FROBENIUS MANIFOLDS, QUANTUM COHOMOLOGY, AND MODULI SPACES (CHAPTERS I,II,III)

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PREFACE

The following notes arose from several lecture courses given at the Max–Planck–Institute in Bonn in 1994–96. I have tried to summarize some results of recent research, multifaceted and fascinating, originated in mathematical physics and quickly crystallizing into a new chapter of geometry.

The first part of the notes is devoted to Frobenius manifolds, both in local and formal versions. The category of formal Frobenius manifolds serves as a receptacle for Quantum Cohomology and its study is closely interwoven with that of moduli spaces of curves, operads and perturbation formalism. The geometric version of this theory was almost singlehandedly created by B. Dubrovin.

The first two Chapters constitute an introduction to Dubrovin's paper [D2]. I have added the basics of superversion, taken from [KM], and some computations related to the quantum cohomology of projective spaces. The treatment of Schlesinger's picture was influenced by [H3], and the presentation of the sixth Painlevé equation was borrowed from [Ma5]. I made every effort to untangle the complex logical structure of the theory and to stress the interconnections which are severed when the presentation is linearly ordered.

The third Chapter is devoted to formal Frobenius manifolds, which in their different guises are related to the moduli spaces of curves and operads. It is a development of the picture presented in [KM] and [KMK].

The second part of the notes is planned. It will be dedicated to the algebraic-geometric construction of the Gromov–Witten invariants which form the foundation of Quantum Cohomology.

Contents

0. Introduction: What is Quantum Cohomology?

0.1. An overview. Let $H = H^*(V, k)$ be the cohomology space of a projective algebraic manifold V with coefficients in a field k of characteristic zero.

The quantum cohomology $H^*_{quant}(V)$ consists of H *plus* an additional piece of data which can be described in at least three seemingly unrelated ways:

i). As a formal series ("potential") Φ in coordinates on H whose third derivatives can be used to define on $K \otimes H$ the structure of a $\mathbf{Z}_2$–graded commutative associative algebra, K being the ring of all formal series in coordinates.

ii). As a family of polylinear cohomological operations $[m] : H^{\otimes n} \to H,\ n \geq 2$, indexed by all homology classes $m \in H_*(\overline{M}_{0,n+1}, k)$. Here $\overline{M}_{0,n+1}$ denotes the moduli space of stable $(n+1)$–marked algebraic curves of genus zero (cf. [Kn] and [Ke].)

iii). As a "completely integrable system" on the tangent sheaf of the formal spectrum Spf(K) (i. e. a formal completion of H considered as a linear supermanifold.) In this context, the system itself consists of one–parametric family of flat connections on the tangent bundle of Spf(K).

The structures i)–iii) can and must be first described abstractly. We will do it in more detail in 0.2–0.4, and then discuss in what sense they are equivalent in 0.5.

A constructive realization of these structures on cohomology spaces, i. e. quantum cohomology of V in the proper sense, involves counting (parametrized) rational curves on V and is thus related to some classical problems of enumerative algebraic geometry. In 0.6 and 0.7, we will give two examples of the potential Φ constructed in this way, for $V = \mathbf{P}^2$ and for $V =$ a quintic hypersurface in $\mathbf{P}^4$. The geometry underlying these constructions leads naturally to the descriptions of the type i) and ii).

Algebraic geometry furnishes also completely integrable systems of the type iii) in a totally different way, related to the periods of algebraic integrals and variations of Hodge structure. We will discuss two examples in 0.8 and 0.9.

If a potential Φ obtained by counting curves on a manifold can be identified with another potential Ψ related to the periods on another manifold, this gives a strong hold on the analytical properties of Φ and behaviour of its coefficients. Existence of such an identification for Calabi–Yau threefolds is the famous Mirror Conjecture. Hopefully, it constitutes a part of a more general mirror pattern.

We will now fix notation for the remaining part of the Introduction. Denote by (H, g) a $\mathbf{Z}_2$–graded finite dimensional k–linear space H endowed with an even non–degenerate graded symmetric bilinear form g. Let $\{\Delta_a\}$ be a basis of H, $g_{ab} = g(\Delta_a, \Delta_b)$, $(g^{ab}) = (g_{ab})^{-1}$, $\Delta = \sum \Delta_a g^{ab} \otimes \Delta_b \in H \otimes H$. Denote by $\{x^a\}$ the dual basis of the dual space of H. We will consider x^a as formal independent graded commuting variables of the same parity as Δ_a. Put $K = k[[x^a]]$; this is the same as the completed symmetric algebra of the dual space. Put $\partial_a = \partial/\partial x^a : K \to K$. We will write Φ_a instead of $\partial_a \Phi$, etc.

0.2. Definition. *A formal solution Φ of the associativity equations on (H, g), or simply a potential, is a formal series $\Phi \in K$ satisfying the following differential equations:*

$$\forall a, b, c, d : \sum_{ef} \Phi_{abe} g^{ef} \Phi_{fcd} = (-1)^{\tilde{x}_a(\tilde{x}_b + \tilde{x}_c)} \sum_{ef} \Phi_{bce} g^{ef} \Phi_{fad} \tag{0.1}$$

where generally $\tilde{x}$ denotes the $\mathbf{Z}_2$-parity of x.

Define a K-linear multiplication $\circ$ on $H_K := K \otimes_k H$ by the rule

$$\Delta_a \circ \Delta_b = \sum_{cd} \Phi_{abc} g^{cd} \Delta_d. \tag{0.2}$$

Clearly, it is supercommutative.

0.2.1. Proposition. *a). $(H_K, \circ)$ is associative iff Φ is a potential. Multiplication $\circ$ does not change if one adds to Φ a polynomial of degree ≤ 2 in x^a.*

b). An element Δ_0 of the basis is a unit with respect to $\circ$ iff it is even and $\Phi_{0bc} = g_{bc}$ for all b, c. Equivalently:

$$\Phi = \frac{1}{6} g_{00}(x^0)^3 + \frac{1}{2} \sum_{c \neq 0} x^0 x^b x^c g_{bc} + \text{terms independent of } x^0. \tag{0.3}$$

If $H = H^*(V, k), g =$ Poincaré pairing ($g_{ab} = \int_V \Delta_a \wedge \Delta_b$), and Φ is obtained via a Gromov–Witten counting of rational curves on V, then $(H_K, \circ)$ is called *the quantum cohomology ring of V*.

0.3. Moduli spaces $\overline{M}_{0n}$. Before giving the next definition, we recall some basic facts about stable curves of genus 0 with $n \geq 3$ labelled pairwise distinct non-singular points $(x_1, \ldots, x_n)$ (cf. [Kn], [Ke].) Such a curve is a tree of $\mathbf{P}^1$'s: any two irreducible components either are disjoint or intersect transversely at one point. Each component must contain at least three special (singular or labelled) points.

The space $\overline{M}_{0n}$ is a smooth projective algebraic manifold of dimension $n - 3$ supporting a universal family $X_n \to \overline{M}_{0n}$ of stable curves whose labelled points are given by n structure sections x_i; $\overline{M}_{0n} \to X_n$. An open subset ("big cell") parametrizes $\mathbf{P}^1$ with n pairwise distinct points on it. The boundary, or infinity, of $\overline{M}_{0n}$ is stratified according to the degeneration type of fibers of X_n: the combinatorics of the incidence tree of the curve and the distribution of labelled points among the components. The number of the components diminished by one is the codimension of the stratum. Of course, the closure of such a stratum includes its own boundary corresponding to further degeneration.

In particular, the irreducible boundary divisors D_σ of $\overline{M}_{0n}$ correspond to the stable (unordered) 2-partitions $\sigma : \{1, \ldots, n\} = S_1 \coprod S_2, |S_i| \geq 2$, describing the distribution of the labelled points among the two $\mathbf{P}^1$'s at the generic point of D_σ. A choice of the ordering of the partition defines an identification of D_σ with

$\overline{M}_{0,n_1+1}\times\overline{M}_{0,n_2+1}, n_i = |S_i|$: on each $\mathbf{P}^1$, add to the labelled points the intersection point of the two components. Thus we have a family of closed embeddings

$$\varphi_\sigma : \ \overline{M}_{0,n_1+1} \times \overline{M}_{0,n_2+1} \to \overline{M}_{0n} \tag{0.4}$$

inducing the restriction morphisms of the cohomology groups with coefficients in k

$$\varphi_\sigma^* : \ H^*(\overline{M}_{0n}) \to H^*(\overline{M}_{0,n_1+1}) \otimes H^*(\overline{M}_{0,n_2+1}) \tag{0.5}$$

Besides, S_n acts on $\overline{M}_{0n}$, $H^*(\overline{M}_{0n})$ and partitions σ by renumbering the labelled points, and (0.5) is compatible with this action.

0.3.1. Definition. *A structure of the Cohomological Field Theory (CohFT) (or an algebra over the operad $H_*\overline{M}_0$, cf. [GeK]) on (H,g) consists of a family of S_n-equivariant $\mathbf{Z}_2$-even polylinear maps*

$$I_n : \ H^{\otimes n} \to H^*(\overline{M}_{0n}, k), \ n \geq 3 \tag{0.6}$$

satisfying the following conditions. For every stable 2-partition σ of $\{1,\dots,n\}$ and all homogeneous $\gamma_1,\dots,\gamma_n \in H$ we have

$$\varphi_\sigma^*(I_n(\gamma_1 \otimes \dots \otimes \gamma_n)) = \epsilon(\sigma)(I_{n_1+1} \otimes I_{n_2+1}) \left(\bigotimes_{i\in S_1} \gamma_i \otimes \Delta \otimes (\bigotimes_{i\in S_2} \gamma_i) \right) \tag{0.7}$$

where $\epsilon(\sigma)$ is the sign of the permutaion induced by σ on the odd-dimensional classes γ_i.

Another way of looking at such a structure is to make a partial dualization with the help of the Poincaré pairing on $\overline{M}_{0,n+1}$ and g on H. Then one can rewrite (0.6_{n+1}) as

$$H_*(\overline{M}_{0,n+1}) \otimes H^{\otimes n} \to H, \ n \geq 2 \tag{0.8}$$

that is, to interpret every class $m \in H_*(\overline{M}_{0,n+1})$ as an n-ary multiplication $[m]$ on H linearly depending on $[m]$. Then (0.7) gives a complex system of quadratic identities between these multiplications which are best described in the operadic formalism (cf. [GeJ], [GeK], [GiK].)

However, the situation simplifies considerably if we restrict ourselves to looking only at those multiplications that correspond to the fundamental classes $[\overline{M}_{0,n+1}] \in H_*(\overline{M}_{0,n+1})$ and denote them simply by

$$[\overline{M}_{0,n+1}] \otimes (\gamma_1 \otimes \dots \otimes \gamma_n) \mapsto (\gamma_1,\dots,\gamma_n), \ n \geq 2. \tag{0.9}$$

These multiplications are supercommutative. Moreover:

0.3.2. Proposition. *The identities (0.7) imply the following generalized associativity equations for these multiplications: for any $\alpha,\beta,\gamma,\delta_1,\dots,\delta_n \in H, \ n \geq 0$ we have*

$$\sum_\sigma \epsilon'(\sigma)((\alpha,\beta,\delta_i|i \in S_1),\gamma,\delta_j|j \in S_2) =$$

$$\sum_{\sigma} \epsilon''(\sigma)(\alpha, (\beta, \gamma, \delta_i | i \in S_1), \delta_j | j \in S_2) \qquad (0.10)$$

where σ runs over 2–partitions $\sigma : \{1, \dots, n\} = S_1 \coprod S_2$ (non–necessarily stable), and ϵ are the standard signs.

In particular, for $n = 0, 1$ we get respectively

$$((\alpha, \beta), \gamma) = (\alpha, (\beta, \gamma)),$$

$$((\alpha, \beta), \gamma, \delta) + (-1)^{\tilde{\gamma}\tilde{\delta}}((\alpha, \beta, \delta), \gamma) = (\alpha, (\beta, \gamma, \delta)) + (\alpha, (\beta, \gamma), \delta). \qquad (0.11)$$

Remarkably, this family of n–ary multiplications is actually equivalent to the whole structure described in 0.3.1: cf. the proof of the Theorem 0.5 below.

In conclusion, let us formally compare the system of operations (0.8) on $H = H^*(V, k)$ (in the situation of quantum cohomology) with the more traditional Steenrod operations.

i). Steenrod powers are defined on the cohomology with coefficients in $\mathbf{F}_p$ whereas we can allow characteristic zero coefficients (perhaps even $\mathbf{Z}$.)

ii). Steenrod powers generate an *algebra* whereas $[m]$, $m \in H_*(\overline{M}_{0,n+1})$ are elements of an *operad.*

iii). Steenrod powers are defined solely in terms of topology of V, whereas to construct $[m]$ we need additionally the structure of algebraic (or symplectic) manifold, in order to be able to define holomorphic curves on V.

0.4. Frobenius manifolds. The term "completely integrable system" is used rather indiscriminately in a wide variety of contexts. The notion relevant here was introduced by B. Dubrovin (cf. [D1], [D2]) under the name of Frobenius manifold. We start with the formal version.

0.4.1. Definition. *a). The structure of a formal Frobenius manifold on (H, g) is a one–parametric system of flat connections on the module of derivations of K/k given by its covariant derivatives*

$$\nabla_{\lambda, \partial_a}(\partial_b) := \lambda \sum_{cd} A_{abc} g^{cd} \partial_d = \lambda \sum_{d} A_{ab}^{d} \partial_d \qquad (0.12)$$

where $A_{abc} \in K$ is a symmetric tensor, λ an even parameter.

b). This structure is called potential one, if the tensor $\partial_d A_{abc}$ is totally symmetric.

More generally, a Frobenius manifold (M, g, A) (in any of the standard geometric categories: smooth, analytic, algebraic (super)manifolds) is a manifold M endowed with a flat metric g and a tensor field A of rank 3 such that if we write the components of A in local g–flat coordinates, the conditions of 0.4.1 a) and eventually b) are satisfied.

0.5. Theorem. *For a given (H, g), there exists a natural bijection between the sets of the additional structures described above:*

i). Formal solutions of the associativity equations on (H, g), modulo terms of degree ≤ 2.

ii). Structures of the CohFT on (H, g).

iii). Structures of the formal potential Frobenius manifold on (H, g).

Easy part of the proof (sketch). We will first describe maps *ii)* $\to$ *i)* $\to$ *iii)*.

ii) $\to$ *i)*.

Assume that we have on (H, g) the structure of CohFT given by some maps I_n as in (0.6). Construct first the symmetric polynomials

$$Y_n : \ H^{\otimes n} \to k, \ Y_n(\gamma_1 \otimes \cdots \otimes \gamma_n) := \int_{\overline{M}_{0n}} I_n(\gamma_1 \otimes \cdots \otimes \gamma_n) \tag{0.13}$$

and form the series

$$\Phi(x) := \sum_{n \geq 3} \frac{1}{n!} Y_n \left((\sum_a x^a \Delta_a)^{\otimes n} \right) \tag{0.14}$$

Keel ([Ke]) has described the linear relations between the cohomology classes of the boundary divisors D_σ defined in 0.3. Namely, choose a quadruple of pairwise distinct indices $i, j, k, l \in \{1, \dots, n\}$, $n \geq 4$. For a stable 2–partition $\sigma = \{S_1, S_2\}$ write $ij\sigma kl$ if $i, j \in S_1$, $k, l \in S_2$ for some ordering of the parts. Then the $\{ijkl\}$–th Keel's relation is

$$\sum_{\sigma : ij\sigma kl} D_\sigma \cong \sum_{\sigma : ik\sigma jl} D_\sigma \quad \text{in } H^*(\overline{M}_{0n}). \tag{0.15}$$

Geometrically, it follows from the fact that the two sides of (0.15) are the two fibers of the projection

$$\overline{M}_{0n} \to \overline{M}_{0,\{ijkl\}} \cong \overline{M}_{0,4} = \mathbf{P}^1$$

forgetting all the labelled points except for x_i, x_j, x_k, x_l. The space $\overline{M}_{0,4}$ has exactly three boundary points corresponding to the three stable partitions of $\{i, j, k, l\}$. In (0.15) we use two of them.

Notice that the existence of the forgetful morphism is a non–trivial geometric fact, because on the level of fibers of X_n (i. e. geometric points of the moduli) it involves contracting those components that become unstable, cf. [Kn].

If we restrict $Y_n(\gamma_1 \otimes \cdots \otimes \gamma_n)$ to D_σ using (0.7) then integrate over D_σ and take into account (0.15), we will get a series of bilinear identities: $\forall i, j, k, l$

$$\sum_{\sigma : ij\sigma kl} \epsilon(\sigma) (Y_{|S_1|+1} \otimes Y_{|S_2|+1}) \left(\bigotimes_{p \in S_1} \gamma_p \otimes \Delta \otimes (\bigotimes_{q \in S_2} \gamma_q) \right) =$$

$$\sum_{\sigma : ik\sigma jl} \epsilon(\sigma) (Y_{|S_1|+1} \otimes Y_{|S_2|+1}) \left(\bigotimes_{p \in S_1} \gamma_p \otimes \Delta \otimes (\bigotimes_{q \in S_2} \gamma_q) \right) \tag{0.16}$$

On the other hand, writing the associativity equations (0.1) for the series (0.14), one can directly show that they reduce to a subfamily of the relations (0.16), which implies the whole family by the standard polarization argument. Thus Φ encodes the same amount of information as $\{Y_n\}$ and (0.16).

i) $\to$ *iii)*.

Given a potential Φ, we simply put $A_{abc} = \partial_a\partial_b\partial_c\Phi$. This is in fact a bijection, because given (H, g, A), the symmetry of A_{abc} and $\partial_d A_{abc}$ implies the existence of Φ with $A_{abc} = \partial_a\partial_b\partial_c\Phi$, and the curvature vanishing equation $\nabla_\lambda^2 = 0$ implies the associativity equations for Φ.

Difficult part of the proof. It remains to show that nothing is lost or gained in the passage from I_n to Y_n, i.e., that the arrow *ii)* $\to$ *i)* is both injective and surjective. Injectivity is again easy, because using (0.7) consecutively one sees that the knowledge of Y_n allows us to reconstruct integrals of I_n along all the boundary strata, whose classes span $H^*(\overline{M}_{0n})$. But surjectivity requires a considerable work. Basically, it reduces to showing that the ad hoc formulas for the integrals over the boundary strata do define a cohomology class, i. e., satisfy all the linear relations between the classes. A remarkable reformulation of this property asserts that the homology of moduli spaces forms a Koszul operad. For details, see the main text.

0.5.1. Remark. What this last argument additionally shows, is that the structure of a CohFT on (H, g) can be replaced by the structure of a $Comm_\infty$–algebra given by a family of n–ary operations, one for each $n \geq 2$, satisfying the generalized associativity relations (0.10). This structure looks simpler because it does not involve the moduli spaces $\overline{M}_{0n}$ which look completely irrelevant also for the remaining two descriptons. However, there are at least three reasons not to eliminate the moduli spaces, and even to consider ii) as the most important structure.

a). In the applications to quantum cohomology, the geometry of the Gromov–Witten invariants naturally involves total maps I_n, not just their top dimensional terms Y_n describing the physicists' correlation functions.

b). The higher genus theory at the moment can be formulated only in terms of the cohomological operations parametrized by the homology classes of the moduli spaces of stable curves $\overline{M}_{gn}$. The analytic part of the theory where an analog of the potential plays the central role is very incompletely understood (cf. [BCOV] and [Ko6].) Besides, it seems that the cohomological operations cannot be reduced to the correlation functions because of the existence of cohomology classes vanishing on the boundary.

c). Returning to the genus zero case, in the abstract framework of $Comm_\infty$–algebras, there exists an operation of their tensor product. It can be defined as follows::

$$(H', g', I'_n) \otimes (H'', g'', I''_n) = (H' \otimes H'', g' \otimes g'', I_n)$$

where I_n are given by

$$I_n(\gamma'_1 \otimes \gamma''_1 \otimes \ldots \otimes \gamma'_n \otimes \gamma''_n) := \epsilon(\gamma', \gamma'') I'_n(\gamma'_1 \otimes \ldots \otimes \gamma'_n) \wedge I''_n(\gamma''_1 \otimes \ldots \otimes \gamma''_n).$$

This is an important and natural operation necessary e. g. for the formulation of the quantum Künneth formula. However, it seems impossible to construct this product without invoking $\overline{M}_{0n}$. In fact, its existence is a reflection of the fact that $H_*(\overline{M}_{0n})$ form an operad of coalgebras, and not just linear spaces.

In particular, consider C_∞–algberas of rank 1 (i. e. dim(H)=1.) In terms of potentials, they correspond to arbitrary power series in one variable $\Phi(x) = \sum_{n\geq 3} \frac{C_n}{n!} x^n$ because the associativity equations in one variable are satisfied identically. Hence we can define a tensor multiplication of such series. It turns out to be given by quite non–trivial polynomials in the coefficients involving a generalization of the Petersson–Weil volumes of $\overline{M}_{0n}$.

We will give now some examples. The fuller treatment will be given in the main body of the text.

0.6. Quantum cohomology of $\mathbf{P}^2$. First, we have

$$H^i(\mathbf{P}^2, k) = k\Delta_i,\ \Delta_i = c_1(\mathcal{O}(1))^i,\ i = 0, 1, 2.$$

Denote by $N(d)$ (for $d \geq 1$) the number of rational curves of degree d in $\mathbf{P}^2$ passing through $3d-1$ points in general position. The first few values of $N(d)$ starting with $d = 1$ are

$$1,\ 1,\ 12,\ 620,\ 87304,\ 26312976,\ 14616808192.$$

The potential $\Phi^{\mathbf{P}^2}$, by definition, is

$$\Phi^{\mathbf{P}^2}(x\Delta_0 + y\Delta_1 + z\Delta_2) = \frac{1}{2}(xy^2 + x^2 z) + \sum_{d=1}^{\infty} N(d) \frac{z^{3d-1}}{(3d-1)!} e^{dy} :=$$

$$\frac{1}{2}(xy^2 + x^2 z) + \varphi(y, z). \tag{0.17}$$

A direct computation shows:

0.6.1. Proposition. *The associativity equations (0.1) for the potential (0.17) are equivalent to one differential equation for φ:*

$$\varphi_{zzz} = \varphi_{yyz}^2 - \varphi_{yyy}\varphi_{yzz} \tag{0.18}$$

which is in turn equivalent to the family of recursive relations uniquely defining $N(d)$ starting with $N(1) = 1$:

$$N(d) = \sum_{k+l=d} N(k)N(l)k^2 l \left[l \binom{3d-4}{3k-2} - k \binom{3d-4}{3k-1} \right],\ d \geq 2. \tag{0.19}$$

0.6.2. Geometry. The identities (0.19) showing that $(H^*(\mathbf{P}^2, \mathbf{Q}), g, \Phi^{\mathbf{P}^2})$ is actually an instance of the structure described above were first proved by M. Kontsevich. He skillfully applied an old trick of enumerative geometry: in order to understand the number of solutions of a numerical problem, try to devise a degenerate case of the problem where it becomes easier. In this setting, Kontsevich starts with a new problem having *one-dimensional space of solutions* and looks at two different degeneration points in the line of solutions.

More precisely, fix $d \geq 2$ and consider a generic configuration in $\mathbf{P}^2$ consisting of two labelled points y_1, y_2, two labelled lines l_1, l_2, and a set of $3d-4$ unlabelled points Y. Look at the space of quintuples $(\mathbf{P}^1, x_1, x_2, x_3, x_4, f)$ where $x_i \in \mathbf{P}^1$ are pairwise distinct points, $f : \mathbf{P}^1 \to \mathbf{P}^2$ is a map of degree d such that $f(x_i) = y_i$ for $i = 1, 2$; $f(x_i) \in l_i$ for i=3,4, and $Y \subset f(\mathbf{P}^1)$. We identify such diagrams if they are isomorphic (identically on $\mathbf{P}^2$.) Then we can assume that $(x_1, x_2, x_3, x_4) = (1, 0, \infty, \lambda)$. If λ is fixed and generic, the number of maps does not depend on it. Kontsevich counts it by letting first $\lambda \to \infty$, and second $\lambda \to 1$. In the stable limit, $\mathbf{P}^1$ degenerates into two projective lines, and we must sum over all possible distributions of $\{x_i\} \cup f^{-1}(Y)$ on these components. Comparison of the two limits furnishes (0.19).

To make all of this rigorous, one must introduce not only the moduli spaces of stable curves, but also the moduli spaces of stable maps $\overline{M}_{0n}(\mathbf{P}^2)$ parametrizing Kontsevich–stable maps to $\mathbf{P}^2$. Then it will become clear that the calculation we sketched above furnishes a particular case of the identities (0.16).

0.7. Quantum cohomology of a three-dimensional quintic. Let $V \subset \mathbf{P}^4$ be a smooth quintic hypersurface. Its even cohomology has rank four and is spanned by the powers of a hyperplane section, the odd cohomology has rank 204 and consists of three-dimensional classes. For a generic even element $\gamma = \sum x^a \Delta_a \in H^*(V)$, denote by y the coefficient at $\Delta_1 := c_1(\mathcal{O}(1))$ and put

$$\Phi^V(\gamma) = \frac{1}{6}(\gamma^3) + \sum_{d \geq 1} n(d) Li_3(e^{dy}) \tag{0.20}$$

where (γ^3) means the triple self intersection index, $Li_3(z) = \sum_{m \geq 1} z^m/m^3$, and $n(d)$ is the appropriately defined number of rational curves of degree d on V.

Before we turn to the definition of $n(d)$, let us notice that in this case the associativity equations are satisfied with whatever choice of these coefficients! This can be checked by a direct calculation. An arguably more enlightening argument runs as follows: in quantum cohomology of any V, the associativity equations must reflect the degeneration properties of rational curves on V as was the case with $\mathbf{P}^2$. Now, on a quintic, the rational curves are typically rigid so that there is nothing to degenerate. (See however the discussion in 0.7.3.)

Algebraically, the quantum cohomology ring of the projective plane with $\circ$–multiplication (cf. 0.2 above) is semisimple whereas that of the quintic is nilpotent. B. Dubrovin has developed a rich theory of the Frobenius manifolds with pointwise semisimple multiplication in tangent sheaf. This should eventually provide analytic tools for the numerical theory of rational curves on Fano varieties. On the contrary, potentials of the Calabi–Yau threefolds are conjecturally constrained by the mirror principle rather than associativity equations.

0.7.1. A definition of the numbers $n(d)$. A naive argument showing that the number of rational curves of degree d on V must be finite runs as follows. The space of maps $f : \mathbf{P}^1 \to \mathbf{P}^4$, $(t_0, t_1) \mapsto (f_0(t_0, t_1), \ldots, f_4(t_0, t_1))$ of degree

d is a Zariski open subset in the space $\mathbf{P}^{5d+4}$ of the coefficients of forms f_i. The condition $F(f_0(t_0,t_1),\dots,f_4(t_0,t_1))=0$ where $F=0$ is the equation of V furnishes $5d+1$ equations on these coefficients. If these equations were independent, the space of colutions would be 3–dimensional. It is acted upon effectively by $\mathrm{Aut}(\mathbf{P}^1)$ (linear reparametrizartions) which leaves us with finitely many equivalence classes of unparametrized curves.

Unfortunately, it is unknown whether there exists a sufficiently generic V for which these equations actually are independent after deleting degenerating maps. The symplectic approach to this problem going back to M. Gromov uses a drastic deformation of the complex structure of V destroying its integrability. In this way the problem is put into general position. More precisely, only isolated non–singular pseudoholomorphic spheres in V with normal sheaf $\mathcal{O}(-1)+\mathcal{O}(-1)$ survive, they can be counted directly, and their number is stable.

Another strategy which we will sketch below does not leave the algebraic geometric framework and even allows one to calculate $n(d)$ using the same degeneration philosophy as in the Example 0.6, although in a rather different setting. This construction is also due to M. Kontsevich ([Ko7]).

Consider a pair (C,f) where C is a connected curve of genus 0 (a tree of $\mathbf{P}^1$'s), $f:\ C\to\mathbf{P}^4$ a map of degree d such that the inverse image of any point in $f(c)$ is either 0–dimensional, or a stable curve of genus zero whose labelled points are intersection points with non–contracted components. Such pairs (C,f) are called (Kontsevich–)stable maps (of genus zero, to $\mathbf{P}^4$.) There exists a diagram

$$\overline{M}(\mathbf{P}^4,d)\leftarrow\overline{C}_d\to\mathbf{P}^4$$

where $\overline{M}(\mathbf{P}^4,d)$ is the moduli space (or rather stack) of stable maps of degree d, $\overline{M}(\mathbf{P}^4,d)$ is the universal curve on it. Denote the right arrow (the universal map) by φ_d, and the left arrow by π. Put $\mathcal{E}_d=\varphi_d^*(\mathcal{O}(5))$, $E_d=\pi_*(\mathcal{E}_d)$.

0.7.2. *a). $\overline{M}(\mathbf{P}^4,d)$ is a smooth orbifold of dimension* $5d+1$.

b). E_d is a locally free sheaf on it of rank $5d+1$.

0.7.3. Definition. $n(d):=c_{5d+1}(E_d)$.

Motivation for this definition is simple: if a quintic V is defined by $s=0$, $s\in\Gamma(\mathbf{P}^4,\mathcal{O}(5))$, then s produces a section $\overline{s}\in\Gamma(\overline{M}(\mathbf{P}^4,d),E_d)$, and

$$c_{5d+1}(E_d)=\text{the number of zeroes of } \overline{s}$$

calculated with appropriated multiplicities. But $\overline{s}([\varphi])=0$ for $[\varphi]\in\overline{M}(\mathbf{P}^4,d)$ iff $\varphi_d(\overline{C}_{d,[\varphi]})\subset V$. Thus we simply avoided the problem of assigning ad hoc multiplicities to actual rational curves on V (which may have a "wrong" normal sheaf, singularities, or come in families) by reducing it to a calculation of Chern numbers on orbifolds.

Moreover, we simultaneously created a setting in which degeneration can easily occur. In fact, instead of considering curves in a fixed quintic V, we are now looking at curves in $\mathbf{P}^4$ lying in V, i. e., treat V as an "incidence condition", similar to

$3d-1$ points in $\mathbf{P}^2$ in 0.6 above. We may now freely change the equation $s=0$ for V and can take, e. g., $s=\prod_{i=0}^4 s_i$ where $s_i \in \Gamma(\mathbf{P}^4, \mathcal{O}(1))$ are coordinates in $\mathbf{P}^4$.

To make sense of the problem of "counting rational curves on the algebraic symplex $V_\infty := \cup_{i=0}^4 \{s_i = 0\}$" Kontsevich proceeds as follows. Consider the G_m–action on the whole setting $(\mathbf{P}^4, \mathcal{O}(5), \overline{M}(\mathbf{P}^4, d))$ given by $s_i \mapsto e^{\lambda_i t} s_i,\ i=0,\dots,4$ where λ_i are the parameters of this action considered as independent variables.

0.7.3. Claim. *a). V_∞ is the only reduced quintic fixed with respect to this action.*

b). Fixed points of this action in $\overline{M}(\mathbf{P}^4, d)$ consist of stable pairs (C, f) where C is a tree of $\mathbf{P}^1$'s mapped by f to the 1–skeleton of V_∞ (consisting of 10 projective lines).

Each such (C, f) has a combinatorial invariant (τ, λ) which is, roughly speaking, the dual tree τ of C each vertex of which is labelled either by zero (if the respective component of C is contracted by f), or by the name of the line in the skeleton to which it is mapped and the degree of this map.

Bott's formula for Chern numbers of a bundle E in a situation where G_m acts upon the whole setting involves a sum of local contributions over the connected components of the set of G_m–fixed points, each contribution depending on the weights of G_m on the normal sheaf of the component and on the restriction of E upon it.

Kontsevich shows that in our case we get a sum

$$n(d) = \sum w(\tau, \lambda) \tag{0.21}$$

where the Bott multiplicities $w(\tau, \lambda)$ of the parametrized curves in the 1-skeleton of V_∞ are explicit but complex rational functions on the parameters λ of the G_m–action. Since $n(d)$ must be a rational or even integral *number*, miraculous cancellations must take place in the r.h.s. of (0.21) which are not at all evident algebraically.

Computer calculations furnish the following values for the first four $n(d)$'s:

$$2875,\ 609250,\ 317206375,\ 242467530000. \tag{0.22}$$

More direct methods of counting rational curves lead to the same numbers.

Although in a sense the potential (0.20) is now explicitly known, it is still difficult to identify it with its conjectural mirror image which we will shortly describe.

0.8. Moduli spaces of Calabi–Yau threefolds as a weak Frobenius manifold. As the discussion in 0.4 and 0.5 shows, the geometry of a Frobenius manifold on M is basically defined by a flat structure and a symmetric cubic tensor which is the third Taylor differential of a potential in flat coordinates. A flat metric is then used in order to raise indices and write the associativity equations.

If we are interested in a class of potentials for which the associativity equations are trivial, like (0.20), we may as well forget about the metric, and call the resulting structure *weakly Frobenius*. This geometry naturally arises from the theory of variation of Hodge structure of Calabi–Yau threefolds.

Let $\pi: W \to Z$ be a complete local family of Calabi–Yau threefolds. Recall that each fiber W_z is a projective algebraic manifold with trivial canonical bundle and $h^{i,0} = 0$ for $i = 1, 2$. Denote by $\mathcal{L} = \pi_* \Omega^3_{W/Z}$ the invertible sheaf of holomorphic volume forms on the fibers of π. We will construct an $\mathcal{L}^{-2}$–valued cubic differential form $G: S^3(\mathcal{T}_Z) \to \mathcal{L}^{-2}$ in the following way. First, according to Bogomolov–Todorov–Tian, the Kodaira–Spencer map (following from $0 \to \mathcal{T}_{W/Z} \to \mathcal{T}_W \to \pi^*(\mathcal{T}_Z) \to 0$)

$$KS: \mathcal{T}_Z \to R^1\pi_*\mathcal{T}_{W/Z}$$

is actually an isomorphism so that the tangent space at $z \in Z$ can be identified with $H^1(W_z, \mathcal{T}_{W_z}) \cong H^1(W_z, \Omega_z^2) \otimes \mathcal{L}(z)^{-1}$. Second, the convolution $i: \mathcal{T}_{W/Z} \times \Omega^p_{W/Z} \to \Omega^{p-1}_{W/Z}$ induces the pairings

$$R^1\pi_*(i): R^1\pi_*\mathcal{T}_{W/Z} \times R^q\pi_*\Omega^p_{W/Z} \to R^{q+1}\pi_*\Omega^{p-1}_{W/Z}$$

or else

$$R^1\pi_*\mathcal{T}_{W/Z} \to \mathcal{E}nd^{(-1,1)}(\oplus_{p,q} R^q\pi_*\Omega^p_{W/Z})$$

which is essentially the graded symbol of the Gauss–Manin connection defined thanks to the Griffiths' transversality condition. Iterating it three times and using Serre's duality we get finally:

$$G: S^3(\mathcal{T}_Z) \cong S^3(R^1\pi_*\mathcal{T}_{W/Z}) \to \mathcal{H}om(\pi_*\Omega^3_{W/Z}, \pi_*\mathcal{O}_N) \cong \mathcal{L}^{-2}.$$

In order to identify $\mathcal{L}^{-2}$ with $\mathcal{O}_Z$ (which we need to define a weak Frobenius structure) we must choose a trivialization of the volume form sheaf. In the context of the mirror conjecture, this is achieved by postulating that Z can be partially compactified by $\dim(Z)$ divisors with normal intersection in such a way that the family W can be extended to a family of "degenerate Calabi–Yau's" and the zero-dimensional stratum of the boundary W_∞ becomes a maximally degenerate manifold, like the simplex V_∞ in the family of quintics. A precise description of this condition is fairly technical, and we omit it here; but see Deligne's paper [De2].

Then the monodromy invariant part of $H_3(W_z, \mathbf{Z})/(tors)$ around zero will be generated by one cycle γ defined up to sign (more or less by the definition of maximal degeneration), and we locally trivialize $\mathcal{L}$ by choosing a volume form ω_z on W_z in such a way that $\int_{\gamma_z} \omega_z = (2\pi i)^3$.

The flat coordinates in which G is the third Taylor differential of a potential Ψ can be constructed in the same context as the action variables of the algebraically completely integrable system whose phase space is the family of Griffiths Jacobians of W_z: cf. [DoM].

A family W is called the mirror family for V if one can identify the weak Frobenius manifold structure on $H^2(V)$ obtained via curve counting on V (A–model) with that corresponding to the variation of Hodge structure for W (B–model).

For the particular case of quintics considered in 0.7 the mirror family depends on one parameter z, and W_z is obtained by resolving singularities of the spaces

$\widetilde{W}_z/(\mathbf{Z}/5\mathbf{Z})^3$ where $\widetilde{W}_z \subset \mathbf{P}^4$ is given by the equation $\sum_{j=1}^5 x_j^5 = z\prod_{j=1}^5 x_j$, and $(\mathbf{Z}/5\mathbf{Z})^3$ acts by $x_j \mapsto \xi_j x_j$, $\xi_j^5 = 1$, $\prod_{j=1}^5 \xi_j = 1$.

All the periods $\psi(z) := \int_{\gamma_z} \nu_z$ of an explicit algebraic volume form along $\gamma_z \in H_3(W_z, \mathbf{Z})$ (any horizontal cycle) satisfy the Picard–Fuchs differential equation $\partial := zd/dz)$:

$$[\partial^4 - 5z(5\partial+1)(5\partial+2)(5\partial+3)(5\partial+4)]\psi(z) = 0.$$

It has four linearly independent solutions near $z = 0$:

$$\psi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5)} z^n,$$

$$\psi_1(z) = \log(z)\psi_0(z) + 5\sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5)}\left(\sum_{k=n+1}^{5n} k^{-1}\right) z^n,$$

and two more for which we give only the top terms

$$\psi_2(z) = \frac{1}{2}(\log z)^2\psi_0(z) + \dots, \psi_3(z) = \frac{1}{6}(\log z)^3\psi_0(z) + \dots$$

An appropriate flat coordinate on the z–line by definition is $\frac{\psi_1}{\psi_0}(z)$. Under the mirror correspondence, it becomes y in (0.20) thus locally identifying $H^2(V, \mathbf{C})$ (where V is a generic quintic) to the moduli space of the dual family W. Putting

$$F(y) := \Phi^V(y) = \frac{5}{6}y^3 + \sum_{d=1}^{\infty} n(d)Li_3(e^{dy}) \tag{0.23}$$

we have the following conjectural mirror identity:

$$(?) \qquad F\left(\frac{\psi_1}{\psi_0}\right) = \frac{5}{2}\frac{\psi_1\psi_2 - \psi_0\psi_3}{\psi_0^2}. \tag{0.24}$$

Since ψ_i are explicitly known, one can check that the first coefficients agree with (0.22).

However, conceptually (0.24) looks baffling. In order to reduce our problem to the proof of an explicit identity, we have oversimplified the geometry. In particular, the mirror pattern must involve some operator of parity change or an odd scalar product on the full Frobenius supermanifold, because an even part of $H^*(V)$ becomes identified with an odd part of $H^*(W)$. E. Witten and M. Kontsevich suggested that generally one should extend the moduli space of the model B rather than restrict (to H^2) the moduli of the problem A. This is crucially important for understanding the mirror picture for the higher–dimensional Calabi–Yau manifolds where rational curves cease to be isolated and a considerably larger (depending on $\dim(V)$) portion of $H^*(V)$ becomes affected by the instanton corrections. According to Kontsevich, one should construct deformations of a Calabi–Yau manifold in a mysterious universe of non–commutative objects like A_∞–categories (cf. [Ko4]).

A. Givental [Giv2] achieved a remarkable progress in proving the Mirror Conjecture for complete intersections in toric varieties where the precise construction of mirrors is due to Batyrev ([Ba1], [BaBo2].) He enriched Kontsevich's approach by passing to the equivariant quantum cohomology. Some work remains to be done in order to complete his arguments.

0.9. Weil–Petersson volumes as rank 1 Cohomological Field Theory. The rank of the CohFT on (H, g) is, by definition, $\dim(H)$. Let it be 1. Assume for simplicity that $g(\Delta_0, \Delta_0) = 1$ for a basis vector $\Delta_0 \in H$ and fix it. Then the whole structure boils down to a sequence of (non-necessarily homogeneous) cohomology classes

$$c_n := I_n(\Delta_0^{\otimes n}) \in H^*(\overline{M}_{0n})^{S_n},\ n \geq 3 \tag{0.25}$$

satisfying the identities

$$\phi_\sigma^*(c_n) = c_{n_1+1} \otimes c_{n_2+1},\ n = n_1 + n_2,\ n_i \geq 2 \tag{0.26}$$

(cf. (0.6) and (0.7)).

