Moduli spaces of sheaves necessary for testing of S-duality conjecture.

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Abstract

In this text we give a geometrical description for several series of moduli spaces of stable sheaves on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, which are interesting for mathematical testing of the S-duality conjecture.

Introduction.

It is known that mathematical testing of the S-duality conjecture for SYM QFT in four dimensions is possible only for scaling invariant theories. It is equivalent that the coefficient of the Gell-Mann – Low β–function on the field configuration of the theory vanishes. In this case there appears a constant

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

as a parameter of our theory, where θ is the vacuum angle and g is the strong coupling constant. The S-duality conjecture predicates that the partition function

$$Z_\tau = \sum_{\delta \geq 0} a_\delta \left(e^{2\pi i \tau}\right)^\delta$$

of our theory admits some modular properties under integral linear fractional transformations of τ. The coefficient a_δ of the partition function is equal to the solution number of field equations (they are enumerated by δ) for a general metric on an algebraic surface S with $K_S \leq 0$ considering as a real 4-manifold. The Kählerian metric is not general and there exists a space $\mathfrak{M}(\delta)$ of gauge orbits of the field equations solutions. In this case the coefficient a_δ is calculated as the top Chern class of an obstruction bundle $\mathcal{H}_\delta$ over $\mathfrak{M}(\delta)$.

From the mathematical viewpoint the field configuration is a collection of representations of Lie algebra for fiber-wise gauge group of our theory. There are known three cases when the β–function coefficient is zero for collections of $\mathfrak{u}(r)$–representations (see [13]).

In the first case the collection consists of one adjoint representation. Testing of S-duality conjecture for this case is suggested in the paper of Vafa and Witten [16]. In the second case, when the coefficient of β-function is zero, the collection consists of $2r$ canonical representations of $\mathfrak{u}(r)$ in $\mathbb{C}^r$. QFT and the field equations with this field configuration are described in [6] for $r = 2$ and in [13] for arbitrary r. The space $\mathfrak{M}(\delta)$ in these papers are interpreted as moduli spaces $\mathfrak{M}(r, c_1, \delta)$ of stable (in the Mumford – Takemoto sense)

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bundles X_δ on a Del Pezzo surface S with fixed $\mathrm{rk}\,(X_\delta) = r,\ c_1(X_\delta) = c_1$ and positive $-\chi(X_\delta) = \delta$ such that X_δ's and $\mathfrak{M}(\delta)$ satisfy the conditions

$$\mathrm{H}^0(X_\delta) = \mathrm{H}^2(X_\delta) = 0, \tag{1}$$

$$\dim \mathfrak{M}(\delta) = 2r\delta. \tag{2}$$

Suppose that the moduli space $\mathfrak{M}(\delta) = \mathfrak{M}(r, c_1, \delta)$ is fine. Then we can consider a universal bundle $\mathcal{H}_\delta$ over $\mathfrak{M}(\delta)$ such that its fiber over a point $X_\delta \in \mathfrak{M}(\delta)$ is $\mathrm{H}^1(X_\delta)$. In this notation the obstruction bundle is $\mathcal{H}_\delta^{\oplus 2r}$. Therefore, the coefficients a_δ of Z_τ is the top Chern class of $\mathcal{H}_\delta^{\oplus 2r}$

$$a_\delta = c_{top}\Big(\mathcal{H}_\delta^{\oplus 2r}\Big).$$

Condition (1) follows from the stability of the sheaf X_δ and the assumption

$$-\mu(\mathcal{O}_S(K_S)) < \mu(X_\delta) < 0. \tag{3}$$

where $= -K_S$ is the anti-canonical divisor of the Del Pezzo surface S (the definition of $\mu(X)$ see in the next section). Condition (2) is equivalent to the fact that X_0 is an exceptional bundle such that

$$\mathrm{Ext}^i(\mathcal{O}_S, X_0) = 0 \quad \forall i, \tag{4}$$

i.e., X_0 is orthogonal to $\mathcal{O}_S$ ([13]). We see that the series of the moduli spaces $\mathfrak{M}(\delta)$ corresponds to the exceptional bundles X_0.

In this preprint, using Ellingsrud's and Strømme's method ([3]), we give a geometrical description for several series of moduli spaces on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, and explicitly construct the bundle $\mathcal{H}_\delta$ on $\mathfrak{M}(\delta)$.

In the case of $\mathbb{P}^2$ we deal with moduli spaces series $\mathfrak{M}_n(\delta)$ corresponding to any exceptional bundle E_n such that E_n satisfies conditions (3) and (4). These bundles can be obtained from $E_0 = \mathcal{O}_{\mathbb{P}^2}(-2)$ and $E_1 = \mathcal{O}_{\mathbb{P}^2}(-1)$ by mutations (see proposition 1.7). As a result, theorem 5.1 gives a geometrical realization of $\mathfrak{M}_n(\delta)$ and $\mathcal{H}_\delta$.

In the case of $\mathbb{P}^1 \times \mathbb{P}^1$ we consider the moduli space series $\mathfrak{M}_{2k}(\delta)$, $\mathfrak{M}'_{2k+1}(\delta)$, and $\mathfrak{M}''_{2k+1}(\delta)$ corresponding to exceptional bundles from three-block collections

$$(Q_{2k}, Q'_{2k+1}, Q''_{2k+1}, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})$$

(see the definition in (10)). In theorems 6.1 and 6.2 we describe these moduli spaces and the corresponding obstruction sheaves.

I am grateful to B. V. Karpov and D. O. Orlov for many valuable discussions, and I would like to thank the MPI for hospitality.

1 Notations and preliminaries..

In this preprint the surface S means either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ over complex field $\mathbb{C}$. Here we deal with coherent sheaves on S only. Because of this we omit the word "coherent". Besides, there will be no notational distinction made between algebraic bundles and the locally free sheaves of their sections.

We denote by K_S the canonical class of S. For a sheaf F we fix the following notation: $\mathrm{rk}\, F$ is rank, $c_1(F)$ and $c_2(F)$ are the first and the second Chern classes of F. The symbol $\deg(F)$ means the degree of F w.r.t the anticanonical class $-K_S$ of the surface S, i.e., $\deg(F) = -\dfrac{(c_1(F), K_S)}{\sqrt{(K_S, K_S)}}$, where $(\cdot,\cdot)$ denotes the cup product of divisors.

The term "stability" means the stability in sense of Mumford – Takemoto w.r.t. $-K_S$. In another words, a torsion free sheaf F on S is stable iff for any its subsheaf of lesser rank E the inequality $\mu(E) < \mu(F)$ holds, where $\mu(F) = \dfrac{\deg(F)}{\mathrm{rk}\, F}$ is a slope of F.

The following remark is often used without explanations

1.1 Remark. *Let G and X be semistable sheaves on S such that*

$$\mu(G(-K_S)) < \mu(X) < \mu(G),$$

then $\mathrm{Ext}^i(G, X) = 0$, *when* $i \neq 1$.

To calculate dimensions of the group $\mathrm{Ext}^i(X, Q)$ for sheaves X and Q on S, we use Riemann-Roch formula in the form

$$\chi(X, Q) = \chi(X)\mathrm{rk}\,(Q) + \chi(Q)\mathrm{rk}\,(X) - \mathrm{rk}\,(X)\mathrm{rk}\,(Q) - s\cdot\deg(X)\mathrm{rk}\,(Q) - (c_1(X), c_1(Q)),$$

where $s = 3$ if $S = \mathbb{P}^2$ and $s = 2$ when $S = \mathbb{P}^1 \times \mathbb{P}^1$. Here

$$\chi(X, Q) = \sum (-1)^i \dim \mathrm{Ext}^i(X, Q) = \sum (-1)^i h^i(X, Q).$$

Further we give some bases definitions and facts about exceptional sheaves and exceptional collections on the surface S ([7], [15]).

1.2 Definition. *1. A sheaf E on the surface S is called exceptional if*

$$\mathrm{Hom}(E, E) = \mathbb{C}, \qquad \text{and} \qquad \mathrm{Ext}^i(E, E) = 0, \quad \text{when} \quad i > 0.$$

2. An ordered pair (E, F) of exceptional sheaves on S is called exceptional, if $\mathrm{Ext}^i(F, E) = 0\ \forall i$.

1.3 Proposition. *1) Any exceptional sheaf on S is locally free and stable;*
2) for any exceptional pair (E, F) on $\mathbb{P}^2$ the groups $\mathrm{Ext}^i(E, F)$ are trivial, when $i > 0$ and $\mathrm{Hom}(E, F) \neq 0$;
3) for any exceptional pair (E, F) on $\mathbb{P}^1 \times \mathbb{P}^1$ $\mathrm{Ext}^2(E, F)$ are trivial, and only one of the groups ether $\mathrm{Ext}^0(E, F)$ or $\mathrm{Ext}^1(E, F)$ does not equal zero.

1.4 Definition. *An exceptional pair (E, F) on S is called a hom-pair if* $\mathrm{Ext}^0(E, F) \neq 0$. *It is called an ext-pair if* $\mathrm{Ext}^1(E, F) \neq 0$. *Provided* $\mathrm{Ext}^0(E, F) = \mathrm{Ext}^1(E, F) = 0$, *the pair (E, F) is called a zero-pair.*

1.5 Remark. *1. An exceptional pair (E, F) on S is a hom-pair iff $\mu(E) < \mu(F)$, it is an ext-pair iff $\mu(E) > \mu(F)$ and it is a zero-pair iff $\mu(E) = \mu(F)$.*
2. All exceptional pair on $\mathbb{P}^2$ are hom-pairs.

For example, the pairs $(\mathcal{O}_{\mathbb{P}^2}(n), \mathcal{O}_{\mathbb{P}^2}(n+1))$ are exceptional hom-pairs and the pair of the line bundles on the quadric $(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0))$ is an exceptional zero-pair.

We can obtain new exceptional sheaves and pairs with help of mutations.

1.6 PROPOSITION. *Let (E, F) be an exceptional hom-pair on S and*

$$lcan : \mathrm{Hom}(E, F) \otimes E \longrightarrow F$$

be the canonical map, corresponding to the identity element of $\mathrm{Hom}(E, F)^* \otimes \mathrm{Hom}(E, F)$, *then lcan is either an epimorphism or a monomorphism. Besides,*
1) if the canonical map is an epimorphism, then the sheaf $L_E F = \ker lcan$ is exceptional and the pair $(L_E F, E)$ is an exceptional hom-pair;
2) if the canonical map is a monomorphism, then the sheaf $L_E F = \mathrm{coker}\, lcan$ is exceptional and the pair $(L_E F, E)$ is an exceptional ext-pair.

1.7 PROPOSITION. *Let (E, F) be an exceptional hom-pair on S and*

$$rcan : E \longrightarrow \mathrm{Hom}(E, F)^* \otimes F$$

be the canonical map, corresponding to the identity element of $\mathrm{Hom}(E, F)^* \otimes \mathrm{Hom}(E, F)$, *then rcan is either an epimorphism or a monomorphism. Besides,*
1) if the canonical map is a monomorphism, then the sheaf $R_F E = \mathrm{coker}\, rcan$ is exceptional and the pair $(F, R_E F)$ is an exceptional hom-pair.
2) if the canonical map is an epimorphism, then the sheaf $R_F E = \ker can$ is exceptional and the pair $(F, R_E F)$ is an exceptional ext-pair.

