} \Sigma_{\bar{n}}\,.$$

The restriction map

$$\rho : f \longmapsto f^*(0) \cdot (PQ), \qquad Q \in Y\,,$$

associates to f a section $\rho(f)$ of $\mathrm{Symm}_e(\xi)$ over Y. It is easily seen that $f \in \mathbb{F}(n+1, e)$ belongs to $\mathbb{F}(Y, e, k)$ if and only if $\rho(f)$ meets Σ_k.

We claim that the set, say, $\Gamma_{e,k}$ of all sections of $\mathrm{Symm}_e(\xi)|_Y$ meeting Σ_k is a Zariski closed subset of $\Gamma(Y, \mathcal{O}(\mathrm{Symm}_e(\xi)|_Y))$. More generally, given projective varieties X and Y and a subvariety $S \subset Y$, the set $\mathcal{M}_S$ of all morphisms $f : X \to Y$ such that the image $f(X)$ meets S is a Zariski closed subset of $\mathrm{Mor}(X, Y)$. Indeed, let us consider the incidence relation

$$I = \{(f, x, y) \in \mathrm{Mor}(X, Y) \times X \times Y \mid f(x) = y\}\,.$$

Then $\mathcal{M}_S = \pi_1(\pi_3^{-1}(S) \cap I)$ is Zariski closed, as claimed.

Consequently, $\mathbb{P}(Y, e, k)$ is Zariski closed in $\mathbb{P}(n+1, e)$, as stated. □

Remark 1.5. Proposition 1.3 asserts that the complement $\mathbb{P}(n+1, e) \setminus \mathbb{P}(Y, e, k)$ is a nonempty Zariski open subset of $\mathbb{P}(n+1, e)$ provided that $e \ge 2 \dim Y + k$. By virtue of Lemma 1.4, this is quite evident if $n = 3$. Indeed, it is easy to see that the union X' of e planes in $\mathbb{P}^3$ in general position belongs to this complement. Presumably the same holds in higher dimensions for unions of e hyperplanes in general position. However the latter is much less evident, so we choose below a different approach.

Proof of Proposition 1.3. We use a coordinate presentation of the morphism ρ as above. We let CY denote the affine cone over Y and $CY^* = CY \setminus \{0\}$ the same cone with the vertex deleted. Let us fix coordinates in $\mathbb{P}^{n+1}$ in such a way that $P = (0 : \ldots : 0 : 1)$ and $Y \subseteq \{z_{n+1} = 0\}$. If $Q = (z_0 : \ldots : z_n : 0) = (z' : 0) \in Y$ then

$$(PQ) = \{(z' : z_{n+1}) \mid z_{n+1} \in \mathbb{C}\} \cup \{P\}\,.$$

For a hypersurface X' in $\mathbb{P}^{n+1}$ of degree e its defining equation $f = 0$ can be written in the form

$$f(z', z_{n+1}) = \sum_{i=0}^{e} a_i(z') z_{n+1}^{e-i} = 0\,, \tag{2}$$

[1]That is the eth Cartesian power factorized by the natural action of the symmetric group of degree e.

where a_i is a homogeneous form in z' of degree i. Assuming that $P \notin X'$ i.e., $a_0 \neq 0$, we can normalize the equation so that $a_0 = 1$. Fixing $z' \in \mathbb{A}^{n+1}$ we specialize f to a monic polynomial $f_{z'} \in \mathbb{C}[z_{n+1}]$ of degree e. In these terms the proposition claims that for $k = e - 2\dim Y$ and for a general $f \in \mathbb{F}(n+1, e)$, the specialization $f_{z'}$ has at least k simple roots whatever is the choice of $z' \in CY^* \subseteq \mathbb{A}^{n+1}$.

The affine chart

$$U = \mathbb{P}(n+1, e) \setminus \{a_0 = 0\}$$

can be identified with the affine space of all sequences of homogeneous forms $a = (a_1, \ldots, a_e)$ with $\deg a_i = i$. The specialization $(f, z') \longmapsto f_{z'}$ defines a morphism

$$\tilde{\rho} : U \times CY \to \mathrm{Poly}_e\,,$$

where Poly_e stands for the affine variety of all monic polynomials of degree e. In turn Poly_e can be identified with $\mathrm{Symm}_e(\mathbb{A}^1) \cong \mathbb{A}^e$.

