Noncommutative spaces and flat descent.

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Abstract. This paper is dedicated to a prevailing during the last fifteen years approach to noncommutative algebraic geometry in which 'spaces' are identified with categories thought as categories of quasi-coherent or coherent sheaves. We introduce a universal for this viewpoint category of 'spaces' and develop basics of algebraic geometry inside of this category; i.e. we define affine objects, classes of morphisms which have geometric meaning, and glue *locally affine spaces* and schemes. One of the main tools for studying and describing thus defined locally affine spaces and morphisms between them is provided by a noncommutative version of flat descent.

Introduction

Spaces considered in commutative geometry, are either locally ringed topological spaces (otherwise called *geometric spaces*), or sheaves of sets on the site **CAff** of commutative affine schemes. Although these two categories of *spaces* have little in common, the categories of schemes obtained by glueing affine objects are naturally equivalent.

Noncommutative algebraic geometry is developing under influence of examples of interest which are felt to be of geometric nature, but appear in different surroundings. These examples suggest the choices of the category of 'spaces'. Thus, noncommutative analogue of *Proj* is defined as a category [M1], [V1], or a category with a fixed object $\mathcal{O}$ (– a structure sheaf) [AZ]. Projective spaces introduced in [KR1] appear as functors from the category Alg_k of associative unital k-algebras to **Sets**.

We studied 'spaces' represented by sheaves of sets in [KR2] and noncommutative stacks in [KR3]. Here we undertake an *elementary* study of 'spaces' represented by categories regarded as categories of quasi-coherent or coherent sheaves on (maybe 'virtual') spaces. Similarly to [KR2], one of the purposes of this work is to provide a detailed background for the construction of non-affine spaces sketched in [KR1]. The paper is turning around several examples and establishes (a part of) a dictionary which extends geometric notions to 'abstract nonsense' environment. This explains the length of the manuscript.

It is important to realize that the setting considered in this work is one of the faces of noncommutative geometry. There are several others. Among them, 'spaces' represented by triangulated categories, or DG-, or A_∞-categories are of increasing importance and require a thorough investigation. We do not discuss them in this paper.

The category of 'spaces'. The minimal starting data for a geometric theory consists of: a category, $\mathfrak{B}$, of 'spaces', a category $\mathcal{A}$ of 'local' (or 'affine') objects, a functor from the category $\mathcal{A}$ to the category $\mathfrak{B}$ which assigns to each affine object a 'space'. Given a set of *covers* for all objects of $\mathfrak{B}$, we define the notion of a *locally affine 'space'* (details of this formalism can be found in sections 9.1 – 9.5).

We add to this minimal data one more ingredient: a map which assigns to each 'space' X a category, C_X, regarded as the category of quasi-coherent modules on X.

This assignment is supposed to be a contravariant pseudo-functor from the category $\mathfrak{B}$ of 'spaces' to the category Cat of categories (which belong to a given universum). In other words, we have a fibered category over the category $\mathfrak{B}$. There is a universal (in a sense which is made precise in the text) for this setting category of 'spaces', $|Cat|^o$. It has same objects as the category Cat^{op} opposite to Cat, morphisms are isomorphism classes of 1-morphisms of Cat^{op}. The universal fibered category of 'spaces' is given by the functor from Cat^{op} to $|Cat|^o$ which is identical on objects and assigns to each 1-morphism of Cat^{op} its isomorphism class. We interpret 1-morphisms of Cat^{op} as inverse image functors of the corresponding morphisms of $|Cat|^o$ (i.e morphisms of 'spaces').

This is the starting point of the work. The immediate purpose is to define and study geometry (or geometries) inside of the category $|Cat|^o$, like commutative scheme theory is defined and studied inside of the category of locally ringed spaces.

What we really study and use is not so much 'spaces', but certain classes of morphisms between 'spaces'. The most important among them are *continuous', flat*, and *affine* morphisms introduced in [R]. A morphism is continuous if its inverse image functor has a right adjoint (called a direct image functor), and flat if, in addition, the inverse image functor is left exact (i.e. preserves finite limits). A continuous morphism is called *affine* if its direct image functor is conservative (i.e. reflects isomorphisms) and has a right adjoint.

Affine and locally affine 'spaces'. Given an object S of the category $|Cat|^o$, we define the category Aff_S of affine S-spaces as the full subcategory of $|Cat|^o/S$ whose objects are affine morphisms to S. The functor from Aff_S to $|Cat|^o$ is the composition of the inclusion functor $Aff_S \hookrightarrow |Cat|^o/S$ and the canonical functor $|Cat|^o/S \longrightarrow |Cat|^o$. The choice of the object S influences drasticly the rest of the story. Thus, if $S = \mathbf{Sp}\mathbb{Z}$ (i.e. C_S is the category of abelian groups), then Aff_S is equivalent to the category opposite to the category $\mathfrak{A}ss$ which is defined as follows: objects of $\mathfrak{A}ss$ are associative unital rings and morphisms are conjugation classes of unital ring morphisms.

If C_S is the category **Sets**, then the category Aff_S is equivalent to the category $\widetilde{\mathfrak{M}on}^{op}$, where objects of $\widetilde{\mathfrak{M}on}$ are monoids and morphisms are conjugation classes of monoid homomorphisms.

Locally affine objects are defined in a straightforward way, once a notion of a cover (a *quasi-pretopology*) is fixed. We introduce several canonical quasi-pretopologies on the category $|Cat|^o$. Their common feature is the following: if a set of morphisms to X is a cover, then the set of their inverse image functors is conservative and all inverse image functors are exact in a certain mild way. If, in addition, morphisms of covers are continuous, X has a finite affine cover, and the category C_S has finite limits, this requirement suffices to recover the object X from the covering data uniquely up to isomorphism (i.e. the category C_X is recovered uniquely up to equivalence) via 'flat descent'.

Semiseparated covers and 'spaces'. We call a cover 'semiseparated' if it is finite and consists of affine morphisms. Semiseparated covers form a quasi-pretopology on $|Cat|^o$. We call a 'space' X 'semiseparated' if it has a semiseparated cover by 'affine spaces' (i.e. images of objects of Aff_S). If the base 'space' S is $\mathbf{Sp}(R)$ for some associative unital ring R (that is C_S coincides with the category $R-mod$ of left R-modules), then semiseparated 'spaces' is the answer to the question 'how much of geometry linear algebra can incorporate?'. Explicitly, if X is semiseparated over $\mathbf{Sp}(R)$, then there exists a ring T and a coalgebra H in the category of R-bimodules such that the category C_X is equivalent to the category of H-comodules, i.e. left R-modules with a coaction of H. This fact is a consequence of *flat descent* which is one of the topics and tools of this work.

Recall that a scheme $\mathcal{X}$ is called *semiseparated* if it has an affine cover $\{U_i \hookrightarrow \mathcal{X} | \ i \in J\}$ such that each morphism $U_i \hookrightarrow \mathcal{X}$ is affine. Any separated scheme is semiseparated; in particular, all classical varieties are semiseparated schemes. One can show that a space $\mathcal{X}$ (geometric or functorial) is a semiseparated scheme iff its image in $|Cat|^o$ (determined by pseudo-functor which assigns to a space $\mathcal{X}$ the category $Qcoh_{\mathcal{X}}$ of quasi-coherent sheaves on $\mathcal{X}$) is a semiseparated scheme over $\mathbf{Sp}\mathbb{Z}$ in the meaning above.

Locally affine morphisms. Even if we consider only semiseparated spaces and schemes over $\mathbb{Z}$ (i.e. $S = \mathbf{Sp}\mathbb{Z}$), the geometry we obtain via the functor $Aff_S \longrightarrow |Cat|^o$ is unusual: our locally affine spaces do not have, in general, a global sections functor. Global sections functor appeares if instead of $|Cat|^o$, we take the full subcategory $|Cat|^o_S$ of the category $|Cat|^o/S$ of S-'spaces' whose objects are pairs (X, f), where f is a continuous morphism $X \longrightarrow S$. The choice of the category of local objects is the same as above, i.e. Aff_S, and the functor from Aff_S to $|Cat|^o_S$ is the inclusion. It follows from our results that this inclusion is a full (hence a fully faithful) functor. Quasi-pretopologies we introduced on $|Cat|^o$ induce quasi-pretopologies on $|Cat|^o_S$. Applying general formalism, we define locally affine 'spaces'; using flat descent (in the case of covers formed by continuous morphisms), we obtain their description via local data.

We apply flat descent consideration to describing morphisms between locally affine 'spaces' in terms of given affine covers of these 'spaces'. A demand to obtain such description immerged in our work [KR1]. Notice that due to the fact that flat covers do not form a pretopology (the base change invariance fails), the standard way to describe morphisms via covers is not available.

Besides of these conceptual facts, the work contains several other things. For instance, we give a useful description of continuous morphisms to the categoric spectrum of a ring. We introduce and study (in 'Complementary facts') noncommutative quasi-affine 'spaces' (spectra of non-unital associative rings) illustrated by some examples coming from representation theory.

The paper is organized as follows.

In Section 1, we introduce first notions of categoric geometry ('spaces' represented by categories, morphisms represented by their inverse image functors, continuous, flat and affine morphisms) and sketch several examples of noncommutative spaces which are among illustrations and motivations of constructions of this work.

In Section 2, we start to view the category Cat^{op} opposite to the category of categories (which belong to a universum $\mathfrak{U}$) as the *bicategory of 'spaces'*: objects are regarded as 'spaces', the corresponding categories as categories of quasi-coherent sheaves, and functors as inverse image functors of morphisms they define. We introduce the *category of spaces*, $|Cat|^o$, by identifying isomorphic inverse image functors. We observe that the natural functor $Cat^{op} \longrightarrow |Cat|^o$ (identical on objects) is a universal object in the category of fibered categories of certain type.

In Section 3, we introduce (or ruther recall the definition of) different classes of morphisms, starting with continuous, flat, and affine morphisms. Their meaning is illustrated by examples. In particular, we consider a functor from the category **Aff** of affine (noncommutative) schemes defined as the category opposite to the category **Rings** of associative rings to the category $|Cat|^o$ of 'spaces' which assigns to each ring R its *categoric spectrum*, $\mathbf{Sp}(R)$ (corresponding to the category of left R-modules) and to each ring morphism an affine morphism between the respective categoric spectra.

In Section 4, we describe continuous morphisms from an arbitrary "space" X to the *categoric spectrum*, $\mathbf{Sp}(R)$, of a ring R. We argue that continuous morphisms $X \longrightarrow \mathbf{Sp}(R)$ are in one-to-one correspondence with isomorphism classes of *right R-modules* $\mathcal{O}$ in the category C_X of quasi-coherent modules on X (i.e. R-modules in the opposite category C_X^{op}). In the case $X = \mathbf{Sp}(T)$ for some ring T, this correspondence expresses the classical fact (a theorem by Eilenberg and Moore) that inverse image functors of continuous morphisms $\mathbf{Sp}(T) \longrightarrow \mathbf{Sp}(R)$ are given by (T, R)-bimodules.

In Section 5, we start to study continuous morphisms via monads and comonads associated with them, using as a main tool the Beck's theorem characterizing the so called *monadic* and *comonadic* morphisms. For a monad $\mathcal{F}$ on a "space" X (i.e. on the category C_X), we define the *spectrum* of $\mathcal{F}$ as the "space" $\mathbf{Sp}(\mathcal{F}/X)$ corresponding to the category of $\mathcal{F}$-modules. The spectrum of a monad is a natural generalization of the categoric spectrum of a ring. Dually, for any comonad $\mathcal{G}$ on X, we define its *cospectrum*, $\mathbf{Sp}^o(X\backslash\mathcal{G})$ as the space corresponding to the category $\mathcal{G} - Comod$ of $\mathcal{G}$-comodules.