1.8 PROPOSITION. *Let (E, F) be an exceptional ext-pair on S and e be the identity element of* $\mathrm{Ext}^1(E, F)^* \otimes \mathrm{Ext}^1(E, F)$, *then*
1) the sheaf $L_E F$ from the universal extension

$$0 \longrightarrow F \longrightarrow L_E F \longrightarrow \mathrm{Ext}^1(E, F) \otimes E \longrightarrow 0,$$

corresponding to e is exceptional and the pair $(L_E F, E)$ is exceptional too;
2) the sheaf $R_F E$ from the universal extension

$$0 \longrightarrow \mathrm{Ext}^1(E, F)^* \otimes F \longrightarrow R_F E \longrightarrow E \longrightarrow 0,$$

corresponding to e is exceptional and the pair $(F, R_F E)$ is exceptional too.

1.9 DEFINITION. *The sheaves $L_E F$ ($R_F E$) defining in propositions 1.6 , 1.7 and 1.8 are called the left (right) shifts of F (E); the pairs $(L_E F, E)$ are called the left mutations of the pair (E, F) and the pairs $(F, R_F E)$ are called the right one.*

The left (right) mutation of a pair (E, F) is called regular provided the pair (E, F) is a hom-pair and the map lcan is an epimorphism (rcan is a monomorphism.)

1.10 REMARK. *If (E, F) is an exceptional pair on $\mathbb{P}^2$, then the left and right mutations of it are regular.*

1.11 PROPOSITION. *let (E, F', F'') be an exceptional collection such that $\mu(E) < \mu(F') = \mu(F'')$. Consider the canonical map*

$$rcan : E \longrightarrow \begin{array}{c} \mathrm{Hom}(E, F')^* \otimes F' \\ \bigoplus \\ \mathrm{Hom}(E, F'')^* \otimes F'' \end{array},$$

corresponding to the element

$$(id, id) \in \mathrm{Hom}(E, F')^* \otimes \mathrm{Hom}(E, F') \oplus \mathrm{Hom}(E, F'')^* \otimes \mathrm{Hom}(E, F'').$$

Then $rcan$ is a monomorphism and $\operatorname{coker} rcan = R_{F''} R_{F'} E$. *We'll denote this sheaf by $R_F E$. In the same way the sheaf $L_E F$ can be define for an exceptional collection (E', E'', F) such that $\mu(E') = \mu(E'') < \mu(F)$.*

1.12 Definition. *An ordered collection $(E_0, \ldots, E_n)$ on S is called exceptional if each its pair (E_i, E_j) $(i < j)$ is exceptional. It is called full, if the minimal triangulated subcategory in the derived category $D^b(S)$ of coherent sheaves on S, containing the collection's objects, coincides with $D^b(S)$.*

1.13 Proposition. *Let $\sigma = (E_0, \ldots, E_n)$ be an exceptional collection on S, then*
1) the collections

$$L_i\sigma = (E_0, \ldots, L_{E_i}E_{i+1}, E_i, E_{i+2}, \ldots, E_n)$$

and

$$R_i\sigma(E_0, \ldots, E_{i-2}, E_i, R_{E_i}E_{i-1}, \ldots, E_n)$$

are exceptional,
2) if σ is full, then $L_i\sigma$ and $R_i\sigma$ are full too,
3) if σ is full, then

$$L_0 L_1 \cdots L_{n-1}\sigma = (E_n(K_S), E_0, \ldots, E_{n-1}),$$
$$R_n R_{n-1} \cdots R_1\sigma = (E_1, \ldots, E_n, E_0(-K_S)).$$

1.14 Definition. *Let $\sigma = (E_0, \ldots, E_k)$ be an exceptional collection on S. The minimal full triangulated subcategory in the derived category of coherent sheaves on S, containing the collection's objects, is called the subcategory generated by σ and is denoted by $\langle E_0, \ldots, E_k\rangle_{cat}$.*

1.15 Lemma *(A. Bondal,[1]) Let $(E_0, \ldots, E_n)$ be a full exceptional collection and X be a sheaf on S, then*

$$X \in \langle E_0, \ldots, E_s\rangle_{cat} \iff \mathrm{Ext}^i(E_j, X) = 0 \quad \forall i,\ j > s;$$

$$X \in \langle E_s, \ldots, E_n\rangle_{cat} \iff \mathrm{Ext}^i(X, E_j) = 0 \quad \forall i,\ j < s.$$

1.16 Definition *Let $\sigma = (E_0, \ldots, E_k)$ be an exceptional collection on S. The collection $({}^\vee E_k, \ldots, {}^\vee E_0)$ is called the left dual collection, whenever*

$${}^\vee E_k = E_k, \quad {}^\vee E_{k-1} = R_{E_k}E_{k-1}, \quad {}^\vee E_{k-2} = R_{E_k}R_{E_{k-1}}E_{k-2}, \quad \ldots$$

1.17 Theorem. *(A. Gorodentsev, [5].) Let $(E_0, \ldots, E_k)$ be an exceptional collection and X be a sheaf on S such that $X \in \langle E_0, \ldots, E_k\rangle_{cat}$. There exists a spectral sequence $\mathbf{E}^{p,q}$ converging to X with the first part of the form*

$$\mathbf{E}^{p,q} = \mathrm{Ext}^{q-\Delta_{p+k}}({}^\vee E_{k+p}, X) \otimes E_{k+p},$$

where Δ_{p+k} is the number of non regular mutations needed for constructing the sheaf ${}^{\vee}E_{k+p}$.

In this preprint we are interesting exceptional bundles E's on S that are orthogonal to $\mathcal{O}_S$, i.e.,

$$\mathrm{Ext}^i(\mathcal{O}_S, E) = 0 \ \forall i. \tag{5}$$

The case of $\mathbb{P}^2$. It is obvious that $\Big(\mathcal{O}_{\mathbb{P}^2}(-2), \mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}\Big)$ is a full exceptional collection. Therefore, if an exceptional bundle E satisfy condition (5), then E belongs to the subcategory $\langle \mathcal{O}_{\mathbb{P}^2}(-2), \mathcal{O}_{\mathbb{P}^2}(-1)\rangle_{cat}$ (1.15). Moreover, E can be obtained from $\mathcal{O}_{\mathbb{P}^2}(-2), \mathcal{O}_{\mathbb{P}^2}(-1)$ by mutations. By definition put,

$$E_0 = \mathcal{O}_{\mathbb{P}^2}(-2),\ E_1 = \mathcal{O}_{\mathbb{P}^2}(-1),\ E_{n+1} = R_{E_n}E_{n-1},\ r_n = \mathrm{rk}\,(E_n), \tag{6}$$

i.e., for each natural n we have the exact triple

$$0 \longrightarrow E_{n-1} \stackrel{rcan}{\longrightarrow} \mathrm{Hom}(E_{n-1}, E_n)^* \otimes E_n \longrightarrow E_{n+1} \longrightarrow 0.$$

It is easy to prove that the following relations are true:

$$\deg(E_n) = -r_{n-1}, \quad r_{n+1} = 3r_n - r_{n-1}, \quad r_n^2 + r_{n-1}^2 + 1 = 3r_nr_{n-1}. \tag{7}$$

Using this relations, we can calculate that

$$\deg(E_{n-1})\mathrm{rk}\,(E_n) - \deg(E_n)\mathrm{rk}\,(E_{n-1}) = r_{n-1}^2 - r_{n-2}r_n = -1. \tag{8}$$

In **the case of** $\mathbb{P}^1 \times \mathbb{P}^1$ we consider so called exceptional three-block collections ([11]).

1.18 DEFINITION. *A full exceptional collection (Q_0, Q_1, Q_2, Q_3) on $\mathbb{P}^1 \times \mathbb{P}^1$ is called three-block collection if for one of $i \in \{0, 1, 2\}$ the pair (Q_i, Q_{i+1}) is a zero-pair, i.e., $\mu(Q_i) = \mu(Q_{i+1})$. It is known that in this case $\mu(Q_j) \leq \mu(Q_{j+1})$ $\forall j$.*

For example,

$$\Big(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,0), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\Big)$$

is the full exceptional three-block collection.

By definition, put

$$Q_0 = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1),\ Q_1' = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,0),\ Q_1'' = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1),$$

$$Q_{2k} = R_{Q_{2k-1}}Q_{2k-2},\ Q_{2k+1}' = R_{Q_{2k}}Q_{2k-1}'',\ Q_{2k+1}'' = R_{Q_{2k}}Q_{2k-1}'; \tag{9}$$

(definition of $R_{Q_{2k-1}}Q_{2k-2}$ see in proposition 1.11)

$$q_{2k} = \mathrm{rk}\,Q_{2k},\ q_{2k+1} = \mathrm{rk}\,Q_{2k+1}' = \mathrm{rk}\,Q_{2k+1}'',$$

It follows from [11], that for any integer k the collections

$$(Q_{2k}, Q_{2k+1}', Q_{2k+1}'', \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}) \quad \text{and} \quad (Q_{2k-1}', Q_{2k-1}'', Q_{2k}, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}) \tag{10}$$

are full exceptional three-block collections. Since,

$$h^0(Q_{2k}, Q_{2k+1}') = h^0(Q_{2k}, Q_{2k+1}'') = h^0(Q_{2k-1}', Q_{2k}) = h^0(Q_{2k-1}'', Q_{2k}) = 2,$$

we obtain,

$$q_{2k+1} = 2q_{2k} - q_{2k-1}, \qquad q_{2k+2} = 4q_{2k+1} - q_{2k} \tag{11}$$

It is not hard to prove the following arithmetic relations

$$c_1(Q_{2k}) = -(q_{2k-1}, q_{2k-1}),\ c_1(Q'_{2k+1}) = -\frac{1}{2}(q_{2k}+1, q_{2k}-1),\ c_1(Q''_{2k+1}) = -\frac{1}{2}(q_{2k}-1, q_{2k}+1);$$

$$\deg(Q_{n-1})\operatorname{rk} Q_n - \deg(Q_n)\operatorname{rk} Q_{n-1} = -1, \tag{12}$$

where Q_n is an exceptional bundle of the sequence of bundles $Q_0, Q'_1, Q''_2, Q_3, \ldots$, moreover, the following analog of Markov equation takes place

$$q_{2k}^2 + 2q_{2k\pm1}^2 + 1 = 4q_{2k}q_{2k\pm1} \tag{13}$$

2 Some useful statements.

2.1 Lemma. *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence of objects from an Abelian category $\mathfrak{A}$ and D be another object from $\mathfrak{A}$. Then for any $\varphi \in \operatorname{Hom}(D, C)$ such that $e_D = \delta_D(\varphi) \neq 0$, where $\delta_D : \operatorname{Hom}(D, C) \longrightarrow \operatorname{Ext}^1(D, A)$ is the coboundary map, there exists a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & A & \longrightarrow & \tilde{B} & \longrightarrow & D & \longrightarrow & 0 \\ & & {\scriptstyle id}\downarrow & & {\scriptstyle \tilde{\varphi}}\downarrow & & {\scriptstyle \varphi}\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}, \tag{14}$$

where the first row of the diagram is determined by the element $\delta_D(\varphi) \in \operatorname{Ext}^1(D, A)$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}(C, C) & \xrightarrow{\varphi_0} & \operatorname{Hom}(D, C) \\ {\scriptstyle \delta_C}\downarrow & & {\scriptstyle \delta_D}\downarrow \\ \operatorname{Ext}^1(C, A) & \xrightarrow{\varphi_1} & \operatorname{Ext}^1(D, A) \end{array},$$

where δ_C, δ_D are coboundary morphisms and φ_0, φ_1 are induced by φ. The second row of diagram(14) is determined by $e_C = \delta_C(id_C) \in \operatorname{Ext}^1(C, A)$. On the other hand, $\varphi_0(id_C) = \varphi$ and $e_D = \delta_D(\varphi) = \varphi_1(e_C)$. Hence there exists a map $\tilde{\varphi} : \tilde{B} \longrightarrow B$ such that diagram (14) commutative.

