Homology 3-spheres bounding acyclic 4-manifolds

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Abstract

We define an integral lift of μ invariant for some class of oriented homology 3-spheres. This invariant is a homology cobordism invariant. If Σ is a homology 3-sphere in this class and the invariant is not zero for Σ, then the connected sum of any number of the copies of Σ cannot be the boundary of an acyclic 4-manifold. The Seifert fibered homology 3-sphere $\Sigma(a_1, a_2, \dots, a_n)$ is in this class if one of the a_i's is even. We use Seiberg-Witten's monopole equation on four-dimensional V-manifolds to show the property of the invariant.

1 Introduction

In this note we give a necessary condition for some homology 3-spheres to be boundaries of acyclic 4-manifolds. R. Stern [26], S. Akbulut and R. Kirby [2] and A. J. Casson and J. L. Harer [3] gave examples of Seifert fibered homology 3-spheres which are the boundaries of acyclic 4-manifolds.

If the μ invariant of a homology 3-sphere is 1 in $\mathbf{Z}_2$, then it cannot be the boundary of an acyclic 4-manifold. The μ-invariant is related to the conjecture that any closed topological manifold of dimension≥ 5 is simplicially triangulable. It is known ([18], [15]) that if a homology 3-sphere has non-vanishing μ-invariant and the connected sum of two copies of it is the boundary of an acyclic 4-manifold, then the conjecture holds.

In this note we show:

Theorem 1 *Let $\Sigma(a_1, a_2, \dots, a_n)$ be a Seifert fibered homology 3-sphere with non-vanishing μ-invariant. If one of the a_i's is even, then every connected sum of any number of its copies cannot be the boundary of an acyclic 4-manifold.*

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For invariants of homology 3-spheres introduced before gauge theory, see [21] and [25]. Some obstructions other than μ-invariant are provided by gauge theory. When a homology 3-sphere is the boundary of a negative definite 4-manifold whose intersection form is not standard, then a theorem of Donaldson implies that this homology 3- sphere cannot be the boundary of a positive definite 4-manifold.

More direct argument for Seifert fibered homology 3-spheres was given by R. Fintushel and R. Stern [5]. They used the fact that Seifert fibered homology 3-sphere is the boundary of a negative definite V-manifold. (Here we are fixing a convention of orientation.) Their argument was generalized in [4] [17] [11] [10] and [7]. In this note we shall also use this V-manifold introduced in [5].

Main difficulty comes from non-compactness of the moduli space of instantons. T. Lawson [17] used a partial classification of instantons on $L(p,q) \times \mathbf{R}$ to control the non-compactness. To measure the non-compactness, Fintushel and Stern [7] and the second author [10] introduced an invariant by using the Chern-Simons invariant of flat connections and they extended the argument to apply to the connected sum of various Seifert fibered homology 3-spheres with both orientations. A. Floer pointed out to the second author that a refinement of this invariant, defined by using a version of Floer homology, is a homology cobordism invariant. As was suggested in [9] for cobordisms among lens spaces, if one makes use of the classification of the instantons on $L(p,q) \times \mathbf{R}$ obtained in [1] [13], then it might be possible to obtain stronger results. However it was not clear how to handle the complicated non-compactness.

K. A. Froyshov investigated definite 4-manifolds with boundaries by using Seiberg-Witten's monopole equation [8]. Since the moduli space of monopole on closed 4-manifold is compact, it is technically easier to use monopole rather than instanton. For manifolds with boundary, or open manifolds, the moduli space is still non-compact and Froyshov defined an invariant to measure the non-compactness. T. Mrowka, P. Ozsvath and B. Yu [19] analyzed monopole equation of the product on the Seifert fibered 3-manifolds and $\mathbf{R}$, and calculated Seiberg-Witten-Floer homology for these manifolds.

L. I. Nicolaescu [20] recently calculated this invariant of Froyshov for Seifert fibered homology 3-spheres and showed that they cannot be the boundary of acyclic or definite 4-manifolds of some type, under some condition.

All the above arguments to use gauge theory could be basically regarded as attempts to extend Donaldson's theorem about definite closed 4-manifold to some theorems for open manifolds or V-manifolds.