Let us consider further the Vieta map

$$\nu : \mathbb{A}^e \to \mathrm{Poly}_e, \qquad (\lambda_1, \ldots, \lambda_e) \longmapsto p(z) = \prod_{i=1}^{e} (z - \lambda_i)\,.$$

This is a ramified covering of degree $e!$. For a multi-index $\bar{n} = (n_1, \ldots, n_s)$ with $\sum_{i=1}^{s} n_i = e$ we let

$$\Sigma'_{\bar{n}} = \nu(D_{\bar{n}}) \subseteq \mathrm{Poly}_e\,,$$

where $D_{\bar{n}}$ is the linear subspace of $\mathbb{A}^e$ given by equations

$$\lambda_1 = \ldots = \lambda_{n_1}, \quad \lambda_{n_1+1} = \ldots = \lambda_{n_1+n_2}, \quad \ldots, \quad \lambda_{n_1+\ldots+n_{s-1}+1} = \ldots = \lambda_e\,.$$

Clearly both $D_{\bar{n}}$ and $\Sigma'_{\bar{n}}$ have pure dimension s. Letting

$$\Sigma'_k = \bigcup_{n_k \geq 2} \Sigma'_{\bar{n}} \subseteq \mathrm{Poly}_e$$

denote the variety of all monic polynomials of degree e with at most $k-1$ simple roots, we have

$$\dim \Sigma'_k = \max_{n_k \geq 2} \{\dim \Sigma'_{\bar{n}}\} = k - 1 + \left[\frac{e-k+1}{2}\right]\,.$$

If $e - k + 1$ is even then the latter maximum is achieved for

$$n_1 = \ldots = n_{k-1} = 1,\ \ n_k = \ldots = n_s = 2\,,$$

and otherwise for

$$n_1 = \ldots = n_{k-2} = 1,\ \ n_{k-1} = \ldots = n_s = 2\,.$$

Anyhow

$$\mathrm{codim}\,(\Sigma'_k, \mathrm{Poly}_e) = 1 + \left[\frac{e-k}{2}\right]\,.$$

Claim 1. The restriction $d\tilde{\rho}|_{TU}$ is surjective at every point $(a, z') \in U \times CY^*$. In particular $d\tilde{\rho}$ has maximal rank e at every such point.

Proof of Claim 1. For a point $(a, z') = (a_1, \ldots, a_e, z_0, \ldots, z_n) \in U \times CY^*$ we let

$$a^0 = (a^0_1, \ldots, a^0_e) \in \mathbb{A}^e, \qquad \text{where} \quad a^0_i = a_i(z'),\ i = 1, \ldots, e\,.$$

Since $z' \neq 0$, for an arbitrary tangent vector $b^0 = (b^0_1, \dots, b^0_e) \in \mathbb{A}^e$ there exists a e-tuple of homogeneous forms $b = (b_1, \dots, b_e)$ with $\deg b_i = i$ such that $b(z') = b^0$. Therefore

$$(a + tb)(z') = a^0 + tb^0 \quad \text{and so} \quad d\tilde{\rho}(a^0, z')(b, 0) = b^0\,.$$

This proves Claim 1. $\square$

By virtue of Claim 1,

$$\operatorname{codim}(\tilde{\rho}^{-1}(\Sigma'_k),\, U \times CY^*) = \operatorname{codim}(\Sigma'_k,\, \mathrm{Poly}_e) = 1 + \left[\frac{e-k}{2}\right]\,.$$

Since

$$f_{\lambda z'}(z_{n+1}) = \lambda . f_{z'}(z_{n+1}) = \lambda^{-e} f_{z'}(\lambda z_{n+1}) \qquad \forall \lambda \in \mathbb{C}^*\,,$$

the subvariety $\tilde{\rho}^{-1}(\Sigma'_k)$ of $U \times CY^*$ is stable under the natural $\mathbb{C}^*$-action on the second factor. Hence

$$\operatorname{codim}\left(\tilde{\rho}^{-1}(\Sigma'_k)/\mathbb{C}^*,\, U \times Y\right) = \operatorname{codim}(\tilde{\rho}^{-1}(\Sigma'_k),\, U \times CY^*) = 1 + \left[\frac{e-k}{2}\right]\,.$$