2.2 Lemma. *Let X and F be stable sheaves on a smooth projective surface S such that their slopes $\mu(X) = \frac{a}{r_1}$, $\mu(F) = \frac{b}{r_2}$ satisfy the condition*

$$ar_2 - br_1 = -1,$$

then a nontrivial extension

$$0 \longrightarrow X \longrightarrow X' \longrightarrow F \longrightarrow 0 \tag{15}$$

is stable if and only if

a) X is locally free;

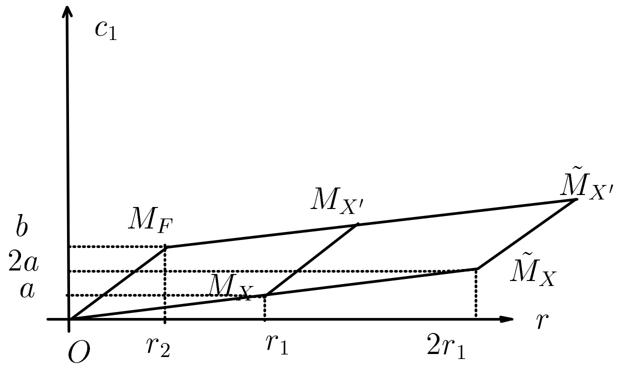
b) X is not locally free, but the element e from $\mathrm{Ext}^1(F,X)$ *determining the extension does not belong to the subspace* $\mathrm{Hom}(X^{**}/X,F) \subset \mathrm{Ext}^1(F,X)$, *where X^{**} is the reflexive span of X.*

PROOF. Let X be locally free and Q is a torsion free quotient of X' with the minimal slope and the minimal rank. By construction Q is stable. Suppose X' is not stable, then $\mu(Q) < \mu(X') < \mu(F)$. This implies that $\mathrm{Hom}(F,Q) = 0$ and there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & X' & \longrightarrow & F & \longrightarrow & 0 \\ & & {\scriptstyle\varphi}\downarrow & & \downarrow & & & & \\ & & Q & \xrightarrow{id} & Q & & & & \end{array}.$$

Since the sheaves X and Q are stable, we have $\mu(X) \le \mu(Q)$.

Consider the following picture



By the assumption of the proposition, area of parallelogram $OM_XM_{X'}M_F$ equals 1. Hence, area of parallelogram $M_X\tilde{M}_X\tilde{M}_{X'}M_{X'}$ equals 1 too. This implies that there are not points with integer coordinates interior parallelogram $O\tilde{M}_X\tilde{M}_{X'}M_F$. On the other hand, $\mathrm{rk}\, Q < \mathrm{rk}\, X'$ and $\mu(X) \le \mu(Q) < \mu(X')$. Therefore, $\mu(Q) = \mu(X)$ and morphism φ is isomorphism in a general point, i.e. we have the exact triple:

$$0 \longrightarrow X \longrightarrow Q \longrightarrow \tilde{Q} \longrightarrow 0,$$

where $\dim \mathrm{Supp}\, \tilde{Q} = 0$.

Using the fact that the group $\mathrm{Ext}^1(\tilde{Q},X) = 0$ for any locally free sheaf X and torsion sheaf $\tilde{Q}$ with zero-dimensional support on a surface ([8]), we obtain that $Q = X \oplus \tilde{Q}$. This contradicts the stability of Q. Hence $Q \cong X$.

Recall that the stability of X implies the fact that X is simple ($\mathrm{Hom}(X,X) \cong \mathbb{C}$) and obtained nontrivial map $X' \longrightarrow X$ shows that sequence (15) splits. This contradiction proves the first statement of proposition.

Now suppose that X is not locally free and consider the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & X & \longrightarrow & X' & \longrightarrow & F & \longrightarrow & 0 \\ & & can\downarrow & & \downarrow & & \downarrow id & & \\ 0 & \longrightarrow & X^{**} & \longrightarrow & \tilde{X} & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & X^{**}/X & & X^{**}/X & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array},$$

where the first line is exact triple (1.8) determined by an element $e \in \mathrm{Ext}^1(F, X)$ and the second one is the extension corresponding to the element $can(e) \in \mathrm{Ext}^1(F, X^{**})$.

Since X^{**} is locally free, in the case $can(e) \neq 0$ $\tilde{X}$ is stable (by the first statement of the proposition). But $\mu(X') = \mu(\tilde{X})$ and any subsheaf of X' is a subsheaf of $\tilde{X}$ too. Hence X' is stable also.

Suppose $can(e) = 0$, i.e. $e \in \mathrm{Hom}(F, X^{**}/X) \subset \mathrm{Ext}^1(F, X)$. Then the following commutative diagram takes place (see lemma 2.1).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & X' & \longrightarrow & F & \longrightarrow & 0 \\ & & id\downarrow & & \psi\downarrow & & e\downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & X^{**} & \longrightarrow & X^{**}/X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & \mathrm{coker}\,\psi & \longrightarrow & \mathrm{coker}\,e & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}.$$

Since $\dim \mathrm{Supp}\,\mathrm{coker}\,\psi = 0$, slope of $\mathrm{im}\,\psi$ is equal to slope of X^{**}, i.e. $\mu(\mathrm{im}\,\psi) < \mu(X')$, and X' is not stable. This concludes the proof of the proposition.

2.3 LEMMA. *Let*

$$V \otimes X \longrightarrow X' \longrightarrow F \longrightarrow 0$$

be an exact sequence of sheaves on a smooth projective complex surface S such that the sheaf X is locally free, stable, the space $\mathrm{Hom}(X', X) = 0$, *and F has no torsion. Suppose that the following conditions is true*

$$\deg(X)\mathrm{rk}\,(F) - \deg(F)\mathrm{rk}\,(X) = -1,$$

then the stability of F is equivalent to the stability of X'.

PROOF. First we note that the assumptions of the lemma implies

$$\mu(X) < \mu(X') < \mu(F).$$

Suppose that the sheaf F is not stable, i.e., it has a quotient sheaf $F \longrightarrow G \longrightarrow 0$ with $\mu(F) \geq \mu(G)$, $\mathrm{rk}\,(G) < \mathrm{rk}\,(F)$. By the same method as in the previous proof we can show that $\mu(X) \geq \mu(G)$. But $\mu(X') > \mu(X)$ and G is the quotient of X'. Therefore, X' is not stable either.

Now suppose that the sheaf F is stable and X' has a quotient sheaf Q such that

$$\mu(Q) \le \mu(X') \qquad \mathrm{rk}\,(Q) < \mathrm{rk}\,(X').$$

Without loss of generality we can assume that Q is stable. Hence, $\mathrm{Hom}(F, Q) = 0$ and, consequently, $\mathrm{Hom}(X, Q) \ne 0$. This yields the inequality $\mu(X) \ge \mu(Q)$.

By the same arguments as before, we obtain, $\mu(X) = \mu(Q)$. Let us recall that Q and X are stable sheaves and X is locally free. Therefore, any nontrivial map $f : X \longrightarrow Q$ is an isomorphism. Thus we have a some nonzero map $X' \longrightarrow X$. This contradiction proves the lemma.

2.4 Remark. *The previous lemma holds in the case when $X = X_1 \oplus X_2$ with stable bundles X_i such that $\mu(X) = \mu(X_1) = \mu(X_2)$.*

2.5 Lemma. *Let X and F be a stable sheaves on a smooth projective complex surface S such that the sheaf X is locally free and*

$$\deg(X)\mathrm{rk}\,(F) - \deg(F)\mathrm{rk}\,(X) = -1.$$

Suppose $\varphi \in \mathrm{Hom}(U \otimes X, F)$ is nontrivial, then
1) either $\mathrm{rk}\,(\mathrm{im}\,\varphi) = \mathrm{rk}\,F$, or $\mathrm{im}\,\varphi$ is a semistable sheaf with a filtration

$$\mathrm{im}\,\varphi = Gr(X, \dots, X),$$

2) provided X is exceptional, $\ker\varphi = U' \otimes X$ for some subspace $U' \subset U$.

Proof. Assume that $\mathrm{rk}\,(\mathrm{im}\,\varphi) < \mathrm{rk}\,(F)$. Taking into account the stability of X and F, we have

$$\mu(X) \le \mu(\mathrm{im}\,\varphi) < \mu(F).$$

This inequality and the assumptions of the lemma imply that

$$\mu(X) = \mu(\mathrm{im}\,\varphi) \tag{16}$$

and the semistability of $\mathrm{im}\,\varphi$ follows from the stability of X.

Let us show that $\mathrm{im}\,\varphi = Gr(X, \dots, X)$ by induction on $\mathrm{rk}\,(\mathrm{im}\,\varphi)$. We have an epimorphism $\varphi : U \otimes X \longrightarrow \mathrm{im}\,\varphi \longrightarrow 0$. Therefore, there exists a nonzero map $\varphi_1 : X \longrightarrow \mathrm{im}\,\varphi$. Since X is a stable bundle and $\mathrm{im}\,\varphi$ is semistable, taking into account (16), we obtain that φ_1 is a monomorphism.

Suppose that $\mathrm{coker}\,\varphi_1$ has a torsion T. It is obvious that $\deg(\mathrm{coker}\,\varphi) = 0$, consequently, its support is a finite set of points, hence $\mathrm{Ext}^1(T, X) = 0$. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & T & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \xrightarrow{\varphi_1} & \mathrm{im}\,\varphi & \longrightarrow & \mathrm{coker}\,\varphi_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{X} & \longrightarrow & \mathrm{im}\,\varphi & \longrightarrow & (\mathrm{coker}\,\varphi_1)/T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & T & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}.$$

We see that $\bar{X} = X \oplus T$ and $X \oplus T \subset \operatorname{im}\varphi$. But $\operatorname{im}\varphi$ has no torsion (as a subsheaf of a torsion free sheaf). Therefore, $T = 0$ and $\operatorname{coker}\varphi_1$ is semistable. Now, application of the inductive hypothesis to $\operatorname{coker}\varphi_1$ proves the first statement of the lemma.

Further, in additional, suppose that X is exceptional, i.e. $\operatorname{Ext}^1(X, X) = 0$. Hence $\operatorname{im}\varphi = U'' \otimes X$ and we have an exact triple

$$0 \longrightarrow \ker\varphi \longrightarrow U \otimes X \xrightarrow{\varphi} U'' \otimes X \longrightarrow 0.$$

Since X is simple, the morphism φ is induces by a linear map of the vector spaces $\varphi' : U \longrightarrow U''$. Therefore, $\ker\varphi = X \otimes \ker\varphi'$. This completes the proof.

