Generalized Harish-Chandra Modules

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Abstract. Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. If $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, we call a $\mathfrak{g}$-module M a strict $(\mathfrak{g}, \mathfrak{k})$-module if $\mathfrak{k}$ coincides with the subalgebra of all elements of $\mathfrak{g}$ which act locally finitely on M. For an intermediate $\mathfrak{k}$, i.e. such that $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$, we construct irreducible strict $(\mathfrak{g}, \mathfrak{k})$-modules. The method of construction is based on the $\mathcal{D}$-module localization theorem of Beilinson and Bernstein. The existence of irreducible strict $(\mathfrak{g}, \mathfrak{k})$-modules has been known previously only for very special subalgebras $\mathfrak{k}$, for instance when $\mathfrak{k}$ is the (reductive) subalgebra of fixed points of an involution of $\mathfrak{g}$. In this latter case strict irreducible $(\mathfrak{g}, \mathfrak{k})$-modules are Harish-Chandra modules.

We give also separate necessary and sufficient conditions on $\mathfrak{k}$ for the existence of an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module of finite type, i.e. an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module with finite $\mathfrak{k}$-multiplicities. In particular, under the assumptions that the intermediate subalgebra $\mathfrak{k}$ is reductive and $\mathfrak{g}$ has no simple components of types B_n for $n > 2$ or F_4, we prove a simple explicit criterion on $\mathfrak{k}$ for the existence of an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module of finite type. It implies that, if $\mathfrak{g}$ is simple of type A or C, for every reductive intermediate $\mathfrak{k}$ there is an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module of finite type.

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Notational conventions

The ground field is $\mathbb{C}$, however all our results can be easily carried out over an algebraically closed field of characteristic zero. $\mathbb{R}_+$ (respectively $\mathbb{Z}_+$) denotes the set of non-negative real numbers (respectively integers), and $\langle\cdot\rangle_{\mathbb{C}}$, $\langle\cdot\rangle_{\mathbb{R}_+}$, $\langle\cdot\rangle_{\mathbb{Z}}$, or $\langle\cdot\rangle_{\mathbb{Z}_+}$ stands for linear span respectively with coefficients in $\mathbb{C}$, $\mathbb{R}_+$, $\mathbb{Z}$, or $\mathbb{Z}_+$. If X is a topological space and $\mathcal{F}$ is a sheaf of abelian groups on X, then $\Gamma(\mathcal{F})$ denotes the global sections of $\mathcal{F}$ on X. If $U \subset X$ is an open subset, $\mathcal{F}_{|U}$ denotes the restriction of $\mathcal{F}$ onto U, and $\mathcal{F}(U) := \Gamma(\mathcal{F}_{|U})$. For $v \in \Gamma(\mathcal{F})$, $v_{|U}$ is the restriction of v to U. If X is an algebraic variety, $\mathcal{O}_X$ stands for the structure sheaf of X, and if $f : X \to Y$ is a morphism of algebraic varieties, f^* denotes an inverse image functor of $\mathcal{O}$-modules. A *multiset* is defined as a map from a set Y into $\mathbb{Z}_+\backslash\{0\}$, or, more informally, as a set whose elements have finite multiplicities.

1 Origin of the problem

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and M be a $\mathfrak{g}$-module. S. Fernando proved in [F] that all elements of $\mathfrak{g}$ which act locally finitely on M [3](i.e. all elements $g \in \mathfrak{g}$ such that $\langle m, g\cdot m, g^2\cdot m, \ldots\rangle_{\mathbb{C}}$ is a finite-dimensional subspace of M for any $m \in M$) form a Lie subalgebra $\mathfrak{g}[M]$ of $\mathfrak{g}$. [4] Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$. We call M a $(\mathfrak{g},\mathfrak{k})$*-module*, or a *generalized Harish-Chandra module for the pair* $(\mathfrak{g},\mathfrak{k})$, if $\mathfrak{k} \subset \mathfrak{g}[M]$. *A strict* $(\mathfrak{g},\mathfrak{k})$*-module*, or a *strict generalized Harish-Chandra module for the pair* $(\mathfrak{g},\mathfrak{k})$, is by definition a $(\mathfrak{g},\mathfrak{k})$-module M for which $\mathfrak{g}[M] = \mathfrak{k}$. Furthermore, we call $\mathfrak{k}$ a *Fernando subalgebra* of $\mathfrak{g}$ if there exists an irreducible strict $(\mathfrak{g},\mathfrak{k})$-module M. (Clearly, for every $\mathfrak{k}$ there exist induced $\mathfrak{g}$-modules which are strict $(\mathfrak{g},\mathfrak{k})$-modules but are not necessarily irreducible.)

As we show at the end of this section, not all subalgebras of $sl(n)$ are Fernando subalgebras. The problem of describing all Fernando subalgebras of a given finite-dimensional Lie algebra $\mathfrak{g}$ is, in our opinion, an important problem in the representation theory of Lie

[3]Fernando assumes that M is a finitely generated $\mathfrak{g}$-module but it is easy to see that this assumption is not necessary.

[4]A combination of arguments due to V. Kac (see [K]) gives a significantly simpler proof of the same result. Moreover, Kac shows that, for any infinite-dimensional Lie algebra $\mathfrak{a}$, the elements of $\mathfrak{a}$ which act locally finitely on an $\mathfrak{a}$-module form a linear subspace of $\mathfrak{a}$.

algebras. Not much is known for non-reductive Lie algebras $\mathfrak{g}$ and the problem is still open in the reductive case. One of our objectives is to desribe all Fernando subalgebras of a reductive Lie algebra $\mathfrak{g}$ containing a Cartan subalgebra. In what follows we will automatically assume that $\mathfrak{g}$ is reductive and that $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{g}$. In Theorem 1 below we prove that any intermediate subalgebra $\mathfrak{k}$, i.e. such that $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$, is a Fernando subalgebra. The corresponding irreducible strict $(\mathfrak{g}, \mathfrak{k})$-modules M are necessarily weight modules and, as it turns out, $\mathfrak{k}$ determines certain essential invariants of M. Here is a brief description of relevant known results about weight modules.

Recall that a $\mathfrak{g}$-module M is a *weight module* if

$$M = \bigoplus_{\nu \in \mathfrak{h}^*} M^\nu \ , \tag{1}$$

where $\mathfrak{h}^*$ stands for the dual space of $\mathfrak{h}$ and $M^\nu := \{m \in M \mid h \cdot m = \nu(h)m \ \forall \ h \in \mathfrak{h}\}$. It is well known that, if M is irreducible, (1) is equivalent to $\mathfrak{h}$ being contained in $\mathfrak{g}[M]$, see for instance Proposition 1 in [PS]. The spaces M^ν are the *weight spaces* of M, and the set of all weights of M, i.e. linear functions $\nu \in \mathfrak{h}^*$ with $M^\nu \neq 0$ is by definition the *support* $\operatorname{supp} M$ *of* M. The *weight multiplicities* are the dimensions $\dim M^\nu$ and they can be finite or infinite. The *formal character of* M is defined as the formal sum

$$\sum_{\nu \in \operatorname{supp} M} (\dim M^\nu) e^\nu \ .$$

This invariant of M has been studied now for about 70 years: the classical Weyl character formula from 1924 computes the formal character of any irreducible finite-dimensional $\mathfrak{g}$-module M as a function of its highest (or extremal) weight(s). In general, the support of an irreducible weight module M carries less information than its formal character, however at the two extremes, when dim $M < \infty$ or when dim $M^\nu = \infty$ for all $\nu \in \operatorname{supp} M$, the support determines the formal character. Furthermore, it is known that, if M is irreducible, all weight multiplicities are simultaneously finite or infinite. In the recent paper [M], O.Mathieu completed the classification of the irreducible weight modules with finite weight multiplicities (see also [BL]), and in particular Mathieu's results lead to a formula for the formal character of any such module. The case of infinite multiplicities is also important and interesting, for most Harish-Chandra weight modules have infinite weight multiplicities. This is the case for which Theorem 1 below is of interest.