By the Theorem 0.5, we see that each such theory is uniquely determined by the coefficients of its potential

$$\Phi(x) := \sum_{n\geq 3} \frac{C_n}{n!}, \quad C_n = \int_{\overline{M}_{0n}} c_n$$

(cf. (0.14)) which can be totally arbitrary because any series in one variable satisfies the associativity equations. Therefore, rank one theories seem to be rather trivial objects. However, this is not so for at least two reasons: first, there are quite interesting specific theories of algebro-geometric origin; second, the behaviour of $\Phi(x)$ with respect to the tensor product of theories is non–trivial.

Here we give an example (the first term of a hierarchy) of algebro-geometric theories.

There is a standard Weil–Petersson hermitian metric on the non-compact moduli spaces M_{0n} parametrizing irreducible curves. On the boundary this metric becomes singular. Neverthelerss, its Kähler form extends to a closed L^2–current on $\overline{M}_{0n}$ thus defining a real cohomology class $\omega_n^{WP} \in H^2(\overline{M}_{0n})^{S_n}$. There is also a purely algebro-geometric definition of this calss (see [AC]). Consider the universal curve $p_n : X_n \to \overline{M}_{0n}$. Let $x_i \subset X_n$ be the divisors corresponding to the structure sections, and $\omega = \omega_{X_n/\overline{M}_{0n}}$ the relative dualizing sheaf. Then

$$\omega_n^{WP} = 2\pi^2 p_{n*}\left(c_1(\omega(\sum_{i=1}^{n} x_i))^2\right) \tag{0.28}$$

The main property of ω_n^{WP} is

$$\varphi_\sigma^*(\omega_n^{WP}) = \omega_{n_1+1}^{WP} \otimes 1 + 1 \otimes \omega_{n_2+1}^{WP}. \tag{0.29}$$

Comparing this with (0.26) one sees that

$$c_n := \exp(\omega_n^{WP}/2\pi^2) \in H^*(\overline{M}_{0n}, \mathbf{Q}) \tag{0.30}$$

is a rank one CohFT. Its potential is a generating function for the Weil–Petersson volumes considered in [Z]:

$$\Phi^{WP}(x) := \sum_{n=3}^{\infty} \frac{v_n}{n!(n-3)!} x^n. \tag{0.31}$$

$$\frac{v_n}{(n-3)!} := \frac{1}{\pi^{2(n-3)}} \int_{\overline{M}_{0n}} \frac{(\omega_n^{WP})^{n-3}}{(n-3)!} \tag{0.32}$$

P. Zograf proved that $v_4 = 1,\ v_5 = 5,\ v_6 = 61,\ v_7 = 1379$, and generally

$$v_n = \frac{1}{2} \sum_{i=1}^{n-3} \frac{i(n-i-2)}{n-1} \binom{n-4}{i-1} \binom{n}{i+1} v_{i+2} v_{n-i},\ n \geq 4. \tag{0.33}$$

This is equivalent to a non–linear differential equation for $\Phi^{WP}(x)$. What is more remarkable, the inverse function for the second derivative of the potential satisfies a linear (modified Bessel) equation:

$$y = \sum_{n=3}^{\infty} \frac{v_n}{(n-2)!\,(n-3)!} x^{n-2} \quad \Longleftrightarrow \quad x = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m!\,(m-1)!} y^m . \tag{0.34}$$

It is tempting to see this as another tiny bit of the general "mirror phenomenon." This can be considerably generalized to the complete description of the tensor product of invertible rank one CohFT's. Thus, in addition to the associativity equations for the quantum cohomology of plane (and other Fano manifolds), the hypergeometric equations for Calabi–Yau (made non–linear by a coordinate change) we have one more differential equation of a seemingly different origin.

0.10. Main themes. From this sketchy overview, it must be clear that the quantum cohomology is an exceptionally rich and tightly woven structure.

In this first part of the notes we devote Chapters I, II to the global geometric and analytic theory of Frobenius manifolds. Chapter III introduces the more algebraic aspects: formal Frobenius manifolds, moduli spaces and operads.

The projected second part of the notes will concentrate upon algebraic geometric constructions of the Gromov–Witten invariants. In this first part they figure only as examples or in axiomatic form.

There is one more structure that keeps appearing in all the ramifications of this subject: trees and more general graphs, eventually with labels. They enumerate the strata and cells of $\overline{M}_{g,n}$, help to visualize the composition laws of operads and operadic algebras, govern the counting of curves on quintics via Kontsevich's construction. Many generating functions and potentials Φ, when they can be explicitly calculated, often appear in the guise of sums over labelled graphs of rather special type, perturbation series, which are well known in statistical physics and quantum field theory.

One can look at graphs as a mere book–keeping device and treat them in *ad hoc* manner whenever they appear. However, I thought it worthwhile to pay them more respect and to use various categories of graphs as a combinatorial skeleton of the theory.

0.11. Problems of higher genus. If we try to count higher genus curves on algebraic manifolds, the general picture becomes less coherent, due to many unsolved problems. Some of the main themes admit a generalization, but they fit together more loosely.

As we mentioned, in the description of modular spaces trees are replaced by modular and/or ribbon graphs of arbitrary topology. There is also a version of modular operads.

A formalism of Gromov–Witten invariants is known, as well as some constructions of them.

Perturbation series become much more complex, roughly speaking, they correspond to the asymptotic expansions of path integrals rather than solutions of classical differential equations.

Of the three descriptions of quantum cohomology suggested in 0.1, only the second one survives in higher genus, involving cohomological operations on $H^*(V)$ parametrized by all classes in $H^*(\overline{M}_{g,n})$. No reduction of this structure to numerical invariants or solutions of differential equations is known.

For the Calabi–Yau threefolds, an extension of the mirror picture (geometry of moduli spaces) is suggested in [BCOV], but it is less well understood and less binding than the genus zero mirror conjecture.

In short, a lot remains to be done.

Our strategy is to throw in some explanations about the higher genus case whenever it looks appropriate, but defer a deeper treatment to the second part.

Bibliography

[AC] E. Arbarello, M. Cornalba. *Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves.* Preprint, 1994 (to appear in Journ. Alg. Geom.)

[AM] P. S. Aspinwall, D. R. Morrison. *Topological field theory and rational curves.* Comm. Math. Phys., 151 (1993), 245–262.

[Ba1] V. Batyrev. *Dual polyhedra and the mirror symmetry for Calabi–Yau hypersurfaces in toric varieties.* Journ. Alg. Geom., 3 (1994), 493–535.

[Ba2] V. Batyrev. *Variation of the mixed Hodge structure of affine hypersurfaces in algebraic tori.* Duke Math. J., 69 (1993), 349–409.

[Ba3] V. Batyrev. *Quantum cohomology ring of toric manifolds.* Astérisque, 218 (1993), 9–34.

[Ba4] V. Batyrev. *Mirror symmetry and canonical flat coordinates on moduli spaces of Calabi–Yau manifolds via the formal deformation theory.* Preprint, 1993.

[Ba5] V. Batyrev. *Quantum cohomology rings of toric manifolds.* Astérisque, 218 (1993), 9–34.

[BaBo1] V. Batyrev, L. Borisov. *On Calabi–Yau complete intersections in toric varieties.* Preprint, 1994.

[BaBo2] V. Batyrev, L. Borisov. *Dual cones and mirror symmetry for generalized Calabi–Yau manifolds.* Preprint alg-geom/9402002.

[BaS] V. Batyrev, D. van Straten. *Generalized hypergeometric functions and rational curves on Calabi–Yau complete intersections in toric varieties.* Comm. Math. Phys., 168 (1995), 493–533.

[Beau] A. Beauville. *Quantum cohomology of complete intersections.* Preprint alg-geom/9501008.

[BeG] A. Beilinson, V. Ginzburg. *Infinitesimal structure of moduli spaces of G-bundles.* Int. Math. Res. Notices, Duke Math. Journ., 4 (1992), 63–74.

[Beh] K. Behrend. *Gromov–Witten invariants in algebraic geometry.* Preprint alg-geom/9601011.

[BehF] K. Behrend, B. Fantechi. *The intrinsic normal cone.* Preprint alg-geom/9601010.

[BM] K. Behrend, Yu. Manin. *Stacks of stable maps and Gromov–Witten invariants.* Preprint MPI/95-56.

[BCOV] M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa. *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes.* Comm. Math. Phys., 165 (1994), 311–427.

[BIZ] D. Bessis, C. Itzykson, J. B. Zuber. *Quantum field theory techniques in graphical enymeration.* Adv. in Appl. Math., 1 (1980), 109–157.

[CK] E. Cattani, A. Kaplan. *Degenerating variations of Hodge structures.* Astérisque, 179–180 (1989), 67–96.

[COGP] Ph. Candelas, X. C. de la Ossa, P. S. Green, L. Parkes. *A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory.* Nucl. Phys. B 359 (1991), 21–74.

[CaH1] L. Caporaso, J. Harris. *Enumerating rational curves: the rational fibration method.* Preprint alg-geom/9608023.

[CaH2] L. Caporaso, J. Harris. *Parameter spaces for curves on surfaces and enumeration of rational curves.* Preprint alg-geom/9608024.

[CaH3] L. Caporaso, J. Harris. *Counting plane curves of any genus.* Preprint alg-geom/9608025.

[C–F] I. Ciocan–Fontanine. *Quantum cohomology of flag varieties.* Int. Math. Res. Notes, 6 (1995), 263–277.

[De1] P. Deligne. *Equations différentiellles á points singuliers réguliers.* Springer Lecture Notes in Math., 163 (1970).

[De2] P. Deligne. *Local behavior of Hodge structures at infinity.* Preprint, 1994.

[DFI] P. Di Francesco, C. Itzykson. *Quantum intersection rings.* In: The Moduli Space of Curves, ed. by R. Dijkgraaf, C. Faber, G. van der Geer, Progress in Math., vol. 129, Birkhäuser 1995, 149–163.

[DoM] R. Donagi, E. Markman. *Cubics, integrable systems, and Calabi–Yau threefolds.* Preprint, 1994.

[D1] B. Dubrovin. *Integrable systems in topological field theory.* Nucl. Phys. B 379 (1992), 627–689.

[D2] B. Dubrovin. *Geometry of 2D topological fielld theories.* In: Springer LNM, 1620 (1996), 120–348.

[DZh] B. Dubrovin, Youjin Zhang. *Extended affine Weyl groups and Frobenius manifolds.* Preprint SISSA 67/96/FM.

[FGK] S. Fomin, S. Gelfand, A. Postnikov. *Quantum Schubert Polynomials.* Preprint, 1996.

[F] R. Fuchs. *Über lineare homogene Differentialgleichungen zweiter Ordnung mit im endlich gelegene wesentlich singulären Stellen.* Math. Ann., 63 (1907), 301–321.

[FuO] K. Fukaya, K. Ono. *Arnold conjecture and Gromov–Witten invariant.* Preprint, 1996.

[FMPh] W. Fulton, R. MacPherson. *A compactification of configuration spaces.* Ann. of Math., 139 (1994), 183–225.

[FP] W. Fulton, R. Pandharipande. *Notes on stable maps and quantum cohomology.* Preprint alg-geom/9608011.

[G] B. Gambier. *Sur les équations différentielles du second ordre et du prémier degré dont l'intégrale générale est à points critiques fixes.* CR Ac. Sci. Paris, 142 (1906), 266–269.

[Ge] E. Getzler. *Operads and moduli spaces of genus zero Riemann surfaces.* In: The Moduli Space of Curves, ed. by R. Dijkgraaf, C. Faber, G. van der Geer, Progress in Math. vol. 129, Birkhäuser, 1995, 199–230.

[GeJ] E. Getzler, J.D.S. Jones. *Operads, homotopy algebra, and iterated integrals for double loop spaces.* Preprint, 1994.

[GeK] E. Getzler, M. M. Kapranov. *Cyclic operads and cyclic homology.* In: "Geometry, Topology, and Physics for Raoul," ed. by B. Mazur, Cambridge, 1995.

[GiK] V. A. Ginzburg, M. M. Kapranov. *Koszul duality for operads.* Duke Math. J., 76:1 (1994), 203–272.

[Giv1] A. Givental. *Homological geometry I: Projective hypersurfaces.* Selecta Math., new ser. 1:2 (1995), 325–345.

[Giv2] A. Givental. *Equivariant Gromov–Witten invariants.* Int. Math. Res. Notes, 13 (1996), 613-663.

[Giv3] A. Givental. *Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture.* Preprint, 1996.

[GivK] A. Givental, B. Kim. *Quantum cohomology of flag manifolds and Toda lattices.* Comm. Math. Phys. 168 (1994), 609-641.

[Gr] M. Gromov. *Pseudoholomorphic curves in symplectic manifolds.* Inv. Math., 82 (1985), 307–447.

[HaZ] J. Harer, D. Zagier. *The Euler characteristic of the moduli space of curves.* Inv. Math. 85 (1986), 457–485.

[Har] J. Harnad. *Dual isomonodromic deformations and moment maps to loop algebras.* Comm. Math. Phys., 166 (1994), 337–365.

[H1] N. Hitchin. *Poncelet polygons and the Painlevé equations.* In: Geometry and Analysis, ed. by S. Ramanan, Oxford University Press, Bombay, 1995, 151–185.

[H2] N. Hitchin. *Twistor spaces, Einstein metrics and isomonodromic deformations.* J. Diff. Geom., 3 (1995), 52–134.

[H3] N. Hitchin. *Frobenius manifolds (notes by D. Calderbank.)* Preprint, 1996.

[HLY1] S. Hosono, B. H. Lian, S. T. Yau. *GKZ-generalized hypergeometric systems in mirror symmetry of Calabi–Yau hypersurfaces.* Preprint alg-geom/9511001

[HLY2] S. Hosono, B. H. Lian, S. T. Yau. *Maximal degeneracy points of GKZ systems.* Preprint alg-geom/9603014.

[I] C. Itzykson. *Counting rational curves on rational surfaces.* Preprint Saclay T94/001.

[JM] M. Jimbo, T. Miwa. *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II.* Physica 2D (1981), 407–448.

[Ka] R. Kaufmann. *The intersection form in $H^*(\overline{M}_{0n})$ and the explicit Künneth formula in quantum cohomology.* Preprint, 1996.

[KaMZ] R. Kaufmann, Yu. Manin, D. Zagier. *Higher Weil–Petersson volumes of moduli spaces of stable n-pointed curves.* Preprint MPI 96–58.

[Ke] S. Keel. *Intersection theory of moduli spaces of stable n-pointed curves of genus zero.* Trans. AMS, 330 (1992), 545–574.

[Ki1] B. Kim. *Quantum cohomology of partial flag manifolds and a residue formula for their intersection pairing.* Int. Math. Res. Notes, 1 (1995), 1–16.

[Ki2] B. Kim. *Quantum cohomology of flag manifolds G/B and Toda lattices.* Preprint alg-geom/9607001.

[Kn] F. Knudsen. *Projectivity of the moduli space of stable curves, II: the stacks* $M_{0,n}$. Math. Scand. 52 (1983), 1225–1265.

[Ko1] M. Kontsevich. *Intersection theory on moduli spaces and matrix Airy function.* Comm. Math. Phys., 147:1 (1992), 1–23.

[Ko2] M. Kontsevich. *Graphs, homotopy Lie algebras and low-dimensional topology.* Preprint, 1992.

[Ko3] M. Kontsevich. *Formal (non-)commutative differential geometry.* In: The Gelfand Mathematical Seminars, 1990–92, ed. by L. Corwin, I. Gelfand, J. Lepowsky. Birkhäuser, Boston, 1993, 173–187.

[Ko4] M. Kontsevich. *A_∞-algebras in mirror symmetry.* Bonn MPI Arbeitstagung talk, 1993.

[Ko5] M. Kontsevich. *Homological algebra of Mirror Symmetry.* Preprint, 1994.

[Ko6] M. Kontsevich. *Mirror symmetry in dimension 3.* Séminaire Bourbaki, n° 801, Juin 1995.

[Ko7] M. Kontsevich. *Enumeration of rational curves via torus actions.* In: The Moduli Space of Curves, ed. by R. Dijkgraaf, C. Faber, G. van der Geer, Progress in Math. vol. 129, Birkhäuser, 1995, 335–368.

[KM] M. Kontsevich, Yu. Manin. *Gromov-Witten classes, quantum cohomology, and enumerative geometry.* Comm. Math. Phys., 164:3 (1994), 525–562.

[KMK] M. Kontsevich, Yu. Manin (with Appendix by R. Kaufmann). *Quantum cohomology of a product.* Inv. Math., 124 (1996), f. 1–3, 313–340.

B. Kostant. *Flag manifold quantum cohomology, the Toda lattice, and the representation with highest weight ρ.* Selecta Mathematica, New Ser., 2:1 (1996), 43–91.

[LT1] J. Li, G. Tian. *Virtual moduli spaces and Gromov–Witten invariants of algebraic varieties.* Preprint alg-geom/9602007.

[LT2] J. Li, G. Tian. *Virtual moduli cycles and Gromov–Witten invariants of general symplectic manifolds.* Preprint alg-geom/9608032.

[Mal1] B. Malgrange. *Déformations de systèmes différentielles et microdifférentielles.* In: Séminaire de l'ENS 1979–1982, Progress in Math. 37, Birkhäuser, Boston (1983), 353–379.

[Mal2] B. Malgrange. *La clasification des connections irrégulieres à une variable.* ibid, 381–399.

[Mal3] B. Malgrange. *Sur les déformations isomonodromiques. I. Singularités régulières.* ibid, 401–426.

[Mal4] B. Malgrange. *Sur les déformations isomonodromiques. II. Singularités irrégulières.* ibid, 427–438.

[Ma1] Yu. Manin. *Rational points of algebraic curves over functional fields.* AMS Translations, ser. 2, vol. 50 (1966), 189–234.

[Ma2] Yu. Manin. *Gauge Field Theory and Complex Geometry.* Springer Verlag, 1988.

[Ma3] Yu. Manin. *Problems on rational points and rational curves on algebraic varieties.* In: Surveys of Diff. Geometry, vol. II, ed. by C. C. Hsiung, S. -T.Yau, Int. Press (1995), 214–245.

[Ma4] Yu. Manin. *Generating functions in algebraic geometry and sums over trees.* In: The Moduli Space of Curves, ed. by R. Dijkgraaf, C. Faber, G. van der Geer, Progress in Math. vol. 129, Birkhäuser, 1995, 401–418.

[Ma5] Yu. Manin. *Sixth Painlevé equation, universal elliptic curve, and mirror of* $\mathbf{P}^2$. Preprint alg–geom/9605010.

[Mo] D. Morrison. *Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians.* J. AMS 6 (1993), 223–247.

[Mu] D. Mumford. *Toward an enumerative geometry of the moduli spaces of curves.* In: Arithmetic and Geometry, ed. by M. Artin and J. Tate, Birkhäuser, Boston 1983, 271–326.

[N] N. Nitsure. *Moduli of semistable logarithmic singularities.* Journ. of the AMS, 6:3 (1993), 597–609.

[Od] T. Oda. *K. Saito's period map for holomorphic functions with isolated critical points.* In: Adv. Studies in Pure Math., 10 (1987), Algebraic Geometry, Sendai, 1985, 591–648.

[O1] K. Okamoto. *Isomonodromic deformation and Painlevé equations, and the Garnier system.* J. Fac. Sci. Univ. Tokyo, Sect. IA Math., 33 (1986), 575–618.

[O2] K. Okamoto. *Studies in the Painlevé equations I. Sixth Painlevé equation PVI.* Annali Mat. Pura Appl., 146 (1987), 337–381.

[O3] K. Okamoto. *Sur les feuilletages associés aux équation du second ordre à points critiques fixes de P. Painlevé. Espaces de conditions initiales.* Japan J. Math., 5:1 (1979), 1–79.

[Pa1] R. Pandharipande. *Intersections of* $\mathbf{Q}$*–divisors on Kontsevich's moduli space* $\overline{M}_{0,n}(\mathbf{P}^r, d)$ *and enumerative geometry.* Preprint, 1995.

[Pa2] R. Pandharipande. *The canonical class of* $\overline{M}_{0,n}(\mathbf{P}^r, d)$ *and enumerative geometry.* Preprint, 1995.

[P1] R. C. Penner. *Perturbative series and the moduli space of Riemann surfaces.* J. Diff. Geo., 27 (1988), 35–53.

[P2] R. C. Penner. *The decorated Teichmüller space of punctured surfaces.* Comm. Math. Phys., 113 (1987), 299–339.

[P3] R. C. Penner. *Calculus on moduli space.* In: Geometry of Group Representations, AMS Contemp. Math., 74 (1988), 277–293.

[P4] R. C. Penner. *Integration over the moduli spaces of Riemann surfaces.* In: Proc. of Superstrings TAMU 1989, Adv. Series in Math. Physics, World Scientific, 1989, 346–353.

[P5] R. C. Penner. *Weil–Petersson volumes.* J. Diff. Geo., 35 (1992), 559–608.

[P6] R. C. Penner. *The Pincaré dual of the Weil–Petersson Kähler form.* Comm. Anal. Geo., 1 (1993), 43–70.

[R] Y. Ruan. *Topological sigma model and Donaldson type invariants in Gromov theory.* Preprint MPI, 1992.

[RT1] Y. Ruan, G. Tian. *A mathematical theory of quantum cohomology.* J. Diff. Geo., 42 (1995), 259–367.

[RT2] Y. Ruan, G. Tian. *Higher genus symplectic invariants and sigma model coupled with gravity.* Preprint alg–geom/9601005.

[Si] B. Siebert. *Gromov–Witten invariants for general symplectic manifolds.* Preprint dg–ga/9608005.

[T] G. Tian. *Quantum cohomology and its associativity.* In: Current Developments in Math., Int. Press, 1995, 231–282.

[V] A. Voronov. *Topological field theories, string backgrounds, and homotopy algebras.* Preprint, 1993.

[Wi] E. Witten. *Two-dimensional gravity and intersection theory on moduli space.* Surveys in Diff. Geom., 1 (1991), 243–310.

[Wo] S. Wolpert. *On the homology of the moduli spaces of stable curves.* Ann. of Math., 118 (1983), 491–523.

[Y] S.–T. Yau, ed. *Essays on Mirror Manifolds.* International Press Co., Hong Cong, 1992.

[Z] P. Zograf. *The Weil–Petersson volume of the moduli spaces of punctured spheres.* In: Contemporary Mathematics, 150 (1993), ed. by R. M. Hain and C. F. Bödigheimer, 367–372.

CHAPTER I. INTRODUCTION TO FROBENIUS MANIFOLDS

§1. Definition of Frobenius manifolds and the structure connection

1.1. Supermanifolds. We will work throughout this section and the next one in the superextension of one of the classical categories of manifolds $\mathcal{M}an$: C^∞, real analytic, or complex analytic. Whenever integration can be avoided, $\mathcal{M}an$ may be even a category of smooth algebraic manifolds over a field of characteristic zero. To fix notation, we briefly recall the basic framework of [Ma2], Ch. 4 and 5.

1.1.1. Definition. *A supermanifold is a locally ringed space $(M, \mathcal{O}_M)$ with the following properties.*

a). $\mathcal{O}_M = \mathcal{O}_{M,0} \oplus \mathcal{O}_{M,1}$ is the structure sheaf of $\mathbf{Z}_2$–graded supercommutative rings.

b). $M_{\mathrm{red}} = (M, \mathcal{O}_{M,\mathrm{red}} := \mathcal{O}_M/(\mathcal{O}_{M,1})$ is a classical manifold, object of the respective classical category.

c). $\mathcal{O}_M$ is locally isomorphic to the exterior algebra $\wedge(E)$ of a free $\mathcal{O}_{M,\mathrm{red}}$–module E.

A morphism of supermanifolds is a morphism of locally ringed spaces extending a classical morphism of underlying reduced manifolds.

We denote $(M, \mathcal{O}_M)$ simply M, when there is no risk of confusion.

1.1.2. Conventions. By $\widetilde{x}$ we denote the $\mathbf{Z}_2$–degree, or parity, of a homogeneous object x (local function, vector field, scalar product etc.)

If M is a supermanifold, local coordinates in a neighbourhood of a point form a family of sections of the structure sheaf which can be obtained as follows. Choose a local isomorphism $\varphi : \wedge(E) \to \mathcal{O}_M$ as above, local coordinates $(\overline{x}^1, \dots, \overline{x}^m)$ on M_{red}, and free local generators $(\overline{x}^{m+1}, \dots, \overline{x}^{m+n})$ of E. Put $x^i = \varphi(\overline{x}^i)$. Then $(x^1, \dots, x^{m+n})$ are local coordinates on M. Any local function on M can be expressed as a polynomial in anticommuting odd coordinates $x^{m+1}, \dots, x^{m+n}$ whose coefficients are classical (C^∞, analytic, etc.) functions of the commuting even coordinates $x^{m+1}, \dots, x^{m+n}$). Odd coordinates are sometimes denoted by Greek letters.

If M is connected, the pair $m|n$ is an invariant of M called its (super)dimension. When $n = 0$, we say that M is pure even. Transition functions between various local coordinate systems, of course, need not be linear in odd coordinates, e. g. $(x, \xi, \eta) \mapsto (x + \xi\eta, x\xi, x^{-1}\eta)$ is a transition function outside $x = 0$.

The De Rham complex of sheaves on M is the universal $(\mathbf{Z}_2, \mathbf{Z})$–graded differential $\mathcal{O}_M$–algebra (Ω^*_M, d) with *odd* differential d. This means that $\widetilde{dx} = \widetilde{x} + 1$, and the Leibniz formula reads

$$d(fg) = df\, g + (-1)^{\widetilde{f}} f\, dg.$$

Notice that as $\mathcal{O}_M$–algebra, Ω^*_M is the *symmetric* algebra of the $\mathcal{O}_M$–module Ω^1_M rather than exterior one. This is the combined effect of our choice of odd d and the rule of signs defining the action of S_n upon $P^{\otimes n}$:

$$\sigma(p_1 \otimes \dots \otimes p_n) = \epsilon(\sigma, p) p_{\sigma^{-1}(1)} \otimes \dots \otimes p_{\sigma^{-1}(n)},$$

where $\epsilon(\sigma, p)$ is the sign of the permutation induced on odd p_i (i.e. when even p_i are simply disregarded).

Given local coordinates (x_a) on M, they determine the local vector fields $\partial_a = \partial/\partial x^a$ by the rule

$$df = \sum dx^a \partial_a f$$

for any f in $\mathcal{O}_M$. Notice that $\widetilde{\partial}_a = \widetilde{x}_a$ and $\partial_a \partial_b = (-1)^{\widetilde{x}_a \widetilde{x}_b} \partial_b \partial_a$ so that the supercommutator, which we denote by the usual square brackets $[\partial_a, \partial_b]$, vanishes. To shorten notation, a sign of the type $(-1)^{\widetilde{x}_a(\widetilde{x}_b + \widetilde{x}_c)}$ will be denoted $(-1)^{a(b+c)}$.

The tangent sheaf $\mathcal{T}_M$ (resp. cotangent sheaf $\mathcal{T}^*_M$) is locally freely generated by (∂_a) (resp. by (dx_a) with *reverse* parity.)

A Riemannian metric on M is an even symmetric pairing $g : S^2(\mathcal{T}_M) \to \mathcal{O}_M$, inducing an isomorphism $g' : \mathcal{T}_M \to \mathcal{T}^*_M$. We put $g_{a,b} := g(\partial_a, \partial_b)$. Clearly, $\widetilde{g}_{ab} = \widetilde{x}_a + \widetilde{x}_b$. No positivity condition is imposed, even in the pure even case over $\mathbf{R}$.

A warning: in many situations it is necessary to consider the relative versions of all these notions, that is, to work with submersions of supermanifolds $M \to S$ considered as a family parametrized by the base S. Functions on S are "constants", and since there are no odd constants in $\mathbf{R}$ or $\mathbf{C}$, the need for a base extension arises in supergeometry more often than in the pure even setting. The necessary changes are routine.

The following structure is important in the theory of Frobenius manifolds.

1.2. Definition. *a). An affine flat structure on the supermanifold M is a subsheaf $\mathcal{T}^f_M \subset \mathcal{T}_M$ of linear spaces of pairwise (super)commuting vector fields, such that $\mathcal{T}_M = \mathcal{O}_M \otimes \mathcal{T}^f_M$ (tensor product over the ground field.)*

Sections of $\mathcal{T}^f_M$ are called flat vector fields.

b). The metric g is compatible with the structure $\mathcal{T}^f_M$, if $g(X, Y)$ is constant for flat X, Y.

In the smooth or analytic case, an affine flat structure can also be equivalently described by a complete atlas whose transition functions are affine linear, because for a maximal commuting set of linearly independent vector fields (X_a) one can find local coordinates such that $X_a = \partial/\partial x^a$, and they are defined up to a constant shift.

If a metric g is compatible with an affine flat structure, it is flat in the sense of the straightforward (not involving spinors) superextension of Riemann geometry. The parallel transport endows $\mathcal{T}^f_M$ with the structure of local system.

We now give the central definition of these notes, due to B. Dubrovin.

1.3. Definition. *Let M be a supermanifold. Consider a triple $(\mathcal{T}_M^f, g, A)$ consisting of an affine flat structure, a compatible metric, and an even symmetric tensor $A: S^3(\mathcal{T}_M) \to \mathcal{O}_M$.*

Define an $\mathcal{O}_M$-bilinear symmetric multiplication $\circ = \circ_{A,g}$ on $\mathcal{T}_M$:

$$\mathcal{T}_M \otimes \mathcal{T}_M \to S^2(\mathcal{T}_M) \xrightarrow{A'} \mathcal{T}_M^* \xrightarrow{g'} \mathcal{T}_M : \ X \otimes Y \to X \circ Y \tag{1.1}$$

where prime denotes a partial dualization, or equivalently,

$$A(X,Y,Z) = g(X \circ Y, Z) = g(X, Y \circ Z). \tag{1.2}$$

This means that the metric is invariant with respect to the multiplication.

a). M endowed with this structure is called a pre-Frobenius manifold.

b). A local potential Φ for $(\mathcal{T}_M^f, A)$ is a local even function such that for any flat local tangent fields X, Y, Z

$$A(X,Y,Z) = (XYZ)\Phi. \tag{1.3}$$

A pre-Frobenius manifold is called potential one, if A everywhere locally admits a potential.

c). A pre-Frobenius manifold is called associative, if the multiplication $\circ$ is associative.

d). A pre-Frobenius manifold is called Frobenius, if it is simultaneously potential and associative.

1.3.1. Remarks. a). If a potential Φ exists, it is unique up to adding a quadratic polynomial in flat local coordinates.

b). In flat local coordinates (x^a) (1.3) becomes $A_{abc} = \partial_a \partial_b \partial_c \Phi$, and (1.2) can be rewritten as

$$\partial_a \circ \partial_b = \sum_c A_{ab}{}^c \partial_c, \tag{1.4}$$

where

$$A_{ab}{}^c := \sum_e A_{abe} g^{ec}, \ (g^{ab}) := (g_{ab})^{-1}.$$

Furthermore,

$$(\partial_a \circ \partial_b) \circ \partial_c = \left(\sum_e A_{ab}{}^e \partial_e \right) \circ \partial_c = \sum_{ef} A_{ab}{}^e A_{ec}{}^f \partial_f,$$

$$\partial_a \circ (\partial_b \circ \partial_c) = \partial_a \circ \sum_e A_{bc}{}^e \partial_e = (-1)^{a(b+c+e)} \sum_{ef} A_{bc}{}^e A_{ae}{}^f \partial_f =$$

$$= (-1)^{a(b+c)} \sum_{ef} A_{bc}{}^e A_{ea}{}^f \partial_f \tag{1.5}$$

(notice our abbreviated notation for signs.)

Comparing the coefficients of ∂_f in (1.5), lowering the superscripts and expressing A_{abc} through a potential, we finally see that the notion of the Frobenius manifold is a geometrization of the following highly non–linear and overdetermined system of PDE:

$$\forall a,b,c,d: \quad \sum_{ef} \Phi_{abe} g^{ef} \Phi_{fcd} = (-1)^{a(b+c)} \sum_{ef} \Phi_{bce} g^{ef} \Phi_{fad}. \tag{1.6}$$

They are called Associativity Equations, or WDVV (Witten–Dijkgraaf–Verlinde–Verlinde) equations.

We will now express (1.6) as a flatness condition.

1.4. Definition. *Let (M, g, A) be a pre-Frobenius manifold (we omit $\mathcal{T}_M^f$ in the notation, since it can be reconstructed from g.) Define the following objects:*

a). The connection $\nabla_0 : \mathcal{T}_M \to \Omega^1_M \otimes \mathcal{T}_M$ well determined by the condition that flat vector fields are ∇_0–horizontal.

Denote its covariant derivative along a vector field X by

$$\nabla_{0,X}(Y) = i_X(\nabla_0(Y)), \; i_X(df \otimes Z) = Xf \otimes Z.$$

b). A pencil of connections depending on an even parameter λ:

$$\nabla_\lambda : \mathcal{T}_M \to \Omega^1_M \otimes \mathcal{T}_M : \; \nabla_{\lambda,X}(Y) := \nabla_{0,X}(Y) + \lambda X \circ Y. \tag{1.7}$$

We will call ∇_λ the structure connection of (M, g, A).

1.4.1. Remark. In flat coordinates (1.7) reads:

$$\nabla_{\lambda,\partial_a}(\partial_b) = \lambda \sum_c A_{ab}{}^c \partial_c = \lambda\, \partial_a \circ \partial_b = (-1)^{ab} \lambda \partial_b \circ \partial_a = (-1)^{ab} \nabla_{\lambda,\partial_b}(\partial_a). \tag{1.8}$$

Therefore ∇_λ has vanishing torsion for any λ. In particular, ∇_0 is the Levi–Civita (super)connection for g.

Notice that the covariant differential ∇_λ is odd. As in the pure even case, it can be naturally extended to all Ω^*_M.

1.5. Theorem. *Let ∇_λ be the structure connection of the pre-Frobenius manifold (M, g, A). Put $\nabla^2_\lambda = \lambda^2 R_2 + \lambda R_1$ (there is no constant term since $\nabla^2_0 = 0$.) Then*

a). $R_1 = 0 \iff (M, g, A)$ is potential.

b). $R_2 = 0 \iff (M, g, A)$ is associative.

Therefore (M, g, A) is Frobenius, iff ∇_λ is flat.

Proof. a). Calculating the λ–terms in

$$[\nabla_{0,\partial_a} + \lambda \partial_a \circ \,, \, \nabla_{0,\partial_b} + \lambda \partial_b \circ](\partial_c)$$

we see that $R_1 = 0$ iff $\forall a,b,c,e,\ \partial_a A_{bc}{}^e = (-1)^{ab}\partial_b A_{ac}{}^e$, or better

$$\forall a,b,c,d, \quad \partial_a A_{bcd} = (-1)^{ab}\partial_b A_{acd}. \tag{1.9}$$

If A is potential, this follows from (1.3). Conversely, assume (1.9). Then for all c, d, the form $\sum_b dx^b A_{bcd}$ is closed, hence locally exact by the superversion of the Poincaré lemma. Thus we can find local functions $B_{cd} = (-1)^{cd}B_{dc}$ such that

$$A_{bcd} = \partial_b B_{cd} = (-1)^{bc}\partial_c B_{bd} = (-1)^{bc}A_{cbd},$$

because A is symmetric. It follows that for all d, $\sum_c dx^c B_{cd}$ is closed. By the same reasoning, we have locally $B_{cd} = \partial_c C_d$ and finally $C_d = \partial_d\Phi$, so that $A_{bcd} = \partial_b\partial_c\partial_d\Phi$.

b). Calculating the λ^2 terms in $[\nabla_{\lambda,X}, \nabla_{\lambda,Y}](Z)$, we find that

$$R_{2,XY}(Z) = X \circ (Y \circ Z) - (-1)^{\tilde{X}\tilde{Y}} Y \circ (X \circ Z).$$

Hence if $\circ$ is associative, $R_2 = 0$, because $\circ$ is always (super)commutative. Conversely, if $R_2 \equiv 0$,

$$X \circ (Y \circ Z) = (-1)^{\tilde{X}\tilde{Y}} Y \circ (X \circ Z) = (-1)^{\tilde{X}(\tilde{Y}+\tilde{Z})} Y \circ (Z \circ X) =$$

$$= (-1)^{\tilde{X}\tilde{Y}+\tilde{X}\tilde{Z}+\tilde{Y}\tilde{Z}} Z \circ (Y \circ X) = (X \circ Y) \circ Z.$$

1.6. Induced structures. Let $M' \to M$ be any morphism of supermanifolds which is an isomorphism locally at any point of M', for instance, an open embedding, or an unramified covering of an open submanifold. Then all structures on M described above induce the respective structures on M'.

Induction on closed submanifolds is less common. However, one can always induce a (pre–) Frobenius structure from M to M_{red}. Functions on M_{red} are obtained by factoring out all nilpotents (their ideal is generated by odd local coordinates.) In the De Rham complex, the differentials of odd coordinates are factored out as well. Under this reduction, the flat even coordinates by definition remain flat; the even–even part of the metric form remains the same; new potential is the reduction of the old one. It is not difficult to check that (1.6) after reduction will become the Associativity Equations for the reduced potential.

In Quantum Cohomology, this will allow us to restrict attention to the pure even dimensional subspace if need be. However some information will be lost thereby.

1.7. Example: cubic potentials. The simplest examples of Frobenius manifolds are furnished by potentials which are cubic forms in flat coordinates with constant coefficients. The algebra of tangent vectors at any point is just a commutative (super)algebra with invariant scalar product, locally independent on the point (flat local fields identify two algebras at a neighborhood of any point.) For more sophisticated examples, see §4 below and the next Chapter.

§2. Identity, Euler field, and the extended structure connection

2.1. Definition. *Let (M, g, A) be a pre–Frobenius manifold. An even vector field e on M is called identity, if $e \circ X = X$ for all X.*

If e exists at all, it is uniquely defined by $\circ$, hence by g and A.

Conversely, given A and e, there can exist at most one metric g making (M, g, A) a pre–Frobenius manifold with this identity:

$$g(X, Y) = A(e, X, Y).$$

This follows from (1.2). If A has a potential Φ, this translates into a non–homogeneous linear differential equation for Φ supplementing the Associativity Equations (1.6):

$$\forall \text{ flat } X, Y, \quad eXY\Phi = g(X, Y). \tag{2.1}$$

In fact, if $e = \sum_a e^a \partial_a$, ∂_a flat, we have from (1.3):

$$A(e, X, Y) = \sum_a e^a \partial_a XY\Phi = eXY\Phi.$$

In most (although not all) important examples e itself is *flat*. If this is the case, one can everywhere locally find a flat coordinate system $(x^0, \dots, x^n)$ such that $e = \partial/\partial x^0 = \partial_0$, and (2.1) becomes

$$\forall\, a, b, \quad \Phi_{0ab} = g_{ab}. \tag{2.2}$$

Since all g_{ab} are constants, we get

2.1.1. Corollary. *On a potential pre–Frobenius manifold with flat identity $e = \partial_0$ (in a flat coordinate system) we have modulo terms of degree ≤ 2:*

$$\Phi(x^0, \dots, x^n) = \frac{1}{2} x^0 \left(\sum_{a,b \ne 0} g_{ab} x^a x^b + \sum_{a \ne 0} g_{0a} x^0 x^a + \frac{1}{3} g_{00} (x^0)^2 \right) + \Psi(x^1, \dots, x^n). \tag{2.3}$$

2.1.2. Co–identity. The metric g identifies $\mathcal{T}_M$ and $\mathcal{T}_M^*$. We will call *the co–identity* and denote ε the 1–form which is the image of e (with reverse parity.) More precisely, ε is defined by

$$\forall X \in \mathcal{T}_M, \quad i_X(\varepsilon) = g(X, e).$$

If (x^a) is a local coordinate system,then

$$\varepsilon = \sum_a dx^a g(\partial_a, e).$$

Finally, if e and (x^a) are flat, then $g(\partial_a, e)$ are constant, and

$$\varepsilon = d\eta,\ \eta = \sum_a x^a g(\partial_a, e). \tag{2.4}$$

2.2. Euler field. We will say that an even vector field E on a manifold with flat metric (M, g) is *conformal*, if $\mathrm{Lie}_E(g) = Dg$ for some constant D. In other words, for all vector fields X, Y we have

$$E(g(X,Y)) - g([E,X],Y) - g(X,[E,Y]) = Dg(X,Y). \tag{2.5}$$

It follows that in flat coordinates we have $E = \sum_a E^a(x)\partial_a$ where $E^a(x)$ are polynomials of degree ≤ 1. In fact, E is a sum of infinitesimal rotation, dilation and constant shift. Hence $[E, \mathcal{T}_M^f] \subset \mathcal{T}_M^f$. Moreover, the operator

$$\mathcal{V}:\ \mathcal{T}_M^f \to \mathcal{T}_M^f,\ \mathcal{V}(X) := [X, E] - \frac{D}{2}X$$

is skewsymmetric:

$$\forall \text{ flat } X, Y:\ g(\mathcal{V}(X), Y) + g(X, \mathcal{V}(Y)) = 0.$$

2.2.1. Definition. *Let E be an even vector field on a pre–Frobenius manifold (M, g, A). It is called an Euler field, if it is conformal, and* $\mathrm{Lie}_E(\circ) = d_0\circ$ *for some constant d_0, that is, for all vector fields X, Y,*

$$[E, X\circ Y] - [E,X]\circ Y - X\circ[E,Y] = d_0 X\circ Y. \tag{2.6}$$

Notice that it suffices to check (2.5) and (2.6) for X, Y in any (local) basis of $\mathcal{T}_M$, because both sides are $\mathcal{O}_M$–bilinear.