3 Monads.

Our method moduli space description is a generalization of the method of G.Ellingsrud and S.A.Strøme ([3]) and uses monads. Here we construct the monads on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

3.1 PROPOSITION. *Let (E, F, G) be a full exceptional collection on $\mathbb{P}^2$. Then*
1) any semistable sheaf X on $\mathbb{P}^2$ satisfying the slope conditions

$$\mu(G) - 3 < \mu(E) < \mu(X) < \mu(G)$$

is the cohomology of the monad $M(X)$

$$0 \longrightarrow U_0(X) \otimes E \longrightarrow U_1(X) \otimes F \longrightarrow U_2(X) \otimes G \longrightarrow 0$$

with $U_0(X) = \operatorname{Ext}^1(E(3), X)$, $U_1(X) = \operatorname{Ext}^1(L_E F(3), X)$, $U_2(X) = \operatorname{Ext}^1(G, X)$;
2) two semistable sheaves X and X' on $\mathbb{P}^2$ are isomorphic iff their monads $M(X)$ and $M(X')$ are.

PROOF. By theorem 1.17 we have the spectral sequence $\mathbf{E}^{pq}$ converging to X with the first part in the form

$$\mathbf{E}_1^{pq} = \operatorname{Ext}^{q-\Delta_{2+p}}({}^\vee G_{2+p}, X) \otimes G_{2+p}.$$

where $G_0 = E$, $G_1 = F$, $G_2 = G$, ${}^\vee G_2 = G$, ${}^\vee G_1 = R_G F$, ${}^\vee G_2 = R_G R_F E$. Taking into account proposition 1.13 and the fact that $\mathcal{O}_{\mathbb{P}^2}(K_{\mathbb{P}^2}) = \mathcal{O}_{\mathbb{P}^2}(-3)$, we obtain,

$$R_G F = L_E F(3) \qquad \text{and} \qquad R_G R_F E = E(3).$$

Besides, all mutations on $\mathbb{P}^2$ are regular (see 1.10), i.e., $\Delta_{2+p} = 0\ \forall p$.

On the other hand, by the slope condition of the proposition and the stability of exceptional bundles (1.3) and X, we get $\mathbf{E}_1^{pq} = 0$, whenever $q \neq 1$. This means that the spectral sequence degenerates into the monad of the proposition.

The second point of the proposition is proved in the same way as lemma 4.1.3. of [9] with help of the spectral sequence functoriality.

3.2 PROPOSITION *Let (E, F', F'', G) be a full exceptional three-block collection on $\mathbb{P}^1 \times \mathbb{P}^1$ $(\mu(F') = \mu(F''))$. Then*
1) any semistable sheaf X on $\mathbb{P}^1 \times \mathbb{P}^1$ satisfying the slope conditions

$$\mu(G) - 4 < \mu(E) < \mu(X) < \mu(G)$$

is the cohomology of the monad $M_1(X)$

$$0 \longrightarrow U_0(X)\otimes E \longrightarrow \begin{array}{c} U_1(X)\otimes F' \\ \oplus \\ U_2(X)\otimes F'' \end{array} \longrightarrow U_3(X)\otimes G \longrightarrow 0$$

with $U_0(X) = \mathrm{Ext}^1(E(2,2),X)$, $U_1(X) = \mathrm{Ext}^1(L_E F'(2,2),X)$,
$U_2(X) = \mathrm{Ext}^1(L_E F''(2,2),X)$, $U_3(X) = \mathrm{Ext}^1(G,X)$;
2) two semistable sheaves X *and* X' *on* $\mathbb{P}^1\times\mathbb{P}^1$ *are isomorphic iff their monads* $M_1(X)$ *and* $M_1(X')$ *are.*

PROOF. This proposition can be proved by the same method as the previous one with help of the Beilinson–Gorodentsev spectral sequence. But here we give another method, using a projections to subcategory generated by sub collection.

At the beginning we note that the stability and our assumptions imply

$$\mathrm{Ext}^i(G,X) = \mathrm{Ext}^i(X,E) = 0, \quad \text{when } i \neq 1. \tag{17}$$

Consider the universal extension

$$0 \longrightarrow \mathrm{Ext}^1(X,E)^* \otimes E \longrightarrow X_1 \longrightarrow X \longrightarrow 0. \tag{18}$$

It follows from the long cohomology sequence, corresponding to (18) that $\mathrm{Ext}^i(X_1,E) = 0\ \forall i$. Therefore (1.15),

$$X_1 \in \langle F', F'', G\rangle_{cat}.$$

Besides,

$$\mathrm{Ext}^i(G,X_1) \cong \mathrm{Ext}^i(G,X)\ \forall i.$$

Further, taking the next universal extension

$$0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \mathrm{Ext}^1(G,X_1)\otimes G \longrightarrow 0, \tag{19}$$

we obtain,

$$X_2 \in \langle F', F''\rangle_{cat}.$$

But (F',F'') is a zero-pair and any sheaf $Y \in \langle F',F''\rangle_{cat}$ has the form $Y = U_1(Y)\otimes F' \oplus U_2(Y)\otimes F''$ for some vector spaces $U_1(Y), U_2(Y)$. This fact can be easily proved with help of the Beilinson–Gorodentsev spectral sequence, associated with the exceptional pair (F',F'').

Note that the Serre duality implies the equality $\mathrm{Ext}^1(X,E)^* \cong \mathrm{Ext}^1(E(2,2),X)$. Therefore, combining (18) and (19) together, we get the monad from the formulation of the proposition. The fact that $U_1(X) = \mathrm{Ext}^1(L_E F'(2,2),X)$, $U_2(X) = \mathrm{Ext}^1(L_E F''(2,2),X)$ is proved by the standard method, using long cohomology sequences. This concludes the proof of the first point of the proposition.

The second point is proved by the same arguments as lemma 4.1.3 of [9] and here we limit the following remark. *Any morphism of sheaves* $f : X \longrightarrow X'$ *induces a morphism of the universal extensions (18) and (19).*

This completes the proposition proof.

3.3 PROPOSITION. *Let* (E',E'',F,G) *be a full exceptional three-block collection on* $\mathbb{P}^1\times\mathbb{P}^1$ $(\mu(E') = \mu(E''))$*. Then*

1) any semistable sheaf X on $\mathbb{P}^1 \times \mathbb{P}^1$ satisfying the slope conditions

$$\mu(G) - 4 < \mu(E') < \mu(X) < \mu(G)$$

is the cohomology of the monad $M_2(X)$

$$0 \longrightarrow \begin{array}{c} U_0(X) \otimes E' \\ \oplus \\ U_1(X) \otimes E'' \end{array} \longrightarrow U_2(X) \otimes F \longrightarrow U_3(X) \otimes G \longrightarrow 0$$

with $U_0(X) = \mathrm{Ext}^1(E'(2,2), X)$, $U_1(X) = \mathrm{Ext}^1(E''(2,2), X)$, $U_2(X) = \mathrm{Ext}^1(L_E F(2,2), X)$, $U_3(X) = \mathrm{Ext}^1(G, X)$;
2) two semistable sheaves X and X' on $\mathbb{P}^1 \times \mathbb{P}^1$ are isomorphic iff their monads $M_2(X)$ and $M_2(X')$ are.

Proof. The proof of this proposition differs from the previous proof at the first step only. Namely, we have to project the sheaf X to the subcategory $\langle F, G\rangle_{cat}$. Consider the universal extension

$$0 \longrightarrow \begin{array}{c} \mathrm{Ext}^1(X, E')^* \otimes E' \\ \oplus \\ \mathrm{Ext}^1(X, E'')^* \otimes E'' \end{array} \longrightarrow X_1 \longrightarrow X \longrightarrow 0,$$

determining by the element

$$(id \times id) \in (\mathrm{Ext}^1(X, E')^* \otimes \mathrm{Ext}^1(X, E')) \oplus (\mathrm{Ext}^1(X, E'')^* \otimes \mathrm{Ext}^1(X, E'')).$$

Then $\mathrm{Ext}^i(X_1, E') = \mathrm{Ext}^i(X_1, E'') = 0\ \forall i$, i.e., $X_1 \in \langle F, G\rangle_{cat}$. Further, taking the universal extension

$$0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \mathrm{Ext}^1(G, X_1) \otimes G \longrightarrow 0,$$

we get $X_2 \in \langle F\rangle_{cat}$. Therefore, $X_2 = U_2(X) \otimes F$ and so on.

3.4 Remark. *In addition to the assumptions of the previous proposition, suppose that $E'' = i_* E'$ and $X = i^* X$, where $i : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is the natural involution, transposing the factors. Then the monad $M_2(X)$ can be rewritten in the form*

$$0 \longrightarrow U_1(X) \otimes \begin{pmatrix} E' \\ \oplus \\ E'' \end{pmatrix} \longrightarrow U_2(X) \otimes F \longrightarrow U_3(X) \otimes G \longrightarrow 0,$$

where $U_1(X) = \mathrm{Ext}^1(X, E')^* \cong \mathrm{Ext}^1(X, E'')^*$.

4 Kronecker moduli and representations of quivers.

To describe required moduli spaces we need Kronecker moduli and moduli space of some quiver representations. We start with Kronecker moduli by [2].

4.1 Definition. *Let U_0, U_1, V be finite dimensional vector spaces with $u_0 = \dim U_0$, $u_1 = \dim U_1$, $v = \dim V$. A linear map $t : U_0 \longrightarrow U_1 \otimes V$ up to the natural action of the reductive group $G = \mathrm{GL}(U_0) \times \mathrm{GL}(U_1)/\mathbb{C} \cdot \{id \times id\}$ is called a Kronecker module.*

4.2 REMARK. *There exists a compact Kronecker moduli space* $\mathfrak{N}(v; u_0, u_1)$*, i.e. the space of the semistable orbits of the* G*-action in the vector space* $U_0^* \otimes U_1 \otimes V$*. Moreover, in the case, when* u_0, u_1 *are coprime, the moduli space* $\mathfrak{N}(v; u_0, u_1)$ *is fine, i.e., there exists a universal Kronecker module* $\mathcal{U}_0^* \otimes \mathcal{U}_1 \otimes V$ *over* $\mathfrak{N}(v; u_0, u_1)$*.*

J.-M. Drezet in [2] shows that we can describe some moduli spaces of bundles on $\mathbb{P}^2$ as Kronecker moduli space. Namely, any morphism of sheaves $\alpha : U_0 \otimes F \longrightarrow U_1 \otimes G$ corresponds to a Kronecker module $t_\alpha : U_0 \longrightarrow U_1 \otimes \mathrm{Hom}(F, G)$. In this notation the following statements are true.