On the other hand, Donaldson also showed that when a closed 4-manifold is *almost* definite and spin, one still has some restriction on its intersection form. It would be natural to try to extend the theorem for open or V-cases. However because of the difficulty of non-compactness, it was hard to realize this idea by using instanton moduli spaces.

In [23] N. Saveliev used a restriction on the intersection form of spin closed 4-manifold, which was proved by using monopole moduli space in [12], to give other examples which cannot be the boundary of acyclic 4-manifolds.

In this note we directly use monopole moduli space to extend the theorem in [12] and to obtain some property of almost definite closed V-manifolds. We do not directly consider monopole equation on 4-manifolds with boundaries. H. Tokui [27] applied a similar argument to 4-manifolds with boundary.

Let $a_1, a_2, \ldots, a_n$ be pairwise-coprime positive integers. Then we have a Seifert fibered homology 3-sphere called $\Sigma(a_1, a_2, \ldots, a_n)$. We define an integer $w(a_1, a_2, \ldots, a_n; m)$ for each integer m using a formula given in Section 5. This number is essentially equal the degree of some class of Seiberg-Witten-Floer homology group.

One of the theorems we show in this note is:

Theorem 2 *1. If the connected sum of some number of copies of $\Sigma(a_1, a_2, \ldots, a_n)$ is the boundary of an acyclic or positive definite 4-manifold, then for any m we have*

$$w(a_1, a_2, \ldots, a_n; m) \leq 0.$$

2. Suppose one of the a_i's is even. If the connected sum of some number of copies of $\Sigma(a_1, a_2, \ldots, a_n)$ is the boundary of an acyclic 4-manifold, then

$$w(a_1, a_2, \ldots, a_n; s) = 0, \qquad \textit{where} \qquad s = \frac{1}{2}\{1 + \alpha(-2 + \sum(1 - \frac{1}{a_i}))\}.$$

When one of the a_i's is even, the integer $w(a_1, a_2, \ldots, a_n; s)$ turns out to be an integral lift of the μ-invariant. Theorem 1 is obtained from this observation and Theorem 2.

Casson's invariant is one of the integral lifts. However the above number does not coincide with Casson's invariant in general.

In Section 2, we collect all the inputs that we need later on. These arguments are shown by using monopole equation on 4-dimensional closed V-manifolds. In Section 3, we consider their direct consequences to obtain some properties of V-manifolds with boundaries. In Section 4, we define some classes $\mathcal{S}(k^+, k^-)$ of oriented homology 3-spheres for non-negative integers k^+ and k^-. The integral lift of μ-invariant is constructed on $\mathcal{S}(k^+, k^-)$ when k^+ and k_- satisfy $k^+ + k^- \leq 2$. In Section 5, we show that the Seifert fibered homology 3-sphere $\Sigma(a_1, a_2, \ldots, a_n)$ is in the class $\mathcal{S}(0, 1)$ and give some examples.

2 Closed V-4-manifolds

In this note we deal with V-manifold with isolated singular points. For general background of V-manifold, see [22] [14] and [19] For general background of V-manifold. Basically there is no problem to extend usual notions of differential geometry to the ones of V-manifolds if they are defined in the equivariant setting under the finite group action. Let X be an oriented closed V-4-manifold which only has isolated singular points. The neighborhood of each singular point is isomorphic to the form of the cone of a lens space or other spherical space form. Let b_i be the i-th Betti number of X and $b_2^+(X)$ (and $b_2^-(X)$) be the maximal dimension of positive (and negative, respectively) definite subspace of $H^2(X, \mathbf{R})$. We assume that

(1) $b_1(X) = 0$ and

(2) a V-spin^c-structure on X is given and fixed.

Take a V-Riemannian metric on X. Let S^+, S^- and L be the associated positive and negative spinor bundles and the associated V-line bundle. Then we have $\wedge^2 S^+ = \wedge^2 S^- = L$. They are V-bundles over X. ¿From now on we omit mentioning "V" since everything is defined in the category of V-manifolds.

Let $D_{\mathrm{ASD}}(X)$ be the elliptic complex

$$D_{\mathrm{ASD}}(X) = d + d^* : \Omega^0(X) \oplus \Omega^2_+(X) \to \Omega^1(X)$$

and $D_{\mathrm{Dirac}}(X)$ be the Dirac operator associated with the spin^c-structure.