Let now $\mathfrak{h} \subset \mathfrak{k}$ and M be an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module. To explain the relationship between $\mathfrak{k}$ and $\operatorname{supp} M$ we need to introduce one more invariant of M, its shadow decomposition. Let $\Delta \subset \mathfrak{h}^*$ denote the root system of $\mathfrak{g}$ and $\Delta_{\mathfrak{k}} \subset \Delta$ be the set of roots of $\mathfrak{k}$. Define $\Gamma_{\mathfrak{k}}$ as the submonoid of $\langle \Delta \rangle_{\mathbb{Z}}$ generated by $\Delta \backslash \Delta_{\mathfrak{k}}$. The *$M$-decomposition of* Δ, or the *shadow decomposition of* Δ *corresponding to* M,

$$\Delta = \Delta_M^- \sqcup \Delta_M^I \sqcup \Delta_M^F \sqcup \Delta_M^+ \ , \tag{2}$$

is defined by setting

$$\begin{array}{lcl}
\Delta_M^I & := & \{\alpha \in \Delta \mid \alpha \in \langle \Gamma_{\mathfrak{k}} \rangle_{\mathbb{R}_+}, \ -\alpha \in \langle \Gamma_{\mathfrak{k}} \rangle_{\mathbb{R}_+}\} \ , \\
\Delta_M^F & := & \{\alpha \in \Delta \mid \alpha \notin \langle \Gamma_{\mathfrak{k}} \rangle_{\mathbb{R}_+}, \ -\alpha \notin \langle \Gamma_{\mathfrak{k}} \rangle_{\mathbb{R}_+}\} \ , \\
\Delta_M^+ & := & \{\alpha \in \Delta \mid \alpha \notin \langle \Gamma_{\mathfrak{k}} \rangle_{\mathbb{R}_+}, \ -\alpha \in \langle \Gamma_{\mathfrak{k}} \rangle_{\mathbb{R}_+}\} \ , \\
\Delta_M^- & := & \{\alpha \in \Delta \mid \alpha \in \langle \Gamma_{\mathfrak{k}} \rangle_{\mathbb{R}_+}, \ -\alpha \notin \langle \Gamma_{\mathfrak{k}} \rangle_{\mathbb{R}_+}\} \ .
\end{array}$$

In particular, the M-decomposition of Δ is determined by $\mathfrak{k}$. The decomposition (2) induces a decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \left(\mathfrak{g}_M^I + \mathfrak{g}_M^F\right) \oplus \mathfrak{g}_M^+ \oplus \mathfrak{g}_M^- \ , \tag{3}$$

where

$$\mathfrak{g}_M^F := \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \Delta_M^F} \mathfrak{g}^\alpha \Big) \ , \qquad \mathfrak{g}_M^I := \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \Delta_M^I} \mathfrak{g}^\alpha \Big) \ , \qquad \mathfrak{g}_M^\pm := \bigoplus_{\alpha \in \Delta_M^\pm} \mathfrak{g}^\alpha \ .$$

It follows from the main results of [DMP] that $\mathfrak{p}_M := (\mathfrak{g}_M^F + \mathfrak{g}_M^I) \oplus \mathfrak{g}_M^+$ is a parabolic subalgebra whose semisimple part is nothing but the direct sum of Lie algebras $[\mathfrak{g}_M^F, \mathfrak{g}_M^F] \oplus [\mathfrak{g}_M^I, \mathfrak{g}_M^I]$. In particular, $[\mathfrak{g}_M^F, \mathfrak{g}_M^F]$ and $[\mathfrak{g}_M^I, \mathfrak{g}_M^I]$ are two commuting semisimple subalgebras of $\mathfrak{p}_M$. Furthermore, if $\mathfrak{k} = \mathfrak{g}[M]$, then

$$\mathfrak{k} = (\mathfrak{g}_M^F + (\mathfrak{k} \cap \mathfrak{g}_M^I)) \oplus \mathfrak{g}_M^+ \ .$$

This is a consequence of the inclusion $\mathfrak{g}_M^F \oplus \mathfrak{g}_M^+ \subset \mathfrak{k}$ and of the fact that $\mathfrak{g}_M^- \cap \mathfrak{k} = 0$. The latter follows easily from some basic representation theory of $sl(2)$.

Note now that the very definition of $\Gamma_{\mathfrak{k}}$ implies

$$\operatorname{supp} M = \operatorname{supp} M^{\mathfrak{k}} + \Gamma_{\mathfrak{k}} \ , \tag{4}$$

where $M^{\mathfrak{k}}$ is any irreducible finite-dimensional $\mathfrak{k}$-submodule of M; see also Proposition 2 in [PS]. Moreover, as $\Delta_M^I \subset \Gamma_{\mathfrak{k}}$ and $(\Delta_M^F + \Delta_M^I) \cap \Delta = \emptyset$, if M^F is any irreducible $\mathfrak{g}_M^F \oplus \mathfrak{g}_M^+$-submodule of $M^{\mathfrak{k}}$, we have also

$$\operatorname{supp} M = \operatorname{supp} M^F + \Gamma_{\mathfrak{k}} \ . \tag{5}$$

M^F always exists, is necessarily finite-dimensional, and is irreducible as a $\mathfrak{g}_M^F$-module. Since $\mathfrak{g}_M^F$ is reductive, $\operatorname{supp} M^F$ is nothing but the intersection of the convex hull of all extremal weights of M^F with the weight lattice of $\mathfrak{g}_M^F$ shifted by any element of $\operatorname{supp} M^F$. Furthermore, as $\mathfrak{g}_M^+$ acts by zero on M^F, the right-hand side of (5) equals simply $\operatorname{supp} M^F + \langle \Delta_M^I \rangle_{\mathbb{Z}} + \langle \Delta_M^- \rangle_{\mathbb{Z}_+}$. Therefore,

$$\operatorname{supp} M = \operatorname{supp} M^F + \langle \Delta_M^I \rangle_{\mathbb{Z}_+} + \langle \Delta_M^- \rangle_{\mathbb{Z}_+}, \tag{6}$$

and $\operatorname{supp} M^F$ is not determined by $\mathfrak{k}$ (and could be arbitrary for a fixed $\mathfrak{k}$) while $\langle \Delta_M^I \rangle_{\mathbb{Z}_+} + \langle \Delta_M^- \rangle_{\mathbb{Z}_+}$ is determined by $\mathfrak{k}$. In other words, the support of an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module is determined by $\mathfrak{k}$ up to adding the support of an arbitrary irreducible finite-dimensional $\mathfrak{g}_M^F$-module. (A more general version of this statement for a not necessarily reductive Lie algebra see in [PS].)

The starting point of this paper is the observation that, when M has infinite weight multiplicities, its formal character (or equivalently its support) does not fully determine the Lie algebra $\mathfrak{g}[M]$ and more precisely its subalgebra $\mathfrak{g}[M] \cap \mathfrak{g}_M^I$. In particular, the existing theory of weight modules provides no answer to the following question. Let $\mathfrak{k} \subset \mathfrak{g}$ be a subalgebra with $\mathfrak{k} \supset \mathfrak{h}$. Is it true that there exists an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module? Theorem 1 below gives an affirmative answer.

Once the existence problem for irreducible strict $(\mathfrak{g}, \mathfrak{k})$-modules is resolved, a further natural question arises. For which $\mathfrak{k}$ does there exist an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module M of finite type, i.e. such that considered as a module over the reductive part $\mathfrak{k}_{red}$ of $\mathfrak{k}$ M has finite-dimensional isotypic components? Our second objective is to give a partial answer to this latter question. Propositions 4 and 5 give respectively a necessary and a sufficient condition on $\mathfrak{k}$ for the existence of an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module of finite type. For a reasonably large class of subalgebras $\mathfrak{k}$ these conditions are directly verifiable and provide a definitive result, see Corollaries 1 and 2. In particular, if $\mathfrak{k}$ is reductive and $\mathfrak{g}$ has no simple components of type B_n for $n > 2$ or F_4, then $\mathfrak{g}$ admits an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module of finite type if and only if the centralizer in $\mathfrak{g}$ of the semisimple part of $\mathfrak{k}$ has simple components only of types A and C.

The $\mathfrak{k}$-finiteness problem has been studied previously in two particular cases: when $\mathfrak{k}$ coincides with the fixed points of an involution on $\mathfrak{g}$, and when $\mathfrak{k}$ is replaced by $\mathfrak{h}$. It is a classical result that in the first case any irreducible $(\mathfrak{g}, \mathfrak{k})$-module has finite type. In the second case finite type means nothing but finite weight multiplicities, and the following Proposition summarizes known results.