In Section 6, we exploit the fact that an affine morphism to X is isomorphic to the canonical morphism $\mathbf{Sp}(\mathcal{F}/X) \longrightarrow X$ for a *continuous* monad $\mathcal{F} = (F, \mu)$. Here 'continuous' means that the functor F has a right adjoint. A consequence of this is that any affine morphism $Y \longrightarrow \mathbf{Sp}(R)$ is equivalent to the morphism $\mathbf{Sp}(T) \longrightarrow \mathbf{Sp}(R)$ corresponding to a ring morphism $R \longrightarrow T$. In particular, a direct image functor of any affine morphism $\mathbf{Sp}(S) \longrightarrow \mathbf{Sp}(R)$ is a composition of a Morita equivalence and the "restriction of scalars" (pull-back) functor corresponding to a ring morphism.

In Section 7, we study *affine flat descent*. If $U \xrightarrow{f} X$ is a flat conservative affine morphism ('conservative' means that f^* reflects isomorphisms), then it follows from Beck's theorem that X is isomorphic to $\mathbf{Sp}^o(U\backslash\mathcal{G}_f)$, where $\mathcal{G}_f = (G_f, \delta_f)$ is a *continuous* comonad. 'Continuous' means that the functor G_f has a right adjoint. In the case $U = \mathbf{Sp}(R)$ for some ring R, continuous comonads are given by coalgebras in the category of R-bimodules. The main commutative example is an arbitrary semiseparated quasi-compact scheme. Recall that a scheme X is semiseparated iff it has an affine cover $\{U_i \hookrightarrow X \mid i \in J\}$ such that all embeddings $U_i \hookrightarrow X$ are affine morphisms.

Section 8 is dedicated to finiteness conditions (locally finitely presentable morphisms and objects) and smooth and étale morphisms. We start with a simple general formalism and then specialize it in the case of 'spaces'.

In Section 9, we introduce a general formalism of glueing locally affine objects and apply it to define locally affine spaces and schemes in the category $|Cat|^o$ and in the categories $|Cat|^o_S$ of S-'spaces'. Here, we define locally affine spaces and schemes in full generality. Note that noncommutative locally affine spaces and schemes defined in earlier works on the subject [R1], [KR] belong to the class of semiseparated spaces which does not include already most of quasi-separated quasi-compact commutative schemes.

Section 10 is dedicated to the flat descent of morphisms.

The content of the second part of the paper, Complementary facts, reflects its title.

In Section C1, we give a description of continuous morphisms to the categoric spectrum of a monoid which is analogous to the description of continuous morphisms to the categoric spectrum of a ring obtained in Section 4.

In Section C2, we discuss compatibility of monads and continuous morphisms with localizations. The facts obtained here are used in the next 3 sections.

Section C3 complements and clarifies Example 1.6. Here we introduce and study the cone of a non-unital monad and, as a special case, the cone of a non-unital associative ring; the latter is regarded as a noncommutative quasi-affine 'space'. Motivated by important constructions of representation theory of classical and quantum groups and enveloping algebras, we consider Hopf actions on non-unital rings and induced actions on the corresponding quasi-affine 'spaces'.

In Section C4, we construct the Proj of graded non-unital monad. As an application, we recover the Proj of a graded associative ring introduced in Example 1.7.

In Section C5, we resume the study of Hopf actions started in C3. Applying general facts to the action of enveloping algebra of a semisimple (or reductive) Lie algebra we realize the category of D-modules on the base affine space and the flag variety as categories of quasi-coherent sheaves on resp. the cone and the Proj of a graded ring naturally associated with the Lie algebra.

This setting is extended to actions of the quantized enveloping algebra of a semisimple Lie algebra on the quantum base affine 'space' and the quantum flag variety. Altogeher is a complement to 1.10.

In Section C6 (which is a complement ot Sections 7 and 9), we study affine flat covers.

In Section C7 (complementing Section 10), we study descent of flat morphisms.

In Sections C8 and C9, we study connection between flat descent and the descent procedure using relations.

Section C10 gathers miscellaneous facts, some of them suggestive.

We introduce connections and interpret comonads as integrable connections.

We introduce *weakly quasi-affine morphisms* and relatively ample morphisms.

We discuss continuous, flat, and affine morphisms in the fibered category of modules on affine (noncommutative) schemes.

Finally, we complement duality of Section 6 (based on the connection between continuous monads and affine morphisms) by producing conditions which garantee the existence of a canonical functor which assigns to each (additive, or arbitrary) monad a continuous monad.

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1. 'Spaces' and categories. Examples.

1.1. Categories and 'spaces'. As usual, Cat, or $Cat_{\mathfrak{U}}$, denotes the bicategory of categories which belong to a fixed universum $\mathfrak{U}$. We call objects of Cat^{op} 'spaces'. For any 'space' X, the corresponding category C_X is regarded as the category of quasi-coherent sheaves on X. We denote by f^* the functor $C_Y \longrightarrow C_X$ corresponding to a 1-morphism $X \xrightarrow{f} Y$ and call it the *inverse image functor* of f. For any $\mathfrak{U}$-category $\mathcal{A}$, we denote by $|\mathcal{A}|$ the corresponding object of Cat^{op} (the underlying 'space') defined by $C_{|\mathcal{A}|} = \mathcal{A}$.

We denote by $|Cat|^o$ the category having same objects as Cat^{op}. Morphisms from X to Y are isomorphism classes of functors $C_Y \longrightarrow C_X$. For a morphism $X \xrightarrow{f} Y$, we denote by f^* any functor $C_Y \longrightarrow C_X$ representing f and call it an *inverse image functor of the morphism* f. We shall write $f = [F]$ to indicate that f is a morphism having an inverse image functor F. The composition of morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ is defined by $g \circ f = [f^* \circ g^*]$. Thus, the map which assigns to each functor $C_Y \xrightarrow{F} C_X$ the morphism $X \xrightarrow{[F]} Y$ is a functor $Cat^{op} \longrightarrow |Cat|^o$ which turns Cat^{op} into a fibred category over $|Cat|^o$.

1.2. Localizations and conservative morphisms. Let Y be an object of $|Cat|^o$ and Σ a class of arrows of the category C_Y. We denote by $\Sigma^{-1}Y$ the object of $|Cat|^o$ such that the corresponding category coincides with (the standard realization of) the quotient of the category C_Y by Σ (cf. [GZ, 1.1]): $C_{\Sigma^{-1}Y} = \Sigma^{-1}C_Y$. The canonical *localization functor* $C_Y \xrightarrow{p_\Sigma^*} \Sigma^{-1}C_Y$ is regarded as an inverse image functor of a morphism, $\Sigma^{-1}Y \xrightarrow{p_\Sigma} Y$.

For any morphism $X \xrightarrow{f} Y$ in $|Cat|^o$, we denote by Σ_f the family of all arrows s of the category C_Y such that $f^*(s)$ is invertible (notice that Σ_f does not depend on the choice of an inverse image functor f^*). Thanks to the universal property of localizations, f^* is represented as the composition of the localization functor $p_f^* = p_{\Sigma_f}^* : C_Y \longrightarrow \Sigma_f^{-1}C_Y$ and a uniquely determined functor $\Sigma^{-1}C_Y \xrightarrow{f_{\mathfrak{c}}^*} C_X$. In other words, $f = p_f \circ f_{\mathfrak{c}}$ for a uniquely determined morphism $X \xrightarrow{f_{\mathfrak{c}}} \Sigma_f^{-1}Y$.

A morphism $X \xrightarrow{f} Y$ is called *conservative* if Σ_f consists of isomorphisms only, or, equivalently, p_f is an isomorphism.

A morphism $X \xrightarrow{f} Y$ is called a *localization* if $f_{\mathfrak{c}}$ is an isomorphism, i.e. the functor $f_{\mathfrak{c}}^*$ is an equivalence of categories.

Thus, $f = p_f \circ f_{\mathfrak{c}}$ is a unique decomposition of a morphism f into a localization and a conservative morphism.

1.3. Continuous, flat, and affine morphisms. A morphism is called continuous if its inverse image functor has a right adjoint (called a direct image functor), and flat if, in addition, the inverse image functor is left exact (i.e. preserves finite limits). A continuous morphism is called 'affine' if its direct image functor is conservative (i.e. reflects isomorphisms) and has a right adjoint.

1.4. Categoric spectrum of a unital ring. For an associative unital ring R, we define the *categoric spectrum* of R as the object $\mathbf{Sp}(R)$ of $|Cat|^o$ such that $C_{\mathbf{Sp}(R)} =$

$R-mod$.

Let $R \xrightarrow{\phi} S$ be a unital ring morphism and $R-mod \xrightarrow{\bar{\phi}^*} S-mod$ the functor $S\otimes_R -$. The canonical right adjoint to $\bar{\phi}^*$ is the pull-back functor by the ring morphism ϕ. A right adjoint to ϕ_* is given by

$$\phi^! : S-mod \longrightarrow R-mod, \quad L \longmapsto Hom_R(\phi_*(S), L).$$

The map

$$\Big(R \xrightarrow{\phi} S\Big) \longmapsto \Big(\mathbf{Sp}(S) \xrightarrow{\bar{\phi}} \mathbf{Sp}(R)\Big)$$

is a functor

$$\mathbf{Sp} : Rings^{op} \longrightarrow |Cat|^o$$

which takes values in the subcategory formed by affine morphisms.

The image $\mathbf{Sp}(R) \xrightarrow{\bar{\phi}} \mathbf{Sp}(T)$ of a ring morphism $T \xrightarrow{\phi} R$ is flat (resp. faithful) iff ϕ turns R into a flat (resp. faithful) right T-module.

1.4.1. Continuous, flat, and affine morphisms from $\mathbf{Sp}(S)$ to $\mathbf{Sp}(R)$. Let R and S be associative unital rings. A morphism $f : \mathbf{Sp}(S) \longrightarrow \mathbf{Sp}(R)$ with an inverse image functor f^* is continuous iff

$$f^* \simeq \mathcal{M}\otimes_R \ : L \longmapsto \mathcal{M} \otimes_R L \tag{1}$$

for an (S,R)-bimodule $\mathcal{M}$ defined uniquely up to isomorphism. The functor

$$f_* = Hom_S(\mathcal{M}, -) : N \longmapsto Hom_S(\mathcal{M}, N) \tag{2}$$

is a direct image of f.

The morphism f with an inverse image functor (1) is conservative iff $\mathcal{M}$ is *faithful* as a right R-module, i.e. the functor $\mathcal{M} \otimes_R -$ is faithful.

The direct image functor (2) is conservative iff $\mathcal{M}$ is a cogenerator in the category of left S-modules, i.e. for any nonzero S-module N, there exists a nonzero S-module morphism $\mathcal{M} \longrightarrow N$.

The morphism f is flat iff $\mathcal{M}$ is flat as a right R-module.

The functor (2) has a right adjoint, $f^!$, iff f_* is isomorphic to the tensoring (over S) by a bimodule. This happens iff $\mathcal{M}$ is a projective S-module of finite type. The latter is equivalent to the condition: the natural functor morphism $\mathcal{M}^*_S \otimes_S - \longrightarrow Hom_S(\mathcal{M}, -)$ is an isomorphism. Here $\mathcal{M}^*_S = Hom_S(M, S)$. In this case, $f^! \simeq Hom_R(\mathcal{M}^*_S, -)$.