We need the indices of $D_{\mathrm{ASD}}(X)$ and $D_{\mathrm{Dirac}}(X)$. Here the index is the difference between the *real* dimension of the kernel and that of the kernel of the operator. Since the kernel and the cokernel of $D_{\mathrm{Dirac}}(X)$ have complex structures, $\mathrm{ind}\, D_{\ Dirac}$ is even. These indices can be calculated by using the index theorem for V- manifold of T. Kawasaki [16]. However the index of D_{ASD} is immediately calculated by using harmonic V-forms and Hodge theory for V-manifold. Since the de Rham cohomology group of any cone, defined by using V-forms, is isomorphic to that of one point, it is a straightforward way to extend the usual de Rham theorem to the V-manifolds. This V-manifold version of de Rham theorem implies the following lemma.

Lemma 1

$$\mathrm{ind}\, D_{\mathrm{ASD}}(X) = 1 + b_2^+(X).$$

Note that we are assuming that $b_1(X) = 0$.

The formal dimension of the monopole moduli space for the spin^c-structure is given by $\mathrm{ind}\, D_{\mathrm{Dirac}}(X) - \mathrm{ind}\, D_{\mathrm{ASD}}(X)$. P. Kronheimer pointed out that for negative definite closed 4-manifolds the formal dimension must be negative. The argument is applicable even for certain non-compact 4-manifolds as long as the monopole moduli space is compact [8]. We use V-manifold version of this statement. Since we are considering closed V-manifolds, the moduli spaces are always compact. Hence we have:

Theorem 3 *If $b_2^+(X) = 0$, then we have* $\mathrm{ind}\, D_{\mathrm{Dirac}} \leq 0$.

The following theorem is proved in [12] for smooth closed spin 4-manifolds. This proof can immediately be applied to closed spin V-4-manifolds as well.

Theorem 4 *Suppose the spin^c-structure comes from a spin structure. Then* $\mathrm{ind}\, D_{\mathrm{Dirac}}(X)$ *is divisible by* 4. *If* $\mathrm{ind}\, D_{\mathrm{Dirac}}(X)$ *is not zero, then we have*

$$1 + \frac{1}{2}\mathrm{ind}\, D_{\mathrm{Dirac}}(X) \leq b_2^+.$$

The r.h.s is equal to $\operatorname{ind} D_{\mathrm{ASD}}(X) - 1$ from Lemma 1. We used this identification.

When the spin^c-structure comes from a spin structure, the kernel and the cokernel of D_{Dirac} have quaternionic structures and hence $\operatorname{ind} D_{\mathrm{Dirac}}(X)$ is divisible by 4.

Let $-X$ be X with its orientation reversed. The spin structure of X canonically induces a spin structure of $-X$. The corresponding Dirac operator is the formal adjoint of the original one. Hence we have $\operatorname{ind} D_{\mathrm{Dirac}}(-X) = -\operatorname{ind} D_{\mathrm{Dirac}}(X)$ and, under the assumption of the above theorem, we obtain:

Corollary 1 *Suppose the spin^c-structure comes from a spin structure.*

1. *If* $\operatorname{ind} D_{\mathrm{Dirac}}(X)$ *is not zero, then we have*

$$1 - \frac{1}{2} \operatorname{ind} D_{\mathrm{Dirac}}(X) \le b_2^-.$$

2. *If* $b_2^+(X), b_2^-(X) \le 2$, *then we have* $\operatorname{ind} D_{\mathrm{Dirac}} = 0$.

Especially, if X is a rational homology spin 4-sphere, then $\operatorname{ind} D_{\mathrm{Dirac}}$ must be zero for the spin structure. This is also shown by using Theorem 3 without assuming the spin structure.

3 V-4-manifolds with boundaries

In this section, let X be a compact oriented V-4-manifold with boundary Σ. We assume that X only has isolated singular points in its interior. Suppose Σ is a homology 3-sphere and X has a spin^c-structure.

Let Y be a spin 4-manifold with boundary $-\Sigma$. Since $\Sigma \times (0,1)$ admits a single spin^c-structure, we have a spin^c-structure on $X \cup_\Sigma Y$ by patching the spin^c-structures on X and Y. Moreover, since the space automorphisms of the spin structure on $\Sigma \times (0,1)$ is connected, the way of patching is unique up to homotopy.