Thus the general fibers of the projection

$$\mathrm{pr}_2 : U \times Y \to U$$

do not meet $\tilde{\rho}^{-1}(\Sigma'_k)/\mathbb{C}^* \subseteq U \times Y$ provided that

$$\dim Y \le \left[\frac{e-k}{2}\right]\,.$$

The latter inequality is equivalent to $k \le e - 2\dim Y$, which fits our assumption. Now the proposition follows. $\square$

2. Examples

Theorem 2.1. *Let Y_0 be a Kobayashi hyperbolic hypersurface in $\mathbb{P}^n$ ($n \ge 2$), where $\mathbb{P}^n$ is realized as the hyperplane $H = \{z_{n+1} = 0\}$ in $\mathbb{P}^{n+1}$. Then a general small deformation $X_\varepsilon \subseteq \mathbb{P}^{n+1}$ of the double cone $2X_0$ over Y_0 is Kobayashi hyperbolic.*

Proof. Suppose the contrary. Then letting X_∞ be a general hypersurface of degree $2d = 2\deg X_0$ and $(X_t)_{t\in\mathbb{P}^1}$ the pencil generated by $2X_0$ and X_∞, we can find a sequence $\varepsilon_n \longrightarrow 0$ and a sequence of Brody curves $\varphi_n : \mathbb{C} \to X_{\varepsilon_n}$ such that $\varphi_n \longrightarrow \varphi$, where $\varphi : \mathbb{C} \to X_0$ is non-constant. We let $\pi : X_0 \dashrightarrow Y_0$ be the cone projection. Since Y_0 is assumed to be hyperbolic we have $\pi \circ \varphi = \mathrm{cst}$. In other words $\varphi(\mathbb{C}) \subseteq l$, where $l \cong \mathbb{P}^1$ is a generator of the cone X_0.

Letting $Y_0 = f_0^*(0)$, where f is a homogeneous form of degree d in $z_0, \dots, z_n$, we note that $\nabla f_0^2|_{X_0} = 0$. If l and X_∞ meet transversally in a point $\varphi(t) \in l \cap X_\infty$ then $d\varphi(t) = 0$ by virtue of Lemma 1.1.

Since $Y_0 \subseteq \mathbb{P}^n$ is hyperbolic and $n \ge 2$ we have $d \ge n+2$. In particular

$$\deg X_\infty = 2d \ge 2n + 4 \ge 2\dim Y + 5\,.$$

By Proposition 1.3, l and X_∞ meet transversally in at least 5 points. Hence the nonconstant meromorphic function $\varphi : \mathbb{C} \to l \cong \mathbb{P}^1$ possesses at least 5 multiple values. Since the defect of a multiple value is $\ge 1/2$, this contradicts the Defect Relation. $\square$

Remark 2.2. Given a hyperbolic hypersurface $Y \subseteq \mathbb{P}^n$ of degree d, Theorem 2.1 provides a hyperbolic hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree $2d$. Iterating the construction yields hyperbolic hypersurfaces in $\mathbb{P}^n$ $\forall n \geq 3$. However, their degrees grow exponentially with n, whereas the best asymptotic achieved so far is $4(n-1)^2$ (see e.g., [11]).

Example 2.3. Let $C \subseteq \mathbb{P}^2$ be a hyperbolic curve of degree $d \geq 4$, and let $X_0 \subseteq \mathbb{P}^3$ be a cone over C. Then a general small deformation of the double cone $2X_0$ is a Kobayashi hyperbolic surface in $\mathbb{P}^3$ of even degree $2d \geq 8$.

The Degeneration Principle of Corollary 1.2 can be combined with the following one, which can be proved along the same lines as Theorem 2.1.