4.3 PROPOSITION. *Let* (E, F, G) *be a full exceptional collection on* $\mathbb{P}^2$ *and* U_0*,* U_1 *be vector spaces with* $u_0 = \dim U_0$*,* $u_1 = \dim U_1$ *such that*

$$1)\ u_0 \mathrm{rk}\,(F) > u_1 \mathrm{rk}\,(G), \qquad 2)\ \mu(E) < \frac{u_0 \deg F - u_1 \deg G}{u_0 \mathrm{rk}\,(F) - u_1 \mathrm{rk}\,(G)},$$

then t_α *is stable iff* α *is an epimorphism and* $\ker\alpha$ *is a stable bundle.*

4.4 THEOREM *In additional to the assumption of the previous proposition suppose that* u_0, u_1 *are coprime, then the moduli space of stable sheaves* Y*'s on* $\mathbb{P}^2$ *with topological invariants, satisfying the conditions*

$$\mathrm{rk}\, Y = u_0 \mathrm{rk}\, F - u_1 \mathrm{rk}\, G,\ \ c_1(Y) = u_0 c_1(F) - u_1 c_1(G),\ \ \chi(Y, E) = 0$$

is isomorphic to the smooth complied fine Kronecker moduli space $\mathfrak{N}(h, u_0, u_1)$*, where* $h = \dim \mathrm{Hom}(F, G)$*. Moreover, all the sheaves* Y *are locally free.*

These Drezet's statements were generalized by V. Karpov to the case of quadric $\mathbb{P}^1 \times \mathbb{P}^1$ [12]

4.5 PROPOSITION. *Let* (D, E, F, G) *be a full exceptional collection on* $\mathbb{P}^1 \times \mathbb{P}^1$ *such that* $\mu(E) < \mu(F) < \mu(G)$ *and the pair* (F, G) *is obtained from a line bundle pair by regular mutations. Suppose that dimensions of vector spaces* $u_0 = \dim U_0$*,* $u_1 = \dim U_1$ *satisfy the conditions*

$$1)\ u_0 \mathrm{rk}\,(F) > u_1 \mathrm{rk}\,(G), \qquad 2)\ \mu(E) < \frac{u_0 \deg F - u_1 \deg G}{u_0 \mathrm{rk}\,(F) - u_1 \mathrm{rk}\,(G)},$$

then t_α *is stable iff* α *is an epimorphism and* $\ker\alpha$ *is a stable bundle.*

4.6 THEOREM. *In additional to the assumption of the previous proposition suppose that* u_0, u_1 *are coprime, then the moduli space of stable sheaves* Y*'s on* $\mathbb{P}^1 \times \mathbb{P}^1$ *with topological invariants, satisfying the conditions*

$$\mathrm{rk}\, Y = u_0 \mathrm{rk}\, F - u_1 \mathrm{rk}\, G,\ \ c_1(Y) = u_0 c_1(F) - u_1 c_1(G),\ \ \chi(Y, E) = \chi(Y, D) = 0$$

is isomorphic to the smooth projective fine Kronecker moduli space $\mathfrak{N}(h, u_0, u_1)$*, where* $h = \dim \mathrm{Hom}(F, G)$*. Moreover, all the sheaves* Y *are locally free.*

To describe moduli spaces of bundles on the quadric we need a some more generalization of Drezet's theorem. At first we formulate results of [14] concerning the quiver Q with three *vertecis* p_0, p_1, p_2, h_{01} arrows from p_0 to p_1, and h_{02} arrows from p_0 to p_2.

4.7 DEFINITION. *A linear map of vector spaces*

$$t : U_0 \longrightarrow \begin{matrix} U_1 \otimes V_{01} \\ \oplus \\ U_2 \otimes V_{02} \end{matrix}$$

with $\dim V_{0i} = h_{0i}$ *is called a representation of the quiver* Q.

The group $G = \mathrm{GL}(U_0) \times \mathrm{GL}(U_1) \times \mathrm{GL}(U_2)$ acts on the vector space $R = U_0^* \otimes \big(U_1 \otimes V_{01} \oplus U_2 \otimes V_{02}\big)$ in the standard way. Obviously, the action orbits coincide with isomorphism classes of the representations.

Let us choose a character of G

$$\chi_\theta(g) = (\det g_0)^{\theta_0} \cdot (\det g_1)^{\theta_1} \cdot (\det g_2)^{\theta_2},$$

where $(g_0, g_1, g_2) \in G$, and let us lift the G-action to the direct product $R \times \mathbb{C}$ as follows $g(v, z) = (g(v), \chi_\theta^{-1}(g) \cdot z)$.

4.8 DEFINITION. *1. A point* $v \in R$ *is* χ_θ*-semistable if there is a relative invariant* $f \in \mathbb{C}[R]^{G,\chi_\theta^n}$ *with* $n \geq 1$*, such that* $f(v) \neq 0$.

2. A point $v \in R$ *is* χ_θ*-stable if it is semistable and* $\dim G \cdot v = \dim G/G_0$, *where* G_0 *is the kernel of the G-action, and the G-action on* $\{v \in R \mid f(v) \neq 0\}$ *is closed.*

Let us fix a dimension vector $\mathbf{u} = (u_0, u_1, u_2)$, where $u_i = \dim U_i$ and let us choose a character of G $\theta = (\theta_0, \theta_1, \theta_2)$ such that

$$\sum \theta_i u_i = 0. \tag{20}$$

This means that the restriction of the character χ_θ to the G-action kernel equals 1.

4.9 DEFINITION. *A line map* $t' : U_0' \longrightarrow (U_1' \otimes V_{01} \oplus U_2' \otimes V_{02})$ *is called a subrepresentation of the representation* t *if* $U_i' \subset U_i$ *and the following commutative diagram takes place:*

$$\begin{array}{ccc} U_0' & \xrightarrow{t'} & (U_1' \otimes V_{01} \oplus U_2' \otimes V_{02}) \\ \downarrow & & \downarrow \\ U_0 & \xrightarrow{t} & (U_1 \otimes V_{01} \oplus U_2 \otimes V_{02}) \end{array}.$$

4.10 PROPOSITION. *A representation* $t \in R$ *is* χ_θ*-(semi)stable if and only if, for every its subrepresentation* t' *with a dimension vector* $\mathbf{u}'$ *the condition* $\sum \theta_i u_i' > 0 \quad (\geq 0\ )$ *holds.*

4.11 THEOREM. *Let* $\mathbf{u} = (u_0, u_1, u_2)$ *be a dimension vector such that* $\gcd(u_0, u_1 + u_2) = 1$, *then the variety*

$$R_{ij}//G = \mathrm{Proj}(\bigoplus_{n \geq 0} \mathbb{C}[R]^G, \chi_\theta^n) = \mathfrak{N}^\theta(h_{01}, h_{02}, \mathbf{u})$$

is a fine moduli space for families of χ_θ*-stable representations. In other words, there exists the universal bundle* $\mathcal{U}_0^* \otimes (\mathcal{U}_1 \otimes V_{01} \oplus \mathcal{U}_2 \otimes V_{02})$ *on* $\mathfrak{N}^\theta(h_{01}, h_{02}, \mathbf{u})$ *such that the global tautological section of it can be interpreted as the bundle morphism*

$$\tau : \mathcal{U}_0 \longrightarrow (\mathcal{U}_1 \otimes V_{01} \oplus \mathcal{U}_2 \otimes V_{01})$$

and the fiber of τ over each point t coincides with the representation t.

Further we fixed θ. Namely, by definition, put $\theta_0 = -(u_1+u_2),\ \theta_1 = u_0,\ \theta_2 = u_0$. Then the inequality of theorem 4.10 has the form

$$\frac{u_0'}{u_1'+u_2'} < \frac{u_0}{u_1+u_2} \qquad (\leq \text{ in semistable case}).$$

Therefore, the fraction $\dfrac{u_0}{u_1+u_2}$ we shall call the slope of a representation t and shall denote it by $\nu(t)$. In these terms, a representation of the quiver Q is χ_θ-stable if and only if any its subrepresentation has a lesser slope.

Later in this text the term "stable representation of the quiver Q" means χ_θ-stable representation with respect to the character χ_θ with $\theta = (-(u_1+u_2), u_0, u_0)$.

Furthermore in this section we construct a moduli space of stable bundles Y's on $\mathbb{P}^1\times\mathbb{P}^1$ such that the monad from proposition 3.2 for Y has the form

$$0 \longrightarrow Y \longrightarrow \begin{array}{c} U_1(Y)\otimes Q'_{2k+1} \\ \oplus \\ U_2(Y)\otimes Q''_{2k+1} \end{array} \xrightarrow{\alpha} U_0(Y)\otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \longrightarrow 0. \tag{21}$$

In another words, the slope of Y satisfies the condition $\mu(Q_{2k}) < \mu(Y) < \mu(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})$, and $\mathrm{Ext}^i(Y, Q_{2k}) = 0\ \forall i$, where $(Q_{2k}, Q'_{2k+1}, Q''_{2k+1}, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})$ is the full exceptional three-block collection on $\mathbb{P}^1\times\mathbb{P}^1$ (see (10)). Namely, we prove the following theorem.

4.12 THEOREM. *Let u_0, u_1, u_2 be natural numbers such that*

$$1)\ \gcd(u_0, u_1+u_2) = 1, \quad 2) (u_1+u_2)q_{2k+1} > u_0, \quad 3)\ \frac{u_0}{u_1+u_2} < \frac{1}{2q_{2k-1}},$$

then the moduli space of stable sheaves Y's from the exact sequence (21) is isomorphic to the fine moduli space of the quiver representations $\mathfrak{N}(2q_{2k}, 2q_{2k}; u_o, u_1, u_2)$, where $2q_{2k} = h^0(Q'_{2k+1}) = h^0(Q''_{2k+1})$

We start with

4.13 LEMMA. *Let*

$$\alpha : \mathbb{C}^{u_0}\otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \longrightarrow \begin{array}{c} \mathbb{C}^{u_1}\otimes Q'^*_{2k+1} \\ \oplus \\ \mathbb{C}^{u_2}\otimes Q''^*_{2k+1} \end{array}$$

be a morphism of bundles and

$$t_\alpha : \mathbb{C}^{u_0} \longrightarrow \begin{array}{c} \mathbb{C}^{u_1}\otimes V_{01} \\ \oplus \\ \mathbb{C}^{u_2}\otimes V_{02} \end{array}$$

be the corresponding representation of the quiver ($V_{01} = H^0(Q'^*_{2k+1}),\ V_{02} = H^0(Q''^*_{2k+1})$). Suppose that t_α is stable and $\nu(t_\alpha) < \dfrac{1}{2q_{2k-1}}$, then α is an embedding of bundles (here $q_{2k-1} = \mathrm{rk}\,(Q'_{2k-1})$).

PROOF. The representation t_α is determined by $(u_0\times(u_1+u_2))$-matrix F with elements $f_{ij}\in V_{01}\cup V_{02}$. Consider the restriction α_x of the morphism α to the fiber at a point x.

The linear map α_x is determined by the matrix F_x, which can be obtained from the matrix F by taking the fibers of F_{ij} at the point x.

Assume that α_x has a nonzero kernel. Without loss of generality it can be assumed that $\alpha_x(e_1) = 0$, where e_1 is the first basis vector of the space $\mathbb{C}^{u_0}$. It means that $f_{i1}(x) = 0\ \forall i$, i.e., $f_{i1} \in H^0(Q'^*_{2k+1} \otimes J_x) \cup H^0(Q''^*_{2k+1} \otimes J_x)$, where J_x is the ideal sheaf of the point x. Therefore, the representation t_α has the subrepresentation $t' : \langle e_1\rangle \longrightarrow U_1 \otimes V_{01} \oplus U_2 \otimes V_{02}$, where

$$\dim U_1 = \dim U_2 = h^0(Q'^*_{2k+1} \otimes J_x) = h^0(Q'^*_{2k+1}) - \operatorname{rk} Q'^*_{2k+1}.$$

It is easily check that $h^0(Q'^*_{2k+1}) = 2\deg(Q'^*_{2k+1}) = 2q_{2k}$. Thus,

$$\nu(t') = \frac{1}{2(2q_{2k} - q_{2k+1})} \overset{11}{=} \frac{1}{2q_{2k-1}} > \nu(t_\alpha).$$

This contradiction concludes the proof.

Because of the following proposition is proved in the same way as the analogous propositions 4.3 and 4.5 from [2] and [12] respectively, we limit its formulation only.