Clearly, any scalar multiple of an Euler field is also an Euler field. One can use this remark in order to normalize E by requiring that some non–vanishing eigenvalue becomes one. A convenient choice is often $d_0 = 1$, if we have reasons to restrict ourselves to the $d_0 \neq 0$ case.

2.2.2. Proposition. *Let E be a conformal vector field on a Frobenius manifold (M, g, Φ). Then E is Euler, iff*

$$E\Phi = (d_0 + D)\Phi + \text{a quadratic polynomial in flat coordinates.} \tag{2.7}$$

Proof. Clearly, (2.7) is equivalent to the following statement: for all flat X, Y, Z

$$XYZE\Phi = (d_0 + D)XYZ\Phi. \tag{2.8}$$

Now

$$XYZE\Phi = EXYZ\Phi - XY[E,Z]\Phi - X[E,Y]Z\Phi - [E,X]YZ\Phi. \tag{2.9}$$

Using (1.3), (1.2), and the fact that $[E, \mathcal{T}_M^f] \subset \mathcal{T}_M^f$, we can rewrite the right hand side of (2.9) as

$$Eg(X \circ Y, Z) - g(X \circ Y, [E, Z]) - g([E, X \circ Y], Z) +$$

$$+ g([E, X \circ Y], Z) - g(X \circ [E, Y], Z) - g([E, X] \circ Y, Z).$$

The first three terms add up to $Dg(X \circ Y, Z) = DXYZ\Phi$. The last three terms add up to $d_0 g(X \circ Y, Z) = d_0 XYZ\Phi$ precisely if E is Euler.

2.3. Gradings induced by E. Put now

$$\mathcal{T}_M(r) := \{X \in \mathcal{T}_M \mid [E, X] = (r - d_0)X\},\ \mathcal{T}_M(*) := \oplus_{r \in \mathbf{C}} \mathcal{T}_M(r). \qquad (2.10)$$

Notice that we are considering not necessarily flat fields, and shift the eigenvalues by d_0. Similarly, put

$$\mathcal{O}_M(s) := \{f \in \mathcal{O}_M \mid Ef = sf\},\ \mathcal{O}_M(*) := \oplus_{s \in \mathbf{C}} \mathcal{O}_M(s). \qquad (2.11)$$

This is a graded sheaf of algebras.

2.3.1. Proposition. *On any pre–Frobenius manifold M with Euler field E, the sheaf $\mathcal{T}_M(*)$ is*

a). A graded $\mathcal{O}_M(*)$-module.

b). A graded supercommutative algebra with multiplication $\circ$.

c). A graded Lie superalgebra with the bracket of degree $-d_0$.

This is proved by a straightforward calculation which is left to the reader.

As a corollary, since $[E, E] = 0$, we have $E \in \mathcal{T}_M(d_0)$, so that $E^{\circ n} \in \mathcal{T}_M(nd_0)$, or

$$[E, E^{\circ n}] = (n-1)d_0 E^{\circ n}. \qquad (2.12)$$

I do not see how to get in this setting the commutation relations between arbitrary $E^{\circ m}$ and $E^{\circ n}$. Later we will obtain them for *semisimple Frobenius manifolds*, and find (for $d_0 = 1$) the algebra of vector fields on a line.

2.4. Case of semisimple $\mathrm{ad}\, E$. We will call the set of eigenvalues of $-\mathrm{ad}\, E$ on $\mathcal{T}_M^f$, together with d_0 and D, *the spectrum* of E. We will say that E is *semisimple*, if $\mathrm{ad}\, E$, acting on flat fields, is. For semisimple E we can construct many homogeneous elements of $\mathcal{O}_M(*)$ and $\mathcal{T}_M(*)$ explicitly.

Let (∂_a) be a local basis of $\mathcal{T}_M^f$ such that

$$[\partial_a, E] = d_a \partial_a \qquad (2.13)$$

where (d_a) form a part of the spectrum of E. (We assume here that the ground field is $\mathbf{C}$ or else complexify the tangent sheaf.) Putting $E = \sum E^a(x)\partial_a$, we find from (2.13) that $\partial_a E^b = \delta_a^b d_a$. Hence if $\partial_a = \partial/\partial x^a$, we have

$$E = \sum_{a:\, d_a \neq 0} (d_a x^a + r^a)\partial_a + \sum_{b:\, d_b = 0} r^b \partial_b.$$

By shifting x^a, we can make $r^a = 0$ for $d_a \neq 0$. Multiplying x^b by a constant, we can make $r^b = 0$ or 1 for $d_b = 0$. So finally we can choose local flat coordinates in such a way that

$$E = \sum_{a:\, d_a \neq 0} d_a x^a \partial_a + \sum_{\text{some } b:\, d_b = 0} \partial_b. \tag{2.14}$$

Clearly, E assigns definite degrees to the following local functions:

$$Ex^a = d_a x^a \text{ for } d_a \neq 0;\ E \exp x^b = \exp x^b \text{ or } 0 \text{ for } d_b = 0. \tag{2.15}$$

Assume now that M has an identity e. From (2.6) we get

$$[e, E] = d_0 e. \tag{2.16}$$

Hence our notation for the spectrum will be consistent, if in the case of flat e we put $e = \partial_0$, and otherwise do not use 0 as one of the subscripts in (2.13).

In more invariant form (2.14) can be written as

$$E = \sum_{s \in \mathbf{C}} E[s\,],$$

where $E[s\,]$ is the part of (2.13) consisting of summands with $d_a = s$ for $s \neq 0$, and the remaining summands for $s = 0$. This decomposition does not depend on the remaining arbitrariness in the choice of local coordinates.

We can now present some of our previous remarks in more concrete form. Put

$$\mathcal{T}_M^f[r] := \{X \in \mathcal{T}_M^f \mid [X, E] = rX\}.$$

(Notice the difference with (2.10).) Then the condition (2.5) is equivalent to the following one:

$\mathcal{T}_M^f[d_a]$ and $\mathcal{T}_M^f[d_b]$ are orthogonal unless $d_a + d_b = D$.

In fact, (2.5) in the basis (2.13) becomes

$$\forall\, a, b:\ g(d_a\partial_a, \partial_b) + g(\partial_a, d_b\partial_b) = Dg_{ab}$$

that is,

$$(d_a + d_b - D)g_{ab} = 0. \tag{2.17}$$

In particular, $g(e, e) = 0$ unless $D = 2d_0$.

2.4.1. Proposition. *(2.6) is equivalent to any one of the following sets of equations written in the basis (2.13):*

$$\forall\, a, b, c:\ EA_{ab}{}^c = (d_0 - d_a - d_b + d_c)A_{ab}{}^c, \tag{2.18}$$

$$\forall\, a, b, c:\ EA_{abc} = (d_0 + D - d_a - d_b - d_c)A_{abc}. \tag{2.19}$$

This follows from the homogeneity of multiplication.

We now have the following supply of homogeneous functions: components of A and mixed monomials in local functions (2.15):

$$\prod_{a:\,d_a\neq 0}(x^a)^{m_a}\prod_{b:\,d_b=0}\exp(n_b x^b)\in\mathcal{O}_M\left(\sum_{a:\,d_a\neq 0}m_a d_a+\sum_{b:\,d_b=0}n_b r^b\right),\qquad(2.20)$$

where $m_a\in\mathbf{Z}$, $n_b\in\mathbf{R}$ (or $\mathbf{C}$.)

2.5. Extended structure connection. Let M be a pre-Frobenius manifold with a conformal vector field E. Put $\widehat{M}:=M\times(\mathbf{P}^1_\lambda\setminus\{0,\infty\})$, where $\mathbf{P}^1_\lambda$ is the completion of Spec $\mathbf{C}[\lambda,\lambda^{-1}]$. Furthermore, put $\widehat{\mathcal{T}}=\mathrm{pr}_M^*(\mathcal{T}_M)$. If X is a vector field on M, it may be lifted to $\widehat{M}$ in two different guises: as a vector field annihilating λ, denoted again X, and as a section of $\widehat{\mathcal{T}}$, then denoted $\widehat{X}$.

Choose a constant d_0 and put $\mathcal{E}:=E-d_0\lambda\dfrac{\partial}{\partial\lambda}\in\mathcal{T}_{\widehat{M}}$. Clearly, $\widehat{X}$ for flat X span $\widehat{\mathcal{T}}$, whereas flat X and $\mathcal{E}$ span $\mathcal{T}_{\widehat{M}}$, *provided* $d_0\neq 0$, which we will assume.

2.5.1. Definition. *Let M be a pre-Frobenius manifold with a conformal field E, and d_0 a non-zero constant. The extended structure connection for M is the connection $\widehat{\nabla}$ on the sheaf $\widehat{\mathcal{T}}$ on $\widehat{M}$, defined by the following formulas for its covariant derivatives: for any local vector fields $X\in\mathcal{T}_M$, $Y\in\mathcal{T}_M^f$,*

$$\widehat{\nabla}_X(\widehat{Y}):=\lambda\widehat{X\circ Y},\qquad(2.21)$$

$$\widehat{\nabla}_{\mathcal{E}}(\widehat{Y}):=\widehat{[E,Y]}.\qquad(2.22)$$

2.5.2. Theorem. *The extended structure connection is flat iff M is Frobenius and E is Euler with* $\mathrm{Lie}_E(\circ)=d_0\circ$.

Proof. From (2.21) it follows that the vanishing of the XY-components of the curvature of $\widehat{\nabla}$ for all flat X,Y is equivalent to the flatness of the structure connection of M.

It remains to calculate the $\mathcal{E}X$-components, i.e. the expression

$$\widehat{\nabla}_{[\mathcal{E},X]}(\widehat{Y})-[\widehat{\nabla}_{\mathcal{E}},\widehat{\nabla}_X](\widehat{Y})\qquad(2.23)$$

for all flat X,Y. Since $[\mathcal{E},X]$ is the lift of the flat field $[E,X]$, from (2.21) and (2.22) it follows that the first term of (2.23) is $\lambda(\widehat{[E,X]\circ Y})$. Furthermore, $\widehat{\nabla}_X(\widehat{Y})=\lambda\widehat{X\circ Y}$, so that

$$\widehat{\nabla}_{\mathcal{E}}\widehat{\nabla}_X(\widehat{Y})=\lambda\widehat{[E,X\circ Y]}-d_0\lambda\widehat{X\circ Y},\ \ \widehat{\nabla}_X\widehat{\nabla}_{\mathcal{E}}(\widehat{Y})=\lambda\widehat{X\circ[E,Y]}.$$

We see that the vanishing of this part of the curvature is equivalent to (2.6). This finishes the proof.

From (2.21) and (2.22) one can derive a formula for the covariant derivative in the λ-direction: if Y is flat, we have

$$\widehat{[E,Y]}=\widehat{\nabla}_{E-d_0\lambda\partial/\partial\lambda}(\widehat{Y})=\widehat{\nabla}_E(\widehat{Y})-d_0\lambda\widehat{\nabla}_{\partial/\partial\lambda}(\widehat{Y})=\lambda\widehat{E\circ Y}-d_0\lambda\widehat{\nabla}_{\partial/\partial\lambda}(\widehat{Y})$$

so that

$$d_0\widehat{\nabla}_{\partial/\partial\lambda}(\widehat{Y})=\widehat{E\circ Y}-\frac{1}{\lambda}\widehat{[E,Y]}.\qquad(2.24)$$

§3. Semisimple Frobenius manifolds

Let (M, g, A) be an associative pre–Frobenius manifold of dimension n. *In this section and the next one we will assume that M is classical, that is, pure even.*

3.1. Definition. *M is called semisimple (resp. split semisimple) if an isomorphism of the sheaves of $\mathcal{O}_M$-algebras*

$$(\mathcal{T}_M, \circ) \xrightarrow{\sim} (\mathcal{O}_M^n, \text{componentwise multiplication}) \tag{3.1}$$

exists everywhere locally (resp. globally.)

This means that in a local (resp. global) basis $(e_1, \dots, e_n)$ of $\mathcal{T}_M$ the multiplication takes form

$$(\sum f_i e_i) \circ (\sum g_j e_j) = \sum f_i g_i e_i,$$

and in particular,

$$e_i \circ e_j = \delta_{ij} e_j. \tag{3.2}$$

Such a family of idempotents is well defined up to renumbering. Another way of saying this is that a semisimple manifold comes with the structure group of $\mathcal{T}_M$ reduced to S_n. Notice that e_i are generally not flat, so that this reduction is not compatible with that induced by $\mathcal{T}_M^f$, with the structure group $GL(n)$.

Hence if M is semisimple, there exists an unramified covering of degree $\le n!$, upon which the induced pre–Frobenius structure is split.

Denote by (ν^i) the basis of 1–forms dual to (e_i). From (1.2) and (3.2) we find

$$g(e_i, e_k) = g(e_i \circ e_i, e_k) = g(e_i, e_i \circ e_k) = \delta_{ik} g_{ii}.$$

We will denote g_{ii} by η_i. We see that the symmetric 2-form representing g is diagonal in the basis (ν^i):

$$g = \sum_i \eta_i (\nu^i)^2. \tag{3.3}$$

Moreover, according to (1.2), $A(e_i, e_j, e_k) = \delta_{ij}\delta_{ik}\eta_i$, so that the symmetric 3-form representing A, is diagonal with the same coefficients:

$$A = \sum_i \eta_i (\nu^i)^3. \tag{3.4}$$

Finally, $e := \sum_i e_i$ is the identity in $(\mathcal{T}_M, \circ)$, and the co–identity, defined in 2.1.2, nicely complements (3.3) and (3.4):

$$\varepsilon = \sum_i \eta_i \nu^i. \tag{3.5}$$

Thus the Definition 3.1 can be restated as follows:

3.2. Definition. *The structure of the semisimple pre–Frobenius manifold on M is determined by the following data:*

a). A reduction of the structure group of $\mathcal{T}_M$ to S_n, specified by a choice of local bases (e_i) and dual bases (ν^i).

b). A flat metric g, diagonal in (e_i), (ν^i).

c). A diagonal cubic tensor A with the same coefficients as g.

Associativity of $(\mathcal{T}_M, \circ)$ is automatic in both descriptions. However, potentiality (and the flatness of g which we postulated) are non–trivial conditions.

3.3. Theorem. *The structure described in the Definition 3.2 is Frobenius iff the following conditions are satisfied:*

a). $[e_i, e_j] = 0$, or equivalently, $e_i = \partial/\partial u^i$, $\nu^i = du^i$ for a local coordinate system (u^i) called canonical one.

b). $\eta_i = e_i\eta$ for a local function η defined up to addition of a constant. Equivalently, ε is closed.

We will call η *the metric potential* of this structure. (Sometimes this term refers to h such that $g_{ab} = \partial_a\partial_b h$; our meaning is different.)

Canonical coordinates are defined up to renumbering and constant shifts.

Proof. Let ∇_λ be the structure connection of the pre–Frobenius manifold M. According to the Theorem 1.5, M is Frobenius iff the curvature ∇_λ^2 vanishes, i. e. iff

$$\forall i, j, k: \quad [\nabla_{\lambda,e_i}, \nabla_{\lambda,e_j}](e_k) = \nabla_{\lambda,[e_i,e_j]}(e_k). \tag{3.6}$$

Since M is associative, and since we assumed that g is flat, we have to worry only about the λ–linear terms in (3.6). Let us start with introducing the Riemannian connection coefficients of g for the basis e_k:

$$\nabla_{0,e_i}(e_k) = \sum_q \Gamma_{ik}{}^q e_q. \tag{3.7}$$

Since $\nabla_{\lambda,X} = \nabla_{0,X} + \lambda X\circ$ (cf. (1.7)), the left hand side of (3.6) produces the λ–terms

$$(\nabla_{0,e_i} + \lambda e_i\circ)(\nabla_{0,e_j} + \lambda e_j\circ)(e_k) - (i \leftrightarrow j) =$$

$$= \lambda e_i \circ \sum_q \Gamma_{jk}{}^q e_q + \lambda \sum_q \delta_{jk}\Gamma_{ik}{}^q e_q - (i \leftrightarrow j) + \dots =$$

$$= \lambda \sum_q (\delta_{iq}\Gamma_{jk}^q + \delta_{jk}\Gamma_{ik}^q - \delta_{jq}\Gamma_{ik}^q - \delta_{ik}\Gamma_{jk}^q)e_q + \dots \tag{3.8}$$

Now introduce the Lie coefficients

$$[e_i, e_j] = \sum_q f_{ij}{}^q e_q.$$

The λ–terms in the right hand side of (3.6) amount to

$$\nabla_{\lambda,[e_i,e_j]}(e_k) = \lambda \sum_q f_{ij}{}^q e_q \circ e_k + \dots = \lambda f_{ij}{}^k e_k + \dots$$

But the coefficient of e_k in (3.8) vanishes. Therefore, if M is Frobenius, then (3.6) is satisfied, so that $f_{ij}{}^k = 0$. Hence e_i pairwise commute, and local canonical coordinates u^i do exist.

Moreover, the left hand side of (3.6) vanishes. Again, it suffices to investigate the meaning of this, looking only at λ-linear terms.

Recall that for any metric $g = \sum g_{ij} du^i du^j$ the coefficients of the Levi-Civita connection are given by the formulas

$$\Gamma_{ij}{}^k = \sum_l \Gamma_{ijl} g^{lk}, \ \Gamma_{ijk} = \frac{1}{2}(e_i g_{jk} - e_k g_{ij} + e_j g_{ki}).$$

The nonvanishing connection coeffitients of $g = \sum \eta_i (du^i)^2$ are $(i \neq j)$:

$$\Gamma_{ii}{}^i = \frac{1}{2}\eta_i^{-1} e_i \eta_i, \ \Gamma_{ii}{}^j = -\frac{1}{2}\eta_j^{-1} e_j \eta_i,$$

$$\Gamma_{ij}{}^i = \Gamma_{ji}{}^i = \frac{1}{2}\eta_i^{-1} e_j \eta_i. \tag{3.9}$$

Hence putting $\nabla := \nabla_0$ (the Levi-Civita connection), $\nabla_i := \nabla_{0,e_i}$, we have

$$\nabla_i(e_i) = \frac{1}{2}\eta_i^{-1} e_i \eta_i \cdot e_i - \sum_{j \neq i} \frac{1}{2}\eta_j^{-1} e_j \eta_i \cdot e_j,$$

$$\nabla_i(e_j) = \frac{1}{2}\eta_i^{-1} e_j \eta_i \cdot e_i + \frac{1}{2}\eta_j^{-1} e_i \eta_j \cdot e_j. \tag{3.10}$$

Now, the vanishing of the λ-terms in the left hand side of (3.6) means that

$$\forall i,j,k: \quad e_i \circ \nabla_j(e_k) + \nabla_i(e_j \circ e_k) = (i \leftrightarrow j). \tag{3.11}$$

Using (3.10), one checks that (3.11) is identically satisfied for $i = j$ and for $i \neq j \neq k \neq i$, whereas the case $i \neq j = k$ gives

$$e_i \eta_j = e_j \eta_i. \tag{3.12}$$

The same condition is obtained for $k = i \neq j$. It follows that $\eta_i = e_i \eta$ for some η, defined at least locally.

Reading this argument in reverse direction, we see that if a) and b) are satisfied, then ∇_λ is flat, and M is Frobenius.

3.4. The Darboux–Egoroff equations. The Theorem 3.3 establishes a (not very explicit) equivalence between the following functional spaces on M (modulo self-evident equivalence):

a). Flat coordinates $(x^1, \dots, x^n)$, flat metric g_{ab}, function $\Phi(x)$ satisfying the Associativity Equations (1.6) and semisimplicity.

b). Canonical coordinates $(u^1, \dots, u^n)$, function $\eta(u)$ such that the metric $g = \sum e_i \eta (du^i)^2$ is flat, where $e_i = \partial/\partial u^i$.

The constraints on η, implicit in b), are called the Darboux–Egoroff equations. In order to write them down explicitly, let us introduce the rotation coefficients of the potential metric:

$$\gamma_{ij} := \frac{1}{2} \frac{\eta_{ij}}{\sqrt{\eta_i \eta_j}} \tag{3.13}$$

where as before, $\eta_i = e_i \eta$, $\eta_{ij} = e_i e_j \eta$.

3.4.1. Proposition. *The diagonal potential metric $g = \sum e_i\eta(du^i)^2$ is flat iff $\forall k \neq i \neq j \neq k$:*

$$e_k\gamma_{ij} = \gamma_{ik}\gamma_{kj} \tag{3.14}$$

and

$$e\gamma_{ij} = 0. \tag{3.15}$$

Proof. This is established by a straightforward calculation, complementing that in the proof of the Theorem 3.3. In fact, we now want to make explicit the condition $\nabla^2 = 0$ where ∇ is the Levi–Civita connection. So we return to (3.6) at $\lambda = 0$, i. e.

$$\nabla_i\nabla_j(e_k) = \nabla_j\nabla_i(e_k).$$

Nonvanishing curvature components can occur only for $i \neq j$. Calculating them directly we arrive to (3.14) and (3.15).

3.5. Proposition. *Let e be the identity, and ε the co–identity of the semisimple Frobenius manifold. Then*

a). $\varepsilon = d\eta$, where η is the metric potential.

b). e is flat iff for all i, $e\eta_i = 0$, or equivalently, $e\eta = g(e,e) =$ const. This condition is satisfied in the presence of an Euler field with $D \neq 2d_0$ (see (2.5), (2.16), (2.17).)

c). If e is flat, and (x^a) is a flat coordinate system, then

$$\eta = \sum_a x^a g(\partial_a, e) + \text{const.} \tag{3.16}$$

The formula (3.16) shows that in the passage from the (x^a, Φ)–description to the (u^i, η)–description the main information is encoded in the transition formulas $u^i = u^i(x)$, at least in the presence of flat identity.

Proof. The first claim follows from (3.5) and the Theorem 3.3 b).

The second one can be obtained directly from (3.10). We have

$$\nabla_i(e) = \nabla_i(e_i + \sum_{j\neq i} e_j) = \frac{1}{2}\frac{e\eta_i}{\eta_i}\cdot e_i.$$

These derivatives vanish iff $e\eta =$ const. But $e\eta = \sum \eta_i = g(e,e)$. From (2.17) it follows that $g(e,e) = 0$ if $D \neq 2d_0$.

Finally, (3.16) is the last formula in 2.1.2.

Notice that the equations $e\eta_i = 0$ imply (3.15).

We will now see that, like the identity, the Euler field is almost uniquely defined by the canonical coordinates, if it exists at all.

3.6. Theorem. *Let E be a vector field on the semisimple Frobenius manifold M, d_0 a constant.*

a). We have $\mathrm{Lie}(\circ) = d_0(\circ)$, *iff*

$$E = d_0 \sum_i (u^i + c^i) e_i, \tag{3.17}$$

where c^i are some constants.

b). For the field of the form (3.17) and a constant D, we have $\mathrm{Lie}_E(g) = Dg$ *iff for all i, $E\eta_i = (D - 2d_0)\eta_i$, or equivalently*

$$E\eta = (D - d_0)\eta + \text{const}. \tag{3.18}$$

Thus in the presence of a non–vanishing Euler field we may and will normalize the canonical coordinates so that $E = d_0 \sum u^i e_i$.

Proof. a). Put $E = \sum_i E^i e_i$ and write (2.6) for $X = e_k, Y = e_l$. Since $[E, e_k] = -\sum_i e_k(E^i) \cdot e_i$, we get $e_k(E^i) = d_0 \delta_{ik}$, so that $E^i = d_0(u^i + c^i)$.

b). Likewise, (2.5) for $X = e_i, Y = e_j$ is identically satisfied for $i \neq j$, and is equivalent to $E\eta_i = (D - 2d_0)\eta_i$ for $i = j$. Since $\eta_i = e_i \eta$ and $Ee_i = e_i E - d_0 e_i$, this is the same as (3.18).

3.6.1. Grading. The semisimplicity of $\mathrm{ad}\, E$ on $\mathcal{T}_M^f$ does not seem to have a good alternate formulation. However, if it holds, then the grading of functions and vector fields defined in 2.3 becomes especially simple in the canonical coordinates. For instance, let $d_0 = 1$; then $Ef = sf$ iff $f(\lambda u^1, \dots, \lambda u^n) = \lambda^s f(u^1, \dots, u^n)$.

Finally, we can complete the commutation relations (2.12).

3.6.2. Proposition. *If $d_0 = 1$, then*

$$[E^{\circ m}, E^{\circ n}] = (n - m) E^{\circ (m+n-1)} \tag{3.19}$$

for $m, n \geq 0$ everywhere on M, and for arbitrary integral m, n outside of $\cup_i (u^i = 0)$ that is, exactly where E is $\circ$–invertible.

In fact, from (3.2) one sees that $E^{\circ m} = \sum u_i^m e_i$.

3.7. A pencil of flat metrics. Equations (3.14) are stable with respect to a semigroup of coordinate changes. Namely, let f_i be arbitrary functions of one variable such that $\check{u}^i := f_i(u^i)$ form a local coordinate system, $\check{e}_i = \partial / \partial \check{u}^i, \check{\eta}_i = \check{e}_i \eta$ etc.

3.7.1. Proposition. *If (e_i, γ_{ij}) satisfy (3.14), then $(\check{e}_i, \check{\gamma}_{ij})$ satisfy (3.14) as well.*

Proof. The rotation coefficients of $\check{g} := \sum_i \check{e}_i \eta (d\check{u}^i)^2$ are (cf. (3.13))

$$\check{\gamma}_{ij} = \frac{1}{2} \check{e}_i \check{e}_j \eta (\check{e}_i \eta \, \check{e}_j \eta)^{-\frac{1}{2}} = \gamma_{ij} (f_i'(u^i) \, f_j'(u^j))^{-\frac{1}{2}}.$$

Hence for $k \neq i \neq j \neq k$ we have, in view of (3.14):

$$\check{e}_k \check{\gamma}_{ij} = \gamma_{ik}\gamma_{kj}(f_i'(u^i)\, f_j'(u^j))^{-\frac{1}{2}} f_k(u^k)^{-1},$$

and

$$\check{\gamma}_{ik}\check{\gamma}_{kj} = \gamma_{ik}\gamma_{kj}(f_i'(u^i)\, f_j'(u^j))^{-\frac{1}{2}} f_k(u^k)^{-1}$$

so that $\check{\gamma}_{ij}$ satisfy (3.14).

In order to satisfy (3.15) as well, we will have to restrict ourselves to the one-parametric family of local coordinate changes

$$\check{u}^i = \log(u^i - \lambda),\ \check{e}_i = (u^i - \lambda)e_i,\ \check{g}_\lambda = \sum_i (u^i - \lambda)^{-1} e_i\eta (du^i)^2 \tag{3.20}$$

which make sense on $M_\lambda := \{x \in M \,|\, \forall i,\ u^i \neq \lambda\}$.

3.7.2. Theorem. *Let M be a semisimple Frobenius manifold with canonical coordinates (u^i) and metric potential η. Then the following statements are equivalent.*

a). For all λ, the structure (3.20) is semisimple Frobenius on M_λ.

b). The same for a particular value of λ.

c). For all $i \neq j$,

$$\sum_k u^k e_k \gamma_{ij} = -\gamma_{ij}. \tag{3.21}$$

Moreover, (3.21) is satisfied if $E = \sum_k u^k e_k$ is the Euler field on M with $d_0 = 1$.

Notice that generally $\check{e} = \sum \check{e}_k$ is not flat for $\check{g}_\lambda$ and $\check{E} = \sum \check{u}^k \check{e}_k$ is not an Euler field.

Proof. Let us start with deducing (3.21). If E is the Euler field with $d_0 = 1$, we have $\sum_k u^k \eta_{ik} = (D-2)\eta_i$ (see Theorem 3.6 b).) Applying e_j we obtain $\sum_k u^k \eta_{ijk} = (D-3)\eta_{ij}$. Hence

$$E\gamma_{ij} = \sum_k u^k e_k \gamma_{ij} = \sum_k u^k \left[\frac{1}{2}\frac{\eta_{ijk}}{\sqrt{\eta_i\eta_j}} - \frac{1}{4}\frac{\eta_{ij}\eta_{ik}}{\eta_i\sqrt{\eta_i\eta_j}} - \frac{1}{4}\frac{\eta_{ij}\eta_{jk}}{\eta_j\sqrt{\eta_i\eta_j}} \right] =$$

$$= -\frac{1}{2}\frac{\eta_{ij}}{\sqrt{\eta_i\eta_j}} = -\gamma_{ij}.$$

Now we turn to the Darboux–Egoroff equations. We know from the assumptions and Proposition 3.7.1 that (3.14) is satisfied both for g and $\check{g}_\lambda$. The second half (3.15) in this situation is equivalent to

$$\forall\, i \neq j,\ \sum_{k \neq i,j} \gamma_{ik}\gamma_{kj} = -(e_i + e_j)\gamma_{ij}, \tag{3.22}$$

so that it remains to see the meaning of (3.22) now written for $\check{\gamma}_{ij}, \check{e}_j$.

We have, using (3.20), $\check{\gamma}_{ij} = \gamma_{ij}(u^i - \lambda)^{1/2}(u^j - \lambda)^{1/2}$. Hence for $i \neq j$

$$\sum_{k\neq i,j} \check{\gamma}_{ik}\check{\gamma}_{kj} = \left[\sum_{k\neq i,j} \gamma_{ik}\gamma_{kj}(u^k - \lambda)\right](u^i - \lambda)^{1/2}(u^j - \lambda)^{1/2} =$$

$$= \left[\sum_{k\neq i,j} u^k\gamma_{ik}\gamma_{kj} + \lambda(e_i + e_j)\gamma_{ij}\right](u^i - \lambda)^{1/2}(u^j - \lambda)^{1/2} =$$

$$= \left[\sum_{k\neq i,j} u^k e_k\gamma_{ij} + \lambda(e_i + e_j)\gamma_{ij}\right](u^i - \lambda)^{1/2}(u^j - \lambda)^{1/2} =$$

$$= \left[E\gamma_{ij} - (u^i - \lambda)e_i\gamma_{ij} - (u^j - \lambda)e_j\gamma_{ij}\right](u^i - \lambda)^{1/2}(u^j - \lambda)^{1/2}. \qquad (3.23)$$

On the other hand,

$$-(\check{e}_i + \check{e}_j)\check{\gamma}_{ij} = -\left[(u^i - \lambda)e_i + (u^j - \lambda)e_j\right]\left[\gamma_{ij}(u^i - \lambda)^{1/2}(u^j - \lambda)^{1/2}\right] =$$

$$= -\left[\gamma_{ij} + (u^i - \lambda)e_i\gamma_{ij} + (u^j - \lambda)e_j\gamma_{ij}\right](u^i - \lambda)^{1/2}(u^j - \lambda)^{1/2}. \qquad (3.24)$$

Comparing (3.23) and (3.24) one sees that their coincidence for one or for all values of λ is equivalent to (3.21). This finishes the proof.

3.7.3. Remarks. If E is Euler, the metric $\check{g}_\lambda$ in (3.20) can be written in coordinate free form:

$$\check{g}_\lambda(X, Y) = g((E - \lambda)^{-1} \circ X, Y). \qquad (3.25)$$

In fact (3.25) is flat on any Frobenius manifold with semisimple Euler field on it, non–necessarily semisimple: cf. [D2].

b). The inverse metrics $\check{g}^t_\lambda$ on the cotangent sheaf form a pencil of flat metrics with two marked points. Conversely, given such a pencil and two metrics g, h in it, we can define the spectrum of such data: zeroes of det $(g - uh)$. If the spectrum (u^i) forms a local coordinate system, the pair (u^i, h) has a chance to define the Frobenius structure: we have to check the potentiality of h written in u^i–coordinates, which is equivalent to the flatness of the structural connection: see Theorem 3.3.

3.8. Summary. We now briefly summarize the two descriptions of semi–simple Frobenius manifolds, stressing their parallelism.

WDVV picture

Flat coordinates $(x^0, \dots, x^{n-1})$, up to affine transformations, can be partially normalized in the presence of E.

Metric with constant coefficients $\sum g_{ab}dx^a dx^b$.

Potential $\Phi(x)$ satisfying the WDVV–equations (1.6), defined up to adding a quadratic polynomial in (x^a).

Flat identity $e = \partial_0$, additional equation $\partial_0 \Phi_{ab} = g_{ab}$.

Euler field $E = \sum E^a(x)\partial_a$, where E^a are of degree ≤ 1. Additional equation $E\Phi = (D + d_0)\Phi$ plus quadratic terms.

Darboux-Egoroff picture

Canonical coordinates $(u^1, \dots, u^n)$, up to renumbering and constant shifts. Shifts can be fixed in the presence of E.

Diagonal potential metric $g = \sum_i e_i\eta\,(du^i)^2,\ e_i = \partial/\partial u^i$.

The metric potential $\eta(u)$ satisfying the Darboux–Egoroff equations (3.14), (3.15), and defined up to adding a constant.

Flat identity $e = \sum_i e_i$, additional equation $e\eta = \text{const}$.

Euler field $E = d_0 \sum_i u^i e_i$. Additional equation $E\eta = (D - d_0)\eta + \text{const}$.

Passage from WDVV to Darboux-Egoroff

In the presence of an Euler field and a flat identity:

$(u^1, \dots, u^n)$ = the spectrum of $E\circ$ acting upon $\mathcal{T}_M$.

Metric potential $\eta = \sum_a x^a g(\partial_a, e)$.

3.9. A problem. It would be important to generalize the notion of semisimplicity to supermanifolds. Here are some scattered observations suggesting that there might be several different versions of it.

a). The main justification for considering Frobenius supermanifolds is the fact that quantum cohomology (theory of Gromov–Witten invariants) provides for any projective algebraic or symplectic V such a structure on an open (or formal) subspace of the conventional cohomology $H^*(V, \mathbf{C})$ considered as a linear superspace. Not many manifolds have pure even–dimensional cohomology, so we need odd coordinates.

b). If we look at the definition 1.2 from the vantage point of, say, supergravity, we will be tempted to replace the metric (g_{ab}) by a more refined structure. The standard nucleus of such a structure consists of a pair of pure odd integrable distributions $\mathcal{T}_l, \mathcal{T}_r \subset \mathcal{T}_M$ such that the supercommutator induces a maximally non–degenerate map $\mathcal{T}_l \otimes \mathcal{T}_r \to \mathcal{T}_M/(\mathcal{T}_l \oplus \mathcal{T}_r)$. There are two drawbacks to it. First, such a structure seems to be nowhere in sight in quantum cohomology. Second, in its natural habitat it is complemented by new constraints depending on dimension, so that there is no dimension independent generalization of Riemannian geometry along these lines.

If one decides against this option, one should keep in mind alternative geometries peculiar to supergeometry, for instance, (a curved version of) Π–symmetry, where Π is the parity switch. For example, in the picture of Calabi–Yau Mirror Symmetry the cohomology spaces of mirror threefolds V, V' are roughly speaking connected by $H(V) = \Pi H(V')$.

Proceeding in this direction, we will have to rethink the ways to construct generating functions from Gromov–Witten invariants.

c). Finally, an extension of the notion of semisimlicity is suggested by the dominant role of the Euler field E, or rather Lie algebra spanned by $E^{\circ n}$. One can imagine a structure, consisting of a supermanifold M, a representation of the Neveu–Schwarz (or Ramond) Lie superalgebra in $\mathcal{T}_M$, and a superversion of the equations $e\eta = \text{const}$, $E\eta = (D - d_0)\eta + \text{const}$. To find a superization of the Darboux–Egoroff equations seems a subtler problem.

§4. Examples

4.1. Dimension one. Let M be a connected simply connected one–dimensional manifold, for definiteness, complex analytic.

The structure of pre–Frobenius manifold on M is given by an arbitrary pair (∂,φ) where ∂ is a vector field without zeroes and φ a function:

$$\mathcal{T}_M^f := \mathbf{C}\partial,\ g(\partial,\partial)=1,\ \partial\circ\partial=\varphi\partial.$$

Two pairs (∂,φ), (∂',φ') define the same structure iff they coincide or differ by common sign.

Such a structure is automatically associative and potential, hence Frobenius. Let M_0 be the complement to the zeroes of φ. On M_0 there is an identity $e=\varphi^{-1}\partial$, which is flat iff $\partial\varphi=0$. If $\partial=d/dx$, co–identity is $\varepsilon=\varphi(x)dx$.

M_0 is also the domain of semisimplicity. Solving the equation $e=d/du$ for u, we get $u=\int\varphi(x)dx=\int\varepsilon$.

A definite choice of u is equivalent to the choice of the would–be Euler field $E=ud/du$ with $d_0=1$. A metric potential is $\eta=u$, hence $E\eta=u$ so that (3.19) is satisfied with $D=2$. Even if e is not flat, we have $[\partial,E]=\partial$, so that E is actually an Euler field.

This rather dull picture will give rise to quite non–trivial problems in the context of *formal* Frobenius manifolds, when we will introduce and calculate the operation of tensor product on them.

4.2. Dimension two. We will give here a local classification of two–dimensional Frobenius structures with flat identity and a semisimple Euler field with $d_0=1$. The multiplication $\circ$ in this situation is automatically associative, so that the WDVV–equations are empty, and it remains to find all potentials satisfying the equations (2.2) and (2.7).

The final answer depends on the spectrum of E.

First, let $(d_0=1,d_1)$ be the spectrum of $-\mathrm{ad}\,E$ on $\mathcal{T}_M^f$, (∂_0,∂_1) the respective flat eigenvectors, $e=\partial_0$. The classification starts branching depending on whether $d_1\neq 0$ or $d_1=0$: this is our first *critical value* of d_1. We choose flat coordinates (x^0,x^1) such that (cf. 2.4)

$$d_1\neq 0:\quad E=x^0\partial_0+d_1x^1\partial_1, \tag{4.1a}$$

(x^1 being defined up to multiplication by a constant),

$$d_1=0:\quad E=x^0\partial_0+2\partial_1 \tag{4.1b}$$

(x^1 being defined up to addition of a constant), or else

$$d_1=0:\quad E=x^0\partial_0. \tag{4.1c}$$

From (2.17) one sees that a compatible non–vanishing flat metric can exist only if $D\in\{2,1+d_1,2d_1\}$, and for $d_1\neq 1$ a non–degenerate flat metric exists only if $D=1+d_1$, so that $D=1+d_1$ always.

If $d_1 \neq 1$, we have $g_{00} = g_{11} = 0$, $g_{01} = \gamma \neq 0$; we can make $\gamma = 1$ by rescaling x_1. If $d_1 = 1$ (this is the second critical value of d_1), (g_{ab}) can be arbitrary symmetric non-degenerate matrix.