Proposition 1 *Let $\mathfrak{k} \subset \mathfrak{g}$ be a subalgebra with $\mathfrak{k} \supset \mathfrak{h}$, and let M be a strict irreducible $(\mathfrak{g}, \mathfrak{k})$-module.*

(a) *If $\mathfrak{k} \cap \mathfrak{g}_M^I \neq \mathfrak{h}$, then M necessarily has infinite weight multiplicities.*

(b) *If $\mathfrak{k} \cap \mathfrak{g}_M^I = \mathfrak{h}$ and M has finite weight multiplicities, then $[\mathfrak{g}_M^I, \mathfrak{g}_M^I]$ is isomorphic to a direct sum of simple Lie algebras of types A and C.*

(c) *If $\mathfrak{g}_M^I = \mathfrak{h}$, then M necessarily has finite weight multiplicities.*

Proof. Fernando has proved in [F] that $\mathfrak{k} \cap \mathfrak{g}_M^I = \mathfrak{h}$ whenever M has finite weight multiplicities. This implies (a). To prove (b), consider a non-zero vector $m \in M$ with $\mathfrak{g}_M^+ \cdot m = 0$. Recall that $\mathfrak{p}_M = (\mathfrak{g}_M^F + \mathfrak{g}_M^I) \oplus \mathfrak{g}_M^+$ is a parabolic subalgebra of $\mathfrak{g}$ and set $M^{FI} := U(\mathfrak{p}_M) \cdot m$. It is straightforward to verify that M^{FI} is irreducible as a $[\mathfrak{g}_M^F, \mathfrak{g}_M^F] \oplus [\mathfrak{g}_M^I, \mathfrak{g}_M^I]$-module. Therefore $M^{FI} \simeq M^F \boxtimes M^I$ for some irreducible finite- dimensional $[\mathfrak{g}_M^F, \mathfrak{g}_M^F]$-module M^F and some irreducible $[\mathfrak{g}_M^I, \mathfrak{g}_M^I]$-module M^I with $[\mathfrak{g}_M^I, \mathfrak{g}_M^I][M^I] = \mathfrak{h} \cap [\mathfrak{g}_M^I, \mathfrak{g}_M^I]$. Furthermore one notes that, as M has finite weight multiplicities, M^I has also finite $\mathfrak{h} \cap [\mathfrak{g}_M^I, \mathfrak{g}_M^I]$-weight multiplicities. Then the main result of Fernando, [F], implies that $[\mathfrak{g}_M^I, \mathfrak{g}_M^I]$ is isomorphic to a direct sum of simple Lie algebras of types A and C only. (b) is proved. Claim (c) follows from the fact that M^{FI} is a finite-dimensional $\mathfrak{p}_M$-module when $\mathfrak{g}_M^I = \mathfrak{h}$. Indeed, then the surjection $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M^{FI} \to M$ gives that M is a highest weight $\mathfrak{g}$-module with respect to any Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ with $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p}_M$. Hence M has finite weight multiplicities. $\square$

We will need the following slightly stronger version of claim (b). If M is a strict $(\mathfrak{g}, \mathfrak{k})$-module which is not necessarily irreducible, we call M *isotropic* if, for every non-zero $m \in M$, $\mathfrak{k}$ coincides with the set of elements $g \in \mathfrak{g}$ which act locally finitely on m. Any irreducible $\mathfrak{g}$-module M is automatically isotropic as a strict $(\mathfrak{g}, \mathfrak{g}[M])$-module.

Lemma 1 *Let $\mathfrak{k}$ be solvable, $\mathfrak{k} \supset \mathfrak{h}$, and let M be an isotropic strict $(\mathfrak{g}, \mathfrak{k})$-module with finite weight multiplicities. Then there exists a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ with $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{p} \cap \mathfrak{k} = \mathfrak{h}$, and such that its semisimple part $\mathfrak{p}_{ss}$ is a direct sum of simple Lie algebras of types A and C.*

Proof. Without loss of generality we can assume that M is generated by some 1-dimensional $\mathfrak{k}$-submodule $E^\lambda \subset M^\lambda$. Then (similarly to (4)) $\operatorname{supp} M = \lambda + \Gamma_{\mathfrak{k}}$. We claim that $\Delta_{\mathfrak{k}} \cap \Gamma_{\mathfrak{k}} = \emptyset$. Indeed, if $\alpha \in \Delta_{\mathfrak{k}} \cap \Gamma_{\mathfrak{k}}$, the solvability of $\mathfrak{k}$ implies that $\mathfrak{g}^\alpha$ acts locally finitely and $\mathfrak{g}^{-\alpha}$ acts freely on M. Furthermore $\{\lambda + \mathbb{Z}\alpha\} \subset \operatorname{supp} M$, and therefore there are infinitely many non-zero vectors $m^\nu \in M^\nu$ for $\nu \in \{\lambda + \mathbb{Z}\alpha\}$ with $\mathfrak{g}^\alpha \cdot m^\nu = 0$. Hence M has infinitely many Verma submodules of the rank one subalgebra $\mathfrak{g}^{-\alpha} \oplus \mathfrak{h} \oplus \mathfrak{g}^\alpha$, and consequently M has infinite weight multiplicities. Contradiction. Therefore $\mathfrak{p} := \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta \cap \Gamma_{\mathfrak{k}}} \mathfrak{g}^\alpha)$ is a parabolic subalgebra with $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{p} \cap \mathfrak{k} = \mathfrak{h}$.

It remains to show that the simple components of $\mathfrak{p}_{ss}$ are of types A and C only. Consider $E := U(\mathfrak{p}_{ss}) \cdot E^\lambda$. Since for every $\alpha \in \Delta_{\mathfrak{p}_{ss}}$ both $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ act freely on M, $\dim E^\mu = \dim E^{\mu \pm \alpha}$ for any $\mu \in \operatorname{supp} M$, i.e. all weight multiplicities of E are equal. Then E is admissible as defined in [M], and thus has finite length ([M], Lemma 3.3). As E is obviously a strict $(\mathfrak{p}_{ss}, \mathfrak{h} \cap \mathfrak{p}_{ss})$-module, any irreducible submodule of E is also a strict $(\mathfrak{p}_{ss}, \mathfrak{h} \cap \mathfrak{p}_{ss})$-module, and by Proposition 1(b) $\mathfrak{p}_{ss}$ is isomorphic to a direct sum of simple Lie algebras of types A and C. $\square$

We conclude this introductory section with an example of a subalgebra $\mathfrak{k}$ which is not a Fernando subalgebra. Let $\mathfrak{g} = sl(n)$ for $n > 2$, let $\mathfrak{b}$ denote the subalgebra of upper triangular matrices, $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$, and let $\mathfrak{k} \subset \mathfrak{b}$ be the subalgebra of all upper triangular matrices with zero first column. Note that $\mathfrak{k}$ contains no Cartan subalgebra. We claim that there is no irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module. Indeed, assume to the contrary that M is such a $\mathfrak{g}$-module. Then, since $\mathfrak{n} = [\mathfrak{k}, \mathfrak{k}]$, one can find a non-zero $m \in M$ such that $\mathfrak{k} \cdot m \subset \langle m \rangle_{\mathbb{C}}$ and $\mathfrak{n} \cdot m = 0$. Choose a basis $h_1, \ldots, h_{n-2}$ in $\mathfrak{h} \cap \mathfrak{k}$. The Casimir operator Ω can be written as

$$\Omega = u^2 + up(h_1, \ldots, h_{n-2}) + q(h_1, \ldots, h_{n-2}) + \sum_i x_i y_i$$

for some non-zero u in the 1-dimensional orthogonal complement of $\mathfrak{h} \cap \mathfrak{k}$ (with respect to the Killing form) in $\mathfrak{h}$, for certain polynomials p and q, $\deg p = 1$, $\deg q = 2$, and for certain strictly upper (respectively strictly lower) triangular matrices y_i (resp. x_i). As $\Omega \cdot m = \mu m$

for some $\mu \in \mathbb{C}$, and $h_i \cdot m = \lambda_i m$ for some $\lambda_i \in \mathbb{C}$, we obtain that $u^2 \cdot m \in \langle m \rangle_{\mathbb{C}} + \langle u \cdot m \rangle_{\mathbb{C}}$. Therefore u acts locally finitely on M, which is a contradiction because $u \notin \mathfrak{k}$.

2 Existence theorem

Theorem 1 *Any subalgebra $\mathfrak{k} \subset \mathfrak{g}$, such that $\mathfrak{h} \subset \mathfrak{k}$, is a Fernando subalgebra.*

The proof splits naturally into two cases: that of a solvable $\mathfrak{k}$ and that of a general $\mathfrak{k}$. We start with some preliminaries, most of which amount to fixing notations.

2.1 Preliminaries needed in the proof. Let G denote a connected reductive algebraic group with Lie algebra $\mathfrak{g}$. All subgroups of G considered will automatically be assumed to be connected and will be denoted by capital letters B, P, etc. The lower case German letters $\mathfrak{b}, \mathfrak{p}$, etc stand for the corresponding Lie subalgebras of $\mathfrak{g}$, and determine the subgroups B, P, etc.

B will always denote a fixed Borel subgroup of G whose Borel subalgebra $\mathfrak{b}$ contains the fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Sometimes $\mathfrak{b}$ will satisfy additional requirements which will be stated explicitly. The choice of $\mathfrak{b}$ fixes a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. By Δ^+ and Δ^- we denote respectively the roots of $\mathfrak{b}$ and $\mathfrak{b}^- := \mathfrak{h} \oplus \mathfrak{n}^-$. N^- is the subgroup of G with Lie algebra $\mathfrak{n}^-$. The big cell U of the flag variety G/B is defined as the (open) orbit $N^- \cdot B$ of the point $B \in G/B$. N^- acts freely on U. This enables us to identify N^- with $\mathfrak{n}^-$, and therefore obtain a set of coordinates $\{x_\alpha\}_{\alpha \in \Delta^-}$ on U which correspond to coordinates on $\mathfrak{n}^-$ arising from a basis of roots vectors in $\mathfrak{n}^-$. The algebra $\mathcal{O}_{G/B}(U)$ is then identified with the polynomial algebra $\mathbb{C}[x_\alpha]_{\alpha \in \Delta^-}$. Note also that $\mathfrak{g}$ acts by derivations on $\mathcal{O}_{G/B}$ via its canonical homomorphism into the tangent bundle $\mathcal{T}_{G/B}$.