4.14 Proposition. *Let $(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}, Q'^*_{2k+1}, Q''^*_{2k+1}, Q'^*_{2k})$ is the full exceptional three-block collection on $\mathbb{P}^1 \times \mathbb{P}^1$ (10) and u_0, u_1, u_2 be natural numbers such that*

$$1)\ (u_1 + u_2)q_{2k+1} > u_0, \quad 2)\frac{u_0}{u_1+u_2} < \frac{1}{2q_{2k-1}},$$

then a representation of the quiver

$$t_\alpha : \mathbb{C}^{u_0} \longrightarrow \begin{array}{c} \mathbb{C}^{u_1} \otimes V_{01} \\ \oplus \\ \mathbb{C}^{u_2} \otimes V_{02} \end{array}$$

is stable iff the corresponding morphism of sheaves

$$\alpha : \mathbb{C}^{u_0} \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \longrightarrow \begin{array}{c} \mathbb{C}^{u_1} \otimes Q'^*_{2k+1} \\ \oplus \\ \mathbb{C}^{u_2} \otimes Q''^*_{2k+1} \end{array}$$

is a monomorphism and $\operatorname{coker}\alpha$ *is a stable bundle.*

As a corollary of this proposition we obtain the following theorem

4.15 Theorem. *Let u_0, u_1, u_2 be natural numbers such that*

$$1)\ \gcd(u_0, u_1+u_2) = 1 \quad 2)\ (u_1+u_2)q_{2k+1} > u_0, \quad 3)\ \frac{u_0}{u_1+u_2} < \frac{1}{2q_{2k-1}},$$

then the moduli space of stable sheaves X's from the exact sequence

$$0 \longrightarrow \mathbb{C}^{u_0} \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \longrightarrow \begin{array}{c} \mathbb{C}^{u_1} \otimes Q'^*_{2k+1} \\ \oplus \\ \mathbb{C}^{u_2} \otimes Q''^*_{2k+1} \end{array} \longrightarrow X \longrightarrow 0,$$

is isomorphic to the smooth compact fine moduli space of the quiver representations

$$\mathfrak{N}(2q_{2k}, 2q_{2k}; u_0, u_1, u_2),$$

where $2q_{2k} = h^0(Q'_{2k+1}) = h^0(Q''_{2k+1})$. Moreover, each of the sheaves X's is locally free.

Since all the X's from this theorem are locally free, we obtain theorem 4.12 as the dual case.

5 Moduli spaces on the projective plane.

In this section we describe a moduli space $\mathfrak{M}_n(\delta) = \mathfrak{M}(r_n, -r_{n-1}, \delta)$ of stable sheaves X_n on $\mathbb{P}^2$ such that their rank and the first Chern class coincide with rank and the first Chern class of the exceptional bundle E_n (see 6) and $-\chi(X_n) = h^1(X_n) = \delta$. Namely, let $\mathfrak{N}_n = \mathfrak{N}(3r_{n-1}, \delta r_{n-1}+1, \delta)$ be the Kronecker moduli space (see remark 4.2) with the universal bundle $\mathcal{U}_1^* \otimes \mathcal{U}_2 \otimes V_{02}$ and $\pi_{\mathfrak{N}} : \mathfrak{G}_n = \mathrm{Grass}_{\delta r_{n-1}}(\mathcal{U}_1 \otimes V_{01}) \longrightarrow \mathfrak{N}_n$ be the Grassman bundle of $(\delta r_{n-1})-$planes in the bundle $\mathcal{U}_1 \otimes V_{01}$ with the universal bundle $\mathcal{S}$ (here $V_{01} = \mathrm{Hom}(E_{n-1}, \mathcal{O}_{\mathbb{P}^2})$, $V_{02} = \mathrm{Hom}(E_{n-1}, E_n)$). Then the following theorem takes place.

5.1 THEOREM. [2] *1. The smooth compact fine moduli space $\mathfrak{M}_n(\delta)$ of stable sheaves on $\mathbb{P}^2$ is isomorphic to the sub-variety $\mathfrak{Z}_n \subset \mathfrak{G}_n$ defined as the zero scheme of a section of the bundle $\mathcal{S}^* \otimes \pi_{\mathfrak{N}}^* \mathcal{U}_2 \otimes V_{02}$.*

2. The universal bundle $\mathcal{H}_\delta$ over $\mathfrak{M}_n(\delta)$, i.e., the bundle with $\mathrm{H}^1(X)$ as the fiber over a point $X \in \mathfrak{M}_n(\delta)$, is isomorphic to the restriction of $\pi_{\mathfrak{N}}^* \mathcal{U}_2$ to the scheme $\mathfrak{Z}_n$.

PROOF. Let X_n be the sheaf as above. It is clear, that $\mu(E_{n-1}) < \mu(X_n) < \mu(\mathcal{O}_{\mathbb{P}^2})$. Therefore, the sheaf X_n is the cohomology of the monad

$$0 \longrightarrow U_0(X_n) \otimes E_{n-1} \xrightarrow{\beta} U_1(X_n) \otimes E_n \xrightarrow{\alpha} U_2(X_n) \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow 0 \tag{22}$$

(see proposition 3.1), where

$$U_0(X_n) = \mathrm{Ext}^1(E_{n-1}(3), X_n), \quad U_1(X_n) = \mathrm{Ext}^1(E_{n-2}(3), X_n), \quad U_2(X_n) = \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}, X_n).$$

By Riemann-Roch formula for $\mathbb{P}^2$ and relations (7), we have,

$$\dim U_0(X) = \delta r_{n-1}, \quad \dim U_1(X) = \delta r_{n-2} + 1, \quad \dim U_2(X) = \delta.$$

Construction of the moduli space base. *1. A map to the Kronecker moduli space.* The monad splits into two exact triples

$$0 \longrightarrow Y \longrightarrow U_1(X_n) \otimes E_n \xrightarrow{\alpha} U_2(X_n) \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow 0,$$

$$0 \longrightarrow U_0(X_n) \otimes E_{n-1} \xrightarrow{\beta} Y \longrightarrow X_n \longrightarrow 0.$$

Since

$$\deg(E_{n-1})\mathrm{rk}\,(X_n) - \deg(X_n)\mathrm{rk}\,(E_{n-1}) = \deg(E_{n-1})\mathrm{rk}\,(E_n) - \deg(E_n)\mathrm{rk}\,(E_{n-1}) = -1$$

(see (8)) and E_{n-1} is a stable bundle (1.3), it follows from lemma 2.3 that Y is stable iff X_n is. Therefore, we have some map from the set of all isomorphism classes of stable sheaves $X_n \in \mathfrak{M}_n(\delta)$ to the set of all isomorphism classes of stable sheaves Y. By theorem 4.4 , there exists a smooth compact fine moduli space of Y's, which isomorphic to the Kronecker moduli space $\mathfrak{N}_n = \mathfrak{N}_n(3r_{n-1}; \delta r_{n-1}+1, \delta)$ $(h^0(E_n, \mathcal{O}_{\mathbb{P}^2}) = 3r_{n-1})$. On $\mathfrak{N}_n$ there exists a universal bundle $\mathcal{U}_1^* \otimes \mathcal{U}_2 \otimes V_{12}$, where $V_{12} = \mathrm{Hom}(E_n, \mathcal{O}_{\mathbb{P}^2})$. Its tautological section can be interpreted as the canonical epimorphism of bundles

$$\tau : \mathcal{U}_1 \otimes V_{12}^* \longrightarrow \mathcal{U}_2 \longrightarrow 0$$

[2]This theorem was proved with A. Gorodentsev.

such that the restriction of τ to a point $t_\alpha \in \mathfrak{N}_n$ gives the stable Kronecker module $t_\alpha : U_1 \otimes V_{12}^* \longrightarrow U_2$, corresponding to the exact triple

$$0 \longrightarrow \ker(\alpha) \longrightarrow U_1 \otimes E_m \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^2} \longrightarrow 0.$$

2. A lift the map to the Grassman bundle. By definition, put

$$V_{01} = \mathrm{Hom}(E_{n-1}, E_n), \qquad V_{02} = \mathrm{Hom}(E_{n-1}, \mathcal{O}_{\mathbb{P}^2}).$$

Let $\gamma \in V_{01}^* \otimes V_{12}^* \otimes V_{02}$ be the tensor, corresponding to the composition $\gamma : V_{01} \otimes V_{12} \longrightarrow V_{02}$. This tensor can be interpreted as the map $\gamma : V_{01} \longrightarrow V_{12}^* \otimes V_{02}$. We denote by ψ the composition of the maps

$$\mathcal{U}_1 \otimes V_{01} \xrightarrow{id\otimes\gamma} \mathcal{U}_1 \otimes V_{12}^* \otimes V_{02} \xrightarrow{\tau\otimes id} \mathcal{U}_2 \otimes V_{02}.$$

Let $\mathfrak{G}_n = \mathrm{Grass}_{\delta r_{n-1}}(\mathcal{U}_1 \otimes V_{01})$ be the Grassman bundle of (δr_{n-1})-planes in the bundle $\mathcal{U}_1 \otimes V_{01}$. Let $\pi_{\mathfrak{N}} : \mathfrak{G}_n \longrightarrow \mathfrak{N}_n$ be the projection of $\mathfrak{G}_n$ to the base and $\widetilde{\mathcal{U}}_i = \pi_{\mathfrak{N}}^* \mathcal{U}_i$. In this notation we can write the sequence of bundles on $\mathfrak{G}_n$.

$$\mathcal{S} \xrightarrow{\eta} \widetilde{\mathcal{U}}_1 \otimes V_{01} \xrightarrow{\widetilde{\psi}} \widetilde{\mathcal{U}}_2 \otimes V_{02}, \tag{23}$$

where $\mathcal{S}$ is the universal subbundle on $\mathfrak{G}_n$. By $\mathfrak{Z}_n$ we denote the zero scheme of the composition $\widetilde{\psi} \circ \eta$.

2. (1–1)-correspondence between $\mathfrak{M}_n(\delta)$ and the zero scheme $\mathfrak{Z}_n$. To show that there exists this one-to-one correspondence it is sufficient to check that each class of isomorphic monads (22) corresponds to a unique point of $\mathfrak{Z}_n$ and vice versa.

Monad (22) is determined by two morphisms β and α. These morphisms correspond to the linear map

$$t_\beta : U_0(X_n) \longrightarrow U_1(X_n) \otimes V_{01}, \qquad t_\alpha : U_1(X_n) \otimes V_{12}^* \longrightarrow U_2(X_n).$$

Since β is a monomorphism, t_β is a monomorphism too. Therefore, monad (22) corresponding to a stable sheaf X_n defines a point $z_{X_n} = (t_\beta(U_0(X_n)), t_\alpha)$ of $\mathfrak{G}_n$ such that $\alpha \circ \beta = 0$ (we'll denote it by (t_β, t_α)).