Proposition 2.4. *Let $(X_t)_{t\in\mathbb{P}^1}$ be a pencil of hypersurfaces in $\mathbb{P}^{n+1}$ generated by two hypersurfaces X_0 and X_∞ of the same degree $d \geq 5$, where $X_0 = kQ$ with $k \geq 2$ for some hypersurface $Q \subseteq \mathbb{P}^{n+1}$, and $X_\infty = \bigcup_{i=1}^d H_{a_i}$, $a_1, \ldots, a_d \in \mathbb{P}^1$, is the union of d distinct hyperplanes from a pencil of hyperplanes $(H_a)_{a\in\mathbb{P}^1}$. If a sequence of Brody curves $\varphi_n : \mathbb{C} \to X_{\varepsilon_n}$, where $\varepsilon_n \to 0$, converges to a Brody curve $\varphi : \mathbb{C} \to X_0$, then $\varphi(\mathbb{C}) \subseteq X_0 \cap H_a$ for some $a \in \mathbb{P}^1$.*

Examples 2.5. Given a pencil of planes $(H_a)_{a\in\mathbb{P}^1}$ in $\mathbb{P}^3$, using Proposition 2.4 one can deform

- $X_0 = 5Q$, where $Q \subseteq \mathbb{P}^3$ is a plane,
- a triple quadric $X_0 = 3Q \subseteq \mathbb{P}^3$, or
- a double cubic, quartic, etc. $X_0 = 2Q \subseteq \mathbb{P}^3$

to an irreducible surface $X_\varepsilon \in \langle X_0, X_\infty \rangle$ of the same degree d, where as before $X_\infty = \bigcup_{i=1}^d H_{a_i}$, so that every limiting Brody curve $\varphi : \mathbb{C} \to X_0$ is contained in a section $X_0 \cap H_a$ for some $a \in \mathbb{P}^1$.

The famous Bogomolov-Green-Griffiths-Lang Conjecture on strong algebraic degeneracy (see e.g., [1, 6]) suggests that every surface S of general type possesses only finite number of rational and elliptic curves and, moreover, the image of any nonconstant entire curve $\varphi : \mathbb{C} \to S$ is contained in one of them. In particular, this should hold for any smooth surface $S \subseteq \mathbb{P}^3$ of degree ≥ 5, which fits the Kobayashi Conjecture. Indeed, by Clemens-Xu-Voisin's Theorem, a general smooth surface $S \subseteq \mathbb{P}^3$ of degree ≥ 5 does not contain rational or elliptic curves, hence it should be hyperbolic provided that the above conjecture holds indeed.

Anyhow, the deformation method leads to the following result, which is an immediate consequence of Proposition 2.4.

Corollary 2.6. *Let $S \subseteq \mathbb{P}^3$ be a surface and $Z \subset S$ be a curve such that the image of any nonconstant entire curve $\varphi : \mathbb{C} \to S$ is contained in Z [2]. Let X_∞ be the union of $d = 2\deg S$ planes from a general pencil of planes in $\mathbb{P}^3$. Then any small enough linear deformation X_ε of $X_0 = 2S$ in direction of X_∞ is hyperbolic.*

Along the same lines, Proposition 2.4 can be applied in the following setting.

Example 2.7. Let us take for X_0 a double cone in $\mathbb{P}^3$ over a plane hyperbolic curve of degree $d \geq 4$, and for X_∞ the union of $2d$ distinct planes from a general pencil of planes $(H_a)_{a\in\mathbb{P}^1}$.

[2]The latter holds, for instance, if S is hyperbolic modulo Z.

Then small deformations X_ε of X_0 in direction of X_∞ provide examples of hyperbolic surfaces of any even degree $2d \geq 8$. In suitable coordinates in $\mathbb{P}^3$ such a surface can be given by equation

$$Q(X_0, X_1, X_2)^2 - P(X_2, X_3) = 0\,, \tag{3}$$

where Q, P are generic homogeneous formes of degree k and $d = 2k$, respectively. The latter are actually the Duval-Fujimoto examples [4, 5].