If $i : Q \hookrightarrow G/B$ is the embedding of a (possibly singular) subvariety of G/B, $\bar{Q}$ will denote the closure of Q in G/B, and $\mathrm{Stab}_{\mathfrak{g}}\, Q \subset \mathfrak{g}$ is defined as the Lie algebra of the subgroup of G which preserves Q.

Throughout the rest of the paper μ will denote a fixed regular $\mathfrak{b}$-dominant weight, sometimes satisfying explicit additional conditions. Following Beilinson and Bernstein, we denote by $\mathcal{D}^\mu$ the sheaf of twisted differential operators (or "tdo" for short) on G/B corresponding to μ, see [BB1]. If Q is an open subvariety we define the tdo $\mathcal{D}^\mu_Q$ simply as $\mathcal{D}^\mu_{|Q}$. If Q is a closed non-singular subvariety then $\mathcal{D}^{\mu}_{Q}$ is defined as the sheaf of left differential endomorphisms of the inverse image $i^*\mathcal{D}^{\mu}$. Clearly, in this way the tdo $\mathcal{D}^{\mu}_{Q}$ is well-defined for any non-singular locally closed subvariety Q of G/B, and $\mathcal{D}^{\mu}_{Q} = \mathcal{D}^{\mu}$ when $Q = G/B$. If $\mu = \rho$ (where $\rho := 1/2\sum_{\alpha\in\Delta^+}\alpha$), then $\mathcal{D} := \mathcal{D}^{\rho}$ is nothing but the sheaf of differential operators on G/B. Furthermore, as U is affine, $\mathcal{D}^{\mu}(U)$ can be identified with the Weyl algebra $\mathbb{C}[u_\alpha, \frac{\partial}{\partial u_\alpha}]_{\alpha\in\Delta^-} = \mathcal{D}(U)$.

A $\mathcal{D}^{\mu}_{Q}$-module $\mathcal{F}$ is by definition a sheaf of $\mathcal{D}^{\mu}_{Q}$-modules on Q which is quasi-coherent as a sheaf of $\mathcal{O}_Q$-modules. The support of $\mathcal{F}$ is the closed subvariety of Q consisting of all closed points for which the sheaf-theoretic fiber of $\mathcal{F}$ is non-zero. We will make extensive use of the $\mathcal{D}^{\mu}$-affinity theorem of A. Beilinson and J. Bernstein, [BB1]. This theorem states that G/B is $\mathcal{D}^{\mu}$-affine , i.e. that every $\mathcal{D}^{\mu}$-module $\mathcal{F}$ is generated over $\mathcal{D}^{\mu}$ by its global sections $\Gamma(\mathcal{F})$ and that all higher cohomology groups of the sheaf $\mathcal{F}$ vanish. Consequently Γ is an equivalence between the category of $\mathcal{D}^{\mu}$-modules and the category of modules over the associative algebra $\Gamma(\mathcal{D}^{\mu})$. Moreover, a further result of Beilinson and Bernstein, [BB1], claims that $\Gamma(\mathcal{D}^{\mu})$ is canonically isomorphic to the quotient $U(\mathfrak{g})/(Ker\theta^{\mu})$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and $(Ker\theta^{\mu})$ is the two sided ideal generated by the kernel of the central character θ^{μ} of a module with $\mathfrak{b}$-highest weight $\mu - \rho$.

The following Lemma relates the Fernando subalgebra of the global sections of a $\mathcal{D}^{\mu}$-module with its support.

Lemma 2 *If Q is the support of a $\mathcal{D}^{\mu}$-module $\mathcal{F}$, then* $\mathfrak{g}[\Gamma(\mathcal{F})] \subset \mathrm{Stab}_{\mathfrak{g}}\, Q$.

Proof. Set $R := (G/B)\backslash Q$. For any $x \in \mathfrak{g}[\Gamma(\mathcal{F})]$, $g_x := \exp x$ is an automorphism of G/B as well as of $\Gamma(\mathcal{F})$. Furthermore, for any $v \in \Gamma(\mathcal{F})$,

$$v|_{g_x(R)} = g_x^{-1}(v)|_R = 0\ . \tag{7}$$

Since $\mathcal{F}$ is generated by $\Gamma(\mathcal{F})$ over $\mathcal{D}^{\mu}$, (7) implies $\mathcal{F}(g_x(R)) = 0$. But then $g_x(R) = R$ and therefore $x \in \mathrm{Stab}_{\mathfrak{g}}Q$. □

Assume that $i : Q \hookrightarrow G/B$ is a locally closed embedding of a non-singular subvariety Q. We denote by i_* the $\mathcal{D}^{\mu}_{Q}$-module direct image functor. By definition, $i_*\mathcal{F} := \mathcal{D}^{\mu}_{\leftarrow} \otimes_{\mathcal{D}^{\mu}_{Q}} \mathcal{F}$, where $\mathcal{D}^{\mu}_{\leftarrow} := i^*(\mathcal{D}^{\mu} \otimes_{\mathcal{O}_{G/B}} \Omega^*_{G/B}) \otimes_{\mathcal{O}_Q} \Omega_Q$ and Ω stands for volume forms. Furthermore the $\mathcal{D}^{\mu}$-module $i_*\mathcal{F}$ has a natural $\mathcal{O}_{G/B}$-module filtration with successive quotients

$$\Lambda^{max}(\mathcal{N}_Q) \otimes_{\mathcal{O}_Q} S^i(\mathcal{N}_Q) \otimes_{\mathcal{O}_Q} \mathcal{F}$$

for $i \in \mathbb{Z}_+$, where $\mathcal{N}_Q$ denotes the normal bundle of Q in G/B, S^i stands for i^{th} symmetric power, and Λ^{max} stands for maximal exterior power. In the case when i is a closed embedding, Kashiwara's theorem, [Ka], claims that i_* is an equivalence between the category of $\mathcal{D}^\mu_Q$-modules and the category of $\mathcal{D}^\mu$-modules with support in Q.

A locally closed embedding i is the composition of a closed embedding $j : Q \to V$ and an open embedding $l : V \to X$. Then $i_* = l_* j_*$, where l_* coincides with the sheaf-theoretic direct image. Under the additional assumptions that $\mathcal{F}$ is irreducible and locally free (of finite rank) as $\mathcal{O}_Q$-module, $i_*\mathcal{F}$ contains a unique irreducible $\mathcal{D}^\mu$-submodule $i_{*!}\mathcal{F}$, see [B], [BB1]. Furthermore, $i_{*!}\mathcal{F}(V) = i_*\mathcal{F}(V)$. In particular the support of $i_{*!}\mathcal{F}$ is $\bar{Q}$. If $i = j$, i.e. if i is closed, then $i_{*!}\mathcal{F}$ simply equals $i_*\mathcal{F}$.

We are now ready to start the proof of the Theorem 1. Throughout the proof $\mathfrak{k}$ is a fixed subalgebra of $\mathfrak{g}$ with $\mathfrak{h} \subset \mathfrak{k}$.

2.2 The case of a solvable $\mathfrak{k}$. Assume first that $\mathfrak{k}$ is solvable and let $\mathfrak{b} \supset \mathfrak{k}$. Set $\tilde{\Delta}_\mathfrak{k} := \Delta^- \backslash \Delta_\mathfrak{k}$.

Lemma 3 *There are global coordinates $\{u_\alpha\}_{\alpha \in \Delta_-}$ on U such that $h(u_\alpha) = \alpha(h) u_\alpha$ for any $h \in \mathfrak{h}$ and any $\alpha \in \Delta_-$, and $k(u_\alpha) = 0$ for all $k \in \mathfrak{k} \cap \mathfrak{n}^-$ and all $\alpha \in \tilde{\Delta}_\mathfrak{k}$.*

Proof. Consider the K-orbit $K \cdot B \subset U$. The polynomial algebra $\mathcal{O}(U) = \mathbb{C}[x_\alpha]_{\alpha \in \Delta^-}$, considered as a $\mathfrak{k}$-module, is a weight module (with respect to $\mathfrak{h}$). The ideal $I_{K \cdot B} := \Gamma(\mathcal{I}_{K \cdot B})$ is a $\mathfrak{k}$-submodule of $\mathcal{O}(U)$, and splits as a direct summand as an $\mathfrak{h}$-submodule. The quotient $\mathcal{O}(U)/I_{K \cdot B}$ is nothing but $\Gamma(\mathcal{O}_{K \cdot B}) = \mathbb{C}[x_\alpha]_{\alpha \in \Delta_\mathfrak{k}}$, and therefore we have an $\mathfrak{h}$-module decomposition

$$\mathcal{O}(U) = \mathbb{C}[x_\alpha]_{\alpha \in \Delta_\mathfrak{k}} \oplus I_{K \cdot B} \ .$$

Set then $u_\alpha := x_\alpha$ for $\alpha \in \Delta_\mathfrak{k}$, and define u_α for $\alpha \in \tilde{\Delta}_\mathfrak{k}$ to be the projection of x_α onto $I_{K \cdot B}$. It is straightforward to check that u_α are as required. $\square$

Lemma 4 *Set $Z := \cup_{\alpha \in \tilde{\Delta}_\mathfrak{k}} Z_\alpha$ where Z_α is the set of zeroes of u_α in U, and $V := U \backslash Z$. Then* $\mathrm{Stab}_\mathfrak{g} V = \mathfrak{k}$.