Let us show that $z_{X_n} \in \mathfrak{Z}_n$. Because of $\eta|_{z_{X_n}} = t_\beta$ and $\widetilde{\psi}|_{z_{X_n}} = \psi|_{t_\alpha}$, it is sufficient to check that the composition

$$U_0(X_n) \xrightarrow{t_\beta} U_1(X_n) \otimes V_{01} \xrightarrow{id\otimes\gamma} U_1(X_n) \otimes V_{12}^* \otimes V_{02} \xrightarrow{t_\alpha\otimes id} U_2(X_n) \otimes V_{02}$$

is trivial. Let us fix bases $\{a_i\}$ in the space $U_0(X_n)$, $\{b_j\}$ in $U_1(X_n)$ and $\{c_s\}$ in $U_2(X_n)$. In these bases the morphism β of the monad is setting by the matrix $F = (f_{ji})$, where $f_{ji} \in V_{01}$. The morphism α is setting by the matrix $G = (g_{sj})$, where $g_{sj} \in V_{12}$. Therefore, the composition $\alpha \circ \beta$ has the form $G \cdot F$. Thus, the condition $\alpha \circ \beta = 0$ is equivalent to the equality

$$\sum_j g_{sj} f_{ji} = 0. \tag{24}$$

At the same time, the linear maps t_β and t_α in the fixed bases are setting by the same matrices. Hence,

$$t_\beta\Big(\sum_i x_i a_i\Big) = \sum_i x_i \sum_j b_j \otimes f_{ji} = \sum_j b_j \otimes \sum_i x_i f_{ji}.$$

If we fix a basis $\{h_k\}$ in the space V_{02}, we can write the element $(id \otimes \gamma) \circ t_\beta\Big(\sum_i x_i a_i\Big)$ as

$$\sum_j b_j \otimes \sum_i x_i \gamma(f_{ji}) = \sum_j b_j \otimes \sum_i x_i \sum_k \lambda^k_{ji} \otimes h_k, \tag{25}$$

where $\lambda^k_{ji} \in V^*_{12}$ and $\sum_k \lambda^k_{ji} \otimes h_k = \gamma(f_{ji})$ is the morphism of V_{12} to V_{02} such that for all $g \in V_{12}$ we have

$$\gamma(f_{ji}) : g \mapsto \sum_k \langle \lambda^k_{ji}, g\rangle \otimes h_k = g \circ f_{ji}, \tag{26}$$

where $\langle \lambda^k_{ji}, g\rangle$ is the natural pairing $V^*_{12} \otimes V_{12} \longrightarrow \mathbb{C}$.

To apply the map $t_\alpha \otimes id$ to (25) let us recall that

$$t_\alpha \sum_j b_j \otimes \lambda_j = \sum_s c_s \sum_j \langle g_{sj}, \lambda_j\rangle.$$

Now we have,

$$t_\alpha \otimes id \sum_k \Big(\sum_j b_j \otimes \sum_i x_i \lambda^k_{ji}\Big) \otimes h_k = \sum_k \Big(\sum_s c_s \sum_j \langle g_{sj}, \sum_i x_i \lambda^k_{ji}\rangle \otimes h_k\Big) =$$

$$= \sum_s c_s \sum_i x_i \sum_j \Big(\sum_k \langle g_{sj}, \lambda^k_{ji}\rangle \otimes h_k\Big) \overset{(26)}{=} \sum_s c_s \sum_i x_i \sum_j g_{sj} \circ f_{ji}.$$

Combining this and (24), we obtain that the point $z_{X_n} = (t_\beta, t_\alpha)$ corresponding to the stable sheaf X_n belongs to $\mathfrak{Z}_n$.

On the other hand, any point $(U_0, t_\alpha) \in \mathfrak{G}_n$ gives a monomorphism t_β and an epimorphism t_α

$$t_\beta : U_0 \longrightarrow U_1 \otimes V_{01}, \qquad t_\alpha : U_1 \otimes V^*_{12} \longrightarrow U_2,$$

i.e., two morphisms of bundles

$$\beta : U_0 \otimes E_{n-1} \longrightarrow U_1 \otimes E_n, \qquad \alpha : U_1 \otimes E_n \longrightarrow U_2 \otimes \mathcal{O}_{\mathbb{P}^2}.$$

Moreover, the previous arguments tell us that if the point (U_0, t_α) belongs to $\mathfrak{Z}_n$, then the composition $\alpha \circ \beta$ is trivial. Thus we proved the following statement:
point of the closed subscheme $\mathfrak{Z}_n \subset \mathfrak{G}_n$ one-to-one correspond to classes of isomorphic complexes (22) such that the morphism α is an epimorphism
(α corresponds to a stable Kronecker module).

Let us check that the morphism β is a monomorphism. We know that the composition $\alpha \circ \beta = 0$. Therefore, we can assume that $\beta : U_0 \otimes E_{n-1} \longrightarrow \ker \alpha$. It is easily to check that the sheaves E_{n-1} and $\ker \alpha$ satisfy the assumptions of lemma 2.5 , consequently, $\ker \beta = U \otimes E_{n-1}$ for some subspace $U \subset U_0$. This contradicts the fact that t_β is a monomorphism. Thus we proved that
point of the closed subscheme $\mathfrak{Z}_n \subset \mathfrak{G}_n$ one-to-one correspond to points of $\mathfrak{M}_n(\delta)$.
In another words we constructed a base of the moduli space $\mathfrak{M}_n(\delta)$.

Construction of the universal sheaf. Denote by $\bar{\mathcal{U}}_i$ and $\bar{\mathcal{S}}$ the restriction of the bundles $\mathcal{U}_i$ and $\mathcal{S}$ to $\mathfrak{Z}_n$ respectively. In this notation the restriction of sequence (23) to $\mathfrak{Z}_n$ is the monad on $\mathfrak{Z}_n$

$$0 \longrightarrow \bar{\mathcal{S}} \longrightarrow \bar{\mathcal{U}}_1 \otimes V_{01} \longrightarrow \bar{\mathcal{U}}_2 \otimes V_{02} \longrightarrow 0. \tag{27}$$

Denote by p_1 and p_2 the projections of the direct product $\mathbb{P}^2 \times \mathfrak{Z}_n$ to the first and the second factors respectively and by definition put,

$$\mathcal{U}'_i = p_2^* \bar{\mathcal{U}}_i \ (i = 1, 2), \qquad \mathcal{U}'_0 = p_2^* \bar{\mathcal{S}}.$$

It can be proved that sequence (27) induces the monad on the direct product $\mathbb{P}^2 \times \mathfrak{Z}_n$

$$0 \longrightarrow \mathcal{U}'_0 \otimes p_1^* E_{n-1} \longrightarrow \mathcal{U}'_1 \otimes p_1^* E_n \longrightarrow \mathcal{U}'_2 \otimes p_1^* \mathcal{O}_{\mathbb{P}^2} \longrightarrow 0 \tag{28}$$

with a cohomology $\mathcal{X}_n$. Note that the restriction of this monad to $\mathbb{P}^2 \times z_{X_n}$ is the monad with the cohomology X_n.

Now we'll prove that the family $\mathcal{X}_n$ over the base $\mathfrak{Z}_n$ is universal, i.e., we'll show that for any family $\mathcal{F}$ of stable sheaves (with the topological invariants from the theorem) over a base $\mathfrak{B}$ there exists a map $\varphi : \mathfrak{B} \longrightarrow \mathfrak{Z}_n$ such that the sheaf $\mathcal{F}$ on $\mathbb{P}^2 \times \mathfrak{B}$ is isomorphic to $(id \times \varphi)^* \mathcal{X}_n$.

On the direct product $\mathbb{P}^2 \times \mathfrak{B}$ there exists a Beilinson-Gorodentsev relative spectral sequence $\mathbf{E}^{p,q}$ converging to the sheaf $\mathcal{F}$ on the diagonal ([10]) with

$$\mathbf{E}_1^{p,q} = \pi_{\mathfrak{B}}^* \mathcal{E}xt^q_{\pi_{\mathfrak{B}}} \Big(\pi^*({}^{\vee}G_{2+p}), \mathcal{F} \Big) \otimes \pi^* G_{2+p}, \quad q = 0, 1, 2; \quad p = -2, -1, 0,$$

where $\pi_{\mathfrak{B}}$ and π are the projections of $\mathbb{P}^2 \times \mathfrak{B}$ to the factors and

$$(G_0, G_1, G_2) = (E_{n-1}, E_n, \mathcal{O}_{\mathbb{P}^2}), \quad ({}^{\vee}G_2, {}^{\vee}G_1, {}^{\vee}G_0) = (\mathcal{O}_{\mathbb{P}^2}, E_{n-2}(3), E_{n-1}(3)).$$

It is not hard to prove, that this spectral sequence is degenerated into a monad

$$0 \longrightarrow \pi_{\mathfrak{B}}^* F_0 \otimes \pi^* E_{n-1} \xrightarrow{\beta} \pi_{\mathfrak{B}}^* F_1 \otimes \pi^* E_n \xrightarrow{\alpha} \pi_{\mathfrak{B}}^* F_2 \otimes \pi^* \mathcal{O}_{\mathbb{P}^2} \longrightarrow 0, \tag{29}$$

with the cohomology $\mathcal{F}$, where

$$F_i = \mathcal{E}xt^1_{\pi_{\mathfrak{B}}} \Big(\pi^*({}^{\vee}G_{2+p}), \mathcal{F} \Big).$$

The restriction of this monad to $\mathbb{P}^2 \times [F]$ is the standard monad (22) on $\mathbb{P}^2$ constructed by the sheaf F. We take each point $[F] \in \mathfrak{B}$ to the Kronecker module $t_\alpha(F)$ corresponding to the restriction of α to $\mathbb{P}^2 \times [F]$. Denote by $f : \mathfrak{B} \longrightarrow \mathfrak{N}_n$ this map. Note that $f^* \mathcal{U}_i = F_i$, $i = 1, 2$, when $\mathcal{U}_i$'s are the universal bundles on $\mathfrak{N}$.

If we tensoring monad (29) by $\pi^* E_{n-1}^*$ and apply the functor $R^0 \pi_{\mathfrak{B}*}$, we obtain an inclusion of bundles

$$F_0 \hookrightarrow F_1 \otimes V_{01} = f^*(\mathcal{U}_1 \otimes V_{01}).$$

Therefore, the map f can be extended to a map $f' : \mathfrak{B} \longrightarrow \mathfrak{G}_n$ (see [4]) such that $F_0 = f'^* \mathcal{S}$.

It is clear, that $f'(\mathfrak{B}) \subset \mathfrak{Z}_n$ and the inverse image of monad (28) under the map $id \times f'$ coincides with (29). Thus $(id \times f')^* \mathcal{X}_n = \mathcal{F}$. This concludes the proof.

6 Moduli spaces on the quadric.

In this section we describe a moduli spaces

$$\mathfrak{M}_{2k}(\delta) = \mathfrak{M}\Big(q_{2k}, -(q_{2k-1}, q_{2k-1}), \delta \Big),$$

$$\mathfrak{M}'_{2k+1}(\delta) = \mathfrak{M}\Big(q_{2k+1}, -\frac{1}{2}(q_{2k}+1, q_{2k}-1), \delta\Big), \quad \mathfrak{M}''_{2k+1}(\delta) = \mathfrak{M}\Big(q_{2k+1}, -\frac{1}{2}(q_{2k}-1, q_{2k}+1), \delta\Big)$$

of stable sheaves X's on the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ such that $-\chi(X) = \delta$, and their rank and the first Chern class coincide with rank and the first Chern class of the exceptional bundles Q_{2k}, Q'_{2k+1}, Q''_{2k+1} respectively (see (9)). Note that the natural involution i on the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ induces the isomorphism of the moduli spaces $i^* : \mathfrak{M}'_{2k+1}(\delta) \longrightarrow \mathfrak{M}''_{2k+1}(\delta)$. Therefore, it is sufficient to construct one of them only. Because of the construction of these moduli spaces is the same as in the section 5, we omit details.