From (2.3) we obtain

$$\Phi(x^0, x^1) = \frac{1}{2}x^0(g_{11}(x^1)^2 + g_{01}x^0x^1 + \frac{1}{3}g_{00}(x^0)^2) + \Psi(x^1), \tag{4.2}$$

and from (2.7)

$$E\Phi = (d_1 + 2)\Phi + \text{a quadratic polynomial}. \tag{4.3}$$

In the case (4.1c) this leads to

$$x^0\partial_0[\frac{\gamma}{2}(x^0)^2x^1 + \Psi(x^1)] = \gamma(x^0)^2x^1 + 2\Psi(x^1) + \text{a quadratic polynomial}$$

so that we can take

$$d_1 = 0,\ E = x^0\partial_0 : \quad \Phi = \frac{\gamma}{2}(x^0)^2x^1. \tag{4.4}$$

The case (4.1b) leads to the equation

$$\partial_1\Psi(x^1) = \Psi(x^1) + \text{a quadratic polynomial in } x^1$$

so that we can take, after rescaling x^1,

$$d_1 = 0,\ E = x^0\partial_0 + 2\partial_1 : \quad \Phi = \frac{\gamma}{2}(x^0)^2x^1 + e^{x^1}. \tag{4.5}$$

In the case (4.1a) with $d_1 = 1$, Φ can be reduced to a cubic form with constant coefficients:

$$d_1 = 0,\ E = x^0\partial_0 + x^1\partial_1 : \quad \Phi = \frac{1}{2}x^0(g_{11}(x^1)^2 + g_{01}x^0x^1 + \frac{1}{3}g_{00}(x^0)^2) + c(x^1)^3. \tag{4.6}$$

Finally, the case (4.1a) produces two more critical values $d_1 = \pm 2$:

$$d_1 = -2,\ E = x^0\partial_0 - 2x^1\partial_1 : \quad \Phi = \frac{1}{2}(x^0)^2x^1 + c\log x^1, \tag{4.7}$$

$$d_1 = 2,\ E = x^0\partial_0 + 2x^1\partial_1 : \quad \Phi = \frac{1}{2}(x^0)^2x^1 + c\,(x^1)^2\log x^1, \tag{4.8}$$

$$d_1 \neq 0, 1, \pm 2,\ E = x^0\partial_0 + d_1x^1\partial_1 : \quad \Phi = \frac{1}{2}(x^0)^2x^1 + c\,(x^1)^{(2+d_1)/d_1}. \tag{4.9}$$

4.3. Dimension three: a promise. This is the first dimension where the Associativity Equations become non-empty even in the presence of the flat identity. The beautiful theory of three dimensional semisimple Frobenius manifolds essentially reduces their study to that of a subfamily of the sixth Painlevé equations. We will address this connection in Chapter II.

4.4. Quantum cohomology: brief encounter. Let V be a smooth projective algebraic manifold over $\mathbf{C}$ (another version of the theory exists for compact symplectic manifolds.)

Denote by H the cohomology space $H^*(V,\mathbf{C})$ considered as *a complex analytic linear supermanifold.* We endow H with its natural flat structure $\mathcal{T}_H^f$, Poincaré form g, and two vector fields e, E which can be described as follows. First, H *as a linear space* can be identified with global flat vector fields. We denote by e the vector field corresponding to the identity in the cohomology ring that is, the dual fundamental class of V. Second, $-\mathrm{ad}\, E$ is the semisimple operator on $\mathcal{T}_H^f$, with eigenvalue $1-p/2$ on $H^p(X,\mathbf{C})$: this determines the first summand in the decomposition (2.14). The second (flat) one is the anticanonical class of V.

Explicitly, let $H^*(V,\mathbf{C}) = \oplus \mathbf{C}\Delta_a$, $\Delta_a \in H^{|\Delta_a|}(V,\mathbf{C})$, Δ_0 the dual fundamental class. Then the coordinates (x^a) in this basis are global flat coordinates on H, and

$$e = \partial_0,\ E = \sum_a (1 - \frac{|\Delta_a|}{2}) x^a \partial_a + \sum_{b:\,|\Delta_b|=2} r^b \partial_b, \tag{4.10}$$

where r^b are defined by

$$c_1(\mathcal{T}_V) = -K_V = \sum_{b:\,|\Delta_b|=2} r^b \Delta_b. \tag{4.11}$$

Moreover, $g_{ab} = \int_V \Delta_a \wedge \Delta_b$ (we imagine cohomology classes as differential forms, and use wedge for the cup product.)

The relations (2.5) (resp. (2.16)) are satisfied with $D = 2 - \dim_{\mathbf{C}} V$ (resp. $d_0 = 1$) so that the total spectrum of E is

$$d_0 = 1,\ d_a = 1 - \frac{\Delta_a}{2} \text{ of multiplicity } \dim H^{|\Delta_a|},\ D = 2 - \dim_{\mathbf{C}} V. \tag{4.12}$$

The remaining and most important structure is the potential Φ. The theory of Gromov–Witten invariants furnishes (at least for manifolds with $K_V \leq 0$) *a formal series* $\Phi(x)$ in flat coordinates satisfying all the axioms of Frobenius structure, with flat identity e and the Euler field E, described above. Moreover, Φ can be actually represented as a series in E–homogeneous monomials (2.20) (notice that they are exponential in codimension two coordinates), with nonnegative integers m_a and n_b, of E–degree $d_0 + D = 3 - \dim_{\mathbf{C}} V$. Coefficients of this series are certain numerical invariants of the space of stable maps of pointed curves of genus 0 to V.

If Φ converges in a subdomain $M \subset H$, it induces a structure of Frobenius manifold on M. Generally, its maximal analytic continuation to an unramified covering of a subdomain of H should be considered as *the* Frobenius manifold representing the quantum cohomology of V.

One approach to the study of Φ consists in the identification of M (physicists' A–model) with a Frobenius manifold constructed by other methods, e. g. from isomonodromic deformations or periods of the families of algebraic manifolds (physicists' B–model.) This can be called a general Mirror Program. The very first step in such an identification is the comparison of spectra. The famous $h_V^{11} = h_{\tilde V}^{12}$ mirror symmetry relation for the Calabi–Yau threefolds expresses such an identification.

As an elementary exercise, let us guess which of the manifolds (4.5)–(4.9) can represent quantum cohomology. Only $\mathbf{P}^1$ has two-dimensional pure even cohomology space, and $-K_{\mathbf{P}^1}$ has degree two, so that E must be of the type (4.1b). In fact, the potential of (the quantum cohomology of) $\mathbf{P}^1$ is given by (4.5) with $\gamma = 1/2$ in the natural basis.

We conclude this brief discussion by describing explicitly the potential Φ for all projective spaces $\mathbf{P}^r$. Put Δ_a = the dual class of the codimension a hyperplane, $\gamma = \sum x^a \Delta_a$, (γ^3) = the triple self-intersection index. Then

$$\Phi^{\mathbf{P}^r}(x) = \frac{1}{6}(\gamma^3) + \sum_{d,n_a \geq 0} N(d; n_2, \dots, n_r) \frac{(x^2)^{n_2} \dots (x^r)^{n_r}}{n_2! \dots n_r!} e^{dx^1} \tag{4.13}$$

where $N(d; n_1, \dots, n_r)$ is the number of rational curves of degree d in $\mathbf{P}^r$ intersecting n_a hyperplanes of codimension a in general position. This number (suitably interpreted in certain boundary cases) can be non-zero only for $\sum_a n_a(a-1) = (r+1)d + r - 3$ which is equivalent to the grading equation (2.7). The Associativity Equations (1.6) follow from a rather sophisticated analysis of degenerations. They allow us to calculate recursively all $N(d; n_1, \dots, n_r)$ starting with a single number $N(1; 0, \dots, 0, 2) = 1$ (there is only one line passing through two different points.)

In fact, the recursive relations obtained from (1.6) form such an overdetermined system that it is not obvious how to prove the existence of a solution to (1.6) and (2.7) formally (i. e., without using the geometric interpretation.) For a roundabout proof, see Ch. II, 4.2 below. The cases $r = 1$ and $r = 2$ are exceptional: we have respectively

$$\Phi^{\mathbf{P}^1}(x\Delta_0 + z\Delta_1) = \frac{1}{2}x^2 z + e^z - (1 + z + \frac{z^2}{2}), \tag{4.14}$$

$$\Phi^{\mathbf{P}^2}(x\Delta_0 + y\Delta_1 + z\Delta_2) = \frac{1}{2}(xy^2 + x^2 z) + \sum_{d=1}^{\infty} N(d) \frac{z^{3d-1}}{(3d-1)!} e^{dy}. \tag{4.15}$$

Here the Associativity equations are equivalent to an explicit recursive formula for $N(d)$ (see Introduction, (0.19).)

All these Frobenius structures are generically (or formally) semisimple. Notice that in the semisimple case the potential η of the Poincaré metric is simply the linear function $\eta : H^*(V, \mathbf{C}) \to \mathbf{C} : \eta(\gamma) = \int_V \gamma$. This is a restatement of (2.4).

4.5. Space of polynomials. The following beautiful example furnishes another series of semisimple Frobenius manifolds of arbitrary dimension. This construction, due to B. Dubrovin and K. Saito, admits various generalizations.

Consider n-dimensional affine space $\mathbf{A}^n$ with coordinate functions $a_1, \dots, a_n$. Identify $\mathbf{A}^n$ with the space of polynomials $p(z) = z^{n+1} + a_1 z^{n-1} + \dots + a_n$. Denote by $\pi : \widetilde{\mathbf{A}}^n \to \mathbf{A}^n$ the covering space of degree $n!$ whose fiber over a point $p(z)$ consists of total orderings of the roots of $p'(z)$. In other words, $\widetilde{\mathbf{A}}^n$ supports functions $\rho_1, \dots, \rho_n$ such that

$$\pi^*(p'(z)) = (n+1) \prod_{i=1}^{n} (z - \rho_i); \tag{4.17}$$

$$\pi^*(a_i) = (-1)^{i+1}\frac{n+1}{n-i}\sigma_{i+1}(\rho_1,\dots,\rho_n),\ i=1,\dots,n-1$$

and $\sigma_1(\rho_1,\dots,\rho_n) = \rho_1+\dots+\rho_n = 0$. We will omit π^* in the notation of lifted functions.

Let $M \subset \widetilde{\mathbf{A}}^n$ be the open dense subspace on which

A. $\forall i,\ p''(\rho_i) \neq 0$ that is, $\rho_i \neq \rho_j$ for $i \neq j$.

B. $u^i := p(\rho_i)$ form local coordinates at any point.

4.5.1. Theorem. *M is a semisimple Frobenius manifold with the following structure data:*

a). Canonical coordinates (u^i), identity $e = \sum_i e_i$, $e_i = \partial/\partial u^i$, Euler field $E = \sum u^i e_i$.

b). Flat metric

$$g := \sum_{i=1}^{n}\frac{(du^i)^2}{p''(\rho_i)} \tag{4.18}$$

with metric potential

$$\eta = \frac{a_1}{n+1} = \frac{1}{n-1}\sum_{i<j}\rho_i\rho_j = \frac{1}{2(n-1)}\sum\rho_i^2. \tag{4.19}$$

Furthermore, e, E and flat coordinates $x^{(1)},\dots,x^{(n)}$ can be calculated through $(a_1,\dots,a_n)$ (which are generically local coordinates as well):

$$e = \partial/\partial a_n,\ \text{i. e., } ea_n = 1, ea_i = 0 \text{ for } i<n. \tag{4.20}$$

$$E = \frac{1}{n+1}\sum_{i=1}^{n}(i+1)a_i\frac{\partial}{\partial a_i}, \tag{4.21}$$

$x^{(i)}$ are the first Laurent coefficients of the inversion of $w = \sqrt[n+1]{p(z)} = z + O(1/z)$ near $z=\infty$:

$$z = w + \frac{x^{(1)}}{w} + \frac{x^{(2)}}{w^2} + \dots + \frac{x^{(n)}}{w^n} + O(w^{-n-1}). \tag{4.22}$$

Finally, the spectrum is $D = \dfrac{n+3}{n+1}$, $d^{(i)} = \dfrac{i+1}{n+1}$, $1\le i\le n$, more precisely, $Ex^{(i)} = \dfrac{i+1}{n+1}x^{(i)}$.

Proof. From (4.17) we find

$$\eta_j := \frac{1}{p''(\rho_j)} = \frac{1}{(n+1)\prod_{i:\,i\neq j}(\rho_i-\rho_j)}. \tag{4.23}$$

Furthermore

$$\delta_{ij} = \frac{\partial u^i}{\partial u^j} = \frac{\partial(p(\rho_i))}{\partial u^j} = \sum_{k=1}^{n}\frac{\partial a_k}{\partial u^j}z^{n-k}\Big|_{z=\rho_i} \tag{4.24}$$

because $p'(\rho_i) = 0$. Therefore the polynomial at the right hand side of (4.24) (depending only on j) must be equal to

$$\prod_{i:\,i\neq j} \frac{z-\rho_i}{\rho_j-\rho_i} \tag{4.25}$$

because it has the same degree $n-1$ and takes the same values at $\rho_1, \ldots, \rho_n$. Comparing (4.23) and (4.25) we see first of all that

$$\frac{\partial a_1}{\partial u^j} = \text{coeff. of } z^{n-1} \text{ in } \prod_{i:\,i\neq j} \frac{z-\rho_i}{\rho_j-\rho_i} = \frac{1}{\prod_{i:\,i\neq j}(\rho_j-\rho_i)} = (n+1)\eta_j. \tag{4.26}$$

This means that $\eta = \dfrac{a_1}{n+1}$ is the metric potential of g (cf. Theorem 3.3b.) Now sum (4.24) for all j. We obtain that $\sum_{k=1}^n ea_k z^{n-k}$ is a polynomial of degree $n-1$ taking value 1 at $z = \rho_1, \ldots, \rho_n$. Hence it is identically 1 that is,

$$ea_n = 1;\ ea_{n-1} = \cdots = ea_1 = 0.$$

This proves (4.20).

Let us now calculate $E\eta$. Multiplying (4.24) by u^j and summing over all j we see that $\sum_k Ea_k z^{n-k}$ is a polynomial of degree $n-1$ taking the value u^j at $z = \rho_j$. We know a polynomial of degree n taking the same values: it is $p(z)$. Hence $p(z) - \sum_k Ea_k z^{n-k}$ is divisible by $p'(z)$ vanishing at all ρ_j. Comparing the top two coefficients we obtain

$$p(z) - \sum_k Ea_k z^{n-k} = \frac{z}{n+1} p'(z)$$

that is,

$$a_k - Ea_k = \frac{n-k}{n+1} a_k$$

which proves (4.21). In particular, $Ea_1 = \dfrac{2}{n+1} a_1$, so that $D = \dfrac{n+3}{n+1}$, because $d_0 = 1$.

We now turn to checking flatness of g. In fact, we can do rather more starting with a neat description of the multiplication $\circ$.

Let $p(z)$ be a point of M or its image in $\mathbf{A}^n$. Using $d\pi$ we can identify the tangent spaces at both points to the Milnor ring $\mathbf{C}[z] \bmod p'(z)$.

4.5.2. Lemma. *$d\pi$ identifies the $\circ$-multiplication with multiplication in the Milnor ring.*

Proof. Explicitly,

$$d\pi\,(e_j|_p) = \frac{\partial p}{\partial u^j} \bmod p'(z).$$

In view of the previous calculations (see (4.24) and (4.25))

$$\frac{\partial p}{\partial u^j} = e_j p = \prod_{i:\,i\neq j} \frac{z-\rho_i}{\rho_j-\rho_i} \bmod p'(z). \tag{4.28}$$

The polynomials at the right hand side of (4.28) are the basic idempotents in the Milnor ring, exactly as e_j in $\mathcal{T}_M$.

4.5.3. Lemma. *The metric (4.18) induces on the Milnor ring the scalar product*

$$g(a(z)|_p, b(z)|_p) = -\mathrm{res}_{z=\infty}\frac{a(z)b(z)}{p'(z)}dz. \tag{4.29}$$

Proof. The right hand side of (4.29) equals

$$\sum_{j=1}^{n}\mathrm{res}_{z=\rho_j}\frac{a(z)b(z)}{p'(z)}dz = \sum_{j=1}^{n}\frac{a(\rho_j)b(\rho_j)}{p''(\rho_j)}. \tag{4.30}$$

Choosing $a(z) = d\pi\,(e_i|_p)$, $b(z) = d\pi\,(e_j|_p)$, we see from (4.28) and (4.26) that the value of the right hand side of (4.30) is

$$\frac{\delta_{ij}}{p''(\rho_j)} = g(e_i|_p, e_j|_p).$$

4.5.4. End of the proof of Theorem 4.5.1. We can now prove simultaneously that g is flat and $x^{(a)}$ are flat coordinates by showing that $g(\partial_a, \partial_b)$ are constant, for $\partial_a := \partial/\partial x^{(a)}$.

In fact, from (4.22) we get, considering z as a function of w and $x^{(a)}$: $p(z(w,x)) = w^{n+1}$ so that

$$\frac{\partial p}{\partial x^{(a)}}(z(w,x)) = -p'(z(w,x))(w^{-a} + O(w^{-n-1})). \tag{4.31}$$

Substituting this into (4.29) we find

$$g(\partial_a|_p, \partial_b|_p) = -\mathrm{res}_{z=\infty}(\partial_a p\,\partial_b p)\frac{dz}{p'(z)} =$$

$$= -\mathrm{res}_{z=\infty}\,p'(z)dz(w^{-a-b} + O(w^{-n-2})).$$

Replacing here the local parameter z at infinity by w and taking into account that $p'(z)dz = (n+1)w^n dw$ we get

$$g(\partial_a, \partial_b) = (n+1)\delta_{a+b,n+1}. \tag{4.32}$$

Finally, $p(z)$ becomes homogeneous of degree 1, if we assign to z the E–degree $\frac{1}{n+1}$. This implies that $x^{(i)}$ is of degree $\frac{i+1}{n+1}$.

4.5.5. Corollary. *The potential Φ is a polynomial of E–degree $D + d_0 = 2 + \frac{2}{n+1}$ and the usual degree $\leq n+3$ in flat coordinates.*

Since Φ is analytic in $x^{(i)}$ and the spectrum of $-\mathrm{ad}\,E$ is strictly positive, the Taylor series can contain only finitely many terms of E–degree $D+d_0$. The maximal usual degree is furnished by $(x^1)^{(n+3)}$.

Notice that quantum cohomology cannot have spectrum of this type.

CHAPTER II. FROBENIUS MANIFOLDS AND ISOMONODROMIC DEFORMATIONS

§1. The second structure connection

1.1. Preparation. Let (M, g, A) be a Frobenius (super)manifold, ∇_0 the Levi–Civita connection (on $\mathcal{T}_M$) of the flat metric g. Recall that the (first) structure connection on M is actually a pencil of flat connections ∇_λ, determined by the formula $\nabla_{\lambda,X}(Y) = \nabla_{0,X}(Y) + \lambda\, X \circ Y$ (see Ch. I, (1.4) and (1.5).) If in addition M is endowed with an Euler field E with $d_0 = 1$, we can define the extended structure connection $\widehat{\nabla}$ on the sheaf $\widehat{\mathcal{T}} = \mathrm{pr}_M^*(\mathcal{T}_M)$ on $\widehat{M} = M \times (\mathbf{P}^1_\lambda \setminus \{0, \infty\})$ such that for $X \in \mathcal{T}_M$, $Y \in \mathcal{T}_M^f$ we have

$$\widehat{\nabla}_X(Y) = \lambda X \circ Y,\ \widehat{\nabla}_{\partial/\partial\lambda}(Y) = E \circ Y - \frac{1}{\lambda}[E, Y] \tag{1.1}$$

(cf. I.2.5, in particular (2.24); we now omit a few extra hats in notation and commit the respective abuses of language.)

In this section and Chapter we will restrict ourselves to the case of semisimple complex Frobenius manifolds with an Euler field with $d_0 = 1$ admitting a global system of canonical coordinates (u^i). We will call *the second structure connection* $\check{\nabla}_\lambda$ the Levi–Civita connection of the flat metric

$$\check{g}_\lambda(X, Y) := g((E - \lambda)^{-1} \circ X, Y)$$

depending on a parameter λ and defined on the open subset $M_\lambda \subset M$ where $u^i \neq \lambda$ for all i. Put $\check{M} := \cup_\lambda (M_\lambda \times \{\lambda\}) \subset M \times \mathbf{P}^1_\lambda$ and denote by $\check{\mathcal{T}}$ the restriction of $\mathrm{pr}_M^*(\mathcal{T}_M)$ to $\check{M}$.

In this section we will construct a flat extension $\check{\nabla}$ of $\check{\nabla}_\lambda$ to $\check{\mathcal{T}}$ which will also be referred to as the second structure connection. Both extensions $\widehat{\nabla}$ and $\check{\nabla}$ will be further studied as isomonodromic deformations of their restrictions to the λ-direction parametrized by M.

More precisely, assume that $\mathcal{T}_M^f$ is a trivial local system (for instance, because M is simply connected.) Put $T := \Gamma(M, \mathcal{T}_M^f)$. Then $\widehat{\nabla}$ (resp. $\check{\nabla}$) induces an integrable family of connections with singularities on the trivial bundle on $\mathbf{P}^1_\lambda$ with the fiber T. The first connection $\widehat{\nabla}$ is singular only at $\lambda = 0$ and $\lambda = \infty$ but whereas 0 is a regular (Fuchsian) singularity, ∞ is irregular one, so that $\widehat{\nabla}$ cannot be an algebraic geometric Gauss–Manin connection, and its monodromy involves the Stokes phenomenon. To the contrary, the second connection $\check{\nabla}$ generally has only regular singularities at infinity and at $\lambda = u^i$ whose positions thus depend on the parameters. It is determined by the conventional monodromy representation and has a chance to define a variation of Hodge structure. For more details, see the next section.

It turns out that both deformations have a common moduli space and deserve to be studied together. In fact, fiberwise they are more or less formal Laplace transforms of each other. More to the point, they form a dual pair in the sense of [Har].

In our calculations the key role will be played by the $\mathcal{O}_M$–linear skew symmetric operator $\mathcal{V} : \mathcal{T}_M \to \mathcal{T}_M$ which is the unique extension of the operator defined in I.2.2 on flat vector fields by the formula

$$\mathcal{V}(X) = [X, E] - \frac{D}{2} X \text{ for } X \in \mathcal{T}_M^f. \tag{1.2}$$

1.1.1. Proposition. *a). We have for arbitrary $X \in \mathcal{T}_M$:*

$$\mathcal{V}(X) = \nabla_{0,X}(E) - \frac{D}{2} X. \tag{1.3}$$

b). Let $e_j = \partial/\partial u^j$, $f_j = e_j/\sqrt{\eta_j}$. Then

$$\mathcal{V}(f_i) = \sum_{j \neq i} (u^j - u^i)\gamma_{ij} f_j. \tag{1.4}$$

Proof. The fact that $-\mathrm{ad}\, E - \frac{D}{2}\mathrm{Id}$ is skew symmetric with respect to g was checked in I.2.2. Formula (1.3) defines an $\mathcal{O}_M$–linear endomorphism of $\mathcal{T}_M$ which coincides with (1.2) on the flat fields, as a calculation in flat coordinates shows.

To check (1.4), we use (1.3) and Ch. I, (3.10):

$$\mathcal{V}(f_i) = \nabla_{0,f_i}(E) - \frac{D}{2} f_i = \nabla_{0,e_i/\sqrt{\eta_i}} \left(\sum_j u^j e_j \right) - \frac{D}{2} \frac{e_i}{\sqrt{\eta_i}} =$$

$$\frac{1}{\sqrt{\eta_i}} \left[e_i + u^i \nabla_{0,e_i}(e_i) + \sum_{j \neq i} u^j \nabla_{0,e_i}(e_j) \right] - \frac{D}{2} \frac{e_i}{\sqrt{\eta_i}} =$$

$$\frac{1}{\sqrt{\eta_i}} \left[e_i + u^i \left(\frac{\eta_{ii}}{2\eta_i} e_i - \sum_{j \neq i} \frac{\eta_{ij}}{2\eta_j} e_j \right) + \sum_{j \neq i} u^j \left(\frac{\eta_{ij}}{2\eta_i} e_i + \frac{\eta_{ij}}{2\eta_j} e_j \right) \right] - \frac{D}{2} \frac{e_i}{\sqrt{\eta_i}}. \tag{1.5}$$

For $j \neq i$, the coefficient of f_j in the right hand side of (1.5) is $(u^j - u^i)\gamma_{ij}$. For $j = i$ it vanishes, because

$$1 + \sum_j u^j \frac{\eta_{ij}}{2\eta_i} = 1 + \frac{E\eta_i}{2\eta_i} = \frac{D}{2}$$

(see Ch. I, Theorem 3.6 b).)

We can now state the main result of this section. In addition to (1.2), define the operator $\mathcal{U} : \mathcal{T}_M \to \mathcal{T}_M$:

$$\mathcal{U}(X) := E \circ X, \tag{1.6}$$

so that $\mathcal{U}(f_i) = u^i f_i$.

1.2. Theorem. *For* $X, Y \in \mathrm{pr}_M^{-1}(\mathcal{T}_M) \subset \mathcal{T}_{\check{M}}$ *(meromorphic vector fields on* $\mathcal{T}_{M\times \mathbf{P}^1_\lambda}$ *independent on* λ*) put*

$$\check{\nabla}_X(Y) = \nabla_{0,X}(Y) - (\mathcal{V} + \frac{1}{2}\,\mathrm{Id})\,(\mathcal{U}-\lambda)^{-1}(X\circ Y), \tag{1.7}$$

$$\check{\nabla}_{\partial/\partial\lambda}(Y) = (\mathcal{V} + \frac{1}{2}\,\mathrm{Id})\,(\mathcal{U}-\lambda)^{-1}(Y). \tag{1.8}$$

Then $\check{\nabla}$ *is a flat connection on* $\check{\mathcal{T}}$ *whose restriction on* $M\times\{\lambda\}$ *defined by (1.7) is the Levi–Civita connection for* $\check{g}_\lambda$*.*

Remark. Rewriting (1.1) in the same notation, we get

$$\widehat{\nabla}_X(Y) = \nabla_{0,X}(Y) + \lambda X\circ Y, \tag{1.9}$$

$$\widehat{\nabla}_{\partial/\partial\lambda}(Y) = \left[\mathcal{U} + \frac{1}{\lambda}(\mathcal{V} + \frac{D}{2}\,\mathrm{Id})\right](Y). \tag{1.10}$$

Proof. We will first apply Ch. I, (3.10) in order to calculate the Levi–Civita connection for $\check{g}_\lambda$ in coordinates $\check{u}^i = \log(u^i-\lambda)$. As in I.3.7 we have

$$\check{e}_i = \frac{\partial}{\partial \check{u}^i} = (u^i-\lambda)e_i,\ \check{\eta}_i = (u^i-\lambda)\eta_i,\ \check{\eta}_{ij} = (u^i-\lambda)(u^j-\lambda)\eta_{ij} + \delta_{ij}(u^i-\lambda)\eta_i,$$

$$\check{\gamma}_{ij} = \gamma_{ij}(u^i-\lambda)^{1/2}(u^j-\lambda)^{1/2}.$$

Then for $i\neq j$

$$\check{\nabla}_{\check{e}_i}(\check{e}_j) = \frac{1}{2}\frac{\check{\eta}_{ij}}{\check{\eta}_i}\check{e}_i + \frac{1}{2}\frac{\check{\eta}_{ij}}{\check{\eta}_j}\check{e}_j = \frac{1}{2}(u^i-\lambda)(u^j-\lambda)\left(\frac{\eta_{ij}}{\eta_i}e_i + \frac{\eta_{ij}}{\eta_j}e_j\right)$$

so that

$$\check{\nabla}_{e_i}(e_j) = \frac{1}{2}\frac{\eta_{ij}}{\eta_i}e_i + \frac{1}{2}\frac{\eta_{ij}}{\eta_j}e_j = \nabla_{0,e_i}(e_j). \tag{1.11}$$

Similarly,

$$\check{\nabla}_{\check{e}_i}(\check{e}_i) = \frac{1}{2}\frac{\check{\eta}_{ii}}{\check{\eta}_i}\check{e}_i - \frac{1}{2}\sum_{j\neq i}\frac{\check{\eta}_{ij}}{\check{\eta}_j}\check{e}_j =$$

$$\frac{1}{2}(u^i-\lambda)^2\left[\frac{\eta_{ii}}{\eta_i} + \frac{1}{u^i-\lambda}\right]e_i - \frac{1}{2}\sum_{j\neq i}(u^i-\lambda)\,(u^j-\lambda)\,\frac{\eta_{ij}}{\eta_j}\,e_j$$

so that

$$\check{\nabla}_{e_i}(e_i) = \frac{1}{2}\left[\frac{\eta_{ii}}{\eta_i} - \frac{1}{u^i-\lambda}\right]e_i - \frac{1}{2}\sum_{j\neq i}\frac{u^j-\lambda}{u^i-\lambda}\,\frac{\eta_{ij}}{\eta_j}\,e_j. \tag{1.12}$$

Subtracting from this (3.10) (Ch. I), we get

$$(\check{\nabla}_{e_i} - \nabla_{0,e_i})(e_i) = -\frac{1}{2}\,\frac{1}{u^i-\lambda}\,e_i - \frac{1}{2}\sum_{j:\,j\neq i}\frac{u^j-u^i}{u^i-\lambda}\,\frac{\eta_{ij}}{\eta_j}\,e_j \tag{1.13}$$

and

$$(\check{\nabla}_{e_i} - \nabla_{0,e_i})(f_i) = -\frac{1}{2}\,\frac{1}{u^i - \lambda}\,f_i - \sum_{j\neq i} \frac{u^j - u^i}{u^i - \lambda}\,\gamma_{ij}\,f_j. \tag{1.14}$$

In view of (1.4), we can write (1.11) and (1.12) together as

$$(\check{\nabla}_{e_i} - \nabla_{0,e_i})(f_j) = -\left(\mathcal{V} + \frac{1}{2}\,\mathrm{Id}\right)(\mathcal{U} - \lambda)^{-1}(e_i \circ f_j) \tag{1.15}$$

because $e_i \circ f_j = \delta_{ij} f_j$. This family of formulas is equivalent to (1.7) so that (1.7) is the Levi–Civita connection for $\check{g}_\lambda$. In particular, it is flat for each fixed λ.

Since $[X, \partial/\partial\lambda] = 0$ for $X \in \mathrm{pr}_M^{-1}(\mathcal{T}_M)$, it remains to show that the covariant derivatives (1.7) and (1.8) commute on $\check{M}$ i. e. , that for all i, j

$$\check{\nabla}_{e_i}\check{\nabla}_{\partial/\partial\lambda}(e_j) = \check{\nabla}_{\partial/\partial\lambda}\check{\nabla}_{e_i}(e_j). \tag{1.16}$$

First of all, from (1.8) and (1.14) we find

$$\check{\nabla}_{\partial/\partial\lambda}(e_j) = \frac{1}{2}\,\frac{1}{u^j - \lambda}\,e_j + \frac{1}{2}\sum_{k\neq j} \frac{u^k - u^j}{u^j - \lambda}\,\frac{\eta_{jk}}{\eta_k}\,e_k. \tag{1.17}$$

Together with (1.11) and (1.12) this gives for $i \neq j$:

$$\check{\nabla}_{\partial/\partial\lambda}\check{\nabla}_{e_i}(e_j) = \frac{1}{2}\,\frac{\eta_{ij}}{\eta_i}\left[\frac{1}{2}\,\frac{1}{u^i - \lambda}\,e_i + \frac{1}{2}\sum_{k\neq i} \frac{u^k - u^i}{u^i - \lambda}\,\frac{\eta_{ik}}{\eta_k}\,e_k\right] + (i \leftrightarrow j), \tag{1.18}$$

$$\check{\nabla}_{e_i}\check{\nabla}_{\partial/\partial\lambda}(e_j) = \frac{1}{2}\,\frac{1}{u^j - \lambda}\left(\frac{1}{2}\frac{\eta_{ij}}{\eta_i}e_i + \frac{1}{2}\frac{\eta_{ij}}{\eta_j}e_j\right) +$$

$$+\frac{1}{2}\sum_{k\neq j} e_i\left(\frac{u^k - u^j}{u^j - \lambda}\,\frac{\eta_{jk}}{\eta_k}\right) e_k + \frac{1}{2}\sum_{k\neq j,i} \frac{u^k - u^j}{u^j - \lambda}\,\frac{\eta_{jk}}{\eta_k}\left(\frac{1}{2}\,\frac{\eta_{ik}}{\eta_i}\,e_i + \frac{1}{2}\,\frac{\eta_{ik}}{\eta_k}\,e_k\right) +$$

$$\frac{1}{2}\,\frac{u^i - u^j}{u^j - \lambda}\,\frac{\eta_{ij}}{\eta_i}\left[\frac{1}{2}\left(\frac{\eta_{ii}}{\eta_i} - \frac{1}{u^i - \lambda}\right)e_i - \frac{1}{2}\sum_{k\neq i} \frac{u^k - \lambda}{u^i - \lambda}\,\frac{\eta_{ik}}{\eta_k}\,e_k\right]. \tag{1.19}$$

The coincidence of coefficients of e_k in (1.18) and (1.19) for $i \neq j \neq k \neq i$ can be checked with the help of the following identity which is equivalent to the Darboux–Egoroff equation Ch. I, (3.14):

$$\eta_{ijk} = \frac{1}{2}\left(\frac{\eta_{ik}\eta_{jk}}{\eta_k} + \frac{\eta_{ij}\eta_{ik}}{\eta_i} + \frac{\eta_{ij}\eta_{jk}}{\eta_j}\right).$$

The coincidence of the coefficients of e_i requires a little more work, and we will give some details, again for the case $i \neq j$.

In (1.18) the coefficient of e_i is

$$\frac{1}{4}\frac{1}{u^i-\lambda}\frac{\eta_{ij}}{\eta_i}+\frac{1}{4}\frac{u^i-u^j}{u^j-\lambda}\frac{\eta_{ij}^2}{\eta_i\eta_j}, \tag{1.20}$$

whereas in (1.19) we get

$$\frac{1}{4}\frac{1}{u^j-\lambda}\frac{\eta_{ij}}{\eta_i}+\frac{1}{2}e_i\left(\frac{u^i-u^j}{u^j-\lambda}\frac{\eta_{ij}}{\eta_i}\right)+$$

$$+\frac{1}{4}\sum_{k\neq i,j}\frac{u^k-u^j}{u^j-\lambda}\frac{\eta_{ik}\eta_{jk}}{\eta_i\eta_k}+\frac{1}{2}\left(\frac{\eta_{ii}}{\eta_i}-\frac{1}{u^i-\lambda}\right)\frac{u^i-u^j}{u^i-\lambda}\frac{\eta_{ij}}{\eta_i}. \tag{1.21}$$

To identify (1.20) and (1.21) we have to get rid of the sum $\sum_k$ in (1.21). This can be done with the help of Ch. I, (3.14), (3.21) and (3.22):

$$\frac{1}{4}\sum_{k\neq i,j}\frac{u^k-u^j}{u^j-\lambda}\frac{\eta_{ik}\eta_{jk}}{\eta_i\eta_k}=\frac{1}{u^j-\lambda}\frac{\eta_j^{1/2}}{\eta_i^{1/2}}\left[\sum_{k\neq i,j}u^k\gamma_{ik}\gamma_{kj}-u^j\sum_{k\neq i,j}\gamma_{ik}\gamma_{kj}\right]=$$

$$=\frac{1}{u^j-\lambda}\frac{\eta_j^{1/2}}{\eta_i^{1/2}}\left[-\gamma_{ij}-u^ie_i\gamma_{ij}-u^je_j\gamma_{ij}+u^j(e_i+e_j)\gamma_{ij}\right]=$$

$$=\frac{1}{2}\frac{1}{u^j-\lambda}\left[-\frac{\eta_{ij}}{\eta_i}+(u^j-u^i)\left(\frac{\eta_{iij}}{\eta_i}-\frac{1}{2}\frac{\eta_{ij}\eta_{ii}}{\eta_i^2}-\frac{1}{2}\frac{\eta_{ij}^2}{\eta_i\eta_j}\right)\right].$$

The remaining part of the calculation is straightforward, and we leave it to the reader, as well as the case $i=j$ which is treated similarly.

1.3. Formal Laplace transform. Assume now that $\mathcal{T}_M^f$ is a trivial local system. This means that if we put $T:=\Gamma(M,\mathcal{T}_M^f)$, there is a natural isomorphism $\mathcal{O}_M\otimes T\to\mathcal{T}_M$.

Formulas (1.8) (resp. (1.10)) define two families of connections with singularities on the trivial vector bundle on $\mathbf{P}^1_\lambda$ with fiber T, parametrized by M. Namely, denote by ∂_λ the covariant derivative along $\partial/\partial\lambda$ on this bundle for which the constant sections are horizontal. Then the two connections are

$$\check{\nabla}_{\partial/\partial\lambda}=\partial_\lambda+\left(\mathcal{V}+\frac{1}{2}\mathrm{Id}\right)(\mathcal{U}-\lambda)^{-1}, \tag{1.22}$$

$$\hat{\nabla}_{\partial/\partial\lambda}=\partial_\lambda+\mathcal{U}+\frac{1}{\lambda}\left(\mathcal{V}+\frac{D}{2}\mathrm{Id}\right). \tag{1.23}$$

Let M,N be two $\mathbf{C}[\lambda,\partial_\lambda]$–modules. *A formal Laplace transform* $M\to N:\ Y\mapsto Y^t$ is a $\mathbf{C}$–linear map for which

$$(-\lambda Y)^t=\partial_\lambda(Y^t),\ (\partial_\lambda Y)^t=\lambda Y^t. \tag{1.24}$$

The archetypal Laplace transform is the Laplace integral

$$Y^t(\mu) = \int e^{-\lambda\mu} Y(\lambda) d\lambda \tag{1.25}$$

taken along a contour (not necessarily closed) in $\mathbf{P}^1(\mathbf{C})$. In an analytical setting we have to secure the convergence of (1.25), the possibility to derivate under the integral sign and the identity

$$\int \partial_\lambda (e^{-\lambda\mu} Y(\lambda)) d\lambda = 0.$$

However, (1.25) may admit other interpretations, for instance, in terms of asymptotic series.

Let now M (resp. N) be two $\mathbf{C}[\lambda, \partial_\lambda]$-modules of local (or formal, or distribution) sections of $\mathbf{P}^1_\lambda \times T$ so that the operators $\check{\nabla} \cdot (\mathcal{U} - \lambda)$ (resp. $\lambda \widehat{\nabla}$) make sense in M (resp. N) (cf. (1.22), resp. (1.23)), and assume that we are given a formal Laplace transform $M \to N$.

1.3.1. Proposition. *We have:*

$$[\check{\nabla}_{\partial/\partial\lambda}((\mathcal{U} - \lambda)Y)]^t = (\lambda \widehat{\nabla}_{\partial/\partial\lambda} + \frac{D-1}{2})\, Y^t = \lambda^{\frac{3-D}{2}} \widehat{\nabla}_{\partial/\partial\lambda} (\lambda^{\frac{D-1}{2}} Y^t).$$

In particular, $\lambda^{\frac{D-1}{2}} Y^t$ *is* $\widehat{\nabla}$*-horizontal, if* $(\mathcal{U} - \lambda)Y$ *is* $\check{\nabla}$*-horizontal.*

Proof. Using (1.22)–(1.24), we find:

$$[\check{\nabla}_{\partial/\partial\lambda}((\mathcal{U} - \lambda)Y)]^t = \left[(\partial_\lambda \cdot (\mathcal{U} - \lambda) + \mathcal{V} + \frac{1}{2}\,\mathrm{Id})Y\right]^t =$$

$$= \left[\lambda\,(\mathcal{U} + \partial_\lambda) + \mathcal{V} + \frac{1}{2}\,\mathrm{Id}\right] Y^t =$$

$$= \left[\lambda \widehat{\nabla}_{\partial/\partial\lambda} + \frac{D-1}{2}\,\mathrm{Id}\right] Y^t = \lambda^{\frac{3-D}{2}} \widehat{\nabla}_{\partial/\partial\lambda} (\lambda^{\frac{D-1}{2}} Y^t).$$

Now we will more systematically review the deformation picture.