Proof. Put $Y := (G/B) \backslash V$. Then Y is the union of irreducible components

$$Y = (\cup_{\alpha \in \tilde{\Delta}_\mathfrak{k}} \bar{Z}_\alpha) \cup Z_1 \cup \cdots \cup Z_n \ ,$$

where $Z_1, \ldots, Z_n$ are all Schubert varieties of codimension 1 in G/B. Furthermore, $\mathrm{Stab}_{\mathfrak{g}} V = \mathrm{Stab}_{\mathfrak{g}} Y$ preserves each irreducible component, in particular

$$\mathrm{Stab}_{\mathfrak{g}} V \subset (\mathrm{Stab}_{\mathfrak{g}} \bar{Z}) \cap \mathfrak{n}^- = \mathfrak{k} \ .$$

On the other hand, the definition of u_α implies that Z_α is $\mathfrak{k}$-invariant. Therefore V is $\mathfrak{k}$-invariant and

$$\mathrm{Stab}_{\mathfrak{g}} V = (\mathrm{Stab}_{\mathfrak{g}} \bar{Z}) \cap \mathfrak{n}^- = \mathfrak{k} \ .$$

□

Fix now a map $\lambda : \tilde{\Delta}_{\mathfrak{k}} \to \mathbb{C}$ and consider the vector space

$$F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu} := \mathcal{O}(V) \otimes_{\mathbb{C}} \langle \prod_{\alpha \in \tilde{\Delta}_{\mathfrak{k}}} u_\alpha^{\lambda(\alpha)} \rangle_{\mathbb{C}} \ .$$

It has a natural structure of a $\mathcal{D}^\mu(V)$-module and therefore of a $\Gamma(\mathcal{D}^\mu)$-module (as $\Gamma(\mathcal{D}^\mu)$ is a subalgebra of $\mathcal{D}^\mu(V)$). Define $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ as the localization of $F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ to a $\mathcal{D}^\mu$-module, i.e. set

$$\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu} := \mathcal{D}^\mu \otimes_{\Gamma(\mathcal{D}^\mu)} F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu} \ .$$

Then $\Gamma(\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}) = F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$.

Proposition 2 *For almost all pairs λ, μ, $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ is an irreducible $\mathcal{D}^\mu$-module and $\mathfrak{g}[F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}] = \mathfrak{k}$.*

Proof. First we show that $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ is an irreducible $\mathcal{D}^\mu$-module whenever μ is generic and $(\mathrm{im}\lambda) \cap \mathbb{Z} = \emptyset$. Note that, as I_Z is generated by u_α for $\alpha \in \tilde{\Delta}_{\mathfrak{k}}$,

$$F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu} = \mathbb{C}[u_\beta]_{\beta \in \Delta_{\mathfrak{k}}} \otimes_{\mathbb{C}} \mathbb{C}[u_\alpha^{\pm 1}]_{\alpha \in \tilde{\Delta}_{\mathfrak{k}}} \otimes_{\mathbb{C}} \langle \prod_{\alpha \in \tilde{\Delta}_{\mathfrak{k}}} u_\alpha^{\lambda(\alpha)} \rangle_{\mathbb{C}} \ .$$

Therefore an immediate verification, using the condition $(\mathrm{im}\lambda) \cap \mathbb{Z} = \emptyset$, shows that as a $\mathcal{D}(U)$-module $F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ is generated by each monomial of the form $\prod_{\alpha \in \Delta^-} u_\alpha^{m_\alpha}$, where $m_\alpha \in \mathbb{Z}_+$ for $\alpha \in \Delta_{\mathfrak{k}}$ and $m_\alpha \in \lambda_\alpha + \mathbb{Z}$ for $\alpha \in \tilde{\Delta}_{\mathfrak{k}}$. On the other hand, $F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ is a semisimple module over the commutative subalgebra generated by $u_\alpha \frac{\partial}{\partial u_\alpha}$ for $\alpha \in \Delta^-$, and the corresponding weight spaces of $F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ are 1-dimensional and are spanned precisely by the above monomials. As any submodule of a weight module is also a weight module, every non-zero $\mathcal{D}(U)$-submodule of

$F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ contains at least one monomial, i.e. $F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ is necessarily an irreducible $\mathcal{D}(U)$-module. Thus $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu|U}$ is an irreducible $\mathcal{D}^{\mu}_{|U}$-module (as U is affine and thus $\mathcal{D}^{\mu}_{|U}$-affine).

To prove the irreducibility of $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$, consider the atlas $\{U_w\}_{w\in W}$ where $U_w := w(U)$ ($U = U_{id}$). It suffices to show that $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(U_w)$ is an irreducible $\mathcal{D}(U_w)$-module for each w (as U_w is affine and thus $\mathcal{D}^{\mu}_{G/B|U_w}$-affine). The crucial point here is that, if w is the reflection corresponding to a simple root α of $\mathfrak{b}$,

$$\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(U_w) = F^{w(\mathfrak{k})\cap\mathfrak{k},w(\mathfrak{b})}_{\lambda',\mu} \ ,$$

where

$$\lambda'(\beta) := \begin{cases} \lambda(\beta) & \text{if } \beta \neq \alpha \\ -\lambda(\beta) + 2\frac{(\mu,\beta)}{(\beta,\beta)} & \text{if } \beta = \alpha \text{ and } -\alpha \in \tilde{\Delta}_{\mathfrak{k}} \\ -2\frac{(\mu,\beta)}{(\beta,\beta)} & \text{if } \beta = \alpha \text{ and } -\alpha \in \Delta_{\mathfrak{k}} \ . \end{cases}$$

The fact that μ is generic ensures that λ' also satisfies the condition $(\mathrm{im}\lambda') \cap \mathbb{Z} = \emptyset$, and thus $F^{w(\mathfrak{k})\cap\mathfrak{k},w(\mathfrak{b})}_{\lambda',\mu}$ is an irreducible $\mathcal{D}(U_w)$-module. Therefore induction on the length of w with respect to $\mathfrak{b}$ enables us to complete the argument.

Finally, it remains to check that $\mathfrak{g}[F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}] = \mathfrak{k}$. Note that the restriction homomorphism $\Gamma(\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}) \to \mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V)$ is an isomorphism. If $x \in \mathfrak{g}[F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}]$ and $V_x := g_x \cdot V$, the restriction homomorphism $\Gamma(\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}) \to \mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V_x)$ is also an isomorphism. Thus $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V)$ and $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V_x)$ are canonically identified. Furthermore, $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V_x) = \mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V \cap V_x)$ by the very definition of $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$. This is sufficient to conclude that $V = V_x$, as otherwise the codimension of $V\backslash(V \cap V_x)$ in V would necessarily be 1 and the restriction map $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V) \to \mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V \cap V_x)$ would not be surjective. Therefore $x \in \mathrm{Stab}_{\mathfrak{g}}V$, and since $\mathrm{Stab}_{\mathfrak{g}}V = \mathfrak{k}$ (Lemma 4), we obtain

$$\mathfrak{g}[F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}] \subset \mathfrak{k} \ .$$

The opposite inclusion $\mathfrak{k} \subset \mathfrak{g}[F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}]$ is verified directly from the definition of $F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$. □

2.3 The case of a general $\mathfrak{k}$. Assume now that $\mathfrak{k}$ is arbitrary. The subalgebra $\mathfrak{k}_{ss}$ of $\mathfrak{g}$ generated by all $\mathfrak{g}^{\alpha}$ for $\alpha \in \Delta_{\mathfrak{k}} \cap -\Delta_{\mathfrak{k}}$ is a Levi subalgebra of $\mathfrak{k}$. Let $\mathfrak{c}$ denote the centralizer of $\mathfrak{k}_{ss}$ in $\mathfrak{g}$. Furthermore, for a basis $h_1, \dots, h_r$ of $\mathfrak{k}_{ss} \cap \mathfrak{h}$ arising from a Chevalley basis of $\mathfrak{k}_{ss}$, set $h := h_1 + \dots + h_r$. Then

$$\mathfrak{p}^h := \mathfrak{h} \oplus (\bigoplus_{\alpha(h)\geq 0} \mathfrak{g}^{\alpha})$$

is a parabolic subalgebra. We call any Borel subalgebra $\mathfrak{b}$ $\mathfrak{k}$-*preferable* if $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p}^h$ for a suitable choice of a Chevalley basis.