1.5. The graded version. Let $\mathcal{G}$ be a monoid and R a $\mathcal{G}$-graded unital ring. We define the 'space' $\mathbf{Sp}_{\mathcal{G}}(R)$ by taking as $C_{\mathbf{Sp}_{\mathcal{G}}(R)}$ the category $gr_{\mathcal{G}}R-mod$ of left $\mathcal{G}$-graded R-modules. There is a natural functor $gr_{\mathcal{G}}R-mod \xrightarrow{\phi_*} R_0-mod$ which assigns to each graded R-module its zero component ('zero' is the unit element of the monoid $\mathcal{G}$). The functor ϕ_* has a left adjoint, ϕ^*, which maps every R_0-module M to the graded R-module $R\otimes_{R_0} M$. The adjunction arrow $Id_{R_0-mod} \longrightarrow \phi_*\phi^*$ is an isomorphism. This means that the functor ϕ^* is fully faithful, or, equivalently, the functor ϕ_* is a localization.

The functors ϕ_* and ϕ^* are regarded as respectively a direct and an inverse image functor of a morphism $\mathbf{Sp}_{\mathcal{G}}(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$. It follows from the above that the morphism ϕ is affine iff ϕ is an isomorphism (i.e. ϕ^* is an equivalence of categories).

In fact, if ϕ is affine, the functor ϕ_* should be conservative. Since ϕ_* is a localization, this means, precisely, that ϕ_* is an equivalence of categories.

1.6. The cone of a non-unital ring. Let R_0 be a unital associative ring, and let R_+ be an associative ring, non-unital in general, in the category of R_0-bimodules; i.e. R_+ is endowed with an R_0-bimodule morphism $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$ satisfying the associativity condition. Let $R = R_0 \oplus R_+$ denote the augmented ring described by this data. Let $\mathcal{T}_{R_+}$ denote the full subcategory of the category $R - mod$ whose objects are all R-modules annihilated by R_+. Let $\mathcal{T}_{R_+}^-$ be the Serre subcategory (that is a full subcategory closed by taking subquotients, extensions, and arbitrary direct sums) of the category $R - mod$ spanned by $\mathcal{T}_{R_+}$.

We define the 'space' *cone of* R_+ by taking as $C_{\mathbf{Cone}(R_+)}$ the quotient category $R - mod/\mathcal{T}_{R_+}^-$. The localization functor $R - mod \xrightarrow{u^*} R - mod/\mathcal{T}_{R_+}^-$ is an inverse image functor of a morphism of 'spaces' $\mathbf{Cone}(R_+) \xrightarrow{u} \mathbf{Sp}(R)$. The functor u^* has a (necessarily fully faithful) right adjoint, i.e. the morphism u is continuous. If R_+ is a unital ring, then u is an isomorphism (see C3.2.1). The composition of the morphism u with the canonical affine morphism $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp}(R_0)$ is a continuous morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Sp}(R_0)$. Its direct image functor is (regarded as) the *global sections functor*.

1.7. The graded version: $\mathbf{Proj}_{\mathcal{G}}$. Let $\mathcal{G}$ be a monoid and $R = R_0 \oplus R_+$ a $\mathcal{G}$-graded ring with zero component R_0. Then we have the category $gr_{\mathcal{G}}R - mod$ of $\mathcal{G}$-graded R-modules and its full subcategory $gr_{\mathcal{G}}\mathcal{T}_{R_+} = \mathcal{T}_{R_+} \cap gr_{\mathcal{G}}R - mod$ whose objects are graded modules annihilated by the ideal R_+. We define the 'space' $\mathbf{Proj}_{\mathcal{G}}(R)$ by setting

$$C_{\mathbf{Proj}_{\mathcal{G}}(R)} = gr_{\mathcal{G}}R - mod/gr_{\mathcal{G}}\mathcal{T}_{R_+}^-.$$

Here $gr_{\mathcal{G}}\mathcal{T}_{R_+}^-$ is the Serre subcategory of the category $gr_{\mathcal{G}}R - mod$spanned by $gr_{\mathcal{G}}\mathcal{T}_{R_+}$. One can show that $gr_{\mathcal{G}}\mathcal{T}_{R_+}^- = gr_{\mathcal{G}}R - mod \cap \mathcal{T}_{R_+}^-$. Therefore, we have a canonical projection

$$\mathbf{Cone}(R_+) \xrightarrow{\mathfrak{p}} \mathbf{Proj}_{\mathcal{G}}(R).$$

The localization functor $gr_{\mathcal{G}}R - mod \longrightarrow C_{\mathbf{Proj}_{\mathcal{G}}(R_+)}$ is an inverse image functor of a continuous morphism $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}_{\mathcal{G}}(R)$. The composition $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}(R_0)$ of the morphism $\mathfrak{v}$ with the canonical morphism $\mathbf{Sp}_{\mathcal{G}}(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$ defines $\mathbf{Proj}_{\mathcal{G}}(R)$ as a 'space' over $\mathbf{Sp}(R_0)$. Its direct image functor is called the *global sections functor*.

1.7.1. Example: cone and Proj of a $\mathbb{Z}_+$-graded ring. Let $R = \oplus_{n \geq 0} R_n$ be a $\mathbb{Z}_+$-graded ring, $R_+ = \oplus_{n \geq 1} R_n$ its 'irrelevant' ideal. Thus, we have the *cone* of R_+, $\mathbf{Cone}(R_+)$, and $\mathbf{Proj}(R) = \mathbf{Proj}_{\mathbb{Z}}(R)$, and a canonical morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Proj}(R)$.

1.8. Example: skew cones and skew projective 'spaces'. Let A be an arbitrary associative k-algebra. And let $\mathbf{q}$ denote a matrix $[q_{ij}]_{i,j \in J}$ with entrees in k such that

$q_{ij}q_{ji} = 1$ for all $i, j \in J$ and $q_{ii} = 1$ for all $i \in J$. Let $R = A_{\mathbf{q}}[\mathbf{x}]$ denote a skew polynomial algebra corresponding to this data. Here $\mathbf{x} = (x_i \mid i \in J)$ is a set of indeterminates satisfying the relations

$$x_i x_j = q_{ij} x_j x_i \quad \text{for all } i, j \in J, \tag{1}$$

$$x_i r = r x_i \quad \text{for all } i \in J \text{ and } r \in R \tag{2}$$

For any $i \in J$, set $S_i = \{x_i^n | n \geq 1\}$. Each of S_i is a left and right Ore set, and the Serre subcategory $\mathcal{T}_{R_+}^-$ is generated by R-modules whose elements are annihilated by some elements of $\bigcup_{i \in J} S_i$. This implies that the localizations $R - mod \longrightarrow S_i^{-1}R - mod$ factor through the localization $R - mod \longrightarrow C_{\mathbf{Cone(R_+)}}$, and the induced localizations $C_{\mathbf{Cone(R_+)}} \xrightarrow{u_i^*} S_i^{-1}R - mod$ form a conservative family. The conservative family $\{\mathbf{Sp}(S_i^{-1}R) \longrightarrow \mathbf{Cone}(R_+) \mid i \in J\}$ is regarded as an *affine cover* of the cone $\mathbf{Cone}(R_+)$. It follows that the algebra $S_i^{-1}R$ is isomorphic to $A_{\mathbf{q}}[\mathbf{x}, x_i^{-1}]$.

Let $\mathcal{G} = \mathbb{Z}^J$; and let γ_i, $i \in J$, denote the canonical generators of the group $\mathcal{G}$. Assigning to each x_i the parity γ_i, we turn the skew polynomial algebra $R = A_{\mathbf{q}}[\mathbf{x}]$ into a $\mathcal{G}$-graded algebra with $R_0 = A$. The localizations $R - mod \longrightarrow S_i^{-1}R - mod$ induce localizations $gr_{\mathcal{G}}R - mod \longrightarrow gr_{\mathcal{G}}S_i^{-1}R - mod$ which factor through the localization

$$C_{\mathbf{Proj}(R)} \xrightarrow{\mathfrak{v}_i^*} gr_{\mathcal{G}}S_i^{-1}R - mod = gr_{\mathcal{G}}A_{\mathbf{q}}[\mathbf{x}, x_i^{-1}] - mod. \tag{3}$$

Let $\mathcal{G}_i$ denote the quotient group $\mathcal{G}/\mathbb{Z}\gamma_i$. The category $gr_{\mathcal{G}}A_{\mathbf{q}}[\mathbf{x}, x_i^{-1}] - mod$ in (3) is naturally equivalent to the category $gr_{\mathcal{G}_i}A_{\mathbf{q}_i}[\mathbf{x}/x_i] - mod$ of left $\mathcal{G}_i$-graded modules over the skew polynomial algebra $A_{\mathbf{q}_i}[\mathbf{x}/x_i]$. Here $\mathbf{x}/x_i$ denotes $\{x_j/x_i | j \in J, j \neq i\}$, and $\mathbf{q}_i$ denotes the matrix $[q_{ni}q_{nm}q_{mi}^{-1}]_{n,m \in J-\{i\}}$ (cf. [R, I.7.2.2.4]). Note that $A_{\mathbf{q}_i}[\mathbf{x}/x_i]$ is the $\mathcal{G}_i$-component of the algebra $A_{\mathbf{q}}[\mathbf{x}, x_i^{-1}]$ of the 'functions' on $\mathbf{Cone}(R)/|\mathbb{S}_i'|$.

Let 'spaces' U_i are defined by $C_{U_i} = gr_{\mathcal{G}_i}A_{\mathbf{q}_i}[\mathbf{x}/x_i] - mod$. Note that if the cardinality of J is greater than one, then the natural morphisms $U_i \xrightarrow{u_i} \mathbf{Proj}_{\mathcal{G}}(R)$ do not form an affine cover of $\mathbf{Proj}_{\mathcal{G}}(R)$ over $\mathbf{Sp}(A)$, because the composition of $\mathfrak{v}_{i*}$ with the direct image of the projection $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\pi} \mathbf{Sp}(A)$ is isomorphic to the functor $gr_{\mathcal{G}_i}A_{\mathbf{q}_i}[\mathbf{x}/x_i] - mod \longrightarrow A - mod$ which assigns to each $\mathcal{G}_i$-graded module (resp. $\mathcal{G}_i$-graded module morphism) its zero component. If the group $\mathcal{G}_i$ is non-trivial, this functor is not faithful, hence the morphism $\pi \circ u_i$ is not affine.

1.8.1. The projective q-'space' $\mathbf{P}_{\mathbf{q}}^r$. Let again $R = A_{\mathbf{q}}[\mathbf{x}]$, $\mathbf{x} = (x_0, x_1, \dots, x_r)$. But, we take $\mathcal{G} = \mathbb{Z}$ with the natural order; and set the parity of each x_i equal to 1. One can repeat with $\mathbf{Cone}_{\mathbb{Z}}(R)$ and $\mathbf{P}_{\mathbf{q}}^r = \mathbf{Proj}_{\mathbb{Z}}(R)$ the same pattern as with $\mathbf{Cone}(R)$ and $\mathbf{P}_{\mathcal{G}}^r = \mathbf{Proj}_{\mathcal{G}}(R)$. Only this time the quotient groups $\mathcal{G}_i$ will be trivial, and we obtain a picture very similar to the classical one: $\mathbf{P}_{\mathbf{q}}^r$ is a $\mathbb{Z}$-scheme covered by $r + 1$ affine spaces $A_{\mathbf{q}_i}[\mathbf{x}/x_i] - mod$, $i = 0, 1, \dots, r$.

1.9. The base affine 'space' and the flag variety of a reductive Lie algebra. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Let $\mathcal{G}$ be the group of integral weights of $\mathfrak{g}$ and $\mathcal{G}_+$ the semigroup of nonnegative integral weights. Let $R = \oplus_{\lambda \in \mathcal{G}_+} R_\lambda$, where R_λ is the vector space of the (canonical) irreducible finite dimensional representation with the highest weight λ. The module R is a $\mathcal{G}$-graded algebra with the multiplication determined by the projections $R_\lambda \otimes R_\nu \longrightarrow R_{\lambda+\nu}$, for all $\lambda, \nu \in \mathcal{G}_+$. It is well known that the algebra R is isomorphic to the algebra of regular functions on the *base affine space* of $\mathfrak{g}$. Recall that G/U, where G is a connected simply connected algebraic group with the Lie algebra $\mathfrak{g}$, and U is its maximal unipotent subgroup.