We denote the spin^c-structure of X as c.

Definition 1

$$w(\Sigma, X, c) := \frac{1}{2} D_{\mathrm{Dirac}}(X \cup_\Sigma Y) + \frac{1}{8}(b_2^+(Y) - b_2^-(Y)).$$

Note that each term of the r.h.s is an integer.

Lemma 2 *The integer $w(\Sigma, X, c)$ is independent of the choice of Y and its spin structure.*

Proof.

This kind of definition is now well known and we only sketch an argument. Consider the elliptic operator $4D_{\mathrm{Dirac}} \oplus D_{\mathrm{sign}}$, where D_{sign} is the signature operator and the coefficient "4" indicates the direct sum of four copies of D_{Dirac}. The Atiyah-Singer index theorem implies that the index of this operator vanishes for closed 4-manifolds. Let Y' be another spin manifold with boundary $-\Sigma$. From an excision argument of the indices of the elliptic operators, we can show that the index for $X \cup_\Sigma Y$ is the sum of the indices of the operators for $X \cup_\Sigma Y'$ and $(-Y') \cup_\Sigma Y$. (One way to show this equality directly is to use Kawasaki's index theorem for V-manifolds.) This implies that the indices for $X \cup_\Sigma Y$ and $X \cup_\Sigma Y'$ are the same. Then the lemma immediately follows.

One way to make the excision argument work is to use the framework of Atiyah-Patodi-Singer and to consider operators on V-manifolds with boundaries. Another approach could be possible by using pseudo-differential operator on V-manifolds. The most direct approach would be to use heat kernel method to calculate the indices for $X \cup_\Sigma Y$ and $X \cup_\Sigma Y'$ to see that the only contribution comes from the singular points.

The same excision or localization argument also implies the following additivity.

Lemma 3

$$w(\Sigma_0 \sharp \Sigma_1, X_0 \natural X_1, c_0 \natural c_1) = w(\Sigma_0, X_0, c_0) + w(\Sigma_1, X_1, c_1).$$

Here $\Sigma_0 \sharp \Sigma_1$ is the connected sum of Σ_0 and Σ_1, and $X_0 \natural X_1$ is the boundary connected sum of X_0 and X_1.

For $i = 1, 2, \ldots, m$, let X_i be a compact oriented V-4-manifold with boundary Σ_i and which only has isolated singular points in its interior. Suppose Σ_i is a homology 3-sphere and X_i has a spin^c-structure c_i.

¿From Theorem 3 and Lemma 3 we obtain:

Theorem 5 *Suppose $X_1, X_2, \ldots, X_m$ are all negative definite. If the connected sum $\Sigma_1 \sharp \Sigma_2 \sharp \ldots \sharp\ Sigma_m$ is the boundary of an acyclic 4-manifold, then we have*

$$\sum_{i=1}^{m} w(\Sigma_i, X_i, c_i) \leq 0$$

Proof.

Let Y be the acyclic 4-manifold, then apply Theorem 3 to

$$(X_1 \natural X_2 \natural \ldots \natural X_m) \cup_{\Sigma_1 \sharp \Sigma_2 \sharp \ldots \sharp \Sigma_m} (-Y).$$

Now we consider spin structures.

Lemma 4 *If c is a spin structure of X, then we have*

$$\mu(\Sigma) = w(\Sigma, X, c) \bmod 2.$$

Especially, if Σ is the boundary of an acyclic 4-manifold, then $w(\Sigma, X, c)$ must be even.

Proof.

Since $\mu(\Sigma) = (b_2^+(Y) - b_2^+(Y))/8 \bmod 2$, it suffices to show that the index of D_{Dirac} is divisible by 4. This follows from the quaternionic structure of the spinor bundles associated to the spin structure.