Fix a $\mathfrak{k}$-preferable Borel subalgebra $\mathfrak{b}$ and set $S := C \cdot B$. Clearly, S is a closed subvariety of G/B isomorphic to the flag variety $C/(C \cap B)$. As $\mathfrak{c} \cap \mathfrak{k}$ is a solvable subalgebra we can consider $U_S, Z_S, V_S \subset S$ as in section 2.2 for the pair $(\mathfrak{c}, \mathfrak{c} \cap \mathfrak{k})$ instead of $(\mathfrak{g}, \mathfrak{k})$. Furthermore, we define Y_S to be the set of all points $s \in V_S$ for which $\dim((\mathfrak{k} + \mathfrak{c}) \cap \mathfrak{b}_s)$ is minimal possible. Obviously, Y_S is an affine open $K \cap C$-invariant subvariety of V_S.

Lemma 5 *$Q := K \cdot Y_S$ is a non-singular subvariety in G/B and* $\mathfrak{k} \subset \mathrm{Stab}_{\mathfrak{g}}(\bar{Q}) \subset \mathfrak{k} + \mathfrak{c}$.

Proof. To prove that Q is non-singular it is sufficient to show that every $s \in Y_S$ is a non-singular point in $K \cdot S$, i.e. that the dimension of the tangent space $\mathcal{T}_{K \cdot S}(s)$ is constant for all $s \in Y_S$. The latter follows immediately from the equality

$$\dim T_{K \cdot S}(s) = \dim(\mathfrak{k} + \mathfrak{c}) - \dim((\mathfrak{k} + \mathfrak{c}) \cap \mathfrak{b}_s),$$

as by the definition Y_S consists of s for which the right-hand side is maximal possible.

It is obvious that $\mathfrak{k} \subset \mathrm{Stab}_{\mathfrak{g}}\bar{Q}$. Furthermore $S = \bar{V}_S$, and thus $K \cdot S \subset \bar{Q}$. If $x \in \mathrm{Stab}_{\mathfrak{g}}\bar{Q}$, then $g_{tx} \cdot B \in K \cdot S$ for sufficiently small $t > 0$ as $K \cdot S$ is locally closed in G/B. This implies that $x \in \mathfrak{k} + \mathfrak{c} + \mathfrak{b}$, i.e.

$$\mathfrak{k} \subset \mathrm{Stab}_{\mathfrak{g}}\bar{Q} \subset \mathfrak{k} + \mathfrak{c} + \mathfrak{b} \ .$$

To show that actually $\mathrm{Stab}_{\mathfrak{g}}\bar{Q} \subset \mathfrak{k} + \mathfrak{c}$, consider $\mathrm{Stab}_{\mathfrak{g}}\bar{Q}$ as a $\mathfrak{k}_{ss}$-module. $\mathrm{Stab}_{\mathfrak{g}}\bar{Q}$ is isomorphic to $\mathfrak{k} \oplus \mathfrak{m}$ for some $\mathfrak{k}_{ss}$-submodule $\mathfrak{m}$ of $\mathfrak{c} + \mathfrak{b}$. Let α be a $\mathfrak{b} \cap \mathfrak{k}_{ss}$-minimal weight of $\mathfrak{m}$. Then $\alpha(h_i) \leq 0$ for $i = 1, \ldots, r$ and, as $\mathfrak{m} \subset \mathfrak{c} + \mathfrak{b}$, we have $\mathfrak{g}^\alpha \subset \mathfrak{b}$. Thus, $\alpha(h) \geq 0$. This is possible only when $\alpha(h_i) = 0$ for each $i = 1, \ldots, r$, i.e. when $\mathfrak{g}^\alpha \in \mathfrak{c}$. Since $\mathfrak{m}$ is generated by $\mathfrak{g}^\alpha$ for all minimal α, we have finally $\mathfrak{m} \subset \mathfrak{c}$. Therefore $\mathrm{Stab}_{\mathfrak{g}}\bar{Q} \subset \mathfrak{k} + \mathfrak{c}$. □

Let $\mathfrak{k}_r$ denote the radical of $\mathfrak{k}$. As an $\mathfrak{h}$-module $\mathfrak{k}_r$ equals $(\mathfrak{k} \cap \mathfrak{c}) \oplus \mathfrak{t}$ for a unique $\mathfrak{h}$-invariant subspace $\mathfrak{t}$ of $\mathfrak{k}_r$. Set T:=exp$\mathfrak{t}$. Since $\mathfrak{k}_r$ is solvable and $\mathfrak{t}$ is contained in the nilpotent radical of $\mathfrak{k}_r$, one can easily show that the multiplication map $m : T \times (K \cap C) \to K_r$ is an isomorphism of algebraic varieties. Therefore $K_{ss} \times T \times (K \cap C) \simeq K$, and furthermore $K_{ss} \times T \times C \simeq K \cdot C$. Hence one can define a projection $K \times C \to C$, and this projection induces the morphism $p : K \cdot C \cdot B \to S$. Obviously $p(Q) = Y_S$.

Recall our construction for the solvable case and apply it to the pair $(\mathfrak{c}, \mathfrak{c}\cap\mathfrak{k})$. Under the assumptions that the the restriction of μ to $\mathfrak{c}\cap\mathfrak{h}$ is generic (and $\mathfrak{b}\cap\mathfrak{c}$-dominant) and that the restriction of μ to $\mathfrak{k}_{ss}\cap\mathfrak{h}$ equals the half-sum $\mathfrak{k}_{ss}\cap\mathfrak{b}$-positive roots of $\mathfrak{k}_{ss}$, this construction yields an irreducible $\mathcal{D}^{\mu}_S$-module $\mathcal{F}_S$. Put $\mathcal{F} := \mathcal{F}_{S|Y_S}$.

Proposition 3 *$M := \Gamma(i_{*!}p^*\mathcal{F})$ is an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module.*

Proof. Obviously $p^*\mathcal{F}$ is an irreducible $\mathcal{D}^{\mu}_Q$-module which is a locally free $\mathcal{O}_Q$-module of finite rank. Therefore $i_{*!}p^*\mathcal{F}$ is an irreducible $\mathcal{D}^{\mu}$-module, and M is an irreducible $\mathfrak{g}$-module. All it remains to show is that $\mathfrak{g}[M] = \mathfrak{k}$.

We have $\mathfrak{k} = \tilde{\mathfrak{k}} \oplus (\mathfrak{c}\cap\mathfrak{h})$ where $\tilde{\mathfrak{k}} := [\mathfrak{k}, \mathfrak{k}]$. As a $\tilde{\mathfrak{k}}$-sheaf $i_{*!}p^*\mathcal{F}$ is isomorphic to $i_{*!}\mathcal{O}_Q$, and thus the $\tilde{\mathfrak{k}}$-action on $i_{*!}p^*\mathcal{F}$ comes from the action of $\tilde{K} \subset G$ on $\mathcal{O}_Q$. This implies that $\tilde{\mathfrak{k}}$ acts locally finitely on M. As a $\mathfrak{c}\cap\mathfrak{h}$-sheaf $i_{*!}p^*\mathcal{F}$ is isomorphic to $\mathcal{O}_Q \otimes L$ for some one-dimensional $\mathfrak{c}\cap\mathfrak{h}$-module L. Therefore $\mathfrak{c}\cap\mathfrak{h}$ acts also locally finitely on M, and $\mathfrak{k} \subset \mathfrak{g}[M]$.

To verify the opposite inclusion, note that, by Lemma 2, $\mathfrak{g}[M] \subset \mathrm{Stab}_{\mathfrak{g}}\bar{Q} \subset \mathfrak{k} + \mathfrak{c}$. Thus we need to check only that $\mathfrak{g}[M] \cap \mathfrak{c} \subset \mathfrak{k}\cap\mathfrak{c}$. By construction, $\Gamma(\mathcal{F})$ is an isotropic strict $(\mathfrak{c}, \mathfrak{k}\cap\mathfrak{c})$-module. Therefore $\Gamma(p^*\mathcal{F}_{|Q})$ is an isotropic strict $(\mathfrak{c}, \mathfrak{k}\cap\mathfrak{c})$-module. Furthermore, $i_*p^*\mathcal{F}$ has a natural $\mathfrak{c}$-sheaf filtration whith successive quotients

$$\Lambda^{max}(\mathcal{N}_Q) \otimes_{\mathcal{O}_Q} S^i(\mathcal{N}_Q) \otimes_{\mathcal{O}_Q} p^*\mathcal{F}.$$

This implies that $\Gamma(i_*p^*\mathcal{F})$ is also an isotropic strict $(\mathfrak{c}, \mathfrak{k}\cap\mathfrak{c})$-module. Since M is a submodule of $\Gamma(i_*p^*\mathcal{F})$, finally $\mathfrak{c}[M] = \mathfrak{g}[M] \cap \mathfrak{c} = \mathfrak{k}\cap\mathfrak{c}$.