The category $C_{\mathbf{Cone}(R)}$ is equivalent to the category of quasi-coherent sheaves on the base affine space Y of the Lie algebra $\mathfrak{g}$. The category $Proj_{\mathcal{G}}(R)$ is equivalent to the category of quasi-coherent sheaves on the flag variety of $\mathfrak{g}$.

1.10. The quantized base affine 'space' and quantized flag variety of a semisimple Lie algebra. Let now $\mathfrak{g}$ be a semisimple Lie algebra over a field k of zero characteristic, and let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$. Define the $\mathcal{G}$-graded algebra $R = \oplus_{\lambda \in \mathcal{G}_+} R_\lambda$ the same way as above. This time, however, the algebra R is not commutative. Following the classical example (and identifying spaces with categories of quasi-coherent sheaves on them), we call $\mathbf{Cone}(R)$ *the quantum base affine 'space'* and $\mathbf{Proj}_{\mathcal{G}}(R)$ the *quantum flag variety* of $\mathfrak{g}$.

1.10.1. Canonical affine covers of the base affine 'space' and the flag variety. Let W be the Weyl group of the Lie algebra $\mathfrak{g}$. Fix a $w \in W$. For any $\lambda \in \mathcal{G}_+$, choose a nonzero w-extremal vector $e^\lambda_{w\lambda}$ generating the one dimensional vector subspace of R_λ formed by the vectors of the weight $w\lambda$. Set $S_w = \{k^* e^\lambda_{w\lambda} | \lambda \in \mathcal{G}_+\}$. It follows from the Weyl character formula that $e^\lambda_{w\lambda} e^\mu_{w\mu} \in k^* e^{\lambda+\mu}_{w(\lambda+\mu)}$. Hence S_w is a multiplicative set. It was proved by Joseph [Jo] that S_w is a left and right Ore subset in R. The Ore sets $\{S_w | w \in W\}$ determine a conservative family of affine localizations

$$\mathbf{Sp}(S_w^{-1}R) \longrightarrow \mathbf{Cone}(R), \quad w \in W, \tag{4}$$

of the quantum base affine 'space' and a conservative family of affine localizations

$$\mathbf{Sp}_{\mathcal{G}}(S_w^{-1}R) \longrightarrow \mathbf{Proj}_{\mathcal{G}}(R), \quad w \in W,$$

of the quantum flag variety. We claim that the category $gr_{\mathcal{G}} S_w^{-1}R - mod$ is naturally equivalent to $(S_w^{-1}R)_0 - mod$. By 1.5, it suffices to verify that the canonical functor $gr_{\mathcal{G}} S_w^{-1}R - mod \longrightarrow S_w^{-1}R)_0 - mod$ which assigns to every graded $S_w^{-1}R$-module its zero component is faithful; i.e. the zero component of every nonzero $\mathcal{G}$-graded $S_w^{-1}R$-module is nonzero. This is, really, the case, because if z is a nonzero element of λ-component of a $\mathcal{G}$-graded $S_w^{-1}R$-module, then $(e^\lambda_{w\lambda})^{-1}z$ is a nonzero element of the zero component of this module.

1.11. Noncommutative Grassmannians and projective spaces. Fix an associative unital k-algebra R. Let $R\backslash Alg_k$ be the category of associative *k-algebras over R* (i.e. pairs $(S, R \to S)$, where S is a k-algebra and $R \to S$ a k-algebra morphism). We call them for convenience R-rings. We denote by R^e the k-algebra $R \otimes_k R^o$. Here R^o is the algebra opposite to R.

1.11.1. The functor $Gr_{M,V}$. Let M, V be left R-modules. Consider the functor, $Gr_{M,V} : R\backslash Alg_k \longrightarrow \mathbf{Sets}$, which assigns to any R-ring $(S, R \xrightarrow{s} S)$ the set of isomorphism classes of epimorphisms $s^*(M) \longrightarrow s^*(V)$ (here $s^*(M) = S \otimes_R M$) and to any R-ring morphism $(S, R \xrightarrow{s} S) \xrightarrow{\phi} (T, R \xrightarrow{t} T)$ the map $Gr_{M,V}(S,s) \longrightarrow Gr_{M,V}(T,t)$ induced by the inverse image functor $S - mod \xrightarrow{\phi^*} T - mod$, $\mathcal{N} \longmapsto T \otimes_S \mathcal{N}$.

1.11.2. The functor $G_{M,V}$. Denote by $G_{M,V}$ the functor $R\backslash Alg_k \longrightarrow$ **Sets** which assigns to any R-ring $(S, R \xrightarrow{s} S)$ the set of pairs of morphisms $s^*(V) \xrightarrow{v} s^*(M) \xrightarrow{u} s^*(V)$ such that $u \circ v = id_{s^*(V)}$ and acts naturally on morphisms. Since V is a projective module, the map

$$\pi = \pi_{M,V} : G_{M,V} \longrightarrow Gr_{M,V}, \ (v, u) \longmapsto [u], \tag{1}$$

is a (strict) functor epimorphism.

1.11.3. Relations. Denote by $\mathfrak{R}_{M,V}$ the "functor of relations" $G_{M,V} \times_{Gr_{M,V}} G_{M,V}$. By definition, $\mathfrak{R}_{M,V}$ is a subfunctor of $G_{M,V} \times G_{M,V}$ which assigns to each R-ring, $(S, R \xrightarrow{s} S)$, the set of all 4-tuples $(u_1, v_1; u_2, v_2) \in G_{M,V} \times G_{M,V}$ such that the epimorphisms u_1, u_2 are equivalent. The latter means that there exists an isomorphism $s^*(V) \xrightarrow{\varphi} s^*(V)$ such that $u_2 = \varphi \circ u_1$, or, equivalently, $\varphi^{-1} \circ u_2 = u_1$. Since $u_i \circ v_i = id$, $i = 1, 2$, these equalities imply that $\varphi = u_2 \circ v_1$ and $\varphi^{-1} = u_1 \circ v_2$. Thus, $\mathfrak{R}_{M,V}(S, s)$ is a subset of all $(u_1, v_1; u_2, v_2) \in G_{M,V}(S, s) \times G_{M,V}(S, s)$ satisfying the following relations:

$$u_2 = (u_2 \circ v_1) \circ u_1, \quad u_1 = (u_1 \circ v_2) \circ u_2 \tag{2}$$

in addition to the relations describing $G_{M,V}(S, s) \times G_{M,V}(S, s)$:

$$u_1 \circ v_1 = id_{S \otimes_R V} = u_2 \circ v_2 \tag{3}$$

Denote by p_1, p_2 the canonical projections $\mathfrak{R}_{M,V} \rightrightarrows G_{M,V}$. It follows from the surjectivity of $G_{M,V} \longrightarrow Gr_{M,V}$ that the diagram

$$\mathfrak{R}_{M,V} \overset{p_1}{\underset{p_2}{\rightrightarrows}} G_{M,V} \xrightarrow{\ \pi\ } Gr_{M,V} \tag{4}$$

is exact.

1.11.4. Proposition. *If both M and V are projective modules of a finite type, then the functors $G_{M,V}$ and $\mathfrak{R}_{M,V}$ are corepresentable.*

Proof. See [KR2, 10.4.3]. ■

1.11.5. Quasi-coherent presheaves on $Gr_{M,V}$. Suppose that M and V are projective modules of a finite type, hence the functors $G_{M,V}$ and $\mathfrak{R}_{M,V}$ are corepresentable by R-rings resp. $(\mathfrak{G}_{M,V}, R \to \mathfrak{G}_{M,V})$ and $(\mathcal{R}_{M,V}, R \to \mathcal{R}_{M,V})$. Then the category $Qcoh(G_{M,V})$ (resp. $Qcoh(\mathfrak{R}_{M,V})$) is equivalent to $\mathfrak{G}_{M,V} - mod$ (resp. $\mathcal{R}_{M,V} - mod$), and the category $Qcoh(Gr_{M,V})$ of quasi-coherent presheaves on $Gr_{M,V}$ is equivalent to the kernel of the diagram

$$Qcoh(G_{M,V}) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} Qcoh(\mathfrak{R}_{M,V}) \tag{5}$$

This means that, after identifying categories of quasi-coherent presheaves in (5) with corresponding categories of modules, quasi-coherent presheaves on $Gr_{_{M,V}}$ can be realized as pairs (L, ϕ), where L is a $\mathfrak{G}_{_{M,V}}$-module and ϕ is an isomorphism $p_1^*(L) \xrightarrow{\sim} p_2^*(L)$. Morphisms $(L, \phi) \longrightarrow (N, \psi)$ are given by morphisms $L \xrightarrow{g} N$ such that the diagram

$$\begin{array}{ccc} p_1^*(L) & \xrightarrow{p_1^*(g)} & p_1^*(N) \\ \phi \downarrow \wr & & \wr \downarrow \psi \\ p_2^*(L) & \xrightarrow{p_2^*(g)} & p_2^*(N) \end{array}$$

commutes. The functor

$$Qcoh(Gr_{_{M,V}}) \xrightarrow{\pi^*} Qcoh(G_{_{M,V}}), \quad (L, \phi) \longmapsto L,$$

is an inverse image functor of the projection $G_{_{M,V}} \xrightarrow{\pi} Gr_{_{M,V}}$ (see 1.11.3(4)).

1.11.6. Noncommutative Grassmannians. Let $\mathfrak{T}$ be a quasi-topology on the category $(R\backslash Alg_k)^{op}$ of affine k-schemes over R. Let $Gr^{\mathfrak{T}}_{_{M,V}}$ be the $\mathfrak{T}$-*Grassmannian* corresponding to $Gr_{_{M,V}}$, i.e. a sheaf of sets (a 'space') associated to the presheaf $Gr_{_{M,V}}$. By [KR3, 2.9.1], if $\mathfrak{T}$ is coarser than the quasi-topology of *effective descent*, then the category $Qcoh(Gr_{_{M,V}})$ of quasi-coherent modules on $Gr_{_{M,V}}$ is naturally equivalent to the category $Qcoh(Gr^{\mathfrak{T}}_{_{M,V}})$ of quasi-coherent modules on the $\mathfrak{T}$-Grassmannian $Gr^{\mathfrak{T}}_{_{M,V}}$.

1.11.7. Noncommutative projective space. Let M be the free R-module of the rank n+1, V the free R-module of the rank 1. In this case, we denote the functor $Gr_{_{M,V}}$ by $\mathcal{P}^n_{_R}$. If a quasi-topology $\mathfrak{T}$ on the category $(R\backslash Alg_k)^{op}$ of affine k-schemes over R is coarser than the quasi-topology of effective descent, than the category $Qcoh(\mathcal{P}^n_{_R})$ of quasi-coherent modules on $\mathcal{P}^n_{_R}$ is equivalent to the category of quasi-coherent modules on the (associated) projective space $\mathbb{P}^n_{_R} = \mathcal{P}^{n^{\mathfrak{T}}}_{_R}$.

2. Fibered categories and 'spaces'. Quasi-coherent modules.

2.1. Initial objects of $|Cat|^o$. The category $\bullet$ with one (identical) morphism (in particular with one object) is an initial object of $|Cat|^o$. A morphism $f : A \longrightarrow B$ in $|Cat|^o$ with an inverse image functor f^* is an isomorphism iff f^* is a category equivalence. In particular, $X \in Ob|Cat|^o$ is an initial object of $|Cat|^o$ iff the category C_X is a connected groupoid; i.e. $HomC_X$ consists of isomorphisms and for any $x, y \in ObC_X$, the set $Hom_{C_X}(x, y)$ is non-empty.