§2. Isomonodromic deformations

2.1. Singularities of meromorphic connections. Let N be a complex manifold, $D \subset N$ a closed complex submanifold of codimension one, $\mathcal{F}$ a locally free sheaf of finite rank on N. A meromorphic connection with singularities on D is given by a covariant differential $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_N((r+1)D)$ for some $r \geq 0$. It is called flat (or integrable) if it is flat outside D. We start with a list of elementary notions and constructions that will be needed later. They depend only on the local behavior of $\mathcal{F}$ and ∇ in a neighborhood of D, so we will assume D irreducible.

i) Order of singularity. We will say that ∇ as above is of order $\leq r+1$ on D if $\nabla_X(\mathcal{F}) \subset \mathcal{F}(rD)$ for any vector field X tangent to D (i. e. satisfying $XJ_D \subset J_D$ where J_D is the ideal of D), and $\nabla_X(\mathcal{F}) \subset \mathcal{F}((r+1)D)$ in general. Locally, if $(t^0, t^1, \dots, t^n)$ is a coordinate system on N such that $t^0 = 0$ is the equation of D, the connection matrix of ∇ in a basis of $\mathcal{F}$ can be written as

$$G_0 \frac{dt^0}{(t^0)^{r+1}} + \sum_{i=1}^{n} G_i \frac{dt^i}{(t^0)^r} \tag{2.1}$$

where $G_i = G_i(t^0, t^1, \dots, t^n)$ are holomorphic matrix functions.

ii) Restriction to a transversal submanifold. Let $i : N' \to N$ be a closed embedding of a submanifold transversal to D, $D' = N' \cap D$, $\mathcal{F}' = i^*(\mathcal{F})$. Then the induced connection $\nabla' = i^*(\nabla)$ on $\mathcal{F}'$ is flat and of order $\leq r+1$ on D' if ∇ has these properties.

iii) Residual connection. Assume now that ∇ is of order ≤ 1 on D. Then one can define a connection without singularities ∇^D on $j^*(\mathcal{F})$ where j is the embedding of D in N. Namely, to define $\nabla^D_{X'}(s')$ where $s' \in j^*(\mathcal{F})$, $X' \in \mathcal{T}_D$, we extend locally s' to a section s of $\mathcal{F}$, X' to a vector field X on N, calculate $\nabla_X(s)$ and restrict it to D. One checks that the result does not depend on the choices made. In the notation of (2.1), the matrix of the residual connection can be written as $(r = 0)$:

$$\sum_{i=1}^{n} G_i(0, t^1, \dots, t^n) dt^i. \tag{2.2}$$

If ∇ is flat, ∇^D is flat.

iv) Principal part of order $r+1$. Similarly to (2.2), we can consider the matrix function on D

$$G_0(0, t^1, \dots, t^n) \tag{2.3}$$

which we will call the principal part of order $r+1$ of ∇. In more invariant terms, it is the $\mathcal{O}_D$-linear map $j^*(\mathcal{F}) \to j^*(\mathcal{F})$ induced by $\mathcal{F} \to j^*(\mathcal{F}) : s \mapsto (t^0)^{r+1} \nabla_{\partial/\partial t^0}(s)\,|_D$. It is well defined, but depends on the choice of local coordinates, and is multiplied by an invertible local function on D when this choice is changed. Hence its spectrum is well defined globally on D.

v) Tameness and resonance. Two general position conditions are important in the study of meromorphic singularities of order $\leq r+1$.

If $r \geq 1$ (irregular case), the singularity is called *tame*, if the spectrum of its principal part at any point of D is simple.

If $r = 0$ (regular case), the singularity is called *non-resonant*, if it is tame and moreover, the difference of any two eigenvalues never takes an integer value on D.

2.1.1. Example: the structure connections. As in 1.3, we will assume that $\mathcal{T}_M^f$ is trivial, and its fibers are identified with the space T of global flat vector fields.

Put $N = M \times \mathbf{P}^1_\lambda$, $\mathcal{F} = \mathcal{O}_N \otimes T$. We can apply the previous considerations to $\widehat{\nabla}$ and $\check{\nabla}$.

Analysis of $\widehat{\nabla}$. Clearly, $\widehat{\nabla}$ has singularity of order 1 at $\lambda = 0$ (i. e. on $D_0 = M \times \{0\}$) and of order 2 at $\lambda = \infty$ (i. e. on $D_\infty = M \times \{\infty\}$): cf. (1.9) and (1.10). Restricting $\widehat{\nabla}$ to $\{y\} \times \mathbf{P}^1_\lambda$ for various $y \in M$ we get a family of meromorphic connections on $\mathbf{P}^1_\lambda$ parametrized by M.

The residual connection is defined on $D_0 = M$ and it coincides with the Levi–Civita connection of g. The principal part of order 1 on D_0 is $\mathcal{V} + \frac{D}{2}\,\mathrm{Id}$. The eigenvalues of this operator do not depend on $y \in D_0$: in Ch. I, 2.4 they were denoted (d_a). Their description for the case of quantum cohomology (see Ch. I, (4.12)) shows that in this case the principal part *is always resonant.*

The principal part of order 2 on $D_\infty = M$ is (proportional to) $\mathcal{U}$ (cf. (1.10), use the local equation $\mu = \lambda^{-1} = 0$ for D_∞.) Its eigenvalues now depend on $y \in M$: they are just the canonical coordinates $u^i(y)$. We will call the point y *tame* if $u^i(y) \neq u^j(y)$ for $i \neq j$. We will call M tame, if all its points are tame. Every M contains the maximum tame subset which is open and dense.

Analysis of $\check{\nabla}$. According to (1.7), (1.8), $\check{\nabla}$ has singularities of order 1 at the divisors $\lambda = u^i$ and $\lambda = \infty$. These divisors do not intersect pairwise iff M is tame. The principal part of order 1 at $\lambda = u^i$ is $-(\mathcal{V} + \frac{1}{2}\,\mathrm{Id}) \cdot (e_i \circ)$.

The residual connection of $\check{\nabla}$ on $\lambda = \infty$ is again the Levi–Civita connection ∇_0 of g. In fact, using (1.15) we find

$$\check{\nabla} = d\lambda \check{\nabla}_{\partial/\partial\lambda} + \sum_i du^i \check{\nabla}_{e_i} =$$

$$= d\lambda \check{\nabla}_{\partial/\partial\lambda} + \sum_i du^i [\nabla_{0,e_i} - (\mathcal{V} + \frac{1}{2}\,\mathrm{Id})\,(\mathcal{U} - \lambda)^{-1}(e_i \circ)].$$

Replacing λ by the local parameter $\mu = \lambda^{-1}$ at infinity, we have

$$\check{\nabla} = d\mu \check{\nabla}_{\partial/\partial\mu} + \sum_i du^i [\nabla_{0,e_i} - \mu\,(\mathcal{V} + \frac{1}{2}\,\mathrm{Id})\,(\mu\,\mathcal{U} - \mathrm{Id})^{-1}(e_i \circ)]$$

so that the expression (2.2) (with $(\mu, u^1, \dots, u^m)$ in lieu of $(t^0, t^1, \dots, t^n)$) becomes $\sum_i du^i \nabla_{0,e_i} = \nabla_0$.

2.2. Versal deformation. We will now review the basic results on the deformation of meromorphic connections on $\mathbf{P}^1_\lambda$, restricting ourselves to the case of singularities of order ≤ 2. This suffices for applications to both structure connections, on the other hand, this is precisely the case treated in full detail by B. Malgrange in [Mal4], Theorem 3.1. It says that the positions of finite poles and the spectra of the principal parts of order 2 form coordinates on the coarse moduli space with tame singularities. To be more precise, one has to rigidify the data slightly.

Let ∇^0 be a meromorphic connection on a locally free sheaf $\mathcal{F}^0$ on $\mathbf{P}^1_\lambda$ of rank p, with $m+1 \geq 2$ tame singularities (including $\lambda = \infty$) of order ≤ 2. Call *the rigidity* for ∇^0 the following data:

a). A numbering of singular points: $a_0^1, \dots, a_0^m$, $a^{m+1} = \infty$.

b). The subset $I \subset \{1, \dots, m+1\}$ such that a_0^j is of order 2 exactly when $j \in I$.

c). For each $j \in I$, a numbering $(b_0^{j1}, \dots, b_0^{jp})$ of the eigenvalues of the principal part at a_0^j.

Construct the space $B = B(m, p, S)$ as the universal covering of

$$(\mathbf{C}^m \setminus \text{diagonals}) \times \prod_{j \in I} (\mathbf{C}^p \setminus \text{diagonals})$$

with the base point $(a_0^i; b_0^{jk})$, let $b_0 \in B$ be its lift. We denote by a^i, b^{jk} the coordinate functions lifted to B. Let $i : \mathbf{P}^1_\lambda \to B \times \mathbf{P}^1_\lambda$ be the embedding $\lambda \mapsto (b_0, \lambda)$, and D_j the divisor $\lambda = a^j$ in $B \times \mathbf{P}^1_\lambda$.

2.2.1. Theorem ([Mal4], Th. 3.1). *For a given $(\nabla^0, \mathcal{F}^0)$ with rigidity, there exists a locally free sheaf $\mathcal{F}$ of rank p on $\mathbf{P}^1_\lambda \times B$, a flat meromorphic connection ∇ on it, and an isomorphism $i^0 : i^*(\mathcal{F}, \nabla) \to (\mathcal{F}^0, \nabla^0)$ with the following properties:*

D_j, $j = 1, \dots, m+1$, are all the poles of ∇, of order 1 (resp. 2) if $j \neq I$ (resp. $j \in I$.) If $j \in I$, then $(b^{j1}, \dots, b^{jp})$ (as functions on D_j) form the spectrum of the principal part of order 2 of ∇ at D_j.

It follows that the restrictions of ∇ to the fibers $\{b\} \times \mathbf{P}^1_\lambda$ are endowed with the induced rigidity, and i^0 is compatible with it.

The data $(\mathcal{F}, \nabla, i^0)$ are unique up to unique isomorphism.

2.2.2. Comments on the proof. a). The case when all singularities are of order 1 is easier. It is treated separately in [Mal3], Th. 2.1; for the thorough study of this case and the treatment of the Gauss–Manin connections see [De1]. Since the second structure connection satisfies this condition, we sketch Malgrange's argument in this case.

Choose base points $a \in U := \mathbf{P}^1_\lambda \setminus \cup_{j=1}^{m+1} \{a_0^j\}$ and $(b_0, a) \in B \times \mathbf{P}^1_\lambda$. Notice that (b_0, a) belongs to $V := B \times \mathbf{P}^1_\lambda \setminus \cup_{j=1}^m D_j$.

The restriction of $(\mathcal{F}^0, \nabla^0)$ to U is determined uniquely up to unique isomorphism by the monodromy action of $\pi_1(U, a)$ on the space F, the geometric fiber $\mathcal{F}^0(a)$ at a, which can be arbitrary. Similarly, there is a bijection between flat connections $(\mathcal{F}, \nabla)$ on V with fixed identification $\mathcal{F}^0(a) \to \mathcal{F}(a) = F$ and actions of $\pi_1(V, (a, b))$ on F. Hence to construct an extension $(\mathcal{F}, \nabla)$ to V together with an isomorphism of its restriction to U with $(\mathcal{F}^0, \nabla^0)$, it suffices to check that i induces an isomorphism $\pi_1(U, a) \to \pi_1(V, (a, b))$, which follows from the homotopy exact sequence and the fact that B is contractible.

This argument explains the term "isomonodromic deformation."

Next, we must extend $(\mathcal{F}, \nabla)$ to $B \times \mathbf{P}^1_\lambda$. It suffices to do this separately in a tubular neighborhood of each D_j disjoint from other D_k. The coordinate change $\lambda \mapsto \lambda - a^j$ (or $\lambda \mapsto \lambda^{-1}$) allows us to assume that the equation of D_j is $\lambda = 0$. Take a neighborhood W of 0 in which $\mathcal{F}^0$ can be trivialized, describe ∇^0 by its connection matrix, lift $(\mathcal{F}^0, \nabla^0)$ to $B \times W$ and restrict to a tubular neighborhood of D_j. On the complement to D_j, this lifting can be canonically identified with $(\mathcal{F}, \nabla)$ through their horizontal sections. Clearly, it is of order ≤ 1 at D_j.

It remains to establish that any two extensions are canonically isomorphic. Outside singularities, an isomorphism exists and is unique. An additional argument which we omit shows that it extends holomorphically to $B \times \mathbf{P}^1_\lambda$.

b). When ∇ admits singularity of order 2, this argument must be completed. The extension of $(\mathcal{F}^0, \nabla^0)$ first to V and then to the singular divisors of order ≤ 1 can be done exactly as before. But both the existence and the uniqueness of the extension to the irregular singularities requires an additional local analysis in order to show that the simple spectrum of the principal polar part determines the singularity. When formulated in terms of the asymptotic behaviour of horizontal sections, this analysis introduces the Stokes data as a version of irregular monodromy, which also proves to be deformation invariant.

2.3. The theta divisor and Schlesinger's equations. In this subsection we will assume that $\mathcal{F}^0 = T \otimes \mathcal{O}_{\mathbf{P}^1_\lambda}$ where T is a finite dimensional vector space which can be identified with the space of global sections of $\mathcal{F}^0$. This is the case of the two structure connections, when the local system $\mathcal{T}^f_M$ is trivial.

Then there exists a divisor Θ, eventually empty, such that the restriction of $\mathcal{F}$ to all fibers $\{b\} \times \mathbf{P}^1_\lambda$, $b \notin \Theta$, is free. This can be proved using the fact that a locally free sheaf $\mathcal{E}$ on $\mathbf{P}^1$ is free iff $H^0(\mathbf{P}^1, \mathcal{E}(-1)) = H^1(\mathbf{P}^1, \mathcal{E}(-1)) = 0$, and that the cohomology of fibers is semi-continuous. For an analytic treatment, see [Mal4], sec. 4 and 5.

Moreover, assume that $\lambda = \infty$ is a singularity of order 1 (to achieve this for the first structure connection, we must replace λ by λ^{-1}.) Then we can identify the inverse image of $\mathcal{F}$ on $B \setminus \Theta \times \mathbf{P}^1_\lambda$ with $T \otimes \mathcal{O}_{B\setminus\Theta\times\mathbf{P}^1_\lambda}$ compatibly with the respective trivialization of $\mathcal{F}^0$. To this end trivialize $\mathcal{F}$ along $\lambda = \infty$ using the residual connection (see 2.1 iii))and then take the constant extension of each residually horizontal section along $\mathbf{P}^1_\lambda$. (If there are no poles of order 1, one can extend this argument using a different version of the residual connection, see [Mal4], p.430, Remarque 1.4.)

Using this trivialization, we can define a meromorphic integrable connection ∂ on $\mathcal{F}$ with the space of horizontal sections T on $B \setminus \Theta \times \mathbf{P}^1_\lambda$. As sections of $\mathcal{F}$, they develop a singularity at Θ. Therefore, the respective connection form $\nabla - \partial$ is a meromorphic matrix one–form with eventual pole at Θ.

The following classical result clarifies the structure of this form in the case *when all poles of* ∇ *are of order 1.*

2.3.1. Theorem. *a). Let $(a^1, \dots, a^m)$ be the functions on B describing the λ-coordinates of finite poles of ∇ (with given rigidity.) Then*

$$\nabla = \partial + \sum_{i=1}^{m} A_i(a^1, \dots, a^m) \frac{d(\lambda - a^i)}{\lambda - a^i} \tag{2.4}$$

where A_i are meromorphic functions $B \to \mathrm{End}\,(T)$ which can be considered as multivalued meromorphic functions of a_i.

b). The connection (2.4) is flat iff A_i satisfy the Schlesinger equations

$$\forall j, \qquad dA_j = \sum_{i \neq j} [A_i, A_j] \frac{d(a^i - a^j)}{a^i - a^j}. \tag{2.5}$$

c). Fix a tame point $a_0 = (a_0^1, \dots, a_0^m)$. Then arbitrary initial conditions $A_i^0 = A_i(a_0)$ define a solution of (2.5) holomorphic on $B \setminus \Theta$, with eventual pole at Θ of order 1.

d). For any such solution ∇ of (2.5), define the meromorphic 1-form on B:

$$\omega_\nabla := \sum_{i<j} \mathrm{Tr}\,(A_i A_j) \frac{d(a^i - a^j)}{a^i - a^j}. \tag{2.6}$$

This form is closed, and for any local equation $t = 0$ of Θ the form $\omega_\nabla - \frac{dt}{t}$ is locally holomorphic.

2.3.2. Corollary. *For any solution ∇ to (2.5), there exists a holomorphic function τ_∇ on B such that $\omega_\nabla = d \log \tau_\nabla$. It is defined uniquely up to a multiplication by a constant.*

In fact, B is simply connected.

For a proof of Theorem 2.3.1, we refer to [Mal3]: a), b), and c) are proved on pp. 406–410, d) on pp. 420–425.

2.4. Hamiltonian structure of Schlesinger's equations. The equations (2.5) can be written in Hamiltonian form, with m times and m time-dependent Hamiltonians.

To be more precise, let X be a manifold with a Poisson structure given by the Poisson bracket $\{\, ,\}$, S a manifold with a coordinate system $(t^1, \dots, t^m)$, $(\mathcal{H}_1, \dots, \mathcal{H}_m)$ a family of functions on $X \times S$ called Hamiltonians. Extend the bracket to $X \times S$ fiberwise. Then we can define m flows on X such that the evolution of any function F is governed by the equations:

$$\frac{\partial F}{\partial t^j} = \{\mathcal{H}_j, F\}. \tag{2.7}$$

These flows commute iff

$$\forall j, k: \qquad \{\mathcal{H}_j, \mathcal{H}_k\} = \frac{\partial \mathcal{H}_k}{\partial t^j} - \frac{\partial \mathcal{H}_j}{\partial t^k}. \tag{2.8}$$

To represent (2.5) in this form, we choose $X = (\mathrm{End}\, T)^m$, $S = B$. The Poisson structure will be the product of m standard Poisson structures on the matrix spaces. If we choose a basis in T and identify $\mathrm{End}\, T$ with the space of matrices $(A_{\alpha\beta})$, the bracket of two matrix elements is

$$\{A_{\alpha\beta}, A_{\gamma\delta}\} = \delta_{\beta\gamma} A_{\alpha\delta} - \delta_{\alpha\delta} A_{\gamma\beta}. \tag{2.9}$$

(I apologize for using the subscript δ in the Kronecker delta symbol.)

Finally, put:

$$\mathcal{H}_j = - \sum_{i:\, i \neq j} \frac{\mathrm{Tr}\,(A_i A_j)}{a^i - a^j}. \tag{2.10}$$

2.4.1. Theorem. *Schlesinger's equations (2.5) are equivalent to the equations*

$$\forall\, i, j, \alpha, \beta : \qquad \frac{\partial A_{j\alpha\beta}}{\partial a^i} = \{\mathcal{H}_i, A_{j\alpha\beta}\}. \tag{2.11}$$

The flows (2.11) pairwise commute.

Proof. Rewrite (2.5) as

$$\frac{\partial A_{j\alpha\beta}}{\partial a^j} = - \sum_{i:\, i \neq j} \frac{[A_i, A_j]_{\alpha\beta}}{a^i - a^j}, \tag{2.12}$$

$$\frac{\partial A_{j\alpha\beta}}{\partial a^i} = \frac{[A_i, A_j]_{\alpha\beta}}{a^i - a^j}, \quad i \neq j. \tag{2.13}$$

On the other hand, in view of (2.10),

$$\{\mathcal{H}_j, A_{j\alpha\beta}\} = - \sum_{i:\, i \neq j} \frac{\{\mathrm{Tr}\,(A_i A_j), A_{j\alpha\beta}\}}{a^i - a^j}, \tag{2.14}$$

$$\{\mathcal{H}_i, A_{j\alpha\beta}\} = \frac{\{\mathrm{Tr}\,(A_i A_j), A_{j\alpha\beta}\}}{a^i - a^j}, \quad i \neq j. \tag{2.15}$$

(Notice that the matrix elements of A_j and A_k pairwise Poisson commute if $j \neq k$.) A straightforward calculation using (2.9) then shows that (2.12) (resp. (2.13)) coincides with (2.14) (resp. (2.15).)

The fact that (2.11) commute means that the trajectories of the flows starting at one point are all contained in a multisection of p which is equivalent to the flatness of ∇ and to (2.5).

§3. Semisimple Frobenius manifolds as special solutions to the Schlesinger equations

3.1. Special solutions. Slightly generalizing (2.5), we will call *a solution to Schlesinger's equations* any data $(M,(u^i),T,(A_i))$ where M is a complex manifold of dimension $m\geq 2$; $(u^1,\dots,u^m)$ a system of holomorphic functions on M such that du^i freely generate Ω^1_M and for any $i\neq j$, $x\in M$, we have $u^i(x)\neq u^j(x)$; T a finite dimensional complex vector space; $A_j : M\to \operatorname{End} T$, $j=1,\dots,m$, a family of holomorphic matrix functions such that

$$\forall j:\qquad dA_j=\sum_{i:\,i\neq j}[A_i,A_j]\,\frac{d(u^i-u^j)}{u^i-u^j}. \tag{3.1}$$

Let such a solution be given. Summing (3.1) over all j, we find $d(\sum_j A_j)=0$. Hence $\sum_j A_j$ is a constant matrix function; denote its value by $\mathcal{W}$.

3.1.1. Definition. *A solution to Schlesinger's equations as above is called special, if* $\dim T=m=\dim M$*; T is endowed with a complex nondegenerate quadratic form g; $\mathcal{W}=-\mathcal{V}-\frac{1}{2}\,\mathrm{Id}$, where $\mathcal{V}\in\operatorname{End}T$ is a skew symmetric operator with respect to g, and finally*

$$\forall j:\qquad A_j=-(\mathcal{V}+\frac{1}{2}\,\mathrm{Id})P_j \tag{3.2}$$

where $P_j : M\to\operatorname{End}T$ is a family of holomorphic matrix functions whose values at any point of M constitute a complete system of orthogonal projectors of rank one with respect to g:

$$P_iP_k=\delta_{ik}P_i,\quad \sum_{i=1}^{m}P_i=\mathrm{Id}_T,\quad g(\operatorname{Im}P_i,\operatorname{Im}P_j)=0 \tag{3.3}$$

if $i\neq j$. Moreover, we require that A_j do not vanish at any point of M.

3.1.2. Comment. We commited a slight abuse of language: the notion of special solution involves a choice of additional data, the metric g. However, when it is chosen, the rest of the data is defined unambiguously if it exists at all.

In fact, assume that $A_j=\mathcal{W}P_j$ as above do not vanish anywhere. Then they have constant rank one. Hence at any point of M we have

$$\operatorname{Ker}A_j=\operatorname{Ker}\mathcal{W}P_j=\operatorname{Ker}P_j=\oplus_{i:\,i\neq j}\operatorname{Im}P_i,$$

so that

$$\operatorname{Im}P_i=\cap_{j:\,j\neq i}\oplus_{k:\,k\neq j}\operatorname{Im}P_k=\cap_{j:\,j\neq i}\operatorname{Ker}A_j.$$

This means that P_j can exist for given A_j only if the spaces $\mathcal{T}_j=\cap_{i:\,i\neq j}\operatorname{Ker}A_j$ are one–dimensional and pairwise orthogonal at any point of M.

Conversely, assume that this condition is satisfied. Define P_j as the orthogonal projector onto $\mathcal{T}_j$. Then $A_iP_j=0$ for $i\neq j$ because $\mathcal{T}_j=\operatorname{Im}P_j\subset\operatorname{Ker}A_i$. Hence

$$A_j=A_j(\sum_{i=1}^{m}P_i)=A_jP_j=(\sum_{i=1}^{m}A_i)P_j=\mathcal{W}P_j.$$

Notice that all A_j are conjugate to diag $(-\frac{1}{2}, 0, \dots, 0)$ and satisfy $A_j^2 + \frac{1}{2} A_j = 0$. These conditions, as well as $\sum_j A_j = -(\mathcal{V} + \frac{1}{2}\,\mathrm{Id})$, are compatible with the equations (3.1) and so must be checked at one point only.

3.2. From Frobenius manifolds to special solutions. Given a semisimple Frobenius manifold with flat identity and an Euler field E with $d_0 = 1$, we can produce a special solution to Schlesinger's equations rephrasing the results of the previous two sections.

Namely, we first pass to a covering M of the subspace of tame points of the initial manifold such that $\mathcal{T}_M^f$ is trivial and a global splitting can be chosen, represented by the canonical coordinates (u^i). Then we put $T = \Gamma(M, \mathcal{T}_M^f)$ and $A_i =$ the coefficients of the second structure connection written as in (2.4).

Since this connection is flat, $(M, (u^i), T, (A_i))$ form a solution of (3.1).

Moreover, this solution is special. In fact, T comes equipped with the metric g. The operator A_i is the principal part of order 1 of $\check{\nabla}$ at $\lambda = u^i$ which is of the form (3.2), with $P_j = e_j \circ$.

Finally, this special solution comes with one more piece of data, the identity $e \in T$. We will axiomatize its properties in the following definition.

3.2.1. Definition. *Consider a special solution to Schlesinger's equations as in the Definition 3.1.1. A vector $e \in T$ is called an identity of weight D for this solution, if*

a). $\mathcal{V}(e) = (1 - \frac{D}{2})\, e$.

b). $e_j := P_j(e)$ do not vanish at any point of M.

For Frobenius manifolds with $d_0 = 1$, a) is satisfied by Ch. I, (2.16) and (1.2).

3.3. From special solutions to Frobenius manifolds. Let $(M, (u^i), T, g, (A_i))$ be a special solution, and $e \in T$ an identity of weight D for it.

3.3.1. Theorem. *If $D \neq 1$, these data come from the unique structure of semisimple split Frobenius manifold on M, with flat identity and Euler field, as it was described in 3.2.*

Remark. I do not know whether the restriction $D \neq 1$ can be removed. (This is the case $d_1 = 0$ in I.4.2.) For quantum cohomology, this excludes only the case of $\mathbf{P}^1$. Very interesting Frobenius manifolds with $D = 1$ are constructed in [D2], Appendix C. They are related to the universal elliptic curve and the Chazy equation, and show that the Painlevé property *in flat coordinates* can fail.

Proof. Proceeding as in 3.2, but in the reverse direction, we are bound to make the following choices.

Put $e_j = P_j(e) \subset \mathcal{O}_M \otimes T$, $j = 1, \dots, m$. Identify $\mathcal{O}_M \otimes T$ with $\mathcal{T}_M$ by setting $e_j = \partial/\partial u^j$. Transfer the metric g from T to $\mathcal{T}_M$. Define the multiplication on $\mathcal{T}_M$ for which $e_i \circ e_j = \delta_{ij} e_j$. Put $\eta_i := g(e_i, e_i)$.

We get a structure of semisimple pre–Frobenius manifold in the sense of Ch. I, Definition 3.2.

To establish that it is Frobenius, it suffices to prove that $e_i\eta_j = e_j\eta_i$ for all i, j: see Ch. I, Theorem 3.3.

We have $\eta_j = g(e, e_j)$. Therefore

$$g(e, A_j(e)) = -g(e, (\mathcal{V} + \frac{1}{2}\,\mathrm{Id})\,P_j e) = g(\mathcal{V}e, e_j) - \frac{1}{2}\,g(e, e_j) = \frac{1-D}{2}\,\eta_j \qquad (3.4)$$

since $\mathcal{V}$ is skewsymmetric, and e is an eigenvector of $\mathcal{V}$. Furthermore, let ∇ be the Levi–Civita connection of the flat metric g. Then derivating (3.4) we find for every i, j:

$$\frac{1-D}{2}\,\frac{\partial}{\partial u^i}\,\eta_j = g(\nabla_{e_i}(e), A_j(e)) + g(e, \nabla_{e_i}(A_j(e))) =$$

$$= g(e, \frac{\partial A_j}{\partial u^i}\,(e)), \qquad (3.5)$$

because $e \in T$ so that $\nabla(e) = 0$. If $i \neq j$, we find from (3.1)

$$\frac{\partial A_j}{\partial u^i} = \frac{[A_i, A_j]}{u^i - u^j} = \frac{\partial A_i}{\partial u^j}. \qquad (3.6)$$

This shows that if $D \neq 1$, $e_i\eta_j = e_j\eta_i$.

It remains to check that $E = \sum_i u^i e_i$ is the Euler field. According to the Theorem 3.6 b) of Ch. I, we must prove that $E\eta_j = (D-2)\eta_j$ for all j. Insert (3.6) into (3.5) and sum over $i \neq j$. We obtain:

$$\frac{1-D}{2}\,E\eta_j = \frac{1-D}{2}\sum_{i:\,i\neq j} u^i\,\frac{\partial \eta_j}{\partial u^i} + \frac{1-D}{2}\,u^j\,\frac{\partial \eta_j}{\partial u^j} =$$

$$= \sum_{i:\,i\neq j} g\left(e, u^i\,\frac{[A_i, A_j]}{u^i - u^j}\,(e)\right) + u^j g(e, \frac{\partial A_j}{\partial u^j}\,(e)). \qquad (3.7)$$

From (3.1) it follows that

$$\frac{\partial A_j}{\partial u^j} = -\sum_{i:\,i\neq j}\frac{[A_i, A_j]}{u^i - u^j}. \qquad (3.8)$$

On the other hand,

$$u^i\,\frac{[A_i, A_j]}{u^i - u^j} = [A_i, A_j] + u^j\,\frac{[A_i, A_j]}{u^i - u^j}. \qquad (3.9)$$

Inserting (3.8) and (3.9) into (3.7), we find

$$\frac{1-D}{2}\,E\eta_j = \sum_{i:\,i\neq j} g(e, [A_i, A_j]\,(e)) + u^j\sum_{i:\,i\neq j} g\left(e, \frac{[A_i, A_j]}{u^i - u^j}\,(e)\right) +$$

$$+u^j g(e, \frac{\partial A_j}{\partial u^j}\,(e)) = g(e, [\sum_{i:\,i\neq j} A_i, A_j]\,(e)) =$$

$$= -g(e, [\mathcal{V} + \frac{1}{2}\,\mathrm{Id}, (\mathcal{V} + \frac{1}{2}\,\mathrm{Id})P_j]\,(e)). \tag{3.10}$$

Using the skew symmetry of $\mathcal{V}$, we see that the last expression in (3.10) equals $\frac{1-D}{2}(D-2)\eta_j$. Hence $E\eta_j = (D-2)\eta_j$ if $D \neq 1$.

3.4. Special initial conditions. Theorem 2.3.1 c) shows that arbitrary initial conditions for Schlesinger's equations determine a global meromorphic solution on the universal covering $B(m)$ of $\mathbf{C}^m \setminus \{\text{diagonals}\}, m \geq 2$.

Fix a base point $b_0 \in B(m)$. Studying the special solutions, we may and will identify T with the tangent space at b_0 thus eliminating the gauge freedom. This tangent space is already coordinatized: we have e_i and e.

We will call a family of matrices $A_1^0, \ldots, A_m^0 \in \mathrm{End}\,T$ *special initial conditions* if we can find a diagonal metric g and a skew symmetric operator $\mathcal{V}$ such that $A_j^0 = -(\mathcal{V} + \frac{1}{2}\,\mathrm{Id})\,P_j$ where P_j is the projector onto $\mathbf{C}\,e_j$.

We will describe explicitly the space $I(m)$ of the special initial conditions.

3.4.1. Notation. Let R be any equivalence relation on $\{1, \ldots, m\}$, $|R|$ the number of its classes. Put $F(m) = (\mathrm{End}\,\mathbf{C}^m)^m$, Furthermore, denote $F_R(m)$ the subset of families $(A_1, \ldots, A_m)$ in $F(m)$ such that R coincides with the minimal equivalence relation for which iRj if $\mathrm{Tr}\,A_iA_j \neq 0$, and put $I_R(m) = F_R(m) \cap I(m)$.

3.4.2. Construction. Denote by $\overline{I}(m) \subset \mathbf{C}^m \times \mathbf{C}^{m(m-1)/2}$ the locally closed subset defined by the equations:

$$\sum_{i=1}^{m} \eta_i = 0, \quad \eta_i \neq 0 \quad \text{for all } i; \tag{3.11}$$

$$v_{ij}\eta_j = -v_{ji}\eta_i \quad \text{for all } i, j; \tag{3.12}$$

$$\sum_{i=1}^{m} v_{ij} := 1 - \frac{D}{2} \quad \text{does not depend on } j. \tag{3.13}$$

Each point of $\overline{I}(m)$ determines the diagonal metric $g(e_i, e_i) = \eta_i$ and the operator $\mathcal{V}:\ e_i \mapsto \sum_i v_{ij}e_j$ which is skew symmetric with respect to g and for which e is an eigenvector. Setting $A_i = -(\mathcal{V} + \frac{1}{2}\,\mathrm{Id})\,P_i$ we get a point in $I(m)$.

This amounts to forgetting (η_i) which furnishes the surjective map $\overline{I}(m) \to I(m)$ because

$$A_i(e_j) = 0 \text{ for } i \neq j, \quad A_i(e_i) = -\frac{1}{2}\,e_i - \sum_{j=1}^{m} v_{ij}e_j.$$

3.4.3. Theorem. *a). The space $\overline{I}(m)$ can be realized as a Zariski open dense subset in $\mathbf{C}^{m+(m-1)(m-2)/2}$.*

b). Inverse image in $\overline{I}(m)$ *of any point in* $I_R(m)$ *is a manifold of dimension* 1 *for* $|R| = 1$, $|R| - 1$ *for* $|R| \geq 2$.

Proof. Fixing η_i, we can solve (3.12) and (3.13) explicitly. Put $w_{ij} = v_{ij}\eta_j$ so that $w_{ij} = -w_{ji}$ and (3.13) becomes

$$\forall j: \qquad \sum_{i=1}^{m} w_{ij} = \eta_j(1 - \frac{D}{2}). \tag{3.14}$$

If we choose arbitrarily the values (w_{ij}) for all $1 \leq i < j \leq m-1$, we can find w_{mj} from the first $m-1$ equations (3.14), and then the last equations will hold automatically:

$$w_{mk} = \eta_k(1 - \frac{D}{2}) - \sum_{i=1}^{m-1} w_{ik},$$

$$\sum_{i=1}^{m} w_{im} = -\sum_{k=1}^{m} w_{mk} = -\sum_{k=1}^{m-1} \eta_k\,(1 - \frac{D}{2}) + \sum_{i,k=1}^{m-1} w_{ik} = \eta_m\,(1 - \frac{D}{2})$$

because of (3.11).

It remains to determine the fiber of the projection onto $I(m)$.

We have for $i \neq j$: $\operatorname{Tr} A_iA_j = v_{ij}v_{ji}$. Hence in the generic case when all these traces do not vanish, we can reconstruct η_i compatible with given v_{ij} from (3.12) uniquely up to a common factor. Generally, for i, j in the same R–equivalence class, (3.12) allows us to determine the value η_i/η_j so that we have $|R|$ overall arbitrary factors constrained by (3.11).

3.4.4. Question. If we choose a special initial condition for the Schlesinger equation, does the solution remain special at every point?

Generically, the answer is positive. If this is the case, we obtain the action of the braid group Bd_m as the group of deck transformations on the space $I(m)$.

3.5. Analytic continuation of the potential. The picture described in this section gives a good grip on the analytic continuation of a germ of semisimple Frobenius manifold (M_0, m_0) in terms of its canonical coordinates. Namely, construct the universal covering M of the subset of the tame points of M_0, then fix at the point $b_0 = (u^i(m_0)) \in B(m)$ the initial conditions of M at m_0. This provides an open embedding $(M, m_0) \subset (B(m), b_0)$. Loosely speaking, we find in this way a maximal tame analytic continuation of the initial germ.

Now construct some global flat coordinates (x^a) on $B(m)$ corresponding to a given Frobenius structure. They map $B(m)$ to a subdomain in $\mathbf{C}^m$. This is the natural domain of the analytic continuation of the potential Φ of this Frobenius structure, which is the most important object for Quantum Cohomology. Unfortunately, its properties are not clear from this description.

§4. Quantum cohomology of projective spaces

In this section we will apply the developed formalism to the study of the quantum cohomology of projective spaces $\mathbf{P}^r$, $r \geq 2$, first introduced in Ch. I, 4.4. Our main goal is the calculation of the initial conditions of the relevant solutions to the Schlesinger's equations.

4.1. Notation. We start with recalling (and somewhat revising) the basic notation. Put $H = H^*(\mathbf{P}^r, \mathbf{C}) = \sum_{a=0}^{r} \mathbf{C}\Delta_a$, Δ_a = the dual class of $\mathbf{P}^{r-a} \subset \mathbf{P}^r$. Denote the dual coordinates on H by $x_0, \dots, x_r$ (lowering indices for visual convenience), $\partial_a = \partial/\partial x_a$. The Poincaré form is $(g_{ab}) = (g^{ab}) = (\delta_{a+b,r})$. The term $\frac{1}{6}(\gamma^3)$ in I. (4.13) is the cubic self–intersection form, the classical part of the Frobenius potential

$$\Phi_{\mathrm{cl}}(x) := \frac{1}{6} \sum_{a_1+a_2+a_3=r} x_{a_1} x_{a_2} x_{a_3}. \tag{4.1}$$

The remaining part of the potential is the sum of physicists' instanton corrections to the self–intersection form:

$$\Phi_{\mathrm{inst}}(x) := \sum_{d=1}^{\infty} \Phi_d(x_2, \dots, x_r) e^{dx_1}, \tag{4.2}$$

where we will now write Φ_d as

$$\Phi_d(x_2, \dots, x_r) = \sum_{n=2}^{\infty} \sum_{\substack{a_1+\dots+a_n= \\ r(d+1)+d-3+n}} I(d; a_1, \dots, a_n) \frac{x_{a_1} \dots x_{a_n}}{n!}. \tag{4.3}$$

This means that if we assign the weight $a-1$ to x_a, $a = 2, \dots, n$, Φ_d becomes the weighted homogeneous polynomial of weight $(r+1)d+r-3$. Moreover, if we assign to e^{dx_1} the weight $-(r+1)$, Φ_{cl} and Φ become weighted homogeneous formal series of weight $r-3$. (Notice that e in the expressions e^{dx_1} and alike is $2,71828\dots$, whereas in other contexts e means the identity vector field. This cannot lead to confusion.)

The starting point of our study in this section will be the following result.

4.2. Theorem. *a). For each $r \geq 2$, there exists a unique formal solution of the Associativity Equations I. (1.6) of the form*

$$\Phi(x) = \Phi_{\mathrm{cl}}(x) + \Phi_{\mathrm{inst}}(x) \tag{4.4}$$

for which $I(1; r, r) = 1$.

b). This solution has a non-empty convergence domain in H on which it defines the structure of semisimple Frobenius manifold $H_{\mathrm{quant}}(\mathbf{P}^r)$ with flat identity $e = \partial_0$ and Euler field

$$E = \sum_{a=0}^{r} (1-a) x^a \partial_a + (r+1)\partial_1 \tag{4.5}$$

with $d_0 = 1, D = 2 - r$.

c). The coefficient $I(d; a_1, \dots, a_n)$ is the number of rational curves of degree d in $\mathbf{P}^r$ intersecting n projective subspaces of codimensions $a_1, \dots a_n \geq 2$ in general position.

Uniqueness of the formal solution can be established by showing that the Associativity Equations imply recursive relations for the coefficients of Φ which allow one to express all of them through $I(1; r, r)$. This is an elementary exercise for $r = 2$ (cf. Introduction, (0.19).) A more general result (stated in the language of Gromov–Witten invariants but of essentially combinatorial nature) is proved in [KM], Theorem 3.1, and applied to the projective spaces in [KM], Claim 5.2.2.

Existence is a subtler fact. The algebraic geometric (or symplectic) theory of the Gromov–Witten invariants provides the numbers $I(d; a_1, \dots a_n)$ satisfying the necessary relations, together with their numerical interpretation: see [KM], [BM], [FuO]. Another approach consists in calculating *ad hoc* the "special initial conditions" for the semisimple Frobenius manifold $H_{\text{quant}}(\mathbf{P}^r)$ in the sense of the previous section and identifying the appropriate special solution to the Schlesinger equations with this manifold. For $r = 2$, direct estimates of the coefficients showing convergence can be found in [D2], p. 185. Probably, they can be generalized to all r.

Our approach in this section consists in taking Theorem 4.2 for granted and investigating the passage to the Darboux–Egoroff picture as a concrete illustration of the general theory. The net outcome are formulas (4.18) and (4.19) for the special initial conditions.

Conversely, starting with them, we can construct the Frobenius structure on the space $B(r+1)$ as was explained in 3.5 above. Expressing the E–homogeneous flat coordinates $(x_0, \dots, x_r)$ on this space satisfying (4.17) in terms of the canonical coordinates and then calculating the multiplication table of the flat vector fields, we can reconstruct the potential which now will be a germ of holomorphic function of (x_a). Because of the unicity, it must have the Taylor series (4.4). So the Theorem 4.2 a),b) can be proved essentially by reading this section in the reverse order. Of course, the last statement is of different nature.

4.3. Tensor of the third derivatives. Most of our calculations in $(\mathcal{T}, \circ)$ will be restricted to the first infinitesimal neighborhood of the plane $x_2 = \dots = x_r = 0$ in H. This just suffices for the calculation of the Schlesinger initial conditions. We denote by J the ideal $(x_2, \dots, x_r)$.

Multiplication by the identity $e = \partial_0$ is described by the components ${\Phi_{0a}}^b = \delta_{ab}$ of the structure tensor. Of the remaining components, we will need only ${\Phi_{1a}}^b$ which allow us to calculate multiplication by ∂_1, and proceed inductively. This is where the Associativity Equations are implicitly used.

Obviously, ${\Phi_{10}}^b = \delta_{1b}$.