Let us start with **the moduli space** $\mathfrak{M}_{2k}(\delta)$. Each sheaf $X \in \mathfrak{M}_{2k}(\delta)$ satisfies the inequality

$$\mu(Q''_{2k-1}) = \mu(Q'_{2k-1}) < \mu(X) < \mu(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}).$$

Therefore, it is the cohomology of the monad

$$0 \longrightarrow U_0(X) \otimes \begin{pmatrix} Q'_{2k-1} \\ \oplus \\ Q''_{2k-1} \end{pmatrix} \xrightarrow{\beta} U_1(X) \otimes Q_{2k} \xrightarrow{\alpha} U_2(X) \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \longrightarrow 0 \tag{30}$$

(see remark 3.4), where

$$U_0(X) = \operatorname{Ext}^1(X, Q'_{2k-1})^* = \operatorname{Ext}^1(X, Q''_{2k-1})^*,$$

$$U_1(X) = \operatorname{Ext}^1(X, L_{Q_{2k-1}} Q_{2k})^* = \operatorname{Ext}^1(X, Q_{2k-2})^*, \ U_2(X) = \mathrm{H}^1(X).$$

By Riemann-Roch formula for $\mathbb{P}^1 \times \mathbb{P}^1$ and relations (11), (12), (13), we have

$$\dim U_0(X) = \delta q_{2k-1}, \ \dim U_1(X) = \delta q_{2k-2} + 1, \ \dim U_2(X) = \delta.$$

Monad (30) splits into two exact sequences

$$0 \longrightarrow Y \longrightarrow U_1(X) \otimes Q_{2k} \xrightarrow{\alpha} U_2(X) \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \longrightarrow 0,$$

$$0 \longrightarrow U_0(X) \otimes \begin{pmatrix} Q'_{2k-1} \\ \oplus \\ Q''_{2k-1} \end{pmatrix} \xrightarrow{\beta} Y \longrightarrow X \longrightarrow 0. \tag{31}$$

It follows from the definition of the sheaf X and (12) that exact triple (31) satisfies the assumptions of lemma 2.3 and remark 2.4 . Hence the sheaf Y is stable iff X is. Whence we obtain a map of $\mathfrak{M}_{2k}(\delta)$ to the Kronecker moduli space $\mathfrak{N}_{2k} = \mathfrak{N}(4q_{2k-1}, \delta q_{2k-2}+1, \delta)$ with the universal Kronecker module $\tau : \mathcal{U}_1 \otimes V^*_{12} \longrightarrow \mathcal{U}_2$, where $4q_{2k-1} = h^0(Q_{2k}, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})$.

By definition, put

$$V_{01} = \operatorname{Hom}(Q'_{2k-1} \oplus Q''_{2k-1}, Q_{2k}), \ V_{12} = \operatorname{Hom}(Q_{2k}, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}),$$

$$V_{02} = \operatorname{Hom}(Q'_{2k-1} \oplus Q''_{2k-1}, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}).$$

Let $\gamma : V_{01} \longrightarrow V^*_{12} \otimes V_{02}$ be the map corresponding to the composition map $V_{01} \otimes V_{12} \longrightarrow V_{02}$. Denote by ψ the composition

$$\mathcal{U}_1 \otimes V_{01} \xrightarrow{id\otimes\gamma} \mathcal{U}_1 \otimes V^*_{12} \otimes V_{02} \xrightarrow{\tau\otimes id} \mathcal{U}_2 \otimes V_{02}.$$

Further, as in the previous section we lift the map $\mathfrak{M}_{2k}(\delta) \longrightarrow \mathfrak{N}_{2k}$ to a map $\mathfrak{M}_{2k}(\delta) \longrightarrow \mathfrak{G}_{2k} = \mathrm{Grass}_{\delta q_{2k-1}}(\mathcal{U}_1 \otimes V_{01})$ and prove that points of $\mathfrak{M}_{2k}(\delta)$ (1–1)-correspond to points of the zero scheme $\mathfrak{Z}_{2k}$ of the composition

$$\mathcal{S} \longrightarrow \pi_{\mathfrak{N}}^* \mathcal{U}_1 \otimes V_{01} \xrightarrow{\tilde{\psi}} \pi_{\mathfrak{N}}^* \mathcal{U}_2 \otimes V_{02},$$

where $\mathcal{S}$ is the universal subbundle over $\mathfrak{G}_{2k} \xrightarrow{\pi_{\mathfrak{N}}} \mathfrak{N}_{2k}$. Repeating the reasons of the previous section, the following theorem can be proved.

6.1 THEOREM. *1. The smooth compact fine moduli space $\mathfrak{M}_{2k}(\delta)$ is isomorphic to the zero scheme $\mathfrak{Z}_{2k}$ of a section of the bundle $\mathcal{S}^* \otimes \pi_{\mathfrak{N}}^* \mathcal{U}_2 \otimes V_{02}$ on the Grassman bundle*

$$\mathrm{Grass}_{\delta q_{2k-1}}(\mathcal{U}_1 \otimes V_{01}) \xrightarrow{\pi_{\mathfrak{N}}} \mathfrak{N}(4q_{2k-1}, \delta q_{2k-2}+1, \delta)$$

over the Kronecker moduli space.

2. The universal bundle $\mathcal{H}$ over $\mathfrak{M}_{2k}(\delta)$, i.e., the bundle with $\mathrm{H}^1(X)$ as the fiber over a point $X \in \mathfrak{M}_{2k}(\delta)$, is isomorphic to the restriction of $\pi_{\mathfrak{N}}^* \mathcal{U}_2$ to the zero scheme $\mathfrak{Z}_{2k}$.

Now, using the moduli space of quiver representation (see theorem 4.11), we construct **the moduli space** $\mathfrak{M}'_{2k+1}(\delta)$. It is clear that each sheaf $X \in \mathfrak{M}'_{2k+1}(\delta)$ satisfies the assumptions of proposition 3.2 with the full exceptional collection $(Q_{2k}, Q'_{2k+1}, Q''_{2k+1}, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1})$. Hence the sheaf X is the cohomology of the monad

$$0 \longrightarrow U_0(X) \otimes Q_{2k} \xrightarrow{\beta} \begin{matrix} U_1(X) \otimes Q'_{2k+1} \\ \oplus \\ U_2(X) \otimes Q''_{2k+1} \end{matrix} \xrightarrow{\alpha} U_3(X) \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow 0,$$

where dimensions of the vector spaces $U_i(X)$ can be calculated by Riemann-Roch formula for $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\dim U_0(X) = \delta q_{2k},\ \dim U_1(X) = \delta q_{2k-1} + 1,\ \dim U_2(X) = \delta q_{2k-1},\ \dim U_3(X) = \delta.$$

It can easily be checked that the sheaves Q_{2k} and X satisfy the assumptions of lemma 2.3 . Therefore, taking each stable sheaf $X \in \mathfrak{M}'_{2k+1}(\delta)$ to the quiver representation

$$\begin{matrix} U_1(X) \otimes \mathrm{Hom}(Q'_{2k+1}, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1})^* \\ \oplus \\ U_2(X) \otimes \mathrm{Hom}(Q''_{2k+1}, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1})^* \end{matrix} \xrightarrow{t_\alpha} U_3(X),$$

we obtain a map of $\mathfrak{M}'_{2k+1}(\delta)$ to the fine smooth compact quiver representation moduli space

$$\mathfrak{N}'_{2k+1} = \mathfrak{N}(2q_{2k}, 2q_{2k}; \delta, \delta q_{2k-1}, \delta q_{2k-1} + 1)$$

(see 4.11) with the universal representation

$$\tau: \begin{matrix} \mathcal{U}_1(X) \otimes V_{13}^* \\ \oplus \\ \mathcal{U}_2(X) \otimes V_{23}^* \end{matrix} \longrightarrow \mathcal{U}_3(X),$$

where

$$V_{13} = \mathrm{Hom}(Q'_{2k+1}, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}), \quad V_{23} = \mathrm{Hom}(Q''_{2k+1}, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}).$$

By definition, put

$$V_{01} = \mathrm{Hom}(Q_{2k}, Q'_{2k+1}),\ V_{02} = \mathrm{Hom}(Q_{2k}, Q''_{2k+1}),\ V_{03} = \mathrm{Hom}(Q_{2k}, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})$$

and denote by γ_i the linear map $\gamma_i : V_{0i} \otimes V_{i3} \longrightarrow V_{03}$ of composition. It can be interpreted as a map

$$\gamma_i : V_{0i} \longrightarrow V_{i3}^* \otimes V_{03}.$$

Denote by ψ the composition of the maps on $\mathfrak{N}'_{2k+1}$:

$$\begin{matrix} \mathcal{U}_1(X) \otimes V_{01} \\ \oplus \\ \mathcal{U}_2(X) \otimes V_{02} \end{matrix} \xrightarrow{\gamma} \begin{matrix} \mathcal{U}_1(X) \otimes V_{13}^* \otimes V_{03} \\ \oplus \\ \mathcal{U}_2(X) \otimes V_{23}^* \otimes V_{03} \end{matrix} \xrightarrow{\tau\otimes id} \mathcal{U}_3 \otimes V_{03},$$

where $\gamma = (id \otimes \gamma_1) \oplus (id \otimes \gamma_2)$.

Let $\mathfrak{G}'_{2k+1} = \mathrm{Grass}_{\delta q_{2k}}(\mathcal{U}_1 \otimes V_{01} \oplus \mathcal{U}_2 \otimes V_{02})$ be the Grassman bundle with the projection $\pi_{\mathfrak{N}}$ to the base and $\mathcal{S}$ as the universal subbundle. Denote by $\mathfrak{Z}'_{2k+1}$ the zero scheme of the composition

$$\mathcal{S} \longrightarrow \pi^*_{\mathfrak{N}} \begin{pmatrix} \mathcal{U}_1(X) \otimes V_{01} \\ \oplus \\ U_2(X) \otimes V_{02} \end{pmatrix} \xrightarrow{\tilde{\psi}} \pi^*_{\mathfrak{N}}\mathcal{U}_3 \otimes V_{03}.$$

It can be shown in the same way as before that the following theorem takes place.

6.2 THEOREM. *1.The smooth compact fine moduli space $\mathfrak{M}'_{2k+1}(\delta)$ is isomorphic to the zero scheme $\mathfrak{Z}'_{2k+1}$ of a section of the bundle $\mathcal{S}^* \otimes \pi^*_{\mathfrak{N}}\mathcal{U}_3 \otimes V_{03}$ on the Grassman bundle*

$$\mathrm{Grass}_{\delta q_{2k}}(\mathcal{U}_1 \otimes V_{01} \oplus \mathcal{U}_2 \otimes V_{02}) \xrightarrow{\pi_{\mathfrak{N}}} \mathfrak{N}(2q_{2k}, 2q_{2k}; \delta, \delta q_{2k-1}, \delta q_{2k-1} + 1)$$

over the quiver representations moduli space.

2. The universal bundle $\mathcal{H}_\delta$ over $\mathfrak{M}'_{2k+1}(\delta)$, i.e., the bundle with $\mathrm{H}^1(X)$ as the fiber over a point $X \in \mathfrak{M}'_{2k+1}(\delta)$, is isomorphic to the restriction of $\pi^*_{\mathfrak{N}}\mathcal{U}_3$ to the zero scheme $\mathfrak{Z}'_{2k+1}$.

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