Notice that for any object X of $|Cat|^o$, the set $|Cat|^o(X, \bullet)$ of morphisms $X \to \bullet$ is isomorphic to the set $|X|$ of isomorphism classes of objects of the category C_X.

The category $|Cat|^o$ has no "real" final objects: the empty category is its unique final object.

2.2. Proposition. *The category $|Cat|^o$ has small limits and colimits.*

Proof. (a) Let $\{X_i \mid i \in J\}$ be a set of objects of $|Cat|^o$. Then $\prod_{i\in J} X_i$ and $\coprod_{i\in J} X_i$ are defined by

$$C_{\prod_{i\in J} X_i} = \coprod_{i\in J} C_{X_i} \quad \text{and} \quad C_{\coprod_{i\in J} X_i} = \prod_{i\in J} C_{X_i}.$$

(b) Every pair of arrows, $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, in $|Cat|^o$ has a cokernel.

Let $f^*, g^* : C_Y \rightrightarrows C_X$ be inverse image functors of resp. f and g. Let C_Z denote the category whose objects are pairs (x, ϕ), where $x \in ObC_Y$ and ϕ is an isomorphism $f^*(x) \xrightarrow{\sim} g^*(x)$. A morphism from (x, ϕ) to (y, ψ) is a morphism $\xi : x \to y$ such that the diagram

$$\begin{array}{ccc} f^*(x) & \xrightarrow{f^*(\xi)} & f^*(y) \\ \phi \big\downarrow & & \big\downarrow \psi \\ f^*(x) & \xrightarrow{f^*(\xi)} & f^*(y) \end{array}$$

commutes. Denote by $\mathfrak{h}^*$ the forgetful functor $C_Z \longrightarrow C_Y,\ (x, \phi) \longmapsto x$. Let $w : Y \longrightarrow W$ be a morphism in $|Cat|^o$ with an inverse image functor w^* such that $w \circ f = w \circ g$. This means that there exists an isomorphism $f^* \circ w^* \xrightarrow{\psi} g^* \circ w^*$. The pair (w^*, ψ) defines a functor $\gamma^*_{w^*,\psi} : C_W \longrightarrow C_Z,\ b \longmapsto (w^*(b), \psi(b))$. A different choice, w_1^*, of the inverse image functor of w and an isomorphism $\psi_1 : w_1^* \circ f^* \xrightarrow{\sim} w_1^* \circ g^*$ produces a functor $\gamma^*_{w_1^*,\psi_1}$ isomorphic to $\gamma^*_{w^*,\psi}$. This shows that the morphism $Y \longrightarrow Z$ having the inverse image $\mathfrak{h}^*$ is the cokernel of the pair (f, g). The existence of cokernels and (small) coproducts is equivalent to the existence of arbitrary (small) colimits.

(c) Every pair of arrows, $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, in $|Cat|^o$ has a kernel.

Let $f^*, g^* : C_Y \rightrightarrows C_X$ be inverse image functors of resp. f and g. Denote by $\mathfrak{D}_{f^*,g^*}$ the diagram scheme defined as follows:

$$Ob\mathfrak{D}_{f^*,g^*} = ObC_Y \coprod ObC_X \quad \text{and} \quad Hom\mathfrak{D}_{f^*,g^*} = HomC_X \coprod \Sigma_{f^*,g^*},$$

where

$$\Sigma_{f^*,g^*} = \{f^*(x) \xrightarrow{s_x} x,\ x \xrightarrow{t_x} g^*(x) \mid x \in ObC_Y\}.$$

Consider the category $\mathcal{P}a\mathfrak{D}_{f^*,g^*}$ of paths of the diagram $\mathfrak{D}_{f^*,g^*}$ together with the natural embeddings $HomC_X \xrightarrow{\tau} Hom\mathcal{P}a\mathfrak{D}_{f^*,g^*} \longleftarrow \Sigma_{f^*,g^*}$ which define the corresponding diagrams. We denote by $\mathcal{P}\mathfrak{D}_{f^*,g^*}$ the quotient of the category $\mathcal{P}a\mathfrak{D}_{f^*,g^*}$ by the minimal equivalence relation such that

$$\tau(\alpha \circ \beta) \sim \tau(\alpha) \circ \tau(\beta) \quad \text{and} \quad \tau(id_x) \sim id_{\tau(x)}$$

for all composable arrows $\alpha,\ \beta$ and for all $x \in ObC_X$.

Finally, we denote by C_W the quotient category $\Sigma^{-1}_{f^*,g^*}\mathcal{P}\mathfrak{D}_{f^*,g^*}$. It follows from the construction that the object W of the category $|Cat|^o$ defined this way is the kernel of the pair (f, g). Details are left to the reader. ■

2.3. Two fibered categories associated with Cat.

2.3.1. The fibered category $(\mathbf{Cat^o}, \pi)$ over Cat^{op}. Consider the category $\mathbf{Cat^o}$ whose objects are pairs (X, M), where $X \in ObCat^{op}$ and $M \in ObC_X$. Morphisms from (X, M) to (Y, L) are pairs (f, ξ), where f is a morphism $X \longrightarrow Y$ and ξ is a morphism $f^*(L) \longrightarrow M$. The pair $(\mathbf{Cat^o}, \pi)$, where $\pi : \mathbf{Cat^o} \longrightarrow Cat^{op}$ is a functor which assigns to each object (X, M) of Cat^o the object X and to every morphism $(f, \xi) : (X, M) \longrightarrow (Y, L)$ the corresponding morphism $X \xrightarrow{f} Y$ in Cat^{op} is a fibered category with the fiber at $X \in ObCat^{op}$ equal to C_X^{op}. This fibered category corresponds to the identical functor $Cat = (Cat^{op})^{op} \longrightarrow Cat$.

2.3.2. The fibered category $\mathfrak{Cat}^o_{\mathfrak{U}}$. Let $|\pi|$ denote the composition of the functor $\pi : \mathbf{Cat^o}_{\mathfrak{U}} \longrightarrow Cat^{op}_{\mathfrak{U}}$ and the canonical functor $Cat^{op}_{\mathfrak{U}} \longrightarrow |Cat|^o_{\mathfrak{U}}$. The pair $(\mathbf{Cat^o_{\mathfrak{U}}}, |\pi|)$ is a fibered category which we denote by $\mathfrak{Cat}^o_{\mathfrak{U}}$.

2.4. The universal property of the fibered category $\mathfrak{Cat}^o_{\mathfrak{U}}$. Fix two universums, $\mathfrak{U}$ and $\mathfrak{V}$ such that $\mathfrak{U} \in \mathfrak{V}$. Denote by $Fib_{\mathfrak{U},\mathfrak{V}}$ the 2-category of fibered categories $\mathfrak{F} \to \mathcal{E}$ such that the 'base' $\mathcal{E}$ belongs to $\mathfrak{V}$ and all fibers, $\mathfrak{F}_x,\ x \in Ob\mathcal{E}$, belong to $\mathfrak{U}$.

Let $\mathfrak{M}Fib_{\mathfrak{U},\mathfrak{V}}$ denote the 2-subcategory of the 2-category $Fib_{\mathfrak{U},\mathfrak{V}}$ formed by cartesian functors (1-morphisms) which induce category equivalence of fibers.

2.4.1. Proposition. *(a) For any fibered category $\mathfrak{F}$ which belongs to $Fib_{\mathfrak{U},\mathfrak{V}}$, there exists a cartesian functor $\psi_{\mathfrak{F}} : \mathfrak{F} \longrightarrow \mathfrak{Cat}^o_{\mathfrak{U}}$ such that for all cartesian functors $F : \mathfrak{F} \longrightarrow \mathfrak{G}$ which belong to $\mathfrak{M}Fib_{\mathfrak{U},\mathfrak{V}}$, the diagram*

$$\begin{tikzcd} \mathfrak{F} \arrow[rr, "F"] \arrow[dr, "\psi_{\mathfrak{F}}"'] & & \mathfrak{G} \arrow[dl, "\psi_{\mathfrak{G}}"] \\ & \mathfrak{Cat}^o_{\mathfrak{U}} & \end{tikzcd}$$

commutes in the 2-category sense.

(b) The union of images of the functors $\psi_{\mathfrak{F}}$ is $\mathfrak{Cat}^o_{\mathfrak{U}}$.

(c) In particular, the fibered category $\mathfrak{Cat}^o_{\mathfrak{U}}$ is a final object of $\mathfrak{M}Fib_{\mathfrak{U},\mathfrak{V}}$ iff $|Cat|^o_{\mathfrak{U}}$ belongs to $\mathfrak{V}$.

Proof. Fix a pseudo-functor $\beta : |Cat|_{\mathfrak{U}} = (|Cat|^o_{\mathfrak{U}})^{op} \longrightarrow Cat_{\mathfrak{U}}$ quasi-inverse to the projection $\mathbf{Cat^o}_{\mathfrak{U}} \xrightarrow{\pi} |Cat|^o_{\mathfrak{U}}$; i.e. β assigns to each object X of $|Cat|$ the category C_X^{op} and to any morphism $X \xrightarrow{f} Y$ the functor $f^{*op} : C_Y^{op} \longrightarrow C_X^{op}$ opposite to a (chosen) inverse image functor of f. This pseudo-functor determines a fibered category $\mathfrak{Cat}_\beta$ equivalent to $\mathfrak{Cat}^o_{\mathfrak{U}}$. Recall that objects of $\mathfrak{Cat}_\beta$ are pairs (X, x), where $X \in Ob\mathcal{B},\ x \in ObC_X$, and morphisms $(X, x) \longrightarrow (Y, y)$ are pairs (f, ξ), where f is a morphism $X \longrightarrow Y$ and ξ is a morphism $x \longrightarrow f^*(y)$. Projection $p_\beta : \mathcal{F}_\beta \longrightarrow \mathcal{B}$ maps (X, x) to X and (f, ξ) to f.

For any fibered category $\mathfrak{F} = (\mathcal{F} \xrightarrow{p} \mathcal{B})$, there exists a natural functor $\Psi_{\mathfrak{F}} : \mathcal{B} \longrightarrow |Cat|^o$ which sends any object X of $\mathcal{B}$ to the class, $|\mathcal{F}_X^{op}|$, of the fiber $\mathcal{F}_X$ over X and any morphism $X \xrightarrow{f} Y$ to the morphism $[f^{*op}] : |\mathcal{F}_X^{op}| \longrightarrow |\mathcal{F}_Y^{op}|$ of $|Cat|^o$. The composition of $\Psi_{\mathfrak{F}}^{op} : \mathcal{B}^{op} \longrightarrow |Cat|_{\mathfrak{U}}$ with the pseudo-functor $\beta : |Cat|_{\mathfrak{U}} \longrightarrow Cat_{\mathfrak{U}}$ is a pseudo-functor, $\beta_{\mathfrak{F}}$, quasi-inverse to $p : \mathcal{F} \longrightarrow \mathcal{B}$. The fibered category $\mathfrak{F}$ is equivalent to the fibered category

$\mathfrak{F}_{\beta_{\mathfrak{F}}} = (\mathcal{F}_{\beta_{\mathfrak{F}}} \xrightarrow{p_{\beta_{\mathfrak{F}}}} \mathcal{B})$. It follows from the construction that the functor $\Psi_{\mathfrak{F}} : \mathcal{B} \longrightarrow |Cat|^o$ defined above gives rise to a functor $\bar{\Psi}_{\mathfrak{F}} : \mathcal{F}_{\beta_{\mathfrak{F}}} \longrightarrow Cat_{\mathfrak{U}}$ such that the pair $(\bar{\Psi}_{\mathfrak{F}}, \Psi_{\mathfrak{F}})$ is a cartesian functor from $\mathcal{F}_{\beta} \xrightarrow{p_{\beta}} \mathcal{B}$ to $\mathfrak{Cat}^o_{\mathfrak{U}}$. Taking the composition with the equivalence $\mathfrak{F} \longrightarrow \mathfrak{F}_{\beta}$, we obtain a cartesian functor $\psi_{\mathfrak{F}} : \mathfrak{F} \longrightarrow \mathfrak{Cat}_{\mathfrak{U}}$.