4.3.1. Claim. *We have*

$$\text{for } 1 \leq a \leq r-1: \qquad {\Phi_{1a}}^b = \delta_{a+1,b} + x_{r+1-a+b} e^{x_1} + O(J^2), \tag{4.6}$$

$${\Phi_{1r}}^b = \delta_{b0} e^{x_1} + x_{b+1} e^{x_1} + O(J^2). \tag{4.7}$$

(Here and below we agree that $x_c = 0$ for $c > r$.

Proof. The term $\delta_{a+1,b}$ in (4.6) comes from $\Phi_{\rm cl}$. The remaining terms are provided by the summands of total degree ≤ 3 in $x_2, \dots, x_r$ in

$$\partial_1 \Phi_{\rm inst} = \sum_{d\ge 1} d e^{dx_1} \left(\sum I(d; a_1, a_2) \frac{x_{a_1} x_{a_2}}{2} + \sum I(d; a_1, a_2, a_3) \frac{x_{a_1} x_{a_2} x_{a_3}}{6} \right) + O(J^4).$$

For $n = 2$, the grading condition means that $d = 1$, $a_1 = a_2 = r$. For $n = 3$, it means that $d = 1$, $a_1 + a_2 + a_3 = 2r + 1$. We know that $I(1; r, r) = 1$. Similarly, $I(1; a_1, a_2, a_3) = 1$ in this range. This can be deduced formally from the Associativity Equations. A nice exercise is to check that this agrees also with the geometric description (for instance, only one line intersects two given generic lines and passes through a given point in the three space.) So finally

$$\partial_1 \Phi_{\rm inst} = \left(\frac{x_r^2}{2} + \frac{1}{6} \sum_{a_1+a_2+a_3=2r+1} x_{a_1} x_{a_2} x_{a_3} \right) e^{x_1} + O(J^4).$$

The term $\delta_{b0} e^{x_1}$ in (4.7) comes from $\dfrac{x_r^2}{2}$. Furthermore,

$$\Phi_{{\rm inst};1ab} = x_{2r+1-a-b} e^{x_1} + O(J^2)$$

and

$$\Phi_{{\rm inst};1a}{}^{b} = \Phi_{{\rm inst};1,a,r-b} = x_{r+1-a+b} e^{x_1} + O(J^2).$$

4.4. Multiplication table. The main formula of this subsection is

$$\partial_1^{\circ(r+1)} = e^{x_1} \left(\partial_0 + \sum_{b=1}^{r-1} (b+1) x_{b+1} \partial_b \right) + O(J^2). \tag{4.8}$$

We will prove it by consecutively calculating the powers $\partial_1^{\circ a}$. The intermediate results will also be used later. (Notice that $O(J^2)$ in (4.8) now means $O(\sum_i J^2 \partial_i)$.)

First, we find from (4.6) and (4.7) for $1 \le a \le r-1$:

$$\partial_1 \circ \partial_a = \sum_{b=0}^{r} \Phi_{1a}{}^{b} \partial_b = \partial_{a+1} + e^{x_1} \sum_{b=0}^{a-1} x_{r+1-a+b} \partial_b + O(J^2), \tag{4.9}$$

$$\partial_1 \circ \partial_r = \sum_{b=0}^{r} \Phi_{1r}{}^{b} \partial_b = e^{x_1} \left(\partial_0 + \sum_{b=1}^{r-1} x_{b+1} \partial_b \right) + O(J^2). \tag{4.10}$$

Then using (4.9) and induction, we obtain

$$\text{for } 1 \le a \le r: \qquad \partial_1^{\circ a} = \partial_a + e^{x_1} \sum_{b=0}^{a-2} (b+1) x_{r+2-a+b} \partial_b + O(J^2). \tag{4.11}$$

Multiplying this formula for $a = r$ by ∂_1 and using (4.10), we finally find (4.8).

From (4.11) it follows that $\partial_1^{\circ a}$ for $0 \le a \le r$ freely span the tangent sheaf.

4.5. Idempotents. Formula (4.8) allows us to calculate all $e_i \bmod J^2$ thus demonstrating semisimplicity. Namely, denote by q the $(r+1)-$th root of the right hand side of (4.8) congruent to $e^{\frac{x_1}{r+1}} \bmod J$ and put $\zeta = \exp\left(\frac{2\pi i}{r+1}\right)$. Then

$$e_i = \frac{1}{r+1}\sum_{j=0}^{r} \zeta^{-ij}(\partial_1 \circ q^{-1})^{\circ j} \tag{4.12}$$

satisfy

$$e_i \circ e_j = \delta_{ij} e_i, \ \sum_i e_i = \partial_0$$

for all $i = 0, \dots r$. A straightforward check shows this.

4.5.1. Proposition. *We have*

$$e_i = \frac{1}{r+1}\sum_{j=0}^{r} \zeta^{-ij} e^{-x_1 \frac{j}{r+1}} \left(e^{x_1} \sum_{b=0}^{j-2} \frac{(b+1-j)(r+1-j)}{r+1} x_{r+b+2-j}\partial_b + \right.$$

$$\left. + \partial_j - \sum_{b=j+1}^{r} \frac{(b+1-j)j}{r+1} x_{b+1-j}\partial_b \right) + O(J^2). \tag{4.13}$$

Proof. We have

$$q^{-1} = e^{-\frac{x_1}{r+1}} \left(\partial_0 - \sum_{b=1}^{r-1} \frac{b+1}{r+1} x_{b+1}\partial_b \right) + O(J^2).$$

Together with (4.9) this gives

$$\partial_1 \circ q^{-1} = e^{-\frac{x_1}{r+1}} \left(\partial_1 - \sum_{b=1}^{r-1} \frac{b+1}{r+1} x_{b+1}\partial_{b+1} \right) + O(J^2).$$

Hence

$$(\partial_1 \circ q^{-1})^j = e^{-\frac{jx_1}{r+1}} \left(\partial_1^{\circ j} - j\partial_1^{\circ(j-1)} \circ \sum_{b=1}^{r-1} \frac{b+1}{r+1} x_{b+1}\partial_{b+1} \right) + O(J^2).$$

Inserting this into (4.12) and using (4.9)–(4.11) once again, we finally obtain (4.13).

4.6. Metric coefficients in canonical coordinates. The metric potential η is simply x^r (see Ch. I, (2.4).) Hence we can easily calculate $\eta_i = e_i x_r$. The answer is

$$\eta_i = \frac{\zeta^i}{r+1} e^{-x_1 \frac{r}{r+1}} - \sum_{b=2}^{r} \frac{\zeta^{ib}}{(r+1)^2} b(r+1-b)\, e^{-x_1 \frac{r+1-b}{r+1}} x_b + O(J^2). \tag{4.14}$$

As an exercise, the reader can check that the same answer results from the (longer) calculation of $\eta_i = g(e_i, e_i)$.

4.7. Derivatives of the metric coefficients. We now see that the chosen precision just suffices to calculate the restriction of η_{ij}, γ_{ij} and the matrix elements of A_j to the plane $x_2 = \cdots = x_r = 0$ any point of which can be taken as initial one.

4.7.1. Claim. *We have*

$$\eta_{ki} = e_k \eta_i = -2 \frac{\zeta^{i-k}}{(\zeta^{i-k}-1)^2} \frac{e^{-x_1}}{(r+1)^2} + O(J). \tag{4.15}$$

Notice that (4.15) is symmetric in i, k as it should be.

This is obtained by a straightforward calculation from (4.13) and (4.14). The numerical coefficient in (4.15) comes as a combination of $\sum_{j=1}^{r} j\zeta^j$ and $\sum_{j=1}^{r} j^2\zeta^j$ which are then summed by standard tricks.

4.8. Canonical coordinates. We find u^i from the formula $E \circ e_i = u^i e_i$. To calculate $E \circ e_i$, use (4.5), (4.13) and (4.9)–(4.11). We omit the details. The result is:

4.8.1. Claim. *We have*

$$u^i = x_0 + \zeta^i (r+1) e^{\frac{x_1}{r+1}} + \sum_{a=2}^{r} \zeta^{ai} e^{\frac{a x_1}{r+1}} x_a + O(J^2). \tag{4.16}$$

The reader can check that $e_i u^j = \delta_{ij} + O(j)$.

4.8. Schlesinger's initial conditions. Recall that the matrix residues A_i of Schlesinger's equations for Frobenius manifolds are

$$A_j(e_i) = 0 \text{ for } i \neq j,$$

$$A_j(e_j) = -\frac{1}{2} e_j - \frac{1}{2} \sum_k (u^k - u^j) \frac{\eta_{jk}}{\eta_k} e_k \tag{4.17}$$

(cf (1.13).) Substituting here (4.14), (4.15) and (4.16), we finally get the main result of this section.

4.8.1. Theorem. *The point* $(x_0, x_1, 0, \dots, 0)$ *has canonical coordinates* $u^i = x_0 + \zeta^i (r+1) e^{\frac{x_1}{r+1}}$.

The special initial conditions at this point (in the sense of 3.4) corresponding to $H_{\text{quant}}(\mathbf{P}^r)$ *are given by*

$$v_{jk} = -\frac{\zeta^{j-k}}{1-\zeta^{j-k}} \tag{4.18}$$

and

$$\eta_i = \frac{\zeta^i}{r+1} e^{-x_1 \frac{r}{r+1}}. \tag{4.19}$$

As an exercise, the reader can check that

$$-\sum_{k:\, k \neq j} \frac{\zeta^{j-k}}{1-\zeta^{j-k}} = 1 - \frac{D}{2} = \frac{r}{2}.$$

§5. Dimension three and Painlevé VI

The equations for the potential Φ or metric potential η generally form a system of PDE. However, in the three-dimensional semisimple case, in the presence of a flat identity and an Euler field, they can be effectively reduced to one nonlinear ODE belonging to the family Painlevé VI. This section contains some details of this study.

5.1. Normalization. i) *Spectrum and normalized flat coordinates.* We start along the lines of Ch. I, 4.2, but with some additional assumptions; see [D2], pp. 127–129 for the general case.

Let M be a connected simply connected Frobenius manifold with flat identity and Euler field with $d_0 = 1$. The most important spectrum point is D.

From the start, *we will exclude from consideration two of the critical values of* D. Namely, we will assume $D \neq 1$ in order to be able to use in the semisimple case Theorem 3.3.1, and $D \neq 2$ which guarantees that the spectrum of $-\mathrm{ad}\, E$ on $\mathcal{T}_M^f$ is simple.

In fact, in the notation of Ch. I, 2.4, this spectrum must be of the form $(d_0, d_1, d_2) = (1, \frac{D}{2}, D-1)$, where the eigenvector for $d_0 = 1$ is $\partial_0 = e$, $g(e,e) = 0$; the eigenvector for $d_2 = D-1$ is uniquely normalized by the condition $g(e, \partial_2) = 1$; and the one for $\frac{D}{2}$ is uniquely up to sign normalized by $g(\partial_1, \partial_1) = 1$. Thus $(g_{ab}) = (g^{ab}) = (\delta_{a+b,2})$.

We can now consider three flat coordinates (x_0, x_1, x_2) such that $\partial_a = \partial/\partial x_a$ defined up to a shift (and sign change for x_1.) Their final normalization will depend on the Euler field.

The spectrum of $\mathcal{V} = -\mathrm{ad}\, E - \frac{D}{2}\mathrm{Id}$ is $\left(1 - \frac{D}{2}, 0, \frac{D}{2} - 1\right)$.

ii) *Euler field and normalized potential.* If $D \neq 0, 1$, then all d_a do not vanish, and we can choose x_a so that

$$D \neq 0: \qquad E = x_0\partial_0 + \frac{D}{2} x_1 \partial_1 + (D-1)\, x_2 \partial_2. \tag{5.1}$$

(Notice that the origin $(x_a) = (0)$ cannot be tame semisimple because E vanishes there.)

For $D = 0$ we obtain an extra parameter (cf. Ch. I, 2.4) which we denote $r+1$ to conform with (4.5); x_1 remains defined only up to a sign change and shift:

$$D = 0: \qquad E = x_0\partial_0 + (r+1)\,\partial_1 - x_2\partial_2. \tag{5.2}$$

We will assume $r + 1 \neq 0$; then the sign can be normalized by $\mathrm{Re}\,(r+1) > 0$.

The potential can be written in the form (Ch. I, (2.3)):

$$\Phi(x_0, x_1, x_2) = \frac{1}{2}\,(x_0 x_1^2 + x_0^2 x_2) + \varphi(x_1, x_2).$$

It is defined up to a quadratic polynomial in (x_a) and must satisfy $E\Phi = (D+1)\Phi + q$, where q is also a quadratic polynomial. We can try to make $q = 0$ by replacing Φ with $\Phi + p$ and solving $(E - 1 - D)p = q$. If $D \neq 0$ and $D \neq -1$, such p exists and is unique. If $D = -1$, we cannot kill a possible constant term c in q which is a new parameter. If $D = 0$, we can unambiguously kill any quadratic polynomial in (x_1, x_2) but the term containing x_0 will remain. So our final normalization is:

$$D \neq \pm 1, 2: \qquad E\varphi = (D+1)\,\varphi, \tag{5.3}$$

$$D = -1: \qquad E\varphi = c.$$

iii) *Associativity Equations.* A straightforward check shows that all the Associativity Equations follow from one of them, which can be written as

$$\varphi_{222} = \varphi_{112}^2 - \varphi_{111}\varphi_{122}. \tag{5.4}$$

In [D2], p.128, equations (5.3) and (5.4) are reduced to an ODE for the function f which is defined in the following way.

If $D \neq 0, \pm 1, 2$, put $\delta = \dfrac{2}{D} - 2$. Then (5.3) means that locally φ can be written as $x_1^4 x_2^{-1} f(x_2 x_1^\delta)$.

If $D = -1$, we can put similarly $\varphi = 2c \log x_1 + f(x_2 x_1^{-4})$.

If $D = 0$, we have $\varphi = x_2^{-1} f(x_1 + (r+1) \log x_2)$. We will copy Dubrovin's equation for f in this case:

$$f'''[(r+1)^3 + 2f' - (r+1)f''] - f''^2 - 6(r+1)^2 f'' + 11(r+1)\,f' - 6f = 0. \tag{5.5}$$

The case $D = 0$ is the most interesting for us because it includes the quantum cohomolgy of $\mathbf{P}^2$. It is not easy to recognize in (5.5) a classical equation. Below we will describe how Dubrovin uses the additional semisimplicity condition in order to reduce it to PVI.

5.2. Semisimplicity and tameness. At a tame semisimple point of M, the operator $E\circ$ has simple spectrum (canonical coordinates of this point.) Conversely, if this is true, one can write down explicitly the idempotents e_i as polynomials in E. This criterium is sufficiently practical for use in flat coordinates.

5.3. Analyticity. Consider now the case when Φ is analytic at the origin. (Recall that if $D = 0$, the origin can be any point along the x_1–axis, so its choice is the same as the choice of x_1.)

5.3.1. Proposition. *a). The origin can be tame semisimple only if $D = 0$. In this case the normalized analytic potential can be written in the form*

$$\Phi(x_0, x_1, x_2) = \frac{1}{2}\,(x_0 x_1^2 + x_0^2 x_2) + \sum_{n=0}^{\infty} \frac{M(n)}{n!}\, e^{\frac{n+1}{r+1} x_1} x_2^n \tag{5.6}$$

so that $E\Phi = \Phi + (r+1)\,x_0 x_1$.

b). The Associativity Equations are equivalent to the following recursive relations for the coefficients $M(n)$*:*

$$M(n+3) = \frac{1}{(r+1)^4} \sum_{\substack{k+l=n \\ k,l\geq 0}} \binom{n}{k} \left[M(k+1)M(l+1)(k+2)^2(l+2)^2 - \right.$$

$$\left. -M(k)M(l+2)(k+1)^3(l+3) \right]. \tag{5.7}$$

Hence any formal solution is uniquely defined by the choice of $M(0), M(1), M(2)$ *which can be arbitrary.*

c). The point (000) *is tame semisimple iff the polynomial*

$$u^3 - \frac{M(0)}{(r+1)^3} u^2 - \frac{8M(1)}{(r+1)^2} u - \frac{3M(2)}{r+1}$$

has no multiple roots.

For the quantum cohomology of $\mathbf{P}^2$, we have $r+1 = 3$, $M(n) = 0$ unless $n = 3d-1$, and $M(2) = 1$. If we put $N(d) := M(3d-1)$, (5.7) becomes (0.19).

Proof. a). As we have already remarked, (000) cannot be tame semisimple with E of the form (5.1) since E vanishes at this point. One easily sees that for $D = 0$, (5.6) is normalized.

b). This is a restatement of (5.4).

c). We will use the criterium of 5.2. From (5.2) one sees that one can look at the spectrum of $\partial_1 \circ$ in lieu of $E\circ$. The multiplication table at the origin is

$$\partial_1 \circ \partial_0 = \partial_1,$$

$$\partial_1 \circ \partial_1 = \frac{4M(1)}{(r+1)^2} \partial_0 + \frac{M(0)}{(r+1)^3} \partial_1 + \partial_2,$$

$$\partial_1 \circ \partial_2 = \frac{3M(2)}{r+1} \partial_0 + \frac{4M(1)}{(r+1)^2} \partial_1.$$

Hence

$$\det(\partial_1 \circ -u\,\mathrm{Id}) = -u^3 + \frac{M(0)}{(r+1)^3} u^2 + \frac{8M(1)}{(r+1)^2} u + \frac{3M(2)}{r+1}.$$

This finishes the proof.

5.3.2. Exercises. a). Calculate formal (at the origin) potentials for $D \neq 0$.

b). Calculate the special Schlesinger's initial conditions at the origin for the potential (5.6).

5.4. Introduction to the PVI equations. These equations form a family $\mathrm{PVI}_{\alpha,\beta,\gamma,\delta}$ depending on four parameters $\alpha, \beta, \gamma, \delta$, and classically written as:

$$\frac{d^2X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt} +$$

$$+\frac{X(X-1)(X-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{X^2}+\gamma\frac{t-1}{(X-1)^2}+\delta\frac{t(t-1)}{(X-t)^2}\right]. \tag{5.8}$$

They were discovered around 1906 and have been approached from at least three different directions.

a. Study of non-linear ordinary differential equations of the second order whose solutions have no movable critical points.

Their classification program was initiated by Painlevé, but he inadvertently omitted (5.8) due to an error in calculations. It was B. Gambier [G] who completed Painlevé's list and found (5.8).

b. Study of the isomonodromic deformations of linear differential equations.

c. Theory of abelian integrals depending on parameters and taken over chains with boundary (not necessarily cycles.)

These two approaches are due to R. Fuchs [F].

In the subsequent development of the theory, relationship with isomonodromic deformations proved to be most fruitful. Briefly speaking, (5.8) can be obtained by a change of variables from Schlesinger's equations with four singular points and the two-dimensional space T. This description can be used in order to connect PVI to the three dimensional Frobenius manifolds. For some recent research and bibliography the reader may consult [JM], [O1], [H1], [H2].

In this section we take up the somewhat neglected approach via abelian integrals and algebraic geometry.

The main outcome of this approach is the representation of (5.8) as an equation on the multisection of an (arbitrary nonconstant) pencil of elliptic curves with marked sections of order two. In particular, passing to the classical uniformization, we will find the following equivalent form of (5.8):

5.4.1. Theorem. *The equation (5.8) is equivalent to*

$$\frac{d^2z}{d\tau^2}=\frac{1}{(2\pi i)^2}\sum_{j=0}^{3}\alpha_j\wp_z(z+\frac{T_j}{2},\tau) \tag{5.9}$$

where

$$(\alpha_0,\dots,\alpha_3):=(\alpha,-\beta,\gamma,\frac{1}{2}-\delta), \tag{5.10}$$

$(T_0,\dots,T_3)=(0,1,\tau,1+\tau)$, *and* $\wp(z,\tau)$ *is the Weierstrass function.*

5.4.2. Theorem. *Any potential of the form (5.6) can be expressed through a solution to (5.9) with* $(\alpha_0,\dots,\alpha_3)=(\frac{1}{2},0,0,0)$ *that is,*

$$\frac{d^2z}{d\tau^2}=-\frac{1}{8\pi^2}\wp_z(z,\tau) \tag{5.11}$$

In particular, the solution corresponding to $\mathbf{P}^2$ *passes through a point of order three on an elliptic curve with complex multiplication by cubic root of unity.*

Below we will give a more detailed version and a proof of both theorems.

The last result gives exact meaning to the statement "mirror of $\mathbf{P}^2$ is a pencil of elliptic curves with marked sections of order two and an additional multisection." It is conceivable that a similar picture will emerge for all homogeneous and toric Fano manifolds and for all Fano complete intersections in them.

An intriguing question about the analytic nature of the particular solution corresponding to $\mathbf{P}^2$ remains open. There are theorems saying that solutions of (5.8) are generically "new" transcendents. There are also many examples of the particular solutions reducible to more classical functions, like hypergeometric ones.

5.5. Painlevé equations and elliptic pencils. We start with the following classical result.

5.5.1. Theorem (R. Fuchs, 1907). *The equation (5.8) can be written in the form*

$$t(1-t)\left[t(1-t)\frac{d^2}{dt^2}+(1-2t)\frac{d}{dt}-\frac{1}{4}\right]\int_{\infty}^{(X,Y)}\frac{dx}{\sqrt{x(x-1)(x-t)}}=$$

$$=\alpha Y+\beta\frac{tY}{X^2}+\gamma\frac{(t-1)Y}{(X-1)^2}+(\delta-\frac{1}{2})\frac{t(t-1)Y}{(X-t)^2} \qquad (5.12)$$

where $Y^2=X(X-1)(X-t)$.

Proof. First, let us clarify the meaning of (5.12). Consider the family of elliptic curves $E\to B$ parametrized by $t\in\mathbf{P}^1\setminus\{0,1,\infty\}:=B$: the curve E_t is the projective closure of $Y^2=X(X-1)(X-t)$. Points at infinity of $\{E_t\}$ form a section D_0 of this family which is the zero section for the standard group law on fibers. Choose in $E_t(\mathbf{C})$ a path from $D_0(t)$ to the point $(X(t),Y(t))$ of a local section. The operator

$$L_t:=t(1-t)\frac{d^2}{dt^2}+(1-2t)\frac{d}{dt}-\frac{1}{4} \qquad (5.13)$$

annihilates the periods $\int\frac{dx}{y}$ along closed paths in $E_t(\mathbf{C})$ because

$$\left[t(1-t)\frac{\partial^2}{\partial t^2}+(1-2t)\frac{\partial}{\partial t}-\frac{1}{4}\right]\frac{d_{E/B}x}{y}=\frac{1}{2}d_{E/B}\frac{y}{(x-t)^2} \qquad (5.14)$$

where we put $\frac{\partial}{\partial t}(x)=0$ and $d_{E/B}t=0$. Applying L_t to $\int_{\infty}^{(X,Y)}\frac{dx}{y}$ we get $\frac{1}{2}\frac{y}{(x-t)^2}\Big|_{\infty}^{(X,Y)}$ plus the contribution of the boundary sections which together with the right hand side of (5.12) amounts to (5.8).

5.5.2. μ–equations. The equation (5.12) is an instance of the general construction which was used in [Ma1] to prove the functional Mordell conjecture. We will briefly describe it now.

A *μ-equation* is a system of non-linear PDE in which independent variables are (local) coordinates on a manifold B and unknown functions are represented by a section s of a family of abelian varieties (or complex tori) $\pi : A \to B$. To write this system explicitly, assume B small enough so that $\pi_*(\Omega^1_{A/B})$ and $\mathcal{D}_B$ (sheaf of differential operators on B) are $\mathcal{O}_B$-free, and make the following choices:

a. An $\mathcal{O}_B$-basis of vertical 1-forms $\omega_1, \dots, \omega_n \in \Gamma(B, \pi_*(\Omega^1_{A/B}))$.

b. A system of generators of the $\mathcal{D}_B$-module of the Picard–Fuchs equations

$$\sum_{i=1}^{n} L_i^{(j)} \int_\gamma \omega_i = 0, \quad j = 1, \dots, N, \tag{5.15}$$

where γ runs over families of closed paths in the fibers spanning $H_1(B_t)$.

c. A family of meromorphic functions $\Phi^{(j)}$, $j = 1, \dots, N$ on A.

The respective μ-equation for a local (multi)-section $s: B \to A$ reads then

$$\sum_{i=1}^{n} L_i^{(j)} \int_0^s \omega_i = s^*(\Phi^{(j)}), \quad j = 1, \dots, N, \tag{5.16}$$

where 0 denotes the zero section.

One drawback of (5.16) is its dependence on arbitrary choices. Clearly, this can be reduced by taking account of the transformation rules with respect to the changes of various generators. For elliptic pencils, the result takes a neat form.

Let again $E \to B$ be a non-constant one-dimensional family of elliptic curves. We temporarily keep the assumption that $\pi_*(\Omega^1_{E/B})$ and the tangent sheaf $\mathcal{T}_B$ are free. For any symbol of order two $\sigma \in S^2(\mathcal{T}_B)$ and any generator ω of $\pi_*(\Omega^1_{E/B})$ denote by $L_{\sigma,\omega}$ the Picard–Fuchs operator on B with the symbol σ annihilating all periods of ω.

5.5.3. Lemma. *For any local section s, the expression $L_{\sigma,\omega} \int_0^s \omega$ is $\mathcal{O}_B$-bilinear in σ and ω.*

Proof. Obviously,

$$L_{f\sigma,\omega} = f L_{\sigma,\omega}, \quad L_{\sigma,g\omega} = g L_{\sigma,\omega} \circ g^{-1},$$

where f, g are functions on B. The lemma follows.

Thus the expression

$$\mu(s) := \left(L_{\sigma,\omega} \int_0^s \omega \right) \otimes \sigma^{-1} \otimes \omega^{-1} \in S^2(\Omega^1_B) \otimes (\pi_* \Omega^1_{E/B})^{-1} \tag{5.17}$$

depends only on s and is compatible with restrictions to open subsets of B. This means that the natural domain of the right hand sides for elliptic μ-equations is the set of meromorphic sections Φ of the sheaf $\pi^* \left[S^2(\Omega^1_B) \otimes (\pi_* \Omega^1_{E/B})^{-1} \right]$.

Notice that the Kodaira–Spencer isomorphism (and eventually a choice of the theta–characteristic of B) allows us to identify Φ with a meromorphic section of $(\Omega^1_{E/B})^3$ or $\pi^*(\Omega^1_B)^{3/2}$ as well.

We will now lift the Fuchs–Painlevé equation (5.12) to the classical covering space, which in particular will make transparent the nature of its right hand side.

5.5.4. Uniformization. Consider the family of elliptic curves parametrized by the upper half–plane H: $E_\tau := \mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau) \mapsto \tau \in H$. Recall that

$$\wp(z,\tau) := \frac{1}{z^2} + \sum{}' \left(\frac{1}{(z+m\tau+n)^2} - \frac{1}{(m\tau+n)^2} \right), \tag{5.18}$$

$$\wp_z(z,\tau) = -2\sum \frac{1}{(z+m\tau+n)^3}. \tag{5.19}$$

We have

$$\wp_z(z,\tau)^2 = 4(\wp(z,\tau)-e_1(\tau))(\wp(z,\tau)-e_2(\tau))(\wp(z,\tau)-e_3(\tau)) \tag{5.20}$$

where

$$e_i(\tau) = \wp(\frac{T_i}{2},\tau),\ (T_0,\dots,T_3) = (0,1,\tau,1+\tau) \tag{5.21}$$

and $e_1+e_2+e_3=0$. Functions $\wp$ and $\wp_z$ are invariant with respect to the shifts $\mathbf{Z}^2 : (z,\tau) \mapsto (z+m\tau+n,\tau)$ and behave in the following way under the full modular group Γ :

$$\wp\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right) = (ct+d)^2 \wp(z,\tau), \tag{5.22}$$

$$\wp_z\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right) = (ct+d)^3 \wp_z(z,\tau). \tag{5.23}$$

Consider now the morphism of families $\varphi :\ \{E_\tau\} \to \{E_t\}$ induced by

$$(z,\tau) \mapsto \left(X = \frac{\wp(z,\tau)-e_1}{e_2-e_1}, Y = \frac{\wp_z(z,\tau)}{2(e_2-e_1)^{3/2}}, t = \frac{e_3-e_1}{e_2-e_1} \right). \tag{5.24}$$

This is a Galois covering with the group $\Gamma(2) \ltimes \mathbf{Z}^2$. We have

$$\varphi^*\left(\frac{d_{E/B}X}{Y} \right) = 2(e_2-e_1)^{1/2} d_{E/H} z. \tag{5.25}$$

In the future formulas of this type we will omit φ^* and denote differentials over a base B by $d_\downarrow$. For instance, $d_\downarrow\left(\frac{z}{c\tau+d}\right) = \frac{d_\downarrow z}{c\tau+d}$, whereas $d\left(\frac{z}{c\tau+d}\right) = \frac{dz}{c\tau+d} - \frac{czd\tau}{(c\tau+d)^2}$.

It follows from (5.25) that if we denote by γ_1 (resp. γ_2) the image of [0,1] (resp. [0,1]τ) in $\{E_t\}$, then

$$\int_{\gamma_1} \frac{d_\downarrow X}{Y} = 2(e_2-e_1)^{1/2}, \quad \int_{\gamma_2} \frac{d_\downarrow X}{Y} = 2\tau(e_2-e_1)^{1/2} \tag{5.26}$$

so that the operator L_t from (5.13) annihilates periods (5.26) as functions of τ.

We can now prove Theorem 5.4.1 in the following form:

5.5.5. Claim. *The lift of (5.12) to the (z,τ)-space $\mathbf{C}\times H$ is (5.9).*

Proof. Following the lead of 5.5.3, we will directly calculate the μ-equation for $\{E_\tau\}$, choosing $\omega = d_\downarrow z$ (instead of $d_\downarrow X/Y$) and $\sigma = \frac{d^2}{d\tau^2}$ (instead of $t^2(1-t)^2\frac{d^2}{dt^2}$.) Since periods of $d_\downarrow z$ are generated by 1 and τ, the relevant Picard–Fuchs operator is simply $\frac{d^2}{d\tau^2}$. From the Lemma 5.5.3 and (5.26) it follows that

$$t(1-t)L_t\circ 2(e_2-e_1)^{1/2} = Z(\tau)\frac{d^2}{d\tau^2}.$$

Using (5.24) and comparing symbols, we see that

$$Z(\tau) = 2\left(\frac{e_3-e_1}{e_2-e_1}\right)^2\left(\frac{e_3-e_2}{e_2-e_1}\right)^2\frac{(e_2-e_1)^4}{9(e_1e_2'-e_2e_1')^2}(e_2-e_1)^{1/2} =$$

$$= \frac{2}{9}\frac{\prod_{i>j}(e_i-e_j)^2}{(e_1e_2'-e_2e_1')^2}(e_2-e_1)^{-3/2}. \tag{5.27}$$

Since $e_1+e_2+e_3=0$, we can replace $(e_1e_2'-e_2e_1')^2$ by $(e_ie_j'-e_je_i')^2$ for any $i\neq j$. It follows that

$$C := \frac{\prod_{i>j}(e_i-e_j)^2}{(e_1e_2'-e_2e_1')^2}$$

is a modular function for the full modular group without zeroes and poles, hence a constant. A calculation with theta-functions, here omitted, shows that $C=-9\pi^2$, so that finally

$$t(1-t)L_t\int_\infty^{(X(t),Y(t))}\frac{d_\downarrow x}{y} = -2\pi^2(e_2-e_1)^{-3/2}\frac{d^2}{d\tau^2}\int_0^{z(\tau)}d_\downarrow z \tag{5.28}$$

for the respective sections. We can now consecutively compare the summands in the right hand side of (5.8) with those in (5.9). The first summand gives

$$\alpha Y = \frac{\alpha}{2}(e_2-e_1)^{-3/2}\wp_z(z,\tau).$$

For the remaining ones we have to use the addition formulas

$$\wp_z(z+\frac{T_i}{2},\tau) = -\frac{(e_i-e_j)(e_i-e_k)}{(\wp(z,\tau)-e_i)^2}\wp_z(z,\tau),\quad \{i,j,k\}=\{1,2,3\},$$

so that, say, for $i=3$ we get

$$(\delta-\frac{1}{2})\frac{t(t-1)Y}{(X-t)^2} = (\delta-\frac{1}{2})\frac{(e_3-e_1)(e_3-e_2)}{(e_2-e_1)^2}\cdot\frac{\wp_z(z,\tau)}{2(e_2-e_1)^{3/2}}\cdot\frac{(e_2-e_1)^2}{(\wp(z,\tau)-e_3)^2} =$$

$$= -\frac{1}{2}\,(\delta-\frac{1}{2})\,(e_2-e_1)^{-3/2}\cdot\frac{-(e_3-e_1)(e_3-e_2)}{(\wp(z,\tau)-e_3)^2}\wp_z(z,\tau) =$$

$$= -\frac{1}{2}(\delta - \frac{1}{2})(e_2 - e_1)^{-3/2}\wp_z(z + \frac{1+\tau}{2}, \tau).$$

The remaining two summands are treated similarly. This finishes the proof.

In [Ma5] Theorem 5.4.1 was used in order to give an algebraic geometric description of the Painlevé VI equations and of their Hamiltonian structure.

5.5.6. S_4–symmetry and the Landin transform. As an application of (5.9) we will construct some natural transformations of PVI.

a. *The classical S_4–symmetry.* Isomorphisms of elliptic pencils with marked sections of order two (E, D_i) which do not conserve the labelling of D_i induce transformations of PVI permuting α_i. In the form (5.9), they act on the solutions as compositions of the transformations of two types: $(z,\tau) \mapsto \left(\frac{z}{cz+\tau}, \frac{a\tau+b}{c\tau+d}\right)$ indexed by cosets $\Gamma/\Gamma(2)$, and $(z,\tau) \mapsto (z + \frac{T_i}{2}, \tau)$ shifting the zero section.

b. *The Landin transform.* From (5.19) one easily deduces Landin's identity

$$\wp_z(z, \frac{\tau}{2}) = -2\left[\sum \frac{1}{(z + 2m\frac{\tau}{2} + n)^3} + \sum \frac{1}{(z + \frac{\tau}{2} + 2m\frac{\tau}{2} + n)^3}\right] =$$

$$= \wp_z(z,\tau) + \wp_z(z + \frac{\tau}{2}, \tau).$$

Hence if $z(\tau)$ is a solution to PVI with parameters $(\alpha_0, \alpha_1, \alpha_0, \alpha_1)$, we have

$$\frac{d^2 z(\tau)}{d\tau^2} = \alpha_0[\wp_z(z,\tau) + \wp_z(z + \frac{\tau}{2}, \tau)] + \alpha_1[\wp_z(z + \frac{1}{2}, \tau) + \wp_z(z + \frac{1+\tau}{2}, \tau)] =$$

$$= \frac{1}{4}\frac{d^2 z(\tau)}{d(\tau/2)^2} = \alpha_0\wp_z(z, \frac{\tau}{2}) + \alpha_1\wp_z(z + \frac{1}{2}, \frac{\tau}{2}),$$

that is, $z(2\tau)$ is a solution to PVI with parameters $(4\alpha_0, 4\alpha_1, 0, 0)$. The converse statement is true as well. In this way we get the following bijections between the sets of solutions to (5.9):

$$(\alpha_0, \alpha_1, \alpha_0, \alpha_1) \leftrightarrow (4\alpha_0, 4\alpha_1, 0, 0) \qquad (5.29)$$

and in particular

$$(\alpha_0, 0, \alpha_0, 0) \leftrightarrow (4\alpha_0, 0, 0, 0). \qquad (5.30)$$

5.5.7. The symmetry group W. Put now $a_i = 2\alpha_i^2, i = 0, \dots, 3$. In [O2], Okamoto found out that the following group W of the transformations of the parameter space (a_i) can be birationally lifted to the group acting on the space of all solutions of all Painlevé VI equations. By definition, W is generated by

a). $(a_i) \mapsto (\varepsilon_i a_i)$, where $\varepsilon_i = \pm 1$.

b). Permutaions of (a_i).

c). $(a_i) \mapsto (a_i + n_i)$, where $n_i \in \mathbf{Z}$ and $\sum_{i=0}^{3} n_i \equiv 0\,(2)$.

This result goes back to Schlesinger who discovered the general discrete symmetries of his equations. It is remarkable however that they act so neatly on a specific reduction represented by PVI. Explicit formulas are quite complicated even for the simplest shift $(a_i) \mapsto (a_i + 2\delta_{i0})$, and composition quickly makes them unmanageable.

5.6. From Frobenius to Painlevé. Following [D2], Appendix E, we will now describe the map which produces a solution to (5.8) for any analytic potential of the form (5.6).

Let $\Phi(x_0, x_1, x_2)$ be the germ of analytic function of the form (5.6), satisfying the Associativity Equations, for which (000) is a tame semisimple point with non zero canonical coordinates, or equivalently, $M(2) \neq 0$. For $a, b = 0, 1, 2$ calculate consecutively the following functions of (x_0, x_1, x_2):

$$G_{ab} := (-1)^a \delta_{a-b,0} \Phi_{ab} + \frac{1}{2}(r+1)\delta_{a+b,1}, \tag{5.31}$$

$$q = \frac{G_{11}G_{22} - G_{12}^2}{G_{22}}, \tag{5.32}$$

$$p = -\frac{G_{11}G_{22}}{G_{12}^3 + G_{02}G_{12}G_{22} - G_{11}G_{12}G_{22} - G_{22}^2 G_{01}}. \tag{5.33}$$

Denote by (u_1, u_2, u_3) the eigenvalues of the operator $E\circ$. Since they are local canonical coordinates, q and p are functions of u_i. Finally, put

$$t = \frac{u_3 - u_1}{u_2 - u_1}, \ X(t) = \frac{q - u_1}{u_2 - u_1}. \tag{5.34}$$

The fact that locally X depends only on t and not on separate u_i follows from the equation $E\Phi = \Phi + (r+1)x_0 x_1$.

5.6.1. Claim. *The function $X(t)$ satisfies the PVI equation (5.8) with parameters*

$$(\alpha, \beta, \gamma, \delta) = (\frac{9}{2}, 0, 0, \frac{1}{2}). \tag{5.35}$$

Moreover, we have

$$\frac{dX}{dt} = \left(2p + \frac{1}{q - u_3}\right) \frac{\prod_{i=1}^3 (q - u_i)}{(u_3 - u_2)(u_3 - u_1)}. \tag{5.36}$$

In fact, Dubrovin in [D2], Appendix E, deduces a more general statement applicable to the case $D \neq 0$ at semisimple points as well. We will restrict ourselves to comparing notation. Our (x_0, x_1, x_2) are Dubrovin's (t^1, t^2, t^3). Our functions G_{ab} correspond to Dubrovin's $g_{\alpha\beta}$ and are calculated with the help of Dubrovin's (3.17), (3.18) and (1.9) (superscripts being lowered with our Poincaré form $(\delta_{a+b,2})$.) Our formulas (5.32) and (5.33) are Dubrovin's (E.8); (5.35) is obtained from the fact that Dubrovin's μ is -1 for $D = 0$. Finally, (5.34) and (5.36) are Dubrovin's (E.16).

Dubrovin also shows how to reconstruct the potential knowing $X(t)$. This involves integration which explains the discrepancy in the numbers of constants (already noticed by the attentive reader.)

5.6.2. Potential for $\mathbf{P}^2$. We can now calculate the Painlevé initial conditions for $\mathbf{P}^2$ at the point $x_a = 0$. According to Theorem 4.8.1, we have (up to renumbering) $(u_1, u_2, u_3) = (3, 3\zeta, 3\zeta^2), \zeta = e^{2\pi i/3}$ at this point. After calculating (5.31), we obtain $q = p = 0$, again at the origin. Then (5.34) and (5.36) give

$$t = \zeta + 1, \; X(\zeta + 1) = \frac{1}{1 - \zeta}, \; X'(t) = \frac{1}{3}. \tag{5.37}$$

Obviously, the elliptic curve $Y^2 = X(X-1)(X-\zeta-1)$ admits complex multiplication by ζ : the q–coordinate can be simply multiplied by ζ. The point $q = 0$ remains invariant, hence it must be of order three on this curve. (I do not see the meaning of the last condition $X'(t) = \frac{1}{3}$.)

It is interesting to remark that the point (5.35) in the parameter space of PVI in a sense also corresponds to the "half period." More precisely, the (a_i)–coordinates of this point are $(a_0, \ldots a_3) = (3, 0, 0, 0)$. By the Schlesinger–Okamoto shift we can reduce this point to $(1, 0, 0, 0)$.

The point $(0, 0, 0, 0)$ corresponds to the equation $d^2z/d\tau^2 = 0$ trivially solvable with two arbitrary constants; all $X(t)$ can be expressed via Weierstrass function. The same is true for the shifted point $(2, 0, 0, 0)$ by Okamoto. The $\mathbf{P}^2$–point lies exactly half–way in between.