□

Example 1 If $\mathfrak{c}$ is abelian, Q is a (possibly non-closed) K-orbit in G/B. In this case our construction is the same as the Beilinson-Bernstein construction of Harish-Chandra modules, see [B], [BB1], [BB2]. In the classical Harish-Chandra setting $\mathfrak{k}$ coincides with the fixed points of an inner involution and $\mathfrak{c}$ is automatically abelian.

3 On $(\mathfrak{g}, \mathfrak{k})$-modules of finite type

Let M be a $\mathfrak{g}$-module and $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}[M]$ whose reductive part $\mathfrak{k}_{red}$ acts semisimply on M. (If M is irreducible then $\mathfrak{k}_{red}$ automatically acts semisimply.) We say that M is of *finite* (respectively, *infinite*) *type* over $\mathfrak{k}$ if each isotypic $\mathfrak{k}_{red}$-component of M is finite-dimensional (resp., infinite-dimensional). When M is a strict $(\mathfrak{g}, \mathfrak{k})$-module we will say simply that M is of *finite* (or, *infinite*) *type*. We define a Fernando subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ to be *of finite type* if there exists an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module of finite type. Otherwise we say that $\mathfrak{k}$ is *of infinite type*. The problem of classifying all Fernando subalgebras of finite type is of obvious significance in the structure theory of $\mathfrak{g}$-modules. In this section we give separate necessary and sufficient conditions for a subalgebra $\mathfrak{k}$ with $\mathfrak{k} \supset \mathfrak{h}$ ($\mathfrak{k}$ is a Fernando subalgebra by Theorem 1) to be of finite type, providing in this way a partial solution of this problem.

Our starting point is the following Lemma which is similar to the fact that the weight multiplicities of an irreducible weight $\mathfrak{g}$-module are either all finite or all infinite.

Lemma 6 *For any reductive subalgebra $\mathfrak{k} \subset \mathfrak{g}$, an irreducible $(\mathfrak{g}, \mathfrak{k})$-module is either of finite or infinite type over $\mathfrak{k}$.*

Proof. Any given irreducible $(\mathfrak{g}, \mathfrak{k})$-module M is a quotient of an induced module

$$M' := U(\mathfrak{g}) \otimes_{U(\mathfrak{g})^{\mathfrak{k}}} M_0,$$

where M_0 is a (any) $\mathfrak{k}$-isotypic component of M and $U(\mathfrak{g})^{\mathfrak{k}} := \{y \in U(\mathfrak{g}) | \mathrm{ad}_{\mathfrak{k}}(y) = 0\}$. If M is not of infinite type over $\mathfrak{k}$, we can choose M_0 to be finite-dimensional. Therefore, to prove that M is then of finite type over $\mathfrak{k}$, it suffices to show that M' is of finite type over $\mathfrak{k}$.

Note that there is an isomorphism of $\mathfrak{k}$-modules

$$M' \simeq (U(\mathfrak{g})/(U(\mathfrak{g}) \cdot U(\mathfrak{g})^{\mathfrak{k}})) \otimes_{\mathbb{C}} M_0$$

($U(\mathfrak{g})$ being $\mathfrak{k}$-module via the adjoint action). As M_0 is finite-dimensional, it is enough to prove that $U(\mathfrak{g})/(U(\mathfrak{g}) \cdot U(\mathfrak{g})^{\mathfrak{k}})$ is of finite type over $\mathfrak{k}$. The Poincaré-Birkhoff-Witt theorem gives an isomorphism of $\mathfrak{k}$-modules

$$U(\mathfrak{g})/(U(\mathfrak{g}) \cdot U(\mathfrak{g})^{\mathfrak{k}}) \simeq S^{\cdot}(\mathfrak{g})/I(\mathfrak{g}),$$

$I(\mathfrak{g})$ being the ideal in $S^{\cdot}(\mathfrak{g})$ generated by the invariants of the coadjoint action of $\mathfrak{k}$ in $\mathfrak{g}^*$. Furthermore, $S^{\cdot}(\mathfrak{g})/I(\mathfrak{g}) \simeq \mathrm{gr}\Gamma(\mathcal{O}_X)$, where X is a generic closed K-orbit in $\mathfrak{g}^*$, $\mathcal{O}_X$ is the sheaf of regular functions on X, and gr stands for graded ring. Since passing to the graded ring commutes with the $\mathfrak{k}$-action, all it remains to show is that $\Gamma(\mathcal{O}_X)$ is of finite type over $\mathfrak{k}$. But, for any $x \in X$, we have $X \simeq K/K^x$, where K^x is the stabilizer of x. Thus $\Gamma(\mathcal{O}_X)$ can be identified with the subspace of regular functions on K which are invariant with respect to right multiplication by elements from K^x. As K is a linear algebraic group, the $\mathfrak{k}$-module of all regular functions on K is of finite type over $\mathfrak{k}$. Therefore $\Gamma(\mathcal{O}_X)$ is also of finite type over $\mathfrak{k}$.

$\square$

In the rest of the paper we assume that $\mathfrak{k} \supset \mathfrak{h}$ and address the question of when $\mathfrak{k}$ is of finite type. If $\mathfrak{k}$ is solvable, a $(\mathfrak{g}, \mathfrak{k})$-module is of finite $\mathfrak{k}$-type if and only if it has finite weight multiplicities. Furthermore there is a simple criterion for a solvable $\mathfrak{k}$ to be of finite type. Namely, $\mathfrak{k}$ is of finite type if and only if $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with $\mathfrak{p} \cap \mathfrak{k} = \mathfrak{h}$ such that $\mathfrak{p}_{ss}$ is a direct sum of simple Lie algebras of types A and C. In one direction this is a direct corollary of Lemma 1, and in the other direction it follows from the fact that, given $\mathfrak{p}$, there always exists an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module M of finite type with $\mathfrak{p}_M = \mathfrak{p}$. M can be taken as the module induced from a suitable $\mathfrak{p}_{ss} + \mathfrak{k}$-module M_0 with trivial action of $[\mathfrak{k}, \mathfrak{k}]$. We leave it to the reader to check this.

The above observation leads to the following necessary condition for a general $\mathfrak{k}$ to be of finite type. Recall that the reductive subalgebra $\mathfrak{c} \subset \mathfrak{g}$ is defined as the centralizer of $\mathfrak{k}_{ss}$ in $\mathfrak{g}$.

Proposition 4 *Let $\mathfrak{k}$ be of finite type. Then the solvable subalgebra $\mathfrak{k} \cap \mathfrak{c}$ is of finite type in $\mathfrak{c}$, i.e. $\mathfrak{c} = (\mathfrak{k} \cap \mathfrak{c}) + \mathfrak{p}$ for some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{c}$ with $\mathfrak{p} \cap \mathfrak{k} = \mathfrak{h}$ and such that $\mathfrak{p}_{ss}$ is a direct sum of simple Lie algebras of types A and C. In addition, $[\mathfrak{k}, \mathfrak{p}_{ss}] \subset \mathfrak{k}$.*

Proof. Let M be an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module of finite type and M' be a $\mathfrak{k}_{ss}$-isotypic component of M. Fix a Borel subalgebra $\mathfrak{b}' \subset \mathfrak{k}_{red}$ with $\mathfrak{b}' \supset \mathfrak{h}$ and set $M'' := \{m' \in M' | [\mathfrak{b}', \mathfrak{b}'] \cdot m' = 0\}$. An immediate verification shows that M'' is an isotropic strict $(\mathfrak{c}, \mathfrak{k} \cap \mathfrak{c})$-module of finite type. Therefore $\mathfrak{c} = (\mathfrak{k} \cap \mathfrak{c}) + \mathfrak{p}$ for a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{c}$ as in Lemma 1.

It remains to show that $[\mathfrak{k}, \mathfrak{p}_{ss}] \subset \mathfrak{k}$. Assume the contrary, i.e. that there are $\gamma \in \Delta_{\mathfrak{k}} \backslash \Delta_{\mathfrak{k}_{red}}$ and $\beta \in \Delta_{\mathfrak{p}_{ss}}$ with $\alpha = \beta + \gamma \in \Delta \backslash \Delta_{\mathfrak{k}}$. Moreover, γ can be assumed to be a $\mathfrak{b}'$-maximal weight of the $\mathfrak{k}_{red}$-module $\mathfrak{k}$. Let $\alpha_1, \ldots, \alpha_s$ be the simple roots of $\mathfrak{b}'$. Then $-\gamma, \alpha_1, \ldots, \alpha_s$ is a system of simple roots of a reductive subalgebra $\mathfrak{s}$ containing $\mathfrak{k}_{red}$. The subalgebra $\mathfrak{q} := \mathfrak{k} \cap \mathfrak{s}$ is a parabolic subalgebra of $\mathfrak{s}$, and its reductive part equals $\mathfrak{k}_{red}$. As $\mathfrak{q} \subset \mathfrak{k}$, there is an irreducible finite-dimensional $\mathfrak{q}$-submodule $L_\lambda \subset M$ of $\mathfrak{b}'$-highest weight λ. Furthermore, since $\Delta_{\mathfrak{q}} \backslash \Delta_{\mathfrak{k}_{red}} \subset \Gamma_{\mathfrak{k}}$, M has an infinite family of finite-dimensional irreducible $\mathfrak{q}$-submodules L_{λ_k} with highest weights λ_k belonging to $\{\lambda + \langle \Delta_{\mathfrak{q}} \backslash \Delta_{\mathfrak{k}_{red}} \rangle_{\mathbb{Z}_+}\}$. Let $M_k := U(\mathfrak{s}) \cdot L_{\lambda_k}$. For almost all k, M_k contains a $\mathfrak{k}_{red}$-submodule isomorphic to L_λ. Thus the multiplicity of L_λ in M is infinite. Contradiction. $\square$

Proposition 4 implies the existence of a large class of reductive subalgebras $\mathfrak{k}$ of infinite type.