Let $\mathfrak{G} = (\mathcal{G} \xrightarrow{q} \mathcal{D})$ be another fibered category. It follows from the construction of functors $\psi_{\mathfrak{F}}$ that for any cartesian functor $F : \mathfrak{F} \longrightarrow \mathfrak{G}$, the diagram

$$\begin{array}{ccc} \mathfrak{F} & \xrightarrow{F} & \mathfrak{G} \\ \psi_{\mathfrak{F}} \searrow & & \swarrow \psi_{\mathfrak{G}} \\ & \mathcal{C}at_2 & \end{array}$$

which commutes in the 2-category sense. ■

2.5. The universal property of the fibered category $(\mathbf{Cat^o}, \pi)$. As in 2.4, fix universums $\mathfrak{U}$ and $\mathfrak{V}$ such that $\mathfrak{U} \in \mathfrak{V}$. Recall that a fibered category over $\mathcal{E}$ is *split* (*scindée*) if it corresponds to a functor $\mathcal{E}^{op} \longrightarrow Cat$ called *splitting* (scindage). Consider the category $\mathcal{S}sin_{\mathfrak{U},\mathfrak{V}}$ formed by split fibered categories $\mathcal{F} \rightleftarrows \mathcal{E}$ such that $\mathcal{E}$ belongs to the universum $\mathfrak{V}$ with all fibers from the universum $\mathfrak{U}$). Morphisms are functors preserving splittings as 1-morphisms. The split category $\mathbf{Cat}_{\mathfrak{U}}$ is a final object of the 2-category $\mathcal{S}cin_{\mathfrak{U},\mathfrak{V}}$ the same sense as $\mathcal{C}at_{2,\mathfrak{U}}$ is a final object of the 2-category $Fib_{\mathfrak{U},\mathfrak{V}}$ (see 2.4.1).

2.6. Quasi-coherent modules. The material of this subsection is borrowed from [KR3]. We refer to [KR3] for more detail and proofs.

2.6.1. Modules and quasi-coherent modules on a category over a category. Let $\mathcal{E}$ be a category (which belongs to some universum $\mathfrak{U}$) and $\mathfrak{F} = (\mathcal{F}, \mathcal{F} \xrightarrow{\pi} \mathcal{E})$ a category over $\mathcal{E}$. Denote by $\mathcal{M}od(\mathfrak{F})$ the category opposite to the category of all sections of $\mathfrak{F}$. We shall call objects of $\mathcal{M}od(\mathfrak{F})$ *modules* on $\mathfrak{F}$.

We denote by $Qcoh(\mathfrak{F})$ the category opposite to the category $Cart_{\mathcal{E}}(\mathcal{E}, \mathfrak{F})$ of cartesian sections of $\mathfrak{F}$. In other words, $Qcoh(\mathfrak{F}) = (Lim\mathfrak{F})^{op}$ (cf. [KR3], A3.5.5). Objects of $Qcoh(\mathfrak{F})$ will be called *quasi-coherent modules* on $\mathfrak{F}$.

Any morphism $\mathfrak{F} \longrightarrow \mathfrak{G}$ of $\mathcal{E}$-categories induces a functor $\mathcal{M}od(\mathfrak{F}) \longrightarrow \mathcal{M}od(\mathfrak{G})$. Thus, we have a functor $\mathcal{M}od : Cat/\mathcal{E} \longrightarrow Cat$ from the category of $\mathcal{E}$-categories to the category of categories.

Similarly, the map $\mathfrak{F} \longmapsto Qcoh(\mathfrak{F})$ extends to a functor $Qcoh : Cart_{\mathcal{E}} \longrightarrow Cat$ from the category of cartesian functors over $\mathcal{E}$ to Cat.

Let $\mathfrak{F}$ be a fibered category corresponding to a pseudo-functor $\mathcal{E}^{op} \longrightarrow Cat$,

$$Ob\mathcal{E} \ni X \longmapsto \mathcal{F}_X, \ Hom\mathcal{E} \ni f \longmapsto f^*, \ Hom\mathcal{E} \times_{Ob\mathcal{E}} Hom\mathcal{E} \ni (f,g) \longmapsto c_{f,g} \qquad (1)$$

(cf. [KR3], A3.7, A3.7.1). Then the category $\mathcal{M}od(\mathfrak{F})$ can be described as follows. An object of $\mathcal{M}od(\mathfrak{F})$ is a function which assigns to each $T \in Ob\mathcal{E}$ an object $M(T)$ of the fiber $\mathcal{F}_T$ and to each morphism $f : T \longrightarrow T'$ a morphism $\xi_f : f^*(M(T')) \longrightarrow M(T)$ such that $\xi_{gf} \circ c_{f,g} = \xi_f \circ f^*(\xi_g)$. Morphisms are defined in a natural way.

An object (M, ξ) of $\mathcal{M}od(\mathfrak{F})$ belongs to the subcategory $Qcoh(\mathfrak{F})$ iff ξ_f is an isomorphism for all $f \in Hom\mathcal{E}$.

2.6.2. The pseudo-functor $Qcoh$**.** Let $\mathfrak{U}$, $\mathfrak{V}$ be two universums such that $\mathfrak{U} \in \mathfrak{V}$. Denote by $Cart_{\mathfrak{U},\mathfrak{V}}$ the 2-category whose objects are categories over categories $\mathfrak{F} = (\mathcal{F} \xrightarrow{\pi} \mathcal{E})$ such that the base $\mathcal{E}$ belongs to $\mathfrak{V}$ and each fiber belongs to $\mathfrak{U}$. Denote by $\mathfrak{M}Cart_{\mathfrak{U},\mathfrak{V}}$ all cartesian functors (1-morphisms of $Cart_{\mathfrak{U},\mathfrak{V}}$)

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{u} & \mathcal{F} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{E}' & \xrightarrow{v} & \mathcal{E} \end{array} \qquad (1)$$

such that the functors induced on fibers are category equivalences. It follows that $\mathfrak{M}Cart_{\mathfrak{U},\mathfrak{V}}$ is a 2-subcategory of the 2-category $Cart_{\mathfrak{U},\mathfrak{V}}$ (and the 2-category $Fib_{\mathfrak{U},\mathfrak{V}}$ introduced in 2.4 is a full 2-subcategory of $\mathfrak{M}Cart_{\mathfrak{U},\mathfrak{V}}$).

2.6.2.1. Proposition. *The map* $\mathfrak{F} \longmapsto Qcoh(\mathfrak{F})$ *extends to a pseudo-functor*

$$Qcoh : \mathfrak{M}Cart^{op}_{\mathfrak{U},\mathfrak{V}} \longrightarrow Cat_{\mathfrak{V}}.$$

Proof. See [KR3, 11.1.4.2]. ∎

The pseudo-functor $Qcoh$ gives rise to a functor $\mathfrak{Sp} : \mathfrak{M}Cart_{\mathfrak{U},\mathfrak{V}} \longrightarrow |Cat|^o_{\mathfrak{V}}$. Thus by definition $C_{\mathfrak{Sp}(\mathfrak{F})} = Qcoh(\mathfrak{F})$ for any category $\mathfrak{F}$ over $\mathcal{E}$. We call $\mathfrak{Sp}(\mathfrak{F})$ the *categoric spectrum* of $\mathfrak{F}$.

2.6.3. Quasi-coherent modules on presheaves of sets. Let X be a presheaf of sets on the base $\mathcal{E}$, i.e. a functor $\mathcal{E}^{op} \longrightarrow \mathbf{Sets}$. Then we have a functor $\mathcal{E}/X \longrightarrow \mathcal{E}$ and the category $\mathfrak{F}/X = \mathfrak{F} \times_{\mathcal{E}} \mathcal{E}/X$ over $\mathcal{E}/X$ obtained via a base change (as usual, we identify $\mathcal{E}$ with a full subcategory of the category $\mathcal{E}^\wedge$ of presheaves of sets on $\mathcal{E}$ formed by representable presheaves). Notice that any morphism of the category $\mathcal{E}/X$ over $\mathcal{E}$ is cartesian. Therefore, by [KR3, 11.1.4.1], the category $Qcoh(\mathfrak{F}/X)$ is equivalent to the category $Cart_{\mathcal{E}}(\mathcal{E}/X, \mathfrak{F})^{op}$ opposite to the category of cartesian functors $\mathcal{E}/X \longrightarrow \mathfrak{F}$.

3. Certain classes of morphisms in $|Cat|^o$ and Cat^{op}.

3.1. Left exact, right exact, and exact morphisms. A morphism $X \xrightarrow{f} Y$ is called *right exact* (resp. *left exact*, resp. *exact*), if its inverse image functor preserves colimits (resp. limits, resp. both limits and colimits) of arbitrary finite diagrams.

The following assertion is a reformulation of Proposition 1.1.4 in [GZ].

3.1.1. Proposition. *Let* $f = p_f \circ f_{\mathfrak{c}}$ *be the canonical decomposition of a morphism* $X \xrightarrow{f} Y$ *into a conservative morphism* $X \xrightarrow{f_{\mathfrak{c}}} \Sigma_f^{-1}Y$ *and a localization* $\Sigma_f^{-1}Y \xrightarrow{p_f} Y$*. Suppose* C_Y *has finite limits (resp. finite colimits). Then* f *is left exact (resp. right exact) iff the class of arrows* Σ_f *satisfies left (resp. right) Ore conditions. In this case both the localization* p_f *and the conservative morphism* $f_{\mathfrak{c}}$ *are left (resp. right) exact.*

In particular, if the category C_Y *has limits and colimits of finite diagrams, then* f *is exact iff both the localization* p_f *and the conservative component* $f_{\mathfrak{c}}$ *are exact. The exactness of* p_f *is equivalent to that* Σ_f *satisfies left and right Ore conditions.*

3.2. Dualization functor and dual notions. The *dualization functor*

$$^o : |Cat|^o \longrightarrow |Cat|^o$$

assigns to each object Y of $|Cat|^o$the object Y^o defined by $C_{Y^o} = C_Y^{op}$, and to each morphism f with an inverse image functor f^*, the morphism f^o having $(f^*)^{op}$ as an inverse image functor. It follows that the dialization functors is an automorphism of the category $|Cat|^o$ and its square is the identical functor.

The dualization functor maps left (resp. right) exact morphisms to right (resp. left) exact morphisms. Conservative morphisms and localizations are stable under the dualization. In particular, the dualization functor preserves the canonical decomposition: $f^o = p_f^o \circ f_{\mathfrak{c}}^o$ and $p_f^o = p_{f^o}$, $f_{\mathfrak{c}}^o = (f^o)_{\mathfrak{c}}$.

3.3. Continuous and cocontinuous morphisms. Duality. A morphism $X \xrightarrow{f} Y$ in $|Cat|^o$ is called *cocontinuous* if f^o is continuous. In other words, f is cocontinuous iff its inverse image functor, f^*, has a left adjoint, $f_!$.

Denote by $|Cat|^o_{\mathfrak{c}}$ the subcategory of $|Cat|^o$ formed by continuous morphisms and by $|Cat|^o_{\mathfrak{coc}}$ the subcategory of $|Cat|^o$ formed by cocontinuous morphisms. The dualization functor induces isomorphisms of categories $|Cat|^o_{\mathfrak{c}} \xrightarrow{\sim} |Cat|^o_{\mathfrak{coc}} \xrightarrow{\sim} |Cat|^o_{\mathfrak{c}}$.