CHAPTER III. FORMAL FROBENIUS MANIFOLDS AND MODULI SPACES OF CURVES

§1. Formal Frobenius manifolds and $Comm_\infty$–algebras

In this Chapter we return to the supergeometric setting of Chapter I, §1 (or rather to its formal version.)

1.1. Formal Frobenius manifolds. Let k be a supercommutative $\mathbf{Q}$–algebra, $H = \oplus_a k\Delta_a$ a free ($\mathbf{Z}_2$–graded) k–module of finite rank, $g : H \otimes H \to k$ an even symmetric pairing which is non–degenerate in the sense that it induces an isomorphism $g' : H \to H^t$ where H^t is the dual module.

Denote by $K = k[[H^t]]$ the completed symmetric algebra of H^t. In other words, if $\sum_a x^a \Delta_a$ is a generic even element of H, then K is the algebra of formal series $k[[x^a]]$.

1.1.2. Definition. *The structure of the formal Frobenius manifold on (H, g) is given by an even potential $\Phi \in K$, defined up to quadratic terms, and satisfying the Associativity Equations (1.6).*

In other words, the multiplication law $\Delta_a \circ \Delta_b = \sum_c \Phi_{ab}{}^c \Delta_c$ turns $H_K = K \otimes_k H$ into a supercommutative K-algebra.

1.1.3. Examples. a). If (M, g, Φ_M) is a Frobenius manifold over $k = \mathbf{R}$ or $\mathbf{C}$, x a point of M, put $H = T_{M,x}$ (the tangent superspace at x identified with the space of local flat tangent fields), $\Phi =$ the image of Φ_M in the completion of the local ring $\mathcal{O}_{M,x}$, (x^a) a system of local flat coordinates vanishing at x.

More generally, we can start with a relative Frobenius manifold M/S where S is affine or Stein, and a section $x : S \to M$ with normal sheaf trivialized by the vertical flat vector fields. The completion along this section will be a formal Frobenius manifold over $k = \Gamma(S, \mathcal{O}_S)$.

b). Quantum cohomology, briefly described in Chapter I, 4.4, furnishes many examples of formal Frobenius structures on the cohomology modules ($H = H^*(V, k)$, $g =$ Poincaré pairing), see e.g. potentials (4.13) of projective spaces.

In this section we will show that the Taylor coefficients of a formal potential Φ can be interpreted as a family of multilinear composition laws on H furnishing a beautiful generalization of the usual commutative algebra. Let (H, g) be as in 1.1.

1.2 Definition. *The structure of the cyclic $Comm_\infty$–algebra on (H, g) is a sequence of even polylinear maps $\circ_n : H^{\otimes n} \to H$, $n = 2, 3, \dots$ satisfying the following conditions:*

a). Higher commutativity: $\circ_n$ are S_n–symmetric (in the sense of superalgebra).

We will denote $\circ_n(\gamma_1 \otimes \dots \otimes \gamma_n)$ by $(\gamma_1, \dots, \gamma_n)$.

b). Cyclicity: the tensors

$$Y_{n+1} : H^{\otimes(n+1)} \to k,\ Y_{n+1}(\gamma_1 \otimes \dots \otimes \gamma_n \otimes \gamma_{n+1}) := g((\gamma_1, \dots, \gamma_n), \gamma_{n+1}) \quad (1.1)$$

are S_{n+1}-symmetric.

c). Higher associativity: for all $m \geq 0$ and $\alpha, \beta, \gamma, \delta_1, \dots, \delta_m \in H$, we have

$$\sum_{\sigma: S_1 \coprod S_2 = \{1,\dots,m\}} \varepsilon'(\sigma)((\alpha, \beta, \delta_i \mid i \in S_1), \gamma, \delta_j \mid j \in S_2) =$$

$$\sum_{\sigma: S_1 \coprod S_2 = \{1,\dots,m\}} \varepsilon''(\sigma)(\alpha, (\beta, \gamma, \delta_i \mid i \in S_1), \delta_j \mid j \in S_2). \qquad (1.2)$$

1.2.1. Comments. In (1.2) σ runs over all ordered partitions of $\{1, \dots, m\}$ into two disjoint subsets. The signs $\varepsilon'(\sigma)$, $\varepsilon''(\sigma)$ are defined as follows: fix an initial ordering, say, $(\alpha, \beta, \gamma, \delta_1, \dots, \delta_m)$, then calculate the sign of the permutation induced by σ on the odd arguments in (1.2).

b). For $m = 0$, (1.2) reads

$$((\alpha, \beta), \gamma) = ((\alpha, (\beta, \gamma)), \qquad (1.3)$$

and for $m = 1$

$$((\alpha, \beta), \gamma, \delta) + (-1)^{\tilde{\gamma}\tilde{\delta}}((\alpha, \beta, \delta), \gamma) = ((\alpha, (\beta, \gamma, \delta)) + (\alpha, (\beta, \gamma), \delta).$$

The general combinatorial structure of (1.2) can be memorized as follows: start with (1.3) and distribute $(\delta_1, \dots, \delta_m)$ in all possible ways between the brackets at both sides, without introducing new brackets and retaining the initial ordering inside each bracketed group.

c). The term "cyclic" comes from cyclic cohomology. One could also say that g must be an invariant scalar product with respect to all multiplications: compare (1.1) to Chapter I, (1.2). Choosing $\circ_n = 0$ for all $n \geq 3$, we will get a conventional commutative algebra with invariant scalar product.

1.3. Abstract Correlation Functions. Clearly, given g, $\circ_n$ and Y_{n+1} uniquely determine each other. It will be useful to axiomatize the functional equations between Y_{n+1} which turn out to be equivalent to the higher associativity laws.

1.3.1. Definition. *A system of Abstract Correlation Functions (ACF) on (H,g) is a family of S_n-symmetric even polynomials $Y_n : H^{\otimes n} \to k$, $n = 3, 4, 5, \dots$ satisfying the following coherence relations:*

for all $n \geq 4$, all pairwise distinct $i, j, k, l \in \{1, \dots, n\}$ and all $\gamma_1, \dots, \gamma_n \in H$, we have

$$\sum_{\sigma: ij\sigma kl} \varepsilon(\sigma)(Y_{|S_1|+1} \otimes Y_{|S_2|+1})(\otimes_{p \in S_1} \gamma_p \otimes \Delta \otimes (\otimes_{q \in S_2} \gamma_q)) = (j \leftrightarrow k), \qquad (1.4)$$

where $\Delta = \sum \Delta_a g^{ab} \otimes \Delta_b$.

Here σ runs over *stable* partitions of $\{1, \dots, n\}$ (this means that $|S_i| \geq 2$), and the notation $ij\sigma kl$ means that either $i, j \in S_1$, $k, l \in S_2$, or $i, j \in S_2$, $k, l \in S_1$.

1.4. Correspondence between formal series, families of multiplications, and families of polynomials. Let $\Phi \in k[[H^t]]$ be a formal series. Disregarding terms of degree ≤ 2, write

$$\Phi = \sum_{n=3}^{\infty} \frac{1}{n!} Y_n \tag{1.5}$$

where $Y_n \in (H^t)^{\otimes n}$ can also be considered as an even symmetric map $H^{\otimes n} \to k$. Having thus produced Y_n, we can define the symmetric polylinear multiplications $\circ_n$ satisfying (1.1). Clearly, both correspondences are bijective.

We can now formally state the main result of this section.

1.5. Theorem. *The correspondence of 1.4 establishes a bijection between the sets of the following structures on (H, g):*

a). Formal Frobenius manifolds.

b). Cyclic $Comm_\infty$-algebras.

c). Abstract Correlation Functions.

Proof. We start with the correspondence a) $\leftrightarrow$ c). The Associativity Equations for Φ can be written as

$$\forall\, a, b, c, d, \quad \sum_{ef} \Phi_{abe} g^{ef} \Phi_{fcd} = (a \mapsto b \mapsto c \mapsto a), \tag{1.6}$$

where the subscripts label a basis of H. Representing Φ as in (1.5) and writing $\gamma = \sum_a x^a \Delta_a$ we see that (1.6) is equivalent to

$$\sum_{n_i \geq 3;\, e,f} \frac{1}{(n_1-3)!} Y_{n_1}(\gamma^{\otimes(n_1-3)} \otimes \Delta_a \otimes \Delta_b \otimes \Delta_e) g^{ef} \frac{1}{(n_2-3)!} Y_{n_2}(\Delta_f \otimes \Delta_c \otimes \Delta_d \otimes \gamma^{\otimes(n_2-3)}) =$$

$$= \sum_{n_i \geq 3} \frac{1}{(n_1-3)!(n_2-3)!} (Y_{n_1} \otimes Y_{n_2})(\gamma^{\otimes(n_1-3)} \otimes \Delta_a \otimes \Delta_b \otimes \Delta \otimes \Delta_c \otimes \Delta_d \otimes \gamma^{\otimes(n_2-3)}) =$$

$$(a \mapsto b \mapsto c \mapsto a). \tag{1.7}$$

In order to deduce (1.7) from the coherence relations (1.4), we proceed as follows. Fix $n \geq 4$, consider in (1.7) only the terms with $n_1 + n_2 - 2 = n$, and multiply them by $(n_1 + n_2 - 6)!$ The resulting identity is a particular case of (1.4), corresponding to the following choices:

$$(\gamma_1, \ldots, \gamma_n) = (\gamma, \ldots, \gamma, \Delta_a, \Delta_b, \Delta_c, \Delta_d),$$

$$(i, j, k, l) = (n_1 + n_2 - 5, n_1 + n_2 - 4, n_1 + n_2 - 3, n_1 + n_2 - 2).$$

Since all the arguments except for the four deltas coincide, summation over the partitions in (1.4) will produce the binomial coefficient which we need. (Actually, we have $\gamma \in H_K$, but this does not violate (1.4).)

Arguing in reverse order, we can deduce (1.4) from (1.7). Then one first obtains (1.4) with a part of γ's coinciding, belonging to H_K and being generic even elements. An easy version of the polarization argument then gives the desired conclusion.

We now turn to the correspondence b) $\leftrightarrow$ c). The relation (1.1) can be rewritten as

$$(\gamma_1, \dots, \gamma_n) = \sum_{ab} Y_{n+1}(\gamma_1 \otimes \dots \otimes \gamma_n \otimes \Delta_a) g^{ab} \Delta_b. \tag{1.8}$$

From here we deduce

$$((\gamma_1, \dots, \gamma_{n_1}), \gamma_{n_1+1}, \dots, \gamma_{n_1+n_2-1}) =$$

$$= \sum_{ab} Y_{n_2+1}(\sum_{cd} Y_{n_1+1}(\gamma_1 \otimes \dots \otimes \gamma_{n_1} \otimes \Delta_c) g^{cd} \Delta_d \otimes \gamma_{n_1+1} \otimes \dots \otimes \gamma_{n_1+n_2-1}) =$$

$$= \sum_{ab} (Y_{n_1+1} \otimes Y_{n_2+1})(\gamma_1 \otimes \dots \otimes \gamma_{n_1} \otimes \Delta \otimes \gamma_{n_1+1} \otimes \dots \otimes \gamma_{n_1+n_2-1} \otimes \Delta_a) g^{ab} \Delta_b. \tag{1.9}$$

The associativity relations (1.2) will exactly match the coherence relations (1.7) rewritten via (1.9) if we put $m + 3 = n_1 + n_2$, $\alpha = \gamma_1, \beta = \gamma_2, \gamma = \gamma_{n_1+1}$; $i = 2, j = 1, k = n_1 + 1, l = m + 3$.

1.6. Identity. If a formal Frobenius manifold (H, g, Φ) admits a flat identity e, it can be identified with a basic element Δ_0. In the respective structure of the cyclic $Comm_\infty$–algebra the formula (2.3) of Chapter I transforms into the following definition of identity, perhaps slightly counter–intuitive:

$$(\Delta_0, \gamma_1, \dots, \gamma_n) = \begin{cases} \gamma_1 & \text{for } n = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

In fact, this formula for $n = 1$ is equivalent to the statement $(\Delta_0, \Delta_a) = \Delta_a$ for all a, or else $g((\Delta_0, \Delta_a), \Delta_b) = g_{ab}$ for all a, b. But in view of (1.1), the left hand side is the same as

$$Y_3(\Delta_0 \otimes \Delta_a \otimes \Delta_b) = \partial_0 \partial_a \partial_b \Phi(\gamma),\ \gamma = \sum_a x^a \Delta_a,$$

which is g_{ab} in view of Chapter I, (2.2).

1.7. The Euler operator. There is not much new to add to the discussion of §2, Chapter I. It is probably worth noting that in the formal situation the grading induced by E interacts with the natural grading on K in which H is of degree 1. If in the semisimple decomposition of E (2.14), Chapter I, the term $\sum \partial_b$ is present, then the grading relation (2.7), Chapter I, connects Y_{n+1} to Y_n, otherwise they become decoupled. This last possibility occurs in quantum cohomology for manifolds with vanishing canonical class so that the general constraints of Frobenius manifolds become less stringent for such manifolds.

1.8. Semisimplicity. (H, g, Φ) is called (formally) semisimple if the k–algebra H with the structure constants $\Phi_{ab}{}^c(0)$ is isomorphic to k^n. One can prove then that H_K is isomorphic to K^n. The basic idempotents $e_i \in H_K$ have the same properties as in the geometric theory.

1.9. Why alternative descriptions? The formal version of Frobenius geometry is natural from the viewpoint of quantum cohomology: the relevant sructure initially is formal, and only after some work Φ can be analytically continued and geometrized.

The reformulation in terms of $\circ_n$ suggests a non–trivial extension of the notion of commutative algebra and combined with operadic formalism (about which later) leads to an unexpected generalization of other classical structures. For example, one can introduce and study the notion of Lie_∞–algebras, given by a family of S_n–skewsymmetric polylinear brackets $[\ ]_n : H^{\otimes n} \to H$, $n = 2, 3, \dots$ satisfying the higher Jacobi identities: for all $k \geq 2$, $l \geq 0$, $a_1, \dots, a_k, b_1, \dots, b_l \in H$ the expression

$$\sum_{i<j} \varepsilon(i,j)[[a_i, a_j], a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_k, b_1, \dots, b_l]$$

must vanish for $l = 0$ and be equal to $[[a_1, \dots, a_k], b_1, \dots, b_l]$ otherwise. This structure was called *gravity algebra* by E. Getzler. It is dual to $Comm_\infty$ in the same sense as Lie algebras are dual to the commutative ones (Quillen, Kontsevich, Kapranov and Ginzburg.)

It would be interesting to find and study a geometric counterpart of this structure (for which Lie_∞ would be a formal version), with an appropriate notion of potential.

Finally, the structure of Abstract Correlation Functions turns out to be a truncated version of an apparently much richer object, consisting of maps $I_n : H^{\otimes n} \to H^*(\overline{M}_{0n}, k)$, $n \geq 3$, where $\overline{M}_{0n}$ are the moduli spaces of stable curves of genus zero with n labelled points. These maps are constrained by the relations coming from the geometry of $\overline{M}_{0n}$ which extend and "explain" the formal identities (1.4).

The most remarkable fact is that this rich structure is in fact equivalent to its truncated version, thus to $Comm_\infty$ and formal Frobenius manifolds. On the other hand, it admits an intrinsic operation of tensor product, quite unexpected in either of the previous descriptions, geometric and formal alike.

The remaining part of this Chapter will be devoted to this structure.

§2. Pointed curves and their graphs

Moduli spaces (orbifolds, stacks) of curves with labelled points are stratified according to their degeneration type. In this section we review the combinatorial structure of this stratification.

2.1. Definition. *A prestable curve over a scheme T is a flat proper morphism $\pi: C \to T$ whose geometric fibers are reduced one-dimensional schemes with at most ordinary double points as singularities. Its genus is a locally constant function on T: $g(t) := \dim H^1(C_t, \mathcal{O}_{C_t})$.*

2.2. Definition. *Let S be a finite set. An S-pointed (equivalently, S-labelled) prestable curve over T is a family $(C, \pi, x_i \,|\, i \in S)$, where $\pi: C \to T$ is a prestable curve, and x_i are sections such that for any geometric point t of T we have $x_i(t) \neq x_j(t)$ for $i \neq j$ and $x_i(t)$ are smooth on C_t. Points $x_i(t), i \in S$, and singular points of C_t are called special.*

Such irreducible curve is called stable if $2g-2+|S| > 0$ and if every non-singular genus zero component of any C_t contains at least three special points. A general prestable pointed curve is called stable if all its connected components are stable.

2.2.1. Remark. Let $(C, \pi, x_i \,|\, i \in S)$ be an S–pointed prestable curve. It is stable iff automorphism groups of its geometric fibers fixing the labelled points are finite.

2.3. Definition. *A (finite) graph τ is the data $(F_\tau, V_\tau, \partial_\tau, j_\tau)$ where F_τ is a (finite) set (of flags), V_τ a finite set (of vertices), $\partial_\tau : F_\tau \to V_\tau$ is the boundary map, and $j_\tau : F_\tau \to F_\tau$ is an involution, $j_\tau^2 = id$.*

An isomorphism $\tau \to \sigma$ consists of two bijections $F_\tau \to F_\sigma$, $V_\tau \to V_\sigma$, compatible with ∂ and j.

Two–element orbits of j_τ form the set E_τ of edges, and one–element orbits form the set S_τ of tails.

It is convenient to think of graphs in terms of their geometric realizations. For each vertex $v \in V_\tau$ put $F_\tau(v) = \partial_\tau^{-1}(v)$ and consider the topological space "star of v" consisting of $|v| := |F_\tau(v)|$ semiintervals having one common boundary point. These semiintervals must be labelled by their respective flags. Then take the union of all stars and replace every two–element orbit of j_τ by a segment joining the respective vertices so that these two flags become halves of the edge, and tails become non–paired flags.

A graph τ is called connected (resp. simply connected) if its geometric realization $\|\tau\|$ is so.

2.4. Definition. *A modular graph is a graph τ together with a map $g: V_\tau \to \mathbf{Z}_{\geq 0}$, $v \mapsto g_v$. An isomorphism of two modular graphs is an isomorphism of the underlying graphs preserving the g–labels of vertices.*

A modular graph (τ, g) is called stable if $|v| \geq 3$ for all v with $g_v = 0$, and $|v| \geq 1$ for all v with $g_v = 1$.

2.5. Definition. *The (dual) modular graph (τ, g) of a prestable S-pointed curve $(C, \pi, x_i \mid i \in S)$ over an algebraically closed field consists of the following data:*

a). F_τ = the set of branches of C passing through special points.

b). V_τ = the set of irreducible components of C, g_v = the genus of the normalization of the component correspondint to v (denoted sometimes C_v.)

c). $\partial_\tau(f) = v$, iff the branch f belongs to the component C_v.

d). $j_\tau(f) = \overline{f}$, $f \neq \overline{f}$, iff the two branches $f, \overline{f}$ intersect at a common double point. Therefore, edges of τ bijectively correspond to the singular points of C.

e). $j_\tau(f) = f$, iff f is a branch passing through a labelled point of C. Thus the tails of τ bijectively correspond to the labelled points of C and to the set S of their labels.

We will sometimes call the isomorphism class of (τ, g) *the combinatorial type of* C.

If C is stable, the combinatorial type of C is stable, and vice versa. Any modular graph represents the combinatorial type of some semistable labelled curve.

2.6. Proposition. *Let (τ, g) be the combinotarial type of a prestable S-pointed connected curve $(C, \pi, x_i \mid i \in S)$, g = genus of C, $n = |S|$. Then we have*

$$g = \sum_{v \in V_\tau} g_v + \dim\, H_1(\|\tau\|), \tag{2.1}$$

$$g - 1 = \sum_{v \in V_\tau} (g_v - 1) + |E_\tau|, \tag{2.2}$$

$$\sum_{v \in V_\tau} |v| = 2|E_\tau| + n. \tag{2.3}$$

Proof. Consider the normalization morphism

$$f : \coprod_{v \in V_\tau} \widetilde{C}_v = \widetilde{C} \to C$$

where $\widetilde{C}_v$ is the normalization of C_v. The exact sequence of sheaves on $\widetilde{C}$

$$0 \to f^*(\mathcal{O}_C) \to \mathcal{O}_{\widetilde{C}} \to \oplus_{y \in C_{sing}} k(y) \to 0$$

generates an exact sequence of linear spaces

$$0 \to k \to k^{V_\tau} \to k^{E_\tau} \to H^1(\widetilde{C}, f^*(\mathcal{O}_C)) \to H^1(\widetilde{C}, \mathcal{O}_C) \to 0. \tag{2.4}$$

Moreover,

$$\dim\, H^1(\widetilde{C}, f^*(\mathcal{O}_C)) = \dim\, H^1(C, \mathcal{O}_C) = g,$$

$$\dim\, H^1(\widetilde{C}, \widetilde{\mathcal{O}}_C) = \sum_{v \in V_\tau} g_v.$$

Hence we get

$$1 - |V_\tau| + |E_\tau| - g + \sum_{v \in V_\tau} g_v = 0.$$

Replacing here $|E_\tau| - |V_\tau|$ by

$$-\chi(\|\tau\|) = \dim H_1(\|\tau\|) - \dim H_0(\|\tau\|) = \dim H_1(\|\tau\|) - 1$$

(since $\|\tau\|$ is connected) we get (2.1). Replacing $\sum_{v \in V_\tau} g_v - |V_\tau|$ by $\sum_{v \in V_\tau} (g_v - 1)$ we get (2.2). Finally, both sides of (2.3) are equal to $|F_\tau|$ counted in two different ways.

2.6.1. Corollary. *For any (g, n) with $2g - 2 + n > 0$ there exist only finitely many isomorphism classes of connected stable modular graphs of genus g with tails $\{1, \ldots, n\}$ (or simply stable (g, n)–graphs).*

More precisely, if (τ, g) is connected and stable, then:

a). $|V_\tau| \leq 2g-2+n$, with equality sign exactly on graphs for which $(g_v, n) = (0, 3)$ or $(1, 1)$ for all vertices v.

b). For $\gamma \geq 2$,

$$\mathrm{card}\{v \,|\, g_v = \gamma\} \leq \frac{2g - 2 + n}{2\gamma - 2}.$$

c). $|E_\tau| \leq 3g - 3 + n$, with equality sign exactly on graphs with $g = 0$, $(g_v, |v|) = (0, 3)$ for all vertices.

Proof. Adding (2.1) to one half of (2.3), we get:

$$\sum_{v \in V_\tau} (g_v - 1 + \frac{1}{2}|v|) = \frac{1}{2}n + g - 1 > 0.$$

Stability implies that $g_v - 1 + \frac{1}{2}|v| \geq \frac{1}{2}$ for $g_v = 0, 1$ and $\geq g_v - 1$ for $g_v \geq 2$. The first two assertions of the Corollary now follow directly.

From (2.2) one sees that

$$|E_\tau| \leq g - 1 + \mathrm{card}\{v \,|\, g_v = 0\} \leq g - 1 + |V_\tau| \leq n + 3g - 3,$$

equality sign corresponding to all $g_v = 0$ and hence $|v| = 3$ in view of a).

2.6.2. Remarks. a). There exist infinitely many unstable (0,2), (0,1) and (0,0) graphs.

b). The stable modular graphs with $g_v = 0$ and $|v| = 3$ for all v describe maximally degenerate pointed curves. Such curves have no moduli because each component is a $\mathbf{P}^1$ with three special points on it.

2.6.3. Corollary. *a). Stable connected modular $(0, n)$-graphs are trees with vertices of valency ≥ 3.*

b). Any isomorphism of such graphs is uniquely defined by its restriction on tails.

c). $|V_\tau| - |E_\tau| = 1$ for such graphs.

2.7. Combinatorics of degeneration. Let $(\tau, g), (\sigma, h)$ be connected stable modular graphs with the same (or explicitly identified) set of tails $S = S_\tau = S_\sigma$. We will write $(\tau, g) \geq (\sigma, h)$ if there exists a family of stable curves with an irreducible base such that the generic geometric fiber has the combinatorial type (τ, g), some other geometric fiber has the type (σ, h), and the specialization of the structure sections induces the given identification of tails.

When the set S is fixed, this relation becomes a partial order, called specialization. If $(\tau, g) > (\sigma, h)$ and any intermediate (ρ, k) coincides with either (τ, g) or (σ, h), we will say that (σ, h) is a *codimension one* specialization of (τ, g).

Any codimension one specialization $(\tau, g) > (\sigma, h)$ can be uniquely specified by the data of one of the two types:

a). *Splitting.* Choose a vertex $v \in V_\tau$ of genus $g_v \geq 0$, a decomposition $g_v = g'_v + g''_v$ and a partition of the set of the flags incident to v: $F_\tau(v) = F'_\tau(v) \cup F''_\tau(v)$, such that both subsets are j_τ-invariant. To obtain (σ, h), replace the vertex v in τ by two vertices v', v'' connected by an edge e, put $g_{v'} = g'_v$, $g_{v''} = g''_v, F_\sigma(v') = F'_\tau(v) \cup \{e'\}, F_\sigma(v'') = F''_\tau(v) \cup \{e''\}$ where e', e'' are the two halves of e. The remaining vertices, flags and incidence relations are the same for τ and σ.

Geometrically, this describes the following degeneration: the irreducible component C_v splits into two irreducible curves, among which the special points of C_v are distributed as specified by the partition of flags. The new edge e "is" the new singular point $C_{v'} \cap C_{v''}$.

b). *Acquisition of a loop/cusp.* Choose a vertex $v \in V_\tau$ of genus $g_v \geq 1$. Put $V_\sigma = V_\tau$, keep all the g-labels of vertices the same except for g_v which is replaced by $g_v - 1$ in σ. Finally, add two new flags forming one j_σ-orbit (a loop) to $F_\tau(v)$.

Geometrically, this corresponds to a degeneration of C_v acquiring a new cusp. The genus of the normalization is thereby reduced by one.

Arbitrary combinatorial specialization of the stable modular graphs can be realized geometrically.

2.8. Stratified moduli spaces. For any (g, n) with $2g - 2 + n > 0$ there exist two basic types of moduli spaces: $M_{g,n}$ and $\overline{M}_{g,n}$. The first one classifies only irreducible stable n-labelled curves, the second one arbitrary ones. The precise definition/construction of these spaces varies depending on the context. There are versions of the type "coarse moduli spaces", "orbifolds","moduli stacks".

In all versions, however, the following intuitive picture can be made precise.

a). $M_{g,0} (g \geq 2), M_{1,1}, M_{0,3}$ are the basic smooth orbifolds of dimension $3g-3, 1, 0$ respectively. Each of them carries the universal curve $C \to M$.

b). $M_{g,n}$ is the n-th (resp. $(n-1)$-th, $(n-3)$-th) relative power of the respective $C \to M$, with partial diagonals (and eventually incidence loci with 1 or 3 basic structure sections) deleted.

We can similarly define $M_{g,S}$ parametrizing S-marked curves.

c). For any stable connected n-labelled graph (τ, g) put

$$M_{(\tau,g)} := \left(\prod_{v \in V_\tau} M_{g_v, F_\tau(v)} \right) / G,$$

where G is the automorphism group of (τ, g) identical on tails. This is the moduli space of stable n-labelled curves of the combinatorial type (τ, g). In fact, deforming such a curve is equivalent to independently deforming its irreducible components keeping track of special points and their incidence relations.

d). Finally, we have a decomposition of $\overline{M}_{g,n}$ into pairwise disjoint locally closed strata indexed by the isomorphism classes of n-graphs:

$$\overline{M}_{g,n} = \coprod_{(\tau,g)} M_{(\tau,g)} = \coprod_{(\tau,g)} \left(\prod_{v \in V_\tau} M_{(g_v, F_\tau(v))} \right) / G.$$

The stratum $M_{(\sigma,h)}$ belongs to the closure of $M_{(\tau,g)}$ exactly when $(\tau, g) > (\sigma, h)$.

In the next section we treat the genus zero case in more detail. An essential simplification is due to the fact that stable n-trees have no non-trivial n-automorphisms (that is, automorphisms identical on leaves.) Therefore moduli spaces of genus zero are actually smooth manifolds.

§3. Moduli spaces of genus 0

This section is a report on the structure of the moduli spaces of curves of genus zero elaborating the general discussion of the previous section. We give precise statements but often omit or only sketch the proofs which can be found in [Kn] and [Ke]. We work in the category of schemes over an arbitrary field (in most cases, Spec $\mathbf{Z}$ would do as well.)

3.1. Theorem. *a). For any $n \geq 3$, there exists a universal n-pointed stable curve $(\overline{\pi}_n : \overline{C}_{0n} \to \overline{M}_{0n}; x_i, i = 1, \dots, n)$ of genus zero. This means that any such curve over a scheme T is induced by a unique morphism $T \to \overline{M}_{0n}$.*

b). $\overline{M}_{0n}$ is a smooth irreducible projective algebraic variety of dimension $n - 3$.

c). For any stable n-tree τ, there exists a locally closed reduced irreducible subscheme $D(\tau) \subset \overline{M}_{0n}$ parametrizing exactly curves of the combinatorial type τ. Its codimension equals the cardinality of the set of edges $|E_\tau|$ that is, the number of singular points of any curve of the type τ. This subscheme depends only on the n-isomorphism class of τ.

d). $\overline{M}_{0n}$ is the disjoint union of all $D(\tau)$. The closure of any of the strata $D(\tau)$ is the union of all strata $D(\sigma)$ such that $\tau > \sigma$ in the sense of 2.7.

Let σ_n be the one-vertex n-tree. We will denote $D(\sigma_n)$ by M_{0n} and the induced stable curve by $\pi_n : C_{0n} \to M_{0n}$. It classifies the irreducible pointed curves. Its geometric points are systems of n pairwise distinct points on $\mathbf{P}^1$ *considered up to a common fractional linear transformation.*

The codimension one strata are labelled by the isomorphism classes of stable one-edge n-graphs σ. Each such class can be identified with an *unordered* partition $\{1, \dots, n\} = S_1 \coprod S_2$, stability means that $|S_i| \geq 2$ for $i = 1, 2$. The curve $\overline{C}_{0n}$ over $D(\sigma)$ has two components, and the partition $S_1 \coprod S_2$ corresponds to the distribution of the structure sections x_i between these components. Of course, with obvious modifications we can replace here $\{1, \dots, n\}$ by any finite set S.

3.2. Examples. The following pictures show the structure of $\overline{M}_{0n}$ with its canonical stratification, and the structure of $\overline{C}_{0n}$, for $n = 3, 4, 5$.

Fig. 1

$M_{03} = \overline{M}_{03}$ is simply a point, and $\overline{C}_{0n}$ is $\mathbf{P}^1$ endowed with three points labelled by 1,2,3 because the fractional linear group acts simply transitively on the ordered triples.

$\overline{M}_{04}$ is as well $\mathbf{P}^1$ with three labelled points, but this time the labels are one-edge stable trees with tails $\{1, 2, 3, 4\}$ corresponding to the divisorial strata, and M_{04} is the complement to these three points. Furthermore, $\overline{C}_{04}$ is a surface fibered over $\mathbf{P}^1$ and endowed with 4 labelled sections. In addition to them, there are six components of degenerate fibers. One can check that all the ten curves are exceptional of the first kind, forming a configuration well known in the theory of the Del Pezzo surfaces. In fact, $\overline{C}_{04}$ is isomorphic to the (rigid) Del Pezzo surface of degree 5, which can be obtained by blowing up four points of $\mathbf{P}^2$ in general position. It is known that S_5, and not just S_4 (renumbering sections) acts on such a surface. In our context this can be explained by the fact that $\overline{C}_{04}$ can be identified with $\overline{M}_{05}$ (non-canonically) or rather with some $\mathbf{M}_{0S}$, $|S| = 5$ canonically; the reader is invited to describe S.

Fig. 2

Of course, M_{05} is the complement to the ten boundary divisors marked by the stable 5-trees with one edge. Each of these divisors contains three 0-dimensional strata marked by the stable 5-trees with two edges.

We see an emerging pattern: $\overline{C}_{0,n}$ is isomorphic to $\overline{M}_{0,n+1}$. It can be explained by the following considerations.

Consider a stable pointed curve $(C, x_1, \dots x_{n+1})$ of genus 0 over a field. We will say that $(C, x_1, \dots, x_n)$ is obtained from it by forgetting the point x_{n+1}. However, $(C, x_1, \dots, x_n)$ may well be unstable. This will happen precisely when the component of C supporting x_{n+1} has only one additional labelled point, say x_j. In this case we can contract this component to its intersection point x'_j with some other component of C, thus getting the n-pointed curve $(C', x_1, \dots, x_{j-1}, x'_j, \dots, x_n)$. We will call the last step *stabilization*, and the resulting construction (forgetting plus stabilization whenever necessary) *stable forgetting*.

3.3. Theorem. *a). There is a canonical morphism $\rho_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0n}$ which acts on the isomorphism classes of $(n+1)$-pointed curves by stably forgetting the last point.*

b). There exists a canonical isomorphism $\mu_n : \overline{M}_{0,n+1} \to \overline{C}_{0n}$ commuting with projections to $\overline{M}_{0n}$.

The first statement is not obvious because it is not clear that collapsing of unstable components can be performed uniformly over a base.

F. Knudsen ([Kn],§2) proves both statements in the following way. He remarks that not only $\overline{M}_{0,n+1}$ but $\overline{C}_{0n}$ as well represents a natural functor, namely

$$\overline{C}_{0n}(T) = \{T\text{–families of stable } (0,n)\text{–curves with an extra section } \Delta\}/(\text{iso}).$$

No restriction is imposed on this extra section. The universal family is $\overline{C}_{0n} \times_{\overline{M}_{0n}} \overline{C}_{0n}$ fibered over $\overline{C}_{0n}$ via the second projection, with relative diagonal as Δ.

Therefore it suffices to produce a functorial bijection between the T–families of the types $(C, x_1, \dots, x_n, x_{n+1})$ and $(D, y_1, \dots, y_n, \Delta)$ respectively. This bijection is defined via two mutually inverse birational maps: a morphism $C \to D$ and a blow up $D \to C$. The first one maps C to the projective spectrum of the sheaf of algebras generated by $\omega_{C/T}(x_1 + \dots + x_n)$ where $\omega_{C/T}$ is the relative dualizing sheaf. One easily sees that it blows down precisely those components of the fibers which become unstable after removing x_{n+1}. We will not describe the second map.

Forgetful morphisms can be used in order to establish relations between the cohomology classes of strata.

For $n \geq 4$, choose pairwise distinct $i, j, k, l \in \{1, \dots, n\}$ and a stable 2-partition σ of $\{1, \dots, n\}$. Recall that we write $ij\sigma kl$ if i, j and k, l belong to the different parts of σ. Let $\mu : \overline{M}_{0n} \to \overline{M}_{0,\{ijkl\}}$ be the iterated forgetful morphism stably forgetting all points except for x_i, x_j, x_k, x_l. The three boundary points of $\overline{M}_{0,\{ijkl\}}$ correspond to the three different stable partitions of the labels; choose one of them, say $\{i, j\} \cup \{k, l\}$.

3.4. Theorem. *The fiber of μ over this point is the scheme theoretical union $\cup_{\sigma: ij\sigma kl} D(\sigma)$.*

For a proof, see [Kn], Theorem 2.7, and [Ke], p. 552, Fact 3.

3.4.1. Corollary. *Let $[D(\sigma)]$ be the cohomology (or Chow) class of $D(\sigma)$. Then for any quadruple $i, j, k, l \in \{1, \dots, n\}$ we have*

$$\sum_{ij\sigma kl} [D(\sigma)] - \sum_{kj\tau il} [D(\tau)] = 0. \tag{3.1}$$

In fact, (3.1) is the difference of two fibers of the forgetful morphism.

In order to state the second corollary, we introduce some notation. For two unordered stable partitions $\sigma = \{S_1, S_2\}$ and $\tau = \{T_1, T_2\}$ of S put

$a(\sigma, \tau) :=$ the number of non-empty pairwise distinct sets among $S_a \cap T_b,\ a, b = 1, 2.$

Clearly, $a(\sigma,\tau) = 2, 3$, or 4. Moreover, $a(\sigma,\tau) = 2$ iff $\sigma = \tau$, and $a(\sigma,\tau) = 4$ iff there exist pairwise distinct $i, j, k, l \in S$ such that simultaneously $ij\sigma kl$ and $ik\tau jl$. If $a(\sigma,\tau) = 3$, we sometimes call σ and τ compatible. A family of 2-partitions $\{\sigma_1, \dots, \sigma_m\}$ is called *good*, if for all $i \neq j$, σ_i and σ_j are compatible.

3.4.2. Corollary. *If* $a(\sigma,\tau) = 4$, *then*

$$\overline{D}(\sigma) \cap \overline{D}(\tau) = \emptyset. \tag{3.2}$$

In fact, $\overline{D}(\sigma)$ and $\overline{D}(\tau)$ belong to two different fibers of an appropriate forgetful morphism to $\mathbf{P}^1$.

3.5. The ring structure of $H^*(\overline{M}_{0S})$. Keel [Ke] has shown that the dual classes of $[\overline{D}(\tau)]$ generate the ring $H^*(\overline{M}_{0S})$, whereas (3.1) and (3.2) generate the ideal of relations.

More precisely, for a given finite set S of cardinality ≥ 3, consider a family of independent commuting variables D_σ indexed by stable unordered 2–partitions of S. Put $F_S = k[D_\sigma]$ ($F_S = k$ for $|S| = 3$.) This is a graded polynomial ring, $\deg D_\sigma = 1$. Define the ideal $I_S \subset F_S$ generated by the following elements:

a). For each ordered quadruple $i, j, k, l \in S$

$$R_{ijkl} := \sum_{ij\sigma kl} D_\sigma - \sum_{kj\tau il} D_\tau \in I_S. \tag{3.3}$$

b). For each pair σ, τ with $a(\sigma,\tau) = 4$:

$$D_\sigma D_\tau \in I_S. \tag{3.4}$$

Finally, put $H^*_S = K[D_\sigma]/I_S$.

3.5.1. Theorem (Keel [Ke]). *The map*

$$D_\sigma \longmapsto \text{dual class of } \overline{D}(\sigma)$$

induces the isomorphism of rings (doubling the degrees)

$$H^*_S \xrightarrow{\sim} H^*(\overline{M}_{0S}, k) = A^*(\overline{M}_{0S})_k. \tag{3.5}$$

Here A^* is the Chow ring.

Keel's presentation (3.5) in principle solves the problem of algorithmic calculations in the cohomology ring. In practice, however, even the most basic properties of this ring are not obvious for H^*_S, e.g. the fact that $H^i_S = 0$ for $i > |S| - 3$, $\dim H^{|S|-3}_S = 1$, and the Poincaré pairing is perfect duality.

In the next section we will need more precise information about the homogeneous components not only of H^*_S, but I_S as well. The remaining part of this subsection is devoted to the preparatory work. We keep notation of the Theorem 3.5.1.

The monomial $D_{\sigma_1} \dots D_{\sigma_a} \in F_S$ is called good, if the family of 2-partitions $\{\sigma_1, \dots, \sigma_a\}$ is good, i.e. $a(\sigma_i, \sigma_j) = 3$ for $i \neq j$. Notice that the relevant divisors are then pairwise distinct. In particular, D_σ and 1 are good.

Consider a stable S–tree τ. Any edge $e \in E_\tau$ defines a stable partition $\sigma(e)$: if one cuts e, the tails of the resulting two trees (except for halves of e) form $\sigma(e)$.

3.5.2. Proposition. *a). The monomial*

$$m(\tau) := \prod_{e\in E_\tau} D_{\sigma(e)}$$

is good.

b). For any $0 \le r \le |S|-3$, the map $\tau \longmapsto m(\tau)$ establishes a bijection between the set of good monomials of degree r in F_S and stable S-trees τ with $|E_\tau| = r$ modulo S-isomorphisms. There are no good monomials of degree $> |S|-3$.

Proof. a). Let f be a flag of a tree τ whose boundary is the vertex v. It defines a subtree of τ which we will call *the branch of f*. If f is itself a tail, its branch consists of f and v. In general, it comprises all vertices, flags and edges that can be reached (in geometric realization) by a no–return path starting with (v, f). Denote by $S(f)$ the set of leaves on this branch (or the set of their labels.)