A nice construction due to J. Duval [3] of a hyperbolic sextic $X_\varepsilon \subseteq \mathbb{P}^3$ uses the deformation method iteratively in 5 steps, so that $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_5)$ has 5 subsequently small enough components. Hence X_ε belongs to a 5-dimensional linear system and the deformation of X_0 to X_ε neither is linear nor very generic. It was suggested in [10] that the union of 6 general planes in $\mathbb{P}^3$ admits a general small linear deformation to an irreducible hyperbolic sextic surface[3]. Let us consider this conjecture in more details.

Example 2.8. Let $X_0 = \bigcup_{i=1}^6 L_i$ be a union of 6 planes in $\mathbb{P}^3$ in general position, and let $X_\infty \subseteq \mathbb{P}^3$ be a general sextic surface. By virtue of Proposition 1, for any Brody curve $\varphi : \mathbb{C} \to X_0$ which is the limit of Brody curves $\varphi_n : \mathbb{C} \to X_{\varepsilon_n}$ ($\varepsilon_n \longrightarrow 0$, $\varepsilon_n \neq 0$), the following hold.

- The entire curve $\varphi(\mathbb{C})$ is contained in one of the planes, say, L_i but in none of the intersection lines $l_{ij} := L_i \cap L_j$ ($i \neq j$), neither in the smooth sextic $q_i = L_i \cap X_\infty$.
- $\varphi(\mathbb{C})$ can meet a line l_{ij} only in the 6 intersection points of l_{ij} with q_i.
- $d\varphi(t) \in T_P q_i$ for any point $P = \varphi(t) \in l_{ij} \cap q_i$. Hence $(f_i \circ \varphi)'(t) = 0$, where $f_i = 0$ is an affine equation of q_i.

Consequently, the general small linear deformations X_ε of X_0 are hyperbolic provided that the following question can be answered affirmatively.

2.9. Question. Consider the union $l = \bigcup_{i=1}^5 l_i$ of 5 lines $l_1, \ldots, l_5$ in general position in $\mathbb{P}^2$, and let $q \subseteq \mathbb{P}^2$ be a general plane sextic. Let in a suitable affine chart in $\mathbb{P}^2$, q be given by equation $f = 0$, where f is a polynomial of degree 6. Consider further an entire curve $\varphi : \mathbb{C} \to \mathbb{P}^2$ whose image is not contained in l. Is it true that $\varphi = \mathrm{cst}$ provided that $(f \circ \varphi)'(t) = 0$ for every point $t \in \mathbb{C}$ such that $\varphi(t) \in l$? Is this true under the additional assumption that the entire curve $\varphi(\mathbb{C})$ does not meet the configuration l outside the intersection $l \cap q$ that is, $\varphi^{-1}(l) \subseteq \varphi^{-1}(q)$?

In other words, we are seeking to strengthen the Borel Lemma, or else the classical Ramification Theorem by replacing the 5 multiple values of $f \circ \varphi$ with the l_i-values of φ, $i = 1, \ldots, 5$.

Example 2.8 can be specified further using Proposition 2.4.

Example 2.10. Let again $X_0 = \bigcup_{i=1}^6 L_i$ be the union of 6 planes in $\mathbb{P}^3$ in general position, and let $X_\infty = \bigcup_{j=1}^6 H_{a_j}$ be a union of 6 planes from a pencil $(H_a = f^*(a))_{a \in \mathbb{P}^1}$ in $\mathbb{P}^3$ in general position with respect to X_0. Let $(X_t)_{t \in \mathbb{P}^1}$ be the pencil generated by X_0 and X_∞. Note that the surface X_t is not hyperbolic since it contains the union of lines $\Gamma = X_0 \cap X_\infty$. We suggest however that X_ε is hyperbolic modulo Γ for all small enough $\varepsilon \neq 0$. This leads to the following uniqueness problem for line configurations.

[3]By [10] hyperbolicity occurs indeed for certain special linear deformations of the union of 15 planes in $\mathbb{P}^3$ in general position.