The map which assigns to every continuous morphism $X \xrightarrow{f} Y$ with a direct image functor f_* the morphism $X \xrightarrow{f^\wedge} Y$ having f_* as an inverse image functor, defines an isomorphism of categories

$$^\wedge : |Cat|_{\mathfrak{c}} = (|Cat|^o_{\mathfrak{c}})^{op} \xrightarrow{\quad\sim\quad} |Cat|^o_{\mathfrak{coc}}.$$

The inverse isomorphism,

$$^\vee : |Cat|_{\mathfrak{coc}} = (|Cat|^o_{\mathfrak{coc}})^{op} \xrightarrow{\quad\sim\quad} |Cat|^o_{\mathfrak{c}},$$

assigns to any cocontinuous morphism $X \xrightarrow{g} Y$ with an inverse image functor g^* the morphism $Y \xrightarrow{g^\vee} X$ having as an inverse image functor a left adjoint, $g_!$, to g^*.

The composition

$$\mathfrak{D}^\wedge : |Cat|_{\mathfrak{c}} \xrightarrow{\quad\sim\quad} |Cat|^o_{\mathfrak{c}}, \tag{1}$$

of the functor $^\wedge$ with the dualization functor ^{o} is a *duality* on $|Cat|^o_{\mathfrak{c}}$, i.e. a contravariant functor whose square is the identical functor.

Similarly, the composition

$$\mathfrak{D}^\vee : |Cat|_{\mathfrak{coc}} \xrightarrow{\quad\sim\quad} |Cat|^o_{\mathfrak{coc}}, \tag{2}$$

of the functor $^\vee$ with the dualization functor ^{o} is a duality on $|Cat|^o_{\mathfrak{coc}}$.

3.4. Flat and coflat morphisms. We call a morphism $X \xrightarrow{f} Y$ *flat* if it is continuous and its inverse image functor is exact. Since f^* preserves colimits of arbitrary small diagrams, the exactness requirement means that f^* preserves limits of finite diagrams, i.e. f is left exact (see (ii) above).

By 3.1.1, a continuous morphism f is flat iff both the localization p_f and the conservative morphism $f_{\mathfrak{c}}$ in the decomposition $f = p_f \circ f_{\mathfrak{c}}$ are exact. One can show that both p_f and $f_{\mathfrak{c}}$ are continuous, hence flat (see [R3]).

3.4.1. Coflat morphisms. A continuous morphism f is called *coflat* if the dual morphism $\mathfrak{D}^\wedge(f)$ is flat; i.e. f is coflat iff its direct image, f_*, is exact, or, equivalently, f_* preserves colimits of finite diagrams.

3.4.2. Weakly flat and coflat morphisms. A pair of arrows $g_1, g_2 : M \longrightarrow L$ is called *coreflexive*, if there exists a morphism $h : L \longrightarrow M$ such that $h \circ g_1 = id_M = h \circ g_2$.

Dually, a pair of arrows $g_1, g_2 : M \longrightarrow L$ is called *reflexive*, if there exists a morphism $h : L \longrightarrow M$ such that $g_1 \circ h = id_M = g_2 \circ h$.

We call a continuous morphism $X \xrightarrow{f} Y$ in $|Cat|^o$ *weakly flat* if its inverse image functor preserves kernels of coreflexive pairs of arrows.

Dually, f is called *weakly coflat* if its direct image functor preserves cokernels of reflexive pairs of arrows.

3.5. Affine morphisms. A continuous morphism $X \xrightarrow{f} Y$ is called *affine* if its direct image functor f_* has a right adjoint, $f^!$, and is conservative.

Applying the duality $\mathfrak{D}^\wedge$, one can identify an affine morphism $X \xrightarrow{f} Y$ with a conservative morphism $Y \longrightarrow X$ with an inverse image functor $f_* : X \longrightarrow Y$ which is both continuous and cocontinuous.

3.5.1. Coaffine morphisms. We call a continuous morphism $X \xrightarrow{f} Y$ *coaffine* iff the dual morphism $\mathfrak{D}^\wedge(f)$ is affine. In other words, f is coaffine iff its inverse image functor f^* is conservative and has a left adjoint, $f_!$.

3.6. Classes of morphisms of Cat^{op}. Given a class $\mathfrak{M}$ of morphisms of $|Cat|^o$, we say that a 1-morphism in Cat^{op} belongs to $\mathfrak{M}$ if its image by the canonical functor $Cat^{op} \longrightarrow |Cat|^o$ belongs to $\mathfrak{M}$. Thus we have continuous, flat, affine etc. 1-morphisms of the category Cat^{op}.

3.7. Example: canonical morphisms of the category $|Cat|^o$. Let $\bullet$ be the initial object of the category $|Cat|^o$ such that $C_\bullet$ is the category with one (hence identical) morphism. For any object X of $|Cat|^o$, denote by f_X the unique morphism $\bullet \longrightarrow X$.

3.7.1. Lemma. *The unique inverse image functor $f_X^* : C_X \longrightarrow C_\bullet$ has a right adjoint, f_{X*}, (i.e. f_X is continuous) iff the category C_X has a final object. In this case, a direct image functor f_{X*} maps the unique object of the category $C_\bullet$ to a final object of the category C_X.*

Dually, the functor f_X^* has a left adjoint, $f_{X!}$ (i.e. f_X is cocontinuous), iff the category C_X has an initial object.

Proof is left to the reader. ■

3.7.2. Futher observations. All functors from $C_\bullet$, in particular f_{X*}, are conservative. Thus $f_X : \bullet \longrightarrow X$ is affine iff there exists a direct image functor f_{X*} (i.e. iff the category C_X has a final object), and the functor f_{X*} has a right adjoint, $f_X{*}^!$.

Since there is only one functor $C_X \longrightarrow C_\bullet$, the functor f_{X*} has a right adjoint iff it is left adjoint to the functor $f_X{}^*$. In this case, f_{X*} is isomorphic to $f_{X!}$, hence it maps the unique object of $C_\bullet$ to an initial object of C_X.

Thus, the unique morphism $\bullet \longrightarrow X$ is affine iff the category C_X has a zero object, i.e. final objects in C_X are initial too.

If the category C_X has an initial object, then there is a continuous morphism $\phi_X : X \longrightarrow \bullet$ whose inverse image functor maps the unique object of $\bullet$ to an initial object of C_X. The direct image functor of ϕ_X is the unique functor $\phi_* : C_X \longrightarrow C_\bullet$ (which coincides with the functor $f_X{}^*$ above).

The functor ϕ_{X*} has a right adjoint, $\phi_X{}^!$, iff the category C_X is marked. Notice, however, that the morphism ϕ_X is affine (that is the functor ϕ_{X*} is conservative) iff all arrows of C_X are invertible, i.e. C_X is a connected groupoid, or, what is the same, $\phi_X{}^*$ is a category equivalence, i.e. $X \simeq \bullet$.

3.8. Example: cocontinuous and coaffine morphisms between spectra of rings. Let R and S be associative unital rings. A morphism $\mathbf{Sp}(S) \xrightarrow{f} \mathbf{Sp}(R)$ is cocontinuous iff its inverse image functor is isomorphic to

$$Hom_R(\mathcal{N}, -) : R - mod \longrightarrow S - mod, \quad L \longmapsto Hom_R(\mathcal{N}, L). \tag{3}$$

for some (R,S)-bimodule $\mathcal{N}$.

The functor (1) has a left adjoint, $f_!$, iff it is isomorphic to the functor (3) for some (R,S)-bimodule $\mathcal{N}$; or, equivalently, the canonical functorial morphism

$$\mathcal{M} \otimes_R L \longrightarrow Hom_R(\mathcal{M}_R^*, L) \tag{4}$$

is an isomorphism for all L (equivalently, (4) is an isomorphism for $L = \mathcal{M}_R^*$). Here $\mathcal{M}_R^* = Hom_R(\mathcal{M}, R)$. This happens iff $\mathcal{M}$ is a projective R-module of a finite type.

Thus, affine morphisms $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp}(S)$ are in bijective correspondence with isomorphism classes of (S,R)-bimodules $\mathcal{M}$ which are *strictly projective* as left S-modules. Recall that *strictly projective* means *projective cogenerator of finite type.*

Coaffine morphisms $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp}(S)$ are in bijective correspondence with isomorphism classes of (S,R)-bimodules $\mathcal{M}$ which are *strictly projective* as right R-modules.

4. Continuous morphisms to the categoric spectrum of a ring and 'structure sheaves'. Let R be an associative unital ring. For a morphism $X \xrightarrow{f} \mathbf{Sp}(R)$ with an inverse image functor f^*, we denote by $\mathcal{O}$ the object $f^*(R)$. It follows that the object $\mathcal{O}$ is determined by the pair (X,f) uniquely up to isomorphism. The functor f^* defines a monoid morphism $End_R(R) \longrightarrow C_X(\mathcal{O},\mathcal{O})$ the whose composition with the canonical ring isomorphism $R^o \xrightarrow{\sim} End_R(R)$ gives a monoid morphism $\phi_f : R \longrightarrow C_X(\mathcal{O},\mathcal{O})^o$. Here $C_X(\mathcal{O},\mathcal{O})^o$ denotes the monoid opposite to $C_X(\mathcal{O},\mathcal{O})$. If the category C_X is preadditive and the functor f^* is additive, the morphism ϕ_f is a unital ring morphism.

In general, the object $\mathcal{O}$ does not determine the morphism f. It does, however, if f is continuous:

4.1. Proposition. *Let $X \xrightarrow{f} \mathbf{Sp}(R)$ be a continuous morphism. Then*
(a) The morphism f is determined by $\mathcal{O} = f^*(R)$ uniquely up to isomorphism.
(b) There exists a coproduct of any small set of copies of $\mathcal{O}$.
(c) The object $\mathcal{O}$ has a structure of an R-module in the category C_X^{op}. In particular, $\mathcal{O}$ is an abelian cogroup in the category C_X (i.e. an abelian group in C_X^{op}) and the canonical map $\phi_f : R \longrightarrow C_X(\mathcal{O}, \mathcal{O})^o$ is a ring morphism.

Proof. (a) Let f_* be a direct image functor of f (i.e. a right adjoint to f^*). we have functorial isomorphisms $C_X(f^*(R), M) \simeq Hom_R(R, f_*(M)) \simeq f_*(M)$ which shows that the direct image functor f_* of f is naturally isomorphic to the functor $M \longmapsto C_X(f^*(R), M)$. Therefore the inverse image functor f^* of f is defined uniqely up to isomorphism (being a left adjoint to f_*) by the object $f^*(R)$. Since f^* preserves colimits, there exists a coproduct of any set of copies of $\mathcal{O}(X, f) = f^*(R)$.

(b) Since f^* preserves colimits, there exists a coproduct of any set of copies of $\mathcal{O}(X, f) = f^*(R)$.

(c) The assertion follows from the isomorphism $f_* \simeq C_X(\mathcal{O}, -)$ and the fact that f_* takes values in the category of R-modules. ■

4.1.1. Global sections functor. The object $\mathcal{O}$ is viewed as the 'structure sheaf' on X. We denote $C_X(\mathcal{O}, \mathcal{O})^o$ by $\Gamma_X\mathcal{O}$. The functor

$$f_{\mathcal{O}*} : C_X \longrightarrow \Gamma_X\mathcal{O} - mod, \quad M \longmapsto C_X(\mathcal{O}, M)$$

will be called the 'global sections functor' on $(X, \mathcal{O})$. In particularly, $\Gamma_X\mathcal{O} = f_{\mathcal{O}*}(\mathcal{O})$ is the ring of global sections of the structure sheaf $\mathcal{O}$.