Let now $e \neq e' \in E_\tau$. There exists a sequence of pairwise distinct edges $e = e'_0, e'_1, \dots, e'_r, e'_{r+1} = e'$, $r \ge 0$, such that e'_j and e'_{j+1} have a common vertex v_j. Let u be the remaining vertex of e, w that of e'. Let S' be the set of all tails of τ belonging to the branches starting at u but not with a flag belonging to e; similarly, let S'' be the set of all tails of τ belonging to the branches that start at w but not with a flag belonging to e'. Finally, let T be the set of all tails on the branches at $v_0, \dots, v_r$ not starting with the flags in $e'_0, \dots, e'_{r+1}$ (we identify tails with their labels). Since τ is stable, all three sets S', S'' and T are non-empty, and

$$\sigma(e) = \{S', S''\coprod T\},\ \sigma(e') = \{S'\coprod T,\ S''\}.$$

It follows that $a(\sigma(e), \sigma(e')) = 3$ so that $m(\tau)$ is a good monomial.

b). For $r = 0, 1$ the assertion is clear. Assume that for some $r \ge 1$ the map $\tau \longmapsto m(\tau)$ is surjective on good monomials of degree r. We will prove then that it is surjective in the degree $r+1$.

Let m' be a good monomial of degree $r+1$. Choose a divisor D_σ of m' which is *extremal* in the following sense: one element, say S_1, of the partition $\sigma = \{S_1, S_2\}$ is minimal in the set of all elements of all 2-partitions σ' such that $D_{\sigma'}$ divides m'. Put $m' = D_\sigma m$. Since m is good of degree r, we have $m = m(\tau)$ for some stable S-tree τ. We will show that $m' = m(\tau')$ where τ' is obtained from τ by inserting a new edge with tails marked by S_1 at an appropriate vertex $v \in V_\tau$. In other words, τ' is a codimension one specialization of τ in the sense of 2.7.

First we must find v in τ. To this end, consider any edge $e \in E_\tau$ and the respective partition $\sigma(e) = \{S'_e, S''_e\}$. Since m' is good, we have $a(\{S_1, S_2\}, \{S'_e, S''_e\}) = 3$. As S_1 is minimal, one sees that exactly one of the sets $\{S'_e, S''_e\}$ strictly contains S_1. Let it be S''_e. Orient e by declaring that the direction from the vertex (corresponding to) S'_e to S''_e is positive. We claim that with this orientation of all edges, for any $w \in V_\tau$ there can be at most one edge outgoing from w. In fact, if τ contains a vertex w with two positively oriented flags $f_1$and f_2, then S_1 must be contained in the two subsets of S, branches $S(f_1)$ and $S(f_2)$. But their intersection is empty.

It follows that there exists exactly one vertex $v \in V_\tau$ having no outgoing edges. Moreover, S_1 is contained in the set of labels of the tails at v by construction. If we now define τ' by inserting a new edge e' at v so that $\sigma(e') = \sigma$, we will clearly have $m' = m(\tau')$. If $r \leq |S| - 4$, the tree τ' cannot be unstable because, first, $|S_1| \geq 2$, and second, at least two more flags converge at v: otherwise the unique incoming edge would produce the partition $\{S_1, S_2\} = \sigma$ which would mean that D_σ divides already $m(\tau)$.

For $r = |S| - 3$, this argument shows that m' cannot exist because all the vertices of τ have valency three.

It remains to check that if $m(\tau_1) = m(\tau_2)$, then τ_1 and τ_2 are S–isomorphic.

Assume that this has been checked in degree $\leq r$ and that deg τ_1 = deg τ_2 = $r + 1$. Choose an extremal divisor D_σ of $m(\tau_1) = m(\tau_2)$ as above and contract the respective edges of τ_1, τ_2 getting the trees τ_1', τ_2'. Since $m(\tau_1') = m(\tau_2') = m(\tau_i)/D_\sigma$, τ_1' and τ_2' are S–isomorphic by the inductive assumption. This isomorphism respects the marked vertices v_1', v_2' corresponding to the contracted edges because as we have seen they are uniquely defined. Hence it extends to an S-isomorphism $\tau_1 \to \tau_2$.

3.5.3. Remark. Since the boundary divisors intersect transversally, the image of $m(\tau)$ in $H^*(\overline{M}_{0S})$ is the dual class of $D(\tau)$.

3.6. Multiplication formulas. In this subsection we will show that good monomials modulo I_S span H_S and therefore, dual classes of strata span $H^*(\overline{M}_{0S})$. This will follow from the more precise formulas (3.6)–(3.9) allowing one to express recursively a product of good monomials modulo I_S as a linear combination of good monomials.

Let σ, τ be two stable S-trees, $|E_\sigma| = 1$. We have to consider the following alternatives.

a). *$D_\sigma m(\tau)$ is a good monomial.* Then

$$D_\sigma m(\tau) = m(\tau') \tag{3.6}$$

where τ is obtained from τ' by contracting the edge in $E_{\tau'}$ whose 2-partition coincides with that of σ.

More generally, if $m(\sigma)m(\tau)$ is a good monomial, then

$$m(\sigma)m(\tau) = m(\sigma \times \tau) \tag{3.7}$$

where the direct product is the categorical one in the category of S-trees and S-morphisms, to be desribed later. We can identify $E_{\sigma\times\tau}$ with $E_\sigma \coprod E_\tau$, and $p_1 : \ \sigma \times \tau \to \sigma$ (resp. $p_2 : \ \rho \times \tau \to \tau$) contracts edges of the second factor (resp. of the first one).

b). *There exists a divisor $D_{\sigma'}$ of $m(\tau)$, $|E_{\sigma'}| = 1$, such that $a(\sigma, \sigma') = 4$.* Then

$$D_\sigma m(\tau) \equiv 0 \bmod I_S \tag{3.8}$$

in view of (3.4).

c). *D_σ divides $m(\tau)$.* Then let $e \in E_\tau$ be the edge corresponding to σ; v_1, v_2 its vertices, (v_i, e) the corresponding flags.

We will write several different expressions for $D_\sigma m(\tau)$ mod I_S, corresponding to various choices of unordered pairs of distinct flags $\{\bar{\imath}, \bar{\jmath}\} \subset F_\tau(v_1)\setminus\{(v_1, e)\}$, $\{\bar{k}, \bar{l}\} \subset F_\tau(v_2) \setminus \{(v_2, e)\}$. For each choice, put

$$T_1 = F_\tau(v_1) \setminus \{\bar{\imath}, \bar{\jmath}, (v_1, e)\},$$
$$T_2 = F_\tau(v_2) \setminus \{\bar{k}, \bar{l}, (v_2, e)\}.$$

Notice that because of stability the set of such choices is non-empty.

3.6.1. Proposition. *For every such choice we have*

$$D_\sigma m(\tau) \equiv - \sum_{\substack{T\subset T_1\\ |T|\geq 1}} m(tr_{T,e}(\tau)) - \sum_{\substack{T\subset T_2\\ |T|\geq 1}} m(tr_{T,e}(\tau)) \bmod I_S \tag{3.9}$$

where $tr_{T,e}(\tau)$ is the tree obtained from τ by "transplanting all branches starting in T to the middle point of the edge e." (An empty sum is zero).

Fig. 3. Transplants: arrows correspond to branches

Remark. We can also describe $tr_{T,e}(\tau)$ as a result of inserting an extra edge instead of the vertex v_1 (resp. v_2) and putting the branches T to the common vertex of the new edge and e. There exists a well defined edge in $tr_{T,e}(\tau)$ whose contraction produces τ.

Proof. We choose pairwise distinct labels on the chosen branches $i \in S(\bar{\imath})$, $j \in S(\bar{\jmath})$, $k \in S(\bar{k})$, $l \in S(\bar{\ell})$ and then calculate the element (see (3.3))

$$R_{ijkl} \cdot m(\tau) = \left(\sum_{ij\rho kl} D_\rho - \sum_{kj\rho il} D_\rho \right) m(\tau) \equiv 0 \bmod I_S. \tag{3.10}$$

Clearly, $ij\sigma kl$, so that for all terms D_ρ of the second sum in (1.5) we have $a(\sigma, \rho) = 4$ so that $D_\rho m(\tau) \in I_S$. Among the terms of the first sum, there is one D_σ. If $ij\rho kl$ and $\rho \neq \sigma$, then D_ρ cannot divide $m(\tau)$. Otherwise ρ would correspond to an edge $e' \neq e$, but the 2-partition of such an edge cannot break $\{i, j, k, l\}$ into $\{i, j\}$ and

$\{k, l\}$ as a glance to a picture of τ shows. It follows that $D_\rho m(\tau) = m(\rho \times \tau)$ as in (3.7). The projection $\rho \times \tau \to \tau$ contracts the extra edge onto a vertex that can be only one of the ends of e, otherwise, as above, the condition $ij\rho kl$ cannot hold. It should be clear by now that $\rho \times \tau$ must be one of the trees $tr_{T,e}(\tau)$, and that each tree of this kind can be uniquely represented as $\rho \times \tau$ for some ρ with $ij\rho kl$. But from (3.10) it follows that

$$D_\sigma m(\tau) \equiv - \sum_{\substack{ij\rho kl \\ \rho \neq \sigma}} D_\rho m(\tau) \bmod I_S$$

which is (3.9).

3.7. Integral over the fundamental class. The functional $\int_{\overline{M}_{0,S}} : H^*(\overline{M}_{0,S}) \to k$ is given by

$$m(\tau) \longmapsto \begin{cases} 1, & \text{if} \quad \deg m(\tau) = |S| - 3, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\deg m(\tau) = |S| - 3$ iff $|v| = 3$ for all $v \in V_\tau$, and $\overline{D}(\tau)$ is a point in this case.

§4. Formal Frobenius manifolds and Cohomological Field Theories

4.1. Definition. *In the notation of 1.1, the structure of the (tree level) Cohomological Field Theory (CohFT) on (H,g) is given by a family of even linear maps (correlators)*

$$I_n : H^{\otimes n} \to H^*(\overline{M}_{0n}, k),\ n = 3,4,\dots \qquad (4.1)$$

satisfying the following conditions:

a). S_n–covariance (with respect to the natural action of S_n on both sides of (4.1).)

b). Splitting, or compatibility with restriction to the boundary divisors: for any stable ordered partition $\sigma : \{1,\dots,n\} = S_1 \coprod S_2$, $n_i = |S_i|$, and the respective map

$$\varphi_\sigma : \overline{M}_{0,n_1+1} \times \overline{M}_{0,n_2+1} \to \overline{D}(\sigma) \subset \overline{M}_{0n}$$

we have

$$\varphi_\sigma^*(I_n(\gamma_1 \otimes \dots \otimes \gamma_n)) = \varepsilon(\sigma)(I_{n_1+1} \otimes I_{n_2+1})(\otimes_{p\in S_1}\gamma_p \otimes \Delta \otimes (\otimes_{q\in S_2}\gamma_q)) \qquad (4.2)$$

where $\Delta = \Sigma \Delta_a g^{ab} \otimes \Delta_b$ is the Casimir element, and $\varepsilon(\sigma)$ is the sign of the permutation induced on the odd arguments $\gamma_1,\dots,\gamma_n$.

Let (H,g,I_*) be a CohFT. Its correlation functions are polylinear functionals

$$Y_n : H^{\otimes n} \to k,\ Y_n(\gamma_1 \otimes \dots \otimes \gamma_n) := \int_{\overline{M}_{0n}} I_n(\gamma_1 \otimes \dots \otimes \gamma_n) \qquad (4.3)$$

where the integral denotes the value of the top dimensional component of I_n on the fundamental cycle of $\overline{M}_{0n}$, cf. 3.7 above.

4.2. Proposition. *Correlation functions of a CohFT satisfy the axioms of Abstract Correlation Functions (Definition 1.3.1.)*

Proof. Clearly, functionals (4.3) are symmetric, because I_n are S_n–covariant.

In order to check (1.4), look at (3.1), this time interpreted as the linear relation between the homology classes of the boundary divisors. This implies

$$\sum_{\sigma : ij\sigma kl} \int_{\overline{M}_{0,n_1+1}\times\overline{M}_{0,n_2+1}} \varphi_\sigma^*(I_n(\gamma_1 \otimes \dots \otimes \gamma_n)) = (j \leftrightarrow k). \qquad (4.4)$$

Substituting (4.2) into (4.4) we obtain (1.4).

We can now state the central result of this section and Chapter:

4.3. Theorem. *Each ACF is the system of correlation functions of the unique CohFT. Thus, the following notions are equivalent:*

a). Formal Frobenius manifolds.

b). Cohomological Field Theories.

Before proving this theorem, we will discuss two related themes.

4.4. Tensor product. Let $\{H', g', I'_n\}$ and $\{H'', g'', I''_n\}$ be two CohFT's. Define $H = H' \otimes H''$ and $g = g' \otimes g''$. Put

$$I_n(\gamma'_1 \otimes \gamma''_1 \otimes \ldots \otimes \gamma'_n \otimes \gamma''_n) := \varepsilon(\gamma', \gamma'') I'_n(\gamma'_1 \otimes \ldots \otimes \gamma'_n) \wedge I''_n(\gamma''_1 \otimes \ldots \otimes \gamma''_n) \quad (4.5)$$

where $\epsilon(\gamma', \gamma'')$ is our standard sign in superalgebra, and $\wedge$ is the cup product in $H^*(\overline{M}_{0n}, k)$.

Claim. (H, g, I_*) *is a CohFT.*

One can easily check S_n–invariance and (4.2).

Thanks to the Theorem 4.3, this tensor product operation can be defined on $Comm_\infty$–algebras and formal Frobenius manifolds. But even if (4.5) looks very simple on the level of the full CohFT's, it cannot be trivially restricted to the former structures. In fact, they are directly formulated in terms of the top components of I_n whereas the tensor product involves components of all degrees.

4.5. Complete Cohomological Field Theories. Complete, as opposed to tree level, CohFT structure on (H, g) is given by a family of maps

$$I_{g,n} : H^{\otimes n} \to H^*(\overline{M}_{g,n}, k)$$

indexed by all stable pairs (g, n). They must satisfy the following extension of the genus zero axioms:

a). S_n–invariance for all g.

b). Splitting: for any g_1, g_2, $g_1 + g_2 = g$, and σ as above, such that $(g_i, n_i + 1)$ are stable, we must have

$$\varphi^*(I_{g,n}(\gamma_1 \otimes \ldots \otimes \gamma_n)) = \varepsilon(\sigma)(I_{g_1,n_1+1} \otimes I_{g_2,n_2+1})(\otimes_{p \in S_1} \gamma_p \otimes \Delta \otimes (\otimes_{q \in S_2} \gamma_q)) \quad (4.7)$$

where $\varphi : \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n}$ is the respective boundary morphism corresponding to the degeneration described in 2.7a).

c). Acquiring a cusp: for $g \geq 1$

$$\psi^*(I_{g,n}(\gamma_1 \otimes \ldots \otimes \gamma_n)) = I_{g-1,n+2}(\gamma_1 \otimes \ldots \otimes \gamma_n \otimes \Delta) \quad (4.8)$$

where $\psi : \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ is the boundary morphism described in 2.7b).

The theory of Gromov–Witten invariants actually furnishes such a structure on the cohomology spaces of projective algebraic and symplectic manifolds. Hence it is very important to study the complete CohFT's. The tensor product formula extends to the complete case and plays the role of the Künneth formula for the

Gromov–Witten invariants. However, it is not clear how to pass from a complete CohFT via formal generation functions to a meaningful geometric picture extending the theory of Frobenius manifolds.

Technically, the difference between the tree level and the complete case reflects our very incomplete understanding of the topology of $\overline{M}_{g,n}$ for $g \geq 1$.

We now start proving Theorem 4.3. We shall first show that if a CohFT with a given system of correlation functions exists at all, then it is unique.

4.6. Proposition. *Let (H, g, I_*) be a CohFT with correlation functions (Y_n). Then for any stable n–tree τ we have*

$$\int_{\overline{D}(\tau)} I_n(\gamma_1 \otimes \ldots \otimes \gamma_n) = (\otimes_{v\in V_\tau} Y_{F_\tau(v)})(\otimes_{i\in S_\tau}\gamma_i \otimes \Delta^{\otimes E_\tau}) \qquad (4.9)$$

Since the homology classes of $\overline{D}(\tau)$ span $H_*(\overline{M}_{0n}, k)$ (cf. 3.6), this establishes the uniqueness.

Proof. Let us explain the meaning of (4.9). We use the extension of the formalism of direct products to the arbitrary finite sets S. Then, say, $Y_n(\gamma_1 \otimes \cdots \otimes \gamma_n)$ can be replaced by $Y_S(\otimes_{i\in S}\gamma_i)$, and the argument of $\otimes_{v\in V_\tau} Y_{F_\tau(v)}$ must be some linear combination of the elements of the form $\otimes_{v\in V_\tau}(\otimes_{f\in F_\tau(v)}\alpha_f)$, $\alpha_f \in H$. If f is a tail marked by i, we choose $\alpha_f = \gamma_i$ in (4.9). Otherwise f is a half of an edge $\{f, \overline{f}\}$, and each such edge contributes Δ.

The formula (4.2) furnishes a particular case of (4.9) for the one–edge tree τ. But we can iterate (4.2) refining the inclusion $\mathbf{D}(\tau) \subset \overline{M}_{0n}$ to a sequence of codimension one boundary embeddings and using (4.2) at each step. A contemplation will convince the reader that (4.9) will be the final answer, independent on the chosen refinement. This proves the Proposition 4.6.

It remains to establish that if (Y_n) is an arbitrary ACF, then the formulas (4.9) actually define a CohFT. The only problem is to check that for any $n \geq 3$ and $\gamma_1, \ldots, \gamma_n \in H$, *there exists a cohomology class $I_n(\gamma_1 \otimes \cdots \otimes \gamma_n) \in H^*(\overline{M}_{0n}, k)$ which as a linear functional on $[\overline{D}(\tau)]$ is defined by* (4.9). Then it will be automatically S_n–invariant, and will satisfy (4.2).

In other words, it remains to show that all linear relations between $[\overline{D}(\tau)]$ are also satisfied by the right hand sides of (4.9). Again, for the codimension one case this is a built–in property: Keel's relations (3.3) between $[\overline{D}(\tau)]$ are precisely reflected in the quadratic relations (1.4) postulated for any ACF. To deal with arbitrary codimension, we will start with a generalization of Keel's relations.

4.7. Basic linear relations. As in 3.6, we will work with classes of boundary strata in $H^*(\overline{M}_{0S})$, represented by the classes of good monomials in $F_S \bmod I_S$.

Let $|S| \geq 4$. Consider a system $(\tau, v, \overline{i}, \overline{j}, \overline{k}, \overline{l})$ where τ is an S-tree, $v \in V_\tau$ is a vertex with $|v| \geq 4$ and $\overline{i}, \overline{j}, \overline{k}, \overline{l} \in F_\tau(v)$ are pairwise distinct flags (taken in this order). Put $T = F_\tau(v) \setminus \{\overline{i}, \overline{j}, \overline{k}, \overline{l}\}$. For any ordered 2-partition of T, $\alpha = \{T_1, T_2\}$, (one or both T_i can be empty) we can define two trees $\tau'(\alpha)$ and $\tau''(\alpha)$. The first one is obtained by inserting a new edge e at $v \in V$ with branches $\{\overline{i}, \overline{j}, T_1\}$ and $\{\overline{k}, \overline{l}, T_2\}$ at its edges. The second one corresponds similarly to $\{\overline{k}, \overline{j}, T_1\}$ and

$\{\bar{i}, \bar{l}, T_2\}$. We remind that $S(\bar{i})$ is the set of labels of tails belonging to the branch of $\bar{i}$: see Figure 4.

4.7.1. Proposition. *We have*

$$R(\tau, v, \bar{i}, \bar{j}, \bar{k}, \bar{l}) := \sum_{\alpha} [m(\tau'(\alpha)) - m(\tau''(\alpha))] \in I_S. \tag{4.10}$$

Proof. Choose $i \in S(\bar{i})$, $j \in S(\bar{j})$, $k \in S(\bar{k})$, $l \in S(\bar{l})$, and calculate $R_{ijkl}m(\tau) \in I_S$, where R_{ijkl} is defined by (3.3). Consider for instance the summands $D_\sigma m(\tau)$ for $ij\sigma kl$.

From the picture of τ it is clear that D_σ does not divide $m(\tau)$. If $D_\sigma m(\tau)$ does not vanish modulo I_S, we must have $D_\sigma m(\tau) = m(\sigma \times \tau)$, and $\sigma \times \tau$ is of the type $\tau'(\alpha)$. Similarly, the summands of $D_\sigma m(\tau)$ with $kj\sigma il$ are of the type $m(\tau''(\alpha))$.

4.8. Theorem. *All linear relations modulo I_S between good monomials of degree $r+1$ are spanned by the relations (4.10) for $|E_\tau| = r$.*

Proof. For $r = 0$ this holds by the definition of I_S. Generally, denote by H_{*S} the linear space, generated by the symbols $\mu(\tau)$ for all S–isomorphism classes τ of stable S–trees satisfying the analog of the relations (4.10)

$$r(\tau, v, \bar{i}, \bar{j}, \bar{k}, \bar{l}) := \sum_{\alpha} [\mu(\tau'(\alpha)) - \mu(\tau''(\alpha))] = 0. \tag{4.11}$$

Denote by 1 the symbol $\mu(\rho)$ where ρ is the one-vertex tree.

4.8.1. Main Lemma. *There exists on H_{*S} a structure of H_S^*–module given by the following multiplication formulas reproducing (3.7), (3.8) and (3.9):*

$$D_\sigma \mu(\tau) = \mu(\sigma \times \tau), \tag{4.12}$$

if $D_\sigma m(\tau)$ is a good monomial;

$$D_\sigma \mu(\tau) = 0, \tag{4.13}$$

if there exists a divisor $D_{\sigma'}$ of $m(\tau)$ such that $a(\sigma,\sigma')=4$;

$$D_\sigma \mu(\tau) = -\sum_{\substack{T\subset T_1\\ |T|\geq 1}} \mu(tr_{T,e}(\tau)) - \sum_{\substack{T\subset T_2\\ |T|\geq 1}} \mu(tr_{T,e}(T)), \tag{4.14}$$

if D_σ divides $M(\tau)$, and e corresponds to σ. The notation in (4.14) is the same as in (3.9).

Deduction of Theorem 4.8 from the Main Lemma. Since the monomials $m(\tau)$ satisfy (4.10), there exists a surjective linear map $a: H_{*S} \to H_S^*$, $\mu(\tau) \mapsto m(\tau)$. On the other hand, from (4.12) it follows that $m(\sigma)\mu(\tau) = \mu(\sigma\times\tau)$ if $m(\sigma)m(\tau)$ is a good monomial. Hence we have a linear map $b: H_S^* \to H_{*S}$: $m(\tau) \mapsto \mu(\tau) = m(\tau)1$ inverse to a. Therefore dim H_{*S} = dim H_S^* so that the Theorem 4.8 follows.

We now start proving the Main Lemma.

4.8.2. (4.14) is well defined. The right hand side of (4.14) formally depends on the choice of $\bar{i},\bar{j},\bar{k},\bar{l}$. We first check that different choices give the same answer modulo (4.11). It is possible to pass from one choice to another by replacing one flag at a time. So let us consider $\bar{i}'\neq\bar{i},\bar{j},\bar{k},\bar{l}$ and write the difference of the right hand sides of the relations (4.14) written for $(\tau,v,\bar{i},\bar{j},\bar{k},\bar{l})$ and $(\tau,v,\bar{i}',\bar{j},\bar{k},\bar{l})$. The terms corresponding to those T that do not contain $\{\bar{i},\bar{i}'\}$ cancel. This includes all terms with $T\subset T_2$. The remaining sum can be rewritten as

$$-\sum_{T\subset T_1\setminus\{\bar{i},\bar{i}',\bar{j}\}} \left[\mu(tr_{T\cup\{\bar{i}'\}}(\tau)) - \mu(tr_{T\cup\{\bar{i}\}}(\tau))\right] \tag{4.15}$$

where now T can be empty.

We contend that (4.15) is of the type (4.11). More precisely, consider any of the trees $tr_{T\cup\{\bar{i}'\}}(\tau)$, $tr_{T\cup\{\bar{i}\}}(\tau)$ and contract the edge whose vertices are incident to the flags $\bar{i},\bar{j},\bar{i}'$. We will get a tree σ and its vertex $v\in V_\tau$. The pair (σ,v) up to a canonical isomorphism does not depend on the transplants we started with. In $F_\sigma(v)$ there are flags $\bar{i},\bar{j},\bar{i}'$ and one more flag whose branch contains both k and l and which we denote $\bar{h}$. Then (4.15) is $-r(\sigma,v,\bar{i},\bar{j},\bar{i}',\bar{h})$. This is illustrated by the Figure 5.

4.8.3. Operators D_σ on H_{*S} pairwise commute. We have to prove the identities

$$D_{\sigma_1}(D_{\sigma_2}\mu(\tau)) = D_{\sigma_2}(D_{\sigma_1}\mu(\tau)). \tag{4.16}$$

Consider several possibilities separately.

i). *There exists a divisor D_σ of $m(\tau)$ such that $a(\sigma_1,\sigma) = 4$, so that $D_{\sigma_1}\mu(\tau) = 0$.*

If $D_{\sigma_2}\mu(\tau) = 0$ as well, (4.16) is true. If $D_{\sigma_2}\mu(\tau) = \mu(\sigma_2 \times \tau)$, then D_σ divides $m(\sigma_2 \times \tau)$, and (4.16) is again true. Finally, if D_{σ_2} divides $m(\tau)$, then $\sigma_2 \neq \sigma$ (otherwise $m(\tau)$ would not be a good monomial). Hence the transplants $tr_{T,e}(\tau)$ involved in the formula of the type (4.14) which we can use to calculate $D_{\sigma_2}\mu(\tau)$ will all contain an edge corresponding to σ so that $D_{\sigma_1}(tr_{T,e}(\tau)) = 0$, and (4.16) again holds.

The same argument applies to the case when $D_{\sigma_2}\mu(\tau) = 0$.

From now on we may and will assume that for any divisor D_σ of $m(\tau)$ we have $a(\sigma,\sigma_1) \leq 3$, $a(\sigma,\sigma_2) \leq 3$, and that $\sigma_1 \neq \sigma_2$.

ii). $a(\sigma_1,\sigma_2) = 4$ *and* D_{σ_2} *divides* $m(\tau)$.

Then D_{σ_1} does not divide $m(\tau)$, so that $D_{\sigma_1}\mu(\tau) = \mu(\sigma_1 \times \tau)$, and $D_{\sigma_2}(D_{\sigma_1}\mu(\tau)) = 0$. On the other hand, $D_{\sigma_2}\mu(\tau)$ is a sum of transplants to the midpoint of the edge, corresponding to σ_2. Each such transplant has an edge giving the 2-partition σ_2, so that $D_{\sigma_1}(D_{\sigma_2}\mu(\tau)) = 0$.

The case $a(\sigma_1,\sigma_2) = 4$ and $D_{\sigma_1}/m(\tau)$ is treated in the same way.

Hence from this point on we can and will in addition assume that $a(\sigma_1,\sigma_2) = 3$.

iii). D_{σ_1} *does not divide* $m(\tau)$.

If D_{σ_2} does not divide $m(\tau)$ as well, then $D_{\sigma_1}(D_{\sigma_2}\mu(\tau)) = D_{\sigma_1}\mu(\sigma_2 \times \tau) = \mu(\sigma_1 \times \sigma_2 \times \tau) = D_{\sigma_2}(D_{\sigma_1}\mu(\tau))$. If D_{σ_2} divides $m(\tau)$, we will use a carefully chosen formulas of the type (4.14) for the calculation of $D_{\sigma_2}\mu(\tau)$. Namely, let v_1 be the (unique) vertex of τ which gets replaced by an edge in $\sigma_1 \times \tau$, and let e_2 be the edge of τ corresponding to D_{σ_2}. Let u_2, u_1 be the vertices of e_2 such that u_1 can be joined to v_1 by a path not passing by e_2.

Consider first the subcase $u_1 \neq v_1$. Choose some $\bar{i}, \bar{j} \in F_\tau(u_2)$ and $\bar{k}, \bar{l} \in F_\tau(u_1)$ in such a way that $\bar{l}$ starts a path leading from u_1 to v_1. Use these $\bar{i}, \bar{j}, \bar{k}, \bar{l}$ in a formula of the type (4.14) to calculate $D_{\sigma_2}\mu(\tau)$ and then $D_{\sigma_1}(D_{\sigma_2}\mu(\tau))$, that will insert an edge instead of the vertex v_1 which survives in all the transplants entering $D_{\sigma_2}\mu(\tau)$. Then calculate $D_{\sigma_2}(D_{\sigma_1}\mu(\tau))$ by first inserting the edge at v_1, and then constructing the transplants not moving $\bar{i}, \bar{j}, \bar{k}, \bar{l}$. Since by our choice of $\bar{l}$ we never transplant the branch containing v_1, the two calculations will give the same result.

Now let $v_1 = u_1$. Let $\{S_1, S_2\}$ be the 2-partition of S corresponding to σ_1. Since $\sigma_1 \times \tau$ exists, $\{S_1, S_2\}$ is induced by a partition of $F_\tau(v_1) = \bar{S}_1 \coprod \bar{S}_2$. We denote by $\bar{S}_2$ the part to which the flag $(v_1 = u_1, e_2)$ belongs. Let $\bar{T} = \bar{S}_2 \setminus (\{(v_1 = u_1, e_2)\} \coprod F_\tau(u_2))$. This set is non-empty because otherwise e_2 would correspond to $\{S_1, S_2\}$ and we would have $\sigma_1 = \sigma_2$. Take $\bar{i}, \bar{j} \in F_\tau(u_2)$, $\bar{k} \in \bar{S}_1$ and $\bar{l} \in T$: see Figure 6.

Now consider $D_{\sigma_2}(D_{\sigma_1}\mu(\tau))$ and $D_{\sigma_1}(D_{\sigma_2}\mu(\tau))$. To calculate the first expression we form a sum of transplants of $\sigma_1 \times \tau$. To calculate the second one, we form transplants of τ, and then insert an edge at $v_1 = u_1$.

The transplants corresponding to the branches at u_2 will be the same in both expressions. The transplants corresponding to the subsets $T \subset \bar{T} \setminus \{\bar{l}\}$ will also be the same. In addition, the second expression will contain the terms $-D_{\sigma_1}(\mu(tr_{T,e_2}(\tau)))$ where $T \cap \bar{S}_1 \neq \emptyset$. But each such term vanishes. In fact, consider the 2–partition $\rho = \{R_1, R_2\}$ of S corresponding to the edge of $tr_{T,e_2}(\tau)$ containing the flag $(v_1 = u_1, e_2)$, and let $k, l \in R_1$. A glance to the third tree of the Figure 6 shows that $a(\rho, \sigma_1) = 4$, because if $\bar{t} \in T \cap \bar{S}_1$, $t \in S(\bar{t})$, then $kt\sigma_1 il$ and $kl\rho it$. Hence the extra terms are irrelevant.

The case when D_{σ_2} does not divide $m(\tau)$ is treated in the same way. It remains to consider the last possibility.

iv). *D_{σ_1} and D_{σ_2} divide $m(\tau)$*, $a(\sigma_1, \sigma_2) = 3$.

Denote by e_1 (resp. e_2) the edge corresponding to σ_1 (resp. σ_2). Let u_1, u_2 (resp. v_1, v_2) be the vertices of e_1 (resp. e_2) numbered in such a way that there is a path from u_2 to v_1 not passing through e_1, e_2 (the case $u_2 = v_1$ is allowed). To calculate the multiplication by D_{σ_1} choose $\bar{i}, \bar{j} \in F_{u_1}(\tau) \setminus \{(u_1, e_1)\}$, $\bar{l}$ on the path from u_2 to v_1 if $u_2 \neq v_1$, and $\bar{l} = (v_1, e_2)$ if $u_2 = v_1$; $\bar{k} \in F_\tau(v_2) \setminus \{\bar{l}\}$. To calculate the product by D_{σ_2}, choose similarly $\bar{k}', \bar{l}' \in F_\tau(v_2) \setminus \{(v_2, e_2)\}$, $\bar{i}' \in F_\tau(v_1)$ on the path from v_1 to u_2, if $v_1 \neq u_2$, and $\bar{i}' = (u_2, e_1)$ if $v_1 = u_2$, $\bar{j}' \in F_\tau(v_1)$ (see Figure 7).

The critical choice here is that of $\bar{l}$ and $\bar{i}'$. It ensures that calculating $D_{\sigma_1}(D_{\sigma_2}\mu(\tau))$ and $D_{\sigma_2}(D_{\sigma_1}\mu(\tau))$ we will get the same sum of transplanted trees. This ends the proof of (4.16).

4.8.4. Compatibility with I_S–generating relations. If $D_{\sigma_1} D_{\sigma_2} = 0$ because $a(\sigma_1, \sigma_2) = 4$, one sees that $D_{\sigma_1}(D_{\sigma_2}\mu(\tau)) = 0$ looking through various subcases in 4.8.3. It remains to show that $R_{ijkl}\mu(\tau) = 0$ where R_{ijkl} is defined by (3.3).

Consider the smallest connected subgraph in τ containing the flags i, j, k, l. The Figure 8 gives the following exhaustive list of alternatives. Paths from i to j and from k to l: i) have at least one common edge; ii) have exactly one common vertex; iii) do not intersect.

Consider them in turn.

i). Let e be an edge common to the paths ij and kl. Denote by ρ the respective

2-partition. Then $ik\rho jl$ or $il\rho kj$. Therefore any summand of R_{ijkl} annihilates D_ρ so that $R_{ijkl}\mu(\tau) = 0$ in view of (4.13).

ii). Let v be the vertex common to the paths ij and kl. Then exactly the same calculation as in the proof of the Proposition 4.7.1 shows that

$$R_{ijkl}\mu(\tau) = \sum_\alpha [\mu(\tau'(\alpha)) - \mu(\tau''(\alpha))] = 0$$

(notation as in (4.10) and (4.11)).

iii). This is the most complex case. Let us draw a more detailed picture of τ in the neighborhood of the subgraph we are considering (Figure 9).

Let v_1 be the vertex on the path ij which is connected by a sequence of edges $e_1, \ldots, e_m$ $(m \geq 1)$ with the vertex v_m on the path kl so that e_a has vertices (v_a, v_{a+1}) in this order. Let T_a be the set of flags at v_a which do not coincide with $\bar{i}, \bar{j}, \bar{k}, \bar{l}$, and do not belong to e_{a-1}, e_a.

Consider any summand D_σ of R_{ijkl}. If $jk\sigma il$, then $D_\sigma\mu(\tau) = 0$ because each edge e_a determines a partition ρ of S such that $ij\rho kl$. From now on we assume that $ij\sigma kl$. Then $D_\sigma\mu(\tau)$ can be nonzero if one of the two alternatives holds:

a). For some v_a, there exists a partition $T_a = T'_a \coprod T''_a$,(with $|T'_a| \geq 1, |T''_a \geq 1$, except for the case $a = 1$ where T'_1 can be empty, and $a = m$ where T'_m can be empty) such that the following two sets

$$S_1 = S(\bar{i}) \coprod S(\bar{j}) \coprod S(T'_1) \coprod \cdots \coprod S(T'_a),$$

$$S_2 = S(T''_a) \coprod S(T_{a+1}) \coprod \cdots \coprod S(T_m) \coprod S(\bar{k}) \coprod S(\bar{l})$$

form the 2-partition corresponding to σ. In this case

$$D_\sigma\mu(\tau) = \mu(\sigma \times \tau),$$

and $\sigma \times \tau$ is obtained by inserting a new edge at v_a and by distributing T'_a and T''_a at different vertices of this edge.

b). For some e_a, the two sets

$$S_1 = S(\bar{i}) \coprod S(\bar{j}) \coprod (\coprod_{i \leq a} S(T_i)),$$

$$S_2 = (\coprod_{i \geq a+1} S(T_i)) \coprod S(\bar{k}) \coprod S(\bar{l})$$

form the 2-partition corresponding to σ.

In this case D_σ divides $m(\tau)$, and in order to calculate $D_\sigma \mu(\tau)$ using a formula of the type (4.14) we must first choose two pairs of flags at the two vertices of v_a.

Contributions from a) and b) come with opposite signs, and we contend that they completely cancel each other.

To see the pattern of the cancellation look first at the case a) at v_1. It brings (with positive sign) the contributions corresponding to the following trees. Form all the partitions $T_1 = T_1' \coprod T_1''$ such that $T_1'' \neq \emptyset$, where $T_1 = F_\tau(v_1) \setminus \{\bar{i}, \bar{j}, (v_1, e_1)\}$. Transplant all T_1''-branches to the midpoint of e_1. Denote the new vertex v_1'. The result is drawn as Figure 10.

Now consider the terms of the type b) for the edge e_1. If $m = 2$, we choose for the calculation of $D_{\sigma_1}\mu(\tau)$ (where σ_1 corresponds to e_1) the flags $\bar{i}, \bar{j}, \bar{k}, \bar{l}$. If $m > 2$, we choose the flags $\bar{i}, \bar{j}, (v_2, e_1), t \in T_2$. Then we get the sum of two contributions. One will consist of the trees obtained by transplanting branches at v_1. They come with negative signs and exactly cancel the previously considered terms of the type a). If $m = 2$, the second group will cancel the terms of the type a) coming from v_2.

Consider a somewhat more difficult case $m > 2$. Then this second group of terms comes from the trees indexed by the partitions $T_2 = T_2' \coprod T_2'', t \in T_2'', T_2' \neq \emptyset$. Branches corresponding to T_2' are transplanted to the midpoint v_1' of the edge e_1. These terms come with negative signs: see Figure 11.

These trees in turn cancel with those coming from the terms of the type a) at the vertex v_2 with positive sign. However, there will be additional terms of the type a) for which $t \in T_2'$. They will cancel with one group of transplants contributing to $D_{\sigma_2}\mu(\tau)$ where σ_2 corresponds to the edge e_2 of the Figure 9, if for the calculation of $D_{\sigma_2}\mu(\tau)$ one uses (4.14) with the following choice of flags: $(v_2, e_1), t$ at one end, (v_3, e_1), some $t' \in T_3$ at the other end (this last choice must be replaced by $\bar{k}, \bar{l}$, if $m = 3$).

The same pattern continues until all the terms cancel.

4.8.5. Compatibility with relations (4.11). By this time we have checked that the action of any element of F_S/I_S on the individual generators $\mu(\tau)$ of H_{*S} is well defined modulo the span I_{*S} of relations (4.11). It remains to show that the subspace in $\oplus_\tau k\mu(\tau)$ spanned by these relations is stable with respect to this action. But the calculation in the proof of the Proposition 4.7.1 shows that

$$r(\tau, v, \bar{i}, \bar{j}, \bar{k}, \bar{l}) \equiv m(\tau) r_{ijkl} \mod I_{*S},$$

where r_{ijkl} is obtained from R_{ijkl} by replacing $m(\sigma)$ with $\mu(\sigma)$. To multiply this by any element of H_S^* we can first multiply it by $m(\tau)$, then represent the result as a linear combination of good monomials, and finally multiply each good monomial by r_{ijkl}. The result will lie in I_{*S}.

This finishes the proof of the Main Lemma and the Theorem 4.8.

4.9. End of proof of the Theorem 4.3. According to the remark at the last paragraph of 4.6, it remains to show that if we start with an ACF $Y_n : H^{\otimes n} \to k$, $n \geq 3$, (Definition 1.3.1) and extend these polynomial maps to all stable trees σ by putting

$$Y_\sigma(\otimes_{i \in S_\sigma} \gamma_i) = (\otimes_{v \in V_\sigma} Y_{F_\sigma(v)})(\otimes_{i \in S_\sigma} \gamma_i \otimes \Delta^{\otimes E_\sigma}) \tag{4.16}$$

then Y_τ will satisfy the following version of the relations (4.10):

$$\sum_\alpha Y_{\tau'(\alpha)} = \sum_\beta Y_{\tau''(\beta)}. \tag{4.17}$$

The notation is explained in the first paragraph of 4.7. Recall that we start with a tree τ, in which a vertex v and four tails $\bar{i}, \bar{j}, \bar{k}, \bar{l}$ are marked. The trees τ', τ'' are obtained from τ by inserting an edge at v. This can be done in many ways, parametrized by 2–partitions of $F_\tau(v)$. They induce 2–partitions of $\{\bar{i}, \bar{j}, \bar{k}, \bar{l}\}$. We put to the left those which break this quadruple into $\{\bar{i}, \bar{j}\} \cap \{\bar{k}, \bar{l}\}$, and to the right those which break it into $\{\bar{i}, \bar{l}\} \cap \{\bar{k}, \bar{j}\}$. The remaining partitions do not contribute.

To prove (4.17), rewrite every summand using (4.16). Look at the factor Δ corresponding to the inserted edge, and represent it as $\sum \Delta_a g^{ab} \otimes \Delta_b$. After some fumbling with indices, one can recognize in the obtained expression a linear combination of identities (1.4) written for various arguments and the one vertex tree with flags S_τ.