¿From Theorem 4 and Lemma 3 we similarly obtain:

Theorem 6 *Suppose all the c_i's are spin structures. If the connected sum $\Sigma_1 \sharp \Sigma_2 \sharp \ldots \sharp$ $Sigma_m$ is the boundary of an acyclic 4-manifold, then we have the following properties of $w(\Sigma_i, X_i, c_i)$.*

1. *The sum $\sum_{i=1}^m w(\Sigma_i, X_i, c_i)$ is even.*
2. *If the above sum is not zero, then we have*

$$1 - \sum_{i=1}^m b_2^-(X_i) \leq \sum_{i=1}^m w(\Sigma_i, X_i, c_i) \leq -1 + \sum_{i=1}^m b_2^+(X_i).$$

Corollary 2 *Suppose c is spin.*

1. *If $b_2^+(X), b_2^-(X)$ is equal to or less than 2 and the connected sum of some odd number of copies of Σ is the boundary of an acyclic 4-manifold, then we have $w(\Sigma, X, c) = 0$.*
2. *If $b_2^+(X), b_2^-(X)$ is equal to or less than 1 and the connected sum of some number of copies of Σ is the boundary of an acyclic 4-manifold, then we have $w(\Sigma, X, c) = 0$.*

4 Invariant of homology 3-spheres

It is convenient to introduce the following notation.

Definition 2

1. *Let $\mathcal{X}$ be the set of isomorphism classes of triples (Σ, X, c) such that*
   (a) *X is a compact oriented spin V-4-manifold with only isolated singularities in its interior,*
   (b) *Σ is the boundary of X and we assume that Σ is a homology 3-sphere and*
   (c) *c is the spin structure of X.*
2. $$\mathcal{X}(k^+, k^-) = \{(\Sigma, X, c) \in \mathcal{X} \mid b_2^+ \leq k^+, b_2^- \leq k^-\}.$$

3. *$\mathcal{S}(k^+, k^-)$ is the set of the isomorphism classes of oriented homology 3-sphere Σ such that (Σ, X, c) is in the class $\mathcal{X}(k^+, k^-)$ for some X and c.*

Note that connected sum induces a map

$$\mathcal{S}(k_0^+, k_0^-) \times \mathcal{S}(k_1^+, k_1^-) \to \mathcal{S}(k_0^+ + k_1^+, k_0^- + k_1^-).$$

The next theorem is a corollary of Lemma 4 and Corollary 2.

Theorem 7 *Suppose Σ is in $\mathcal{S}(1,1)$. If $\mu(\Sigma) \equiv 1$, then the connected sum of any number of copies of Σ cannot be the boundary of an acyclic 4-manifold.*

Now we define an integral lift of μ-invariant for some classes including $\mathcal{S}(1,1)$. The invariant w is defined on $\mathcal{X}$.

Theorem 8 *Suppose (k^+, k^-) is equal to $(2,0)$, $(1,1)$ or $(0,2)$, and two oriented homology 3-spheres Σ_0 and Σ_1 in the class $\mathcal{S}(k^+, k^-)$ are homology cobordant to each other. By definition there are some triples (Σ_0, X_0, c_0) and (Σ_1, X_1, c_1) in $\mathcal{X}(k^+, k^-)$. Then we have*

$$w(\Sigma_0, X_0, c_0) = w(\Sigma_1, X_1, c_1).$$

Proof.Recall that Σ_0 and Σ_1 are homology cobordant to each other if and only if the connected sum of Σ_0 and $-\Sigma_1$ is the boundary of an acyclic 4-manifold. Since $X_0 \natural - X_1$ satisfies the assumption of Corollary 2, we obtain

$$w(\Sigma_0, X_0, c_0) + w(-\Sigma_1, -X_1, -c_1) = 0.$$

In particular we have

$$w(\Sigma_1, X_1, c_1) + w(-\Sigma_1, -X_1, -c_1) = 0.$$

Compare these two to get the required equality.

As an immediate corollary of the above theorem, we obtain:

Theorem 9 *We have a well-defined map*

$$w(k^+, k^-) : \mathcal{S}(k^+, k^-) \to \mathbf{Z},$$

for (k^+, k^-) satisfying $k^+ + k^- = 2$. The map $w(k^+, k^-)$ is a homology cobordism invariant.

Remark. The authors do not know if the invariant is a homology cobordism invariant on the union of the three $\mathcal{S}(k^+, k^-)$'s with $k^+ + k^- = 2$.

The main theorem of this note is the following theorem, which is just a consequence of the theorems above.