4.2. Right R-modules. We call R-modules in C_X^{op} *right R-modules in C_X*, or *right R-modules on X*. Right R-modules on X form a category which we denote by $R^o - \mathcal{M}od_X$. A morphism from an R^o-module (M, ϕ) to an R^o-module (M', ϕ') is a morphism $h : M \longrightarrow M'$ of (abelian) groups in the category C_X such that the diagram

$$\begin{array}{ccc} C_X(M, M) & \xrightarrow{C_X(M,h)} & C_X(M, M') \\ \phi \uparrow & & \uparrow C_X(h, M) \\ R & \xrightarrow{\phi'} & C_X(M', M') \end{array} \qquad (1)$$

commutes. The composition is defined in an obvious way.

4.3. Proposition. *For every continuous morphism $X \xrightarrow{f} Y$, its inverse image functor induces a functor $R^o - \mathcal{M}od_Y \longrightarrow R^o - \mathcal{M}od_X$.*

Proof. In fact, the functorial isomorphism $C_X(f^*(M), T) \simeq C_Y(M, f_*(T))$ implies that if M has a structure of an abelian group in the category $C_Y^{op} = C_{Y^o}$ (i.e. the functor $C_Y(M, -) : C_Y^{op} \longrightarrow Sets$, has a lifting to a functor $C_Y^{op} \longrightarrow \mathbb{Z} - mod$), then $f^*(M)$

has a structure of an abelian group in the category $C_X^{op} = C_{X^o}$. In particular, for any abelian group M in C_Y^{op} and for any object T in C_Y, the morphism $f^*_{M,T} : C_Y(M,T) \longrightarrow C_X(f^*(M), f^*(T)) \simeq C_Y(M, f_*f^*(T))$ is an abelian group morphism. It follows that $f^*_{M,M} : C_Y(M,M) \longrightarrow C_X(f^*(M), f^*(M))$ is a ring morphism. Thus if $R^o \longrightarrow C_Y(M,M)$ is a ring morphism (i.e. a right R-module structure on M), its composition with $f^*_{M,M}$ is a right R-module structure on $f^*(M)$. This extends to a functor $R^o-\mathcal{M}od_Y \longrightarrow R^o-\mathcal{M}od_X$, hence the assertion. ■

Thus, for any associative ring R, we have a pseudo-functor

$$\beta_R : |Cat|_{\mathfrak{c}} = (|Cat|^o_{\mathfrak{c}})^{op} \longrightarrow Cat$$

which defines the fibered category $(R^o - \mathcal{M}od \longrightarrow |Cat|^o_{\mathfrak{c}})$. Here $|Cat|^o_{\mathfrak{c}}$ is the subcategory of the category $|Cat|^o$ formed by the continuous morphisms.

4.4. Right R-modules and continuous morphisms to $\mathbf{Sp}(R)$: functorial picture. We denote by $|Cat|^o(R)$ the *category of right R-modules*. Its objects are triples $(X, \mathcal{O}_X, \phi)$, where X is an object of $|Cat|^o$, and $(\mathcal{O}_X, \phi)$ is a right R-module in the category C_X (cf. 4.2). Morphisms from $(X, \mathcal{O}_X, \phi)$ to $(Y, \mathcal{O}_Y, \psi)$ are morphisms $X \xrightarrow{f} Y$ such that there exists an isomorphism $\lambda : f^*(\mathcal{O}_Y) \xrightarrow{\sim} \mathcal{O}_X$ making the diagram

$$\begin{array}{ccc} R^o & \xrightarrow{\phi} & C_X(\mathcal{O}_X, \mathcal{O}_X) \\ \psi \downarrow & & \downarrow \mathfrak{c}_\lambda \\ C_Y(\mathcal{O}_Y, \mathcal{O}_Y) & \xrightarrow{f^*_{\mathcal{O}_Y,\mathcal{O}_Y}} & C_X(f^*(\mathcal{O}_Y), f^*(\mathcal{O}_Y)) \end{array} \qquad (2)$$

commute. Here $\mathfrak{c}_\lambda$ denotes the conjugation by λ, $h \longmapsto \lambda^{-1} \circ h \circ \lambda$.

Let $Cat^{op}(R)$ denote the category whose objects are same as objects of $|Cat|^o(R)$. Morphisms from $(X, \mathcal{O}_X, \phi)$ to $(Y, \mathcal{O}_Y, \psi)$ are given by pairs (f^*, λ), where f^* is a functor $C_Y \longrightarrow C_X$, λ an isomorphism $f^*(\mathcal{O}_Y) \xrightarrow{\sim} \mathcal{O}_X$ such that the diagram (2) commutes. Composition is defined in an obvious way. The canonical functor $Cat^{op}(R) \longrightarrow |Cat|^o(R)$ is a fibered category.

4.4.1. Note. Morphisms of the categories $Cat^{op}(R)$ and $|Cat|^o(R)$ (more precisely, the meaning of the component λ of a morphism in $Cat^{op}(R)$, see above) might be viewed as follows. For any object X of $|Cat|^o$, let $Cat^{op}(R)_X$ denote the fiber of the category $Cat^{op}(R)$ over X; its objects are triples $(X, \mathcal{O}_X, \phi)$, where ϕ is a right action of the ring R on $\mathcal{O}_X$, morphisms are morphisms of actions. An inverse image functor, f^*, of a morphism $X \xrightarrow{f} Y$ induces a functor

$$f^*_R : Cat^{op}(R)_Y \longrightarrow Cat^{op}(R)_X \qquad (3)$$

which maps an object $(Y, \mathcal{O}_Y, \psi)$ to the object $(X, f^*(\mathcal{O}_Y), \psi_{f^*})$, where the action ψ_{f^*} is the composition of $\psi : R^o \longrightarrow C_Y(\mathcal{O}_Y, \mathcal{O}_Y)$ and the ring morphism

$$f^*_{\mathcal{O}_Y,\mathcal{O}_Y} : C_Y(\mathcal{O}_Y, \mathcal{O}_Y) \longrightarrow C_X(f^*(\mathcal{O}_Y), f^*(\mathcal{O}_Y)).$$

The commutativity of the diagram (2) means exactly that $\lambda : f^*(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X$ is an isomorphism $f_R^*(Y, \mathcal{O}_Y, \psi) \longrightarrow (X, \mathcal{O}_X, \phi)$.

We denote by $|Cat|^o(R)_{\mathfrak{c}}$ the full subcategory of the category $|Cat|^o(R)$ whose objects are triples $(X, \mathcal{O}_X, \phi)$ such that the following conditions hold:

(a) $(\mathcal{O}_X, \phi)$ is a right R-module in the category C_X, i.e. $\mathcal{O}_X$ is an abelian group in C_X^{op} and ϕ a ring morphism $R^o \longrightarrow C_X(\mathcal{O}_X, \mathcal{O}_X)$ (cf. 4.1(c) and 4.2).

(b) There exists a coproduct of any set of copies of $\mathcal{O}_X$.

(c) Let Φ_ϕ denote the functor from the subcategory $\mathfrak{L}_R$ of free R-modules to C_X which is uniquely defined by the action ϕ (thanks to (b) above). The image by Φ_ϕ of any pair of arrows $X_1 \rightrightarrows X_0$ has a cokernel.

Let $Cat^{op}(R)_{\mathfrak{c}}$ denote the preimage of $|Cat|^o(R)_{\mathfrak{c}}$ in $Cat^{op}(R)$. On the other hand, let $(|Cat|^o/\mathbf{Sp}(R))_{\mathfrak{c}}$ denote the full subcategory of the category $|Cat|^o/\mathbf{Sp}(R)$ whose objects are continuous morphisms to $\mathbf{Sp}(R)$, and let $(Cat^{op}/\mathbf{Sp}(R))_{\mathfrak{c}}$ denote its preimage in $Cat^{op}/\mathbf{Sp}(R)$.

4.4.2. Proposition. *The functor (3) induces an equivalence of the fibered categories*

$$\begin{pmatrix} (Cat^{op}/\mathbf{Sp}(R))_{\mathfrak{c}} \\ \downarrow \\ (|Cat|^o/\mathbf{Sp}(R))_{\mathfrak{c}} \end{pmatrix} \longrightarrow \begin{pmatrix} (Cat^{op}(R)_{\mathfrak{c}} \\ \downarrow \\ |Cat|^o(R)_{\mathfrak{c}} \end{pmatrix}. \qquad (4)$$

Proof. By 4.1, the cartesian morphism (3) induces a cartesian morphism (4).

Thanks to 4.1, in order to prove that (4) is an equivalence, it suffices to show that the functor f^* can be reconstructed (uniquely up to isomorphism) from the right R-module $(\mathcal{O}, R \stackrel{\phi_f}{\to} C_X(\mathcal{O}, \mathcal{O})^o)$ associated with f. In fact, the right R-module $(\mathcal{O}, \phi_f)$ gives raise to a functor, Φ_f, from the category $\mathfrak{L}_R$ of free left R-modules (a full subcategory of $R-mod$) to C_X which is isomorphic to the restriction of f^* to $\mathfrak{L}_R$: direct sums of copies of R are maped to direct sums of copies of $\mathcal{O}$ and ϕ_f (regarded as $End_R(R) \longrightarrow C_X(\mathcal{O}, \mathcal{O})$) determines map on morphisms. Every R-module M is a cokernel of a pair of morphisms $L_1 \rightrightarrows L_0$, where L_1, L_0 are free R-modules. Since the functor f^* preserves cokernels of pairs of arrows, $f^*(M)$ is isomorphic to the cokernel of the pair $\Phi_f(L_1) \rightrightarrows \Phi_f(L_0)$. ∎

4.5. $\mathbb{Z}$-'spaces'. Let $X \xrightarrow{f} \mathbf{Sp}(R)$ be a continuous morphism with an inverse image functor f^*, and let $\mathcal{O} = f^*(R)$. It follows from 4.1(c) that the functor $f_* = C_X(\mathcal{O}, -)$ is naturally decomposed into

$$\begin{array}{ccc} C_X & \xrightarrow{f_{\mathcal{O}^*}} & \Gamma_X\mathcal{O} - mod \\ & f_* \searrow \quad \swarrow \bar{\phi}_{f^*} & \\ & R - mod & \end{array} \qquad (5)$$

where $\bar{\phi}_{f^*}$ is the pull-back by the ring morphism $\phi_f : R \longrightarrow \Gamma_X\mathcal{O}$ defining a right R-module structure on $\mathcal{O}$.

4.5.1. Lemma. *The global sections functor $f_{\mathcal{O}^*}$ is a direct image functor of a continuous morphism, $f_{\mathcal{O}} : X \longrightarrow \mathbf{Sp}(\Gamma_X\mathcal{O})$, iff any pair of arrows $\mathcal{O}^{\oplus I} \rightrightarrows \mathcal{O}^{\oplus J}$ between coproducts of copies of $\mathcal{O}$ has a cokernel in C_X.*

Proof. The fact follows from (the argument of) 4.4.2 applied to the case when $R = \Gamma_X\mathcal{O}$: the inverse image functor $f^*_{\mathcal{O}}$ assigns to a free $\Gamma_X\mathcal{O}$-module $\Gamma_X\mathcal{O}^{\oplus J}$ the coproduct $\mathcal{O}^{\oplus J}$ of J copies of the object $\mathcal{O}$. ∎

4.5.2. The category of $\mathbb{Z}$-spaces. Denote by $|Cat|^o_{\mathbb{Z}}$ the category whose objects are all pairs $(X, \mathcal{O})$, where X is an object of $|Cat|^o$ and $\mathcal{O}$ is an abelian group in C_X^{op} such that there exist coproducts of small sets of copies of $\mathcal{O}$ and any pair of arrows $\mathcal{O}^{\oplus I} \rightrightarrows \mathcal{O}^{\oplus J}$ between coproducts of copies of