Corollary 1 *Let $\mathfrak{g}$ be simple of types B_n for $n > 4$, D_n for $n > 4$, E_7, E_8, and let $\mathfrak{k}$ be reductive and such that $\mathfrak{c}_{ss}$ has a simple component not of type A or C. Then $\mathfrak{k}$ is of infinite type.*

Example 2 The Lie algebra $\mathfrak{g} = sp(6)$ is the simple Lie algebra of smallest dimension which admits a subalgebra $\mathfrak{k}$ of infinite type with non-zero nilpotent radical and non-zero semisimple part. Let $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha} \oplus \mathfrak{g}^\beta$ where α and β are respectively a long and a short orthogonal roots. Then there is no $\mathfrak{p}$ as in Proposition 5, and hence $\mathfrak{k}$ is of infinite type.

Our final step is to state a sufficient condition for a reductive $\mathfrak{k}$ to be of finite type.

Proposition 5 *Let $\mathfrak{k}$ be reductive and such that the simple components of $\mathfrak{c}_{ss}$ are of types A and C only. If in addition $\langle \Delta_{\mathfrak{c}} \rangle_{\mathbb{C}} \cap \Delta = \Delta_{\mathfrak{c}}$, then $\mathfrak{k}$ is of finite type.*

Proof. Let $\mathfrak{h}'$ be the centralizer of $\mathfrak{c}_{ss}$ in $\mathfrak{h}$, and $\mathfrak{k}' := \mathfrak{k}_{ss} + \mathfrak{h}'$. The condition $\langle \Delta_{\mathfrak{c}} \rangle_{\mathbb{C}} \cap \Delta = \Delta_{\mathfrak{c}}$ enables us to choose a $\mathfrak{k}$-preferable Borel subalgebra $\mathfrak{b}$ for which the projection of $\Delta^- \backslash \Delta'_{\mathfrak{c}}$ onto $(\mathfrak{h}')^*$ is contained in some open half-space of some real subspace of $(\mathfrak{h}')^*$. Put $Q_1 := K'/(K' \cap B)$, $Q_2 := C_{ss}/(C_{ss} \cap B)$ and $Q := (K' \times C_{ss}) \cdot B$. Then $Q = Q_1 \times Q_2$ is a closed subvariety of G/B, and Lemma 5 implies that $\mathrm{Stab}_{\mathfrak{g}} Q = \mathfrak{k}' \oplus \mathfrak{c}_{ss}$. Let F_2 be an irreducible strict $(\mathfrak{c}_{ss}, \mathfrak{h}'')$-module with finite weight multiplicities, where $\mathfrak{h}'' := \mathfrak{c}_{ss} \cap \mathfrak{h}$. The existence of F_2 is well-known, see for instance [M]. Denote by $\mathcal{F}_2$ the localization of F_2 on Q_2. Set finally $\mathcal{F} := \mathcal{O}_{Q_1} \boxtimes \mathcal{F}_2$, $\mathcal{M} := i_* \mathcal{F}$ and $M := \Gamma(\mathcal{M})$, i being the closed embedding of Q into G/B.

Arguments similar to ones in the proof of Proposition 3 imply that M is a strict irreducible $(\mathfrak{g}, \mathfrak{k})$-module. We will show that M is of finite type. Consider the filtration of $\mathcal{M}$ with successive quotients

$$\Lambda^{max}(\mathcal{N}_Q) \otimes_{\mathcal{O}_Q} S^i(\mathcal{N}_Q) \otimes_{\mathcal{O}_Q} \mathcal{F},$$

and choose a finer filtration of $\mathcal{M}$ such that its successive quotients are all sheaves $\mathcal{O}(\lambda)\otimes_{\mathcal{O}_Q}\mathcal{F}$, where λ runs over the multiset D of sums of roots from $\Delta^-\backslash\Delta_{\mathfrak{k}+\mathfrak{c}}$ and $\mathcal{O}(\lambda)$ stands for the invertible $K'\times C_{ss}$-sheaf on Q induced by λ. Then $\mathcal{O}(\lambda)\otimes_{\mathcal{O}_Q}\mathcal{F}\simeq\mathcal{O}(\lambda')\boxtimes\mathcal{F}(\lambda'')$, where λ' (respectively, λ'') is the projection of λ on $(\mathfrak{h}')^*$ (resp., $(\mathfrak{h}'')^*$) and $\mathcal{F}(\lambda''):=\mathcal{F}_2\otimes_{\mathcal{O}_{Q_2}}\mathcal{O}(\lambda'')$. Thus M is a $\mathfrak{k}+\mathfrak{c}$-submodule of $\oplus_{\lambda\in D}(\Gamma(\mathcal{O}(\lambda'))\boxtimes\Gamma(\mathcal{F}(\lambda'')))$. The Borel-Weil-Bott theorem implies that $\Gamma(\mathcal{O}(\lambda'))\neq 0$ if and only if λ' is $\mathfrak{b}\cap\mathfrak{k}$-antidominant. In the latter case $\Gamma(\mathcal{O}(\lambda'))=V(-\lambda')^*$, where $V(-\lambda')$ is the irreducible finite-dimensional $\mathfrak{k}'$-module of $\mathfrak{b}\cap\mathfrak{k}'$-highest weight $-\lambda'$. Furthermore $\Gamma(\mathcal{F}(\lambda''))$ is a $(\mathfrak{c}_{ss},\mathfrak{h}'')$-module of finite type over $\mathfrak{h}''$. Therefore to show that M has finite type it suffices to check that for each $\omega\in(\mathfrak{h}')^*$ the set $D_\omega:=\{\lambda\in D|\lambda'=\omega\}$ is finite. The latter is the direct corollary of the fact that the projection of D onto $(\mathfrak{h}')^*$ is contained in an open half-space of a real subspace of $(\mathfrak{h}')^*$. □

Corollary 2 *Let $\mathfrak{k}$ be reductive.*

(a) *If $\mathfrak{c}$ is abelian, then $\mathfrak{k}$ is of finite type.*

(b) *Assume that $\mathfrak{g}$ has no simple components of type B_n for $n>2$ or F_4. Then $\mathfrak{k}$ is of finite type if and only if all simple components of $\mathfrak{c}_{ss}$ are of types A and C.*

Proof. (a) is obvious. In proving (b) one can assume that $\mathfrak{g}$ is simple. If $\mathfrak{g}$ is not of type C_n, B_n for $n>2$ or F_4, the condition that $\langle\Delta_\mathfrak{c}\rangle_\mathbb{C}\cap\Delta=\Delta_\mathfrak{c}$ is automatic, and therefore Propositions 4 and 5 imply the statement. To check the condition $\langle\Delta_\mathfrak{c}\rangle_\mathbb{C}\cap\Delta=\Delta_\mathfrak{c}$ for $\mathfrak{g}$ of type C_n we will use the fact that $\mathfrak{c}$ is the centralizer $\mathfrak{k}_{ss}$. Assume, to the contrary, that $\langle\Delta_\mathfrak{c}\rangle_\mathbb{C}\cap\Delta\neq\Delta_\mathfrak{c}$. Then there are two orthogonal long roots $\alpha,\beta\in\Delta_\mathfrak{c}$ such that $(\alpha+\beta)/2\notin\Delta_\mathfrak{c}$. Furthermore, for $\gamma\in\Delta_{\mathfrak{k}_{ss}}$, $\alpha\pm\gamma,\beta\pm\gamma\notin\Delta_\mathfrak{c}$ but $(\alpha+\beta)/2+\gamma\in\Delta_\mathfrak{c}$. This is impossible. □

In the classical Harish-Chandra setting, when $\mathfrak{k}$ coincides with the fixed points of an inner involution, every irreducible strict $(\mathfrak{g},\mathfrak{k})$-module is of finite type. The latter is not true just under the assumption that $\mathfrak{k}$ is of finite type. It is an open problem to characterize all subalgebras $\mathfrak{k}$ for which any irreducible strict $(\mathfrak{g},\mathfrak{k})$-module is of finite type.

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