Theorem 10 *Suppose Σ is in $\mathcal{S}(k^+, k^-)$ for some (k^+, k^-) satisfying $k^+ + k^- \leq 2$.*

1. $$\mu(\Sigma) \equiv w(k^+, k^-)(\Sigma) \bmod 2.$$

2. *If $w(k^+, k^-)$ is not zero, then the connected sum of any odd number of Σ cannot be the boundary of an acyclic 4-manifold.*

3. *If $w(k^+, k^-)$ is not zero and $k^+ + k^-$ is equal to or less than 1, then the connected sum of any number of Σ cannot be the boundary of an acyclic 4-manifold.*

5 Seifert fibered homology 3-spheres

In this section we give some examples.

Seifert fibered homology 3-spheres are classified as follows.

Let Z be a V-Riemann surface whose underlying space is $\mathbf{CP}^1$ with n marked points for $n \geq 3$. Let $a_1, a_2, \dots, a_n$ be the order of the cyclic isotropy groups at the singular points. Assume that a_i's are pairwise coprime to each other. Then the abelian group consisting of all the isomorphism classes of V-line bundles is an infinite order cyclic group. The first Chern class of one of the two generators is equal to $1/\alpha$ for $\alpha = a_1 a_2 \dots a_n$. Let L_0 be a V-line bundle with $c_1(L_0) = -1/\alpha$. Then other V-line bundles are of the form L_0^k for some integer k.

We consider the disk bundle $X = D(L_0)$ as an oriented V-4-manifold. When the isotropy representation at each singular point of Z acts freely on the fiber of a V-line bundle except for the origin, the associated V-circle bundle is actually smooth. The V-line bundle L_0 satisfies this condition. Hence we have a smooth oriented 3-manifold $S(L_0) = \partial X$. We write $\Sigma(a_1, a_2, \dots, a_n)$ for this oriented 3-manifold. It is easy to see that this is a homology 3-sphere. It is known that every oriented Seifert fibered homology 3-sphere is of the form $\pm\Sigma(a_1, a_2, \dots, a_n)$. See [14] for details. The canonical line bundle of the total space of L_0 is isomorphic to the dual of the pull-back of $L_0 \otimes TZ$. Note that TZ is isomorphic to L_0^k for some k. Then the canonical bundle is the dual of the pullback of L_0^{k+1}.

The total space of L_0 has a canonical spin^c-structure coming from its complex structure. As a subset of L_0, we have a canonical spin^c-structure of X.

Theorem 11 *The V-4-manifold X has a spin structure if and only if one of the a_i's is even.*

Proof.

Note that the total space of L_0 is spin if and only if k is odd and hence L_0^{k+1} is a square. A calculation of k follows by looking at the Euler characteristic number of Z [14]:

$$k = \frac{\chi(Z)}{-1/\alpha} = \alpha(-2 + \sum(1 - \frac{1}{a_i})).$$

Then we see the theorem from the above expression of k.

Corollary 3 *If one of the a_i's is even, then*

$$\Sigma(a_1, a_2, \ldots, a_n) \in \mathcal{S}(0,1),$$

hence it lies in $\mathcal{S}(1,1)$ and $\mathcal{S}(0,2)$.

To obtain the spin structure, we have to twist the spinor bundle for the canonical spin^c-structure. We need the twisting by the square root of the dual of the canonical bundle, i.e., $L_0^{(k+1)/2}$.

Let $c(m)$ be the spin^c-structure defined by the twisting of the canonical spin^c-structure by L_0^m. Then $c(0)$ is the canonical spin^c-structure and the spin structure is given by $c((k+1)/2)$.

Now the invariant $w(\Sigma(a_1, a_2, \ldots, a_n), X, c(m))$ is calculated by using V-version of the Atiyah-Singer index theorem [16]. We shall give detailed arguments and also comparison with other invariants elsewhere. In this note we only state the result of calculation.

Theorem 12

$$\begin{aligned} & w(\Sigma(a_1, a_2, \ldots, a_n), X, c(m)) \\ = \; & \frac{1}{8}\left[-\frac{1}{\alpha}\left\{ \left(2 - \sum_{i=1}^{n}(1 - \frac{1}{a_i}) \right) \alpha + 2m - 1 \right\}^2 + 1 \right. \\ & \quad - \sum_{i=1}^{n} \frac{1}{a_i} \sum_{l=1}^{a_i - 1} \left\{ \cot\left(\frac{\pi l}{a_i}\right) \cot\left(\frac{\pi b_i l}{a_i}\right) \right. \\ & \qquad \left.\left. + \; 2\cos\left(\frac{\pi(1 + (-2m+1)b_i)l}{a_i}\right) \operatorname{cosec}\left(\frac{\pi l}{a_i}\right) \operatorname{cosec}\left(\frac{\pi b_i l}{a_i}\right) \right\} \right] \end{aligned}$$

where

$$b_i \frac{\alpha}{a_i} \equiv -1 \pmod{a_i}, \;\; 0 \le b_i \le a_i - 1, \;\; b_i \in \mathbf{Z}.$$

In Introduction we used the notation $w(a_1, a_2, \ldots, a_n; m)$ to denote this number.

Theorem 13 *Suppose one of the a_i's is even. Then we have*

$$w(\Sigma(a_1, a_2, \ldots, a_n)) = w(a_1, a_2, \ldots, a_n; s),$$

for

$$s = \frac{1}{2}\left\{ 1 + \alpha\left(-2 + \sum_{i=1}^{n}(1 - \frac{1}{a_i}) \right) \right\}.$$

Here $w = w(1,1)$ or $w(0,2)$.

More explicitly, we can write:

$$
\begin{aligned}
&w(\Sigma(a_1, a_2, \ldots, a_n)) \\
&= \frac{1}{8}\left[1 - \sum_{i=1}^{n} \frac{1}{a_i} \sum_{l=1}^{a_i - 1} \left\{ \cot\left(\frac{\pi l}{a_i}\right) \cot\left(\frac{\pi b_i l}{a_i}\right) \right.\right. \\
&\qquad \left.\left. + \ 2\cos\left(\frac{\pi(1 + (-2s+1)b_i)l}{a_i}\right) \operatorname{cosec}\left(\frac{\pi l}{a_i}\right) \operatorname{cosec}\left(\frac{\pi b_i l}{a_i}\right) \right\}\right].
\end{aligned}
$$

Examples

We give some examples of Brieskorn homology 3-spheres such that the connected sum of any number of the copies cannot be the boundary of an acyclic 4-manifold. The following is a list of Brieskorn homology 3-spheres for which Theorem 5 is not applied, but Theorem 6 can be applied.

Brieskorn	max{ w }	spin case	Casson's invariant
$\Sigma(2,3,7)$	0	$w(1) = -1$	-1
$\Sigma(4,5,7)$	0	$w(29) = -1$	-5
$\Sigma(7,9,10)$	0	$w(204) = -1$	-25
$\Sigma(2,5,11)$	0	$w(12) = -1$	-3
$\Sigma(5,8,11)$	0	$w(129) = -1$	-17
$\Sigma(4,9,11)$	0	$w(109) = -1$	-15
$\Sigma(5,7,12)$	0	$w(121) = -2$	-16
$\Sigma(7,11,12)$	0	$w(316) = -1$	-37
$\Sigma(3,4,13)$	0	$w(27) = -1$	-5
$\Sigma(5,6,13)$	0	$w(109) = -1$	-15
$\Sigma(3,8,13)$	0	$w(73) = -1$	-11

In the above list, only $\Sigma(5,7,12)$ has trivial μ-invariant.

Remark.

1. There are other examples in $\mathcal{S}(k^+, k^-)$ $(k^+ + k^- = 2)$. For $i = 1, 2, \ldots, m$ suppose L_i is a V-line bundle over a V-Riemann surface Z_i. Take any tree graph with m vertices. Write v_i $(i = 1, 2, \ldots, m)$ for the vertices. Then one can construct a plumbing of L_i's corresponding to this tree graph. Truncate this V-4-manifold to get a V-4-manifold with boundary. If this boundary is a smooth homology 3-sphere, then it lies in the class $\mathcal{S}(k^+, k^-)$ for some k^+ and k^- satisfying $k^+ + k^- = m$.

2. The class $\mathcal{S}(k^+, k^-)$ $(k^+ + k^- = 2)$ is defined by oriented homology 3-spheres which are the boundaries of almost definite spin V-manifolds. In other words, we used cobordism between the homology 3-sphere and spherical space forms. It is possible to extend the argument by replacing spherical space forms with oriented closed rational homology 3-spheres which

admit positive scalar curvature metrics. Moreover when they do not admit positive scalar curvature metric, presumably certain condition about Seiberg-Witten Floer homology could be used instead. By finding this condition, it might be possible to generalize the theorems in this note.

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References

[1] D. M. Austin, $SO(3)$-instantons on $L(p,q) \times \mathbf{R}$, J. Differ. Geom. 32 (1990), 383-413.

[2] S. Akbulut and R. Kirby, Mazur manifolds, Michigan Math. J. 26 (1979), 259-284.

[3] A. J. Casson and J. L. Harer, Some homology lens spaces which bound rational homology balls, Pacific J. Math. 96 (1981), 23-36.

[4] R. Fintushel and T. Lawson, Compactness of moduli spaces for orbifold instantons, Topology Appl. 23 (1986), 305-312.

[5] R. Fintushel and R. Stern, Pseudofree orbifolds, Ann. Math. II Ser. 122 (1985), 335-364.

[6] R. Fintushel and R. Stern, Rational homology cobordisms of spherical space forms, Topology 26 (1987), 385-393.

[7] R. Fintushel and R. Stern, Instanton homology of Seifert fibred homology three spheres, Proc. London Math. Soc. 61 (1990), 109-137.

[8] K. A. Froyshov, The Seiberg-Witten equations and four manifolds with boundary, Math. Res. Lett. 3 (1996), 373-390.

[9] M. Furuta, $\mathbf{Z}_a$-invariant $SU(2)$-instantons on S^4, Geometry of low dimensional manifolds1, (1990) London Math. Soc. LNS, Cambridge, 161-174.

[10] M. Furuta, Homology cobordism group of homology 3-spheres, Invent. Math. 100 (1990), 339-355.

[11] M. Furuta, On self-dual pseudo-connections on some orbifolds, Math. Z. 209 (1992), 319-337.

[12] M. Furuta, Monopole equation and the 11/8-conjecture preprint.

[13] M. Furuta and Y. Hashimoto, Invariant instantons on S^4, J. Fac. Sci. Univ. Tokyo Sec. IA, Math. 37 (1990), 585-600.

[14] M. Furuta and B. Steer, Seifert fibred homology 3-spheres and the Yang Mills equations on Riemann surfaces with marked points, Adv. in Math. 96 (1992), 38-102.

[15] D. E. Galenski and R. J. Stern, Classification of simplicial triangulations of topological manifolds, Ann. Math., II. Ser. 111,(1980) 1-34.

[16] T. Kawasaki, The index of elliptic operators over V-manifolds, Nagoya Math. J. 84 (1981), 135-157.

[17] T. Lawson, Compactness results for orbifold instantons, Math. Z. 200 (1988), 123-140.

[18] T. Matumoto, Triangulation of manifolds, Proc. Sympos. Pure Math. 32 (1978), Part II, 3-7.

[19] T. Mrowka, P. Ozsvath and B. Yu, Seiberg-Witten monopoles on Seifert fibered spaces, MSRI preprint, 1996-93.

[20] L. Nicolaescu Finite energy Seiberg-Witten moduli spaces on 4-manifolds bounding Seifert fibrations. dg-da/9711006.

[21] W. Neumann, An invariant of plumbed homology spheres, Lecture Notes in Math. 788, Springer-Verlag, 1980, 125-144.

[22] I. Satake, The Gauss-Bonne theorem for V-manifolds, J. of the Math. Soc. of Japan, 9 (1957) 464-492.

[23] N. Saveliev, On the homology cobordism group of homology 3-spheres, MSRI preprint 1996-63.

[24] N. Saveliev, Floer homology of Brieskorn homology spheres: solution to Atiyah's problem, MSRI preprint 1997-6.

[25] L. Siebenmann, On vanishing of the Rohlin invariant and nonfinitely amphicheiral homology 3-spheres, Lecture Notes in Math. 788, Springer-Verlag, 1980, 172-222.

[26] R. Stern, Some more Brieskorn spheres which bound contractible manifolds, Notices Amer. Math. Soc. 25 (1978), A448.

[27] H. Tokui, Msc Thesis, 1997 January, University of Tokyo.