2.11. Question. Consider as before the union $l = \bigcup_{i=1}^{5} l_i$ of 5 lines in general position in $\mathbb{P}^2$, and let $q = \bigcup_{j=1}^{6} h_j$ be the union of 6 distinct lines $h_i = f^*(a_i)$, $i = 1 \ldots, 6$, in $\mathbb{P}^2$ through a common point, where f is a (general) linear function in a suitable affine chart. Let an entire curve $\varphi : \mathbb{C} \to \mathbb{P}^2$ satisfies the following conditions:

- $\varphi(\mathbb{C}) \not\subseteq l$,
- $\varphi^{-1}(l) \subseteq \varphi^{-1}(q)$,
- $(f \circ \varphi)'(t) = 0 \ \forall t \in \varphi^{-1}(l)$.

Is then necessarily $f \circ \varphi = a_i$ for some $i \in \{1, \ldots, 6\}$?

Let us finally turn to the Kobayashi problem on hyperbolicity of complements of general hypersurfaces. By virtue of Kiernan-Kobayashi-M. Green's version of Borel's Lemma, the complement $\mathbb{P}^n \setminus L$ of the union $L = \bigcup_{i=1}^{2n+1} L_i$ of $2n + 1$ hyperplanes in $\mathbb{P}^n$ in general position is Kobayashi hyperbolic. In particular, this applies to the union l of 5 lines in $\mathbb{P}^2$ in general position. Moreover [13] l can be deformed to a smooth quintic curve with hyperbolic complement via a small deformation. This deformation proceeds in 5 steps and neither is linear nor very generic. So the following question arises.

2.12. Question. Let L stands as before for the union of $2n + 1$ hyperplanes in $\mathbb{P}^n$ in general position. Is the complement of a general small linear deformation of L Kobayashi hyperbolic? In particular, does the union of 5 lines in $\mathbb{P}^2$ in general position admit a general small linear deformation to an irreducible quintic curve with hyperbolic complement?

References

[1] Bogomolov F., De Oliveira B. Hyperbolicity of nodal hypersurfaces. J. Reine Angew. Math. 596 (2006), 89–101.

[2] Demailly J.-P., El Goul J. Hyperbolicity of generic surfaces of high degree in projective 3-space. Amer. J. Math. 122 (2000), 515–546.

[3] Duval J. Une sextique hyperbolique dans $\mathbb{P}^3(\mathbb{C})$. Math. Ann. 330 (2004), 473–476.

[4] Duval J. Letter to J.-P. Demailly, October 30, 1999 (unpublished).

[5] Fujimoto H. A family of hyperbolic hypersurfaces in the complex projective space. The Chuang special issue. Complex Variables Theory Appl. 43 (2001), 273–283.

[6] Green M., Griffiths Ph. Two applications of algebraic geometry to entire holomorphic mappings. The Chern Symposium 1979, 41–74, Springer, New York-Berlin, 1980.

[7] McQuillan M. Holomorphic curves on hyperplane sections of 3-folds. Geom. Funct. Anal. 9 (1999), 370–392.

[8] Rousseau B. Equation différentielles sur les hypersurfaces de $\mathbb{P}^4$. J. Mathém. Pure Appl. 86 (2006), 322–341.

[9] Shiffman B., Zaidenberg M. New examples of Kobayashi hyperbolic surfaces in $\mathbb{CP}^3$. (Russian) Funktsional. Anal. i Prilozhen. 39 (2005), 90–94; English translation in Funct. Anal. Appl. 39 (2005), 76–79.

[10] Shiffman B., Zaidenberg M. Constructing low degree hyperbolic surfaces in $\mathbb{P}^3$. Special issue for S. S. Chern. Houston J. Math. 28 (2002), 377–388.

[11] Shiffman B., Zaidenberg M. Hyperbolic hypersurfaces in $\mathbb{P}^n$ of Fermat-Waring type. Proc. Amer. Math. Soc. 130 (2002), 2031–2035.

[12] Siu Y.-T. Hyperbolicity in Complex Geometry, in: The legacy of Niels Henric Abel, Springer-Verlag, Berlin, 2004, 543–566.

[13] Zaidenberg M. Stability of hyperbolic embeddedness and construction of examples. (Russian) Matem. Sbornik 135 (177) (1988), 361–372; English translation in Math. USSR Sbornik 63 (1989), 351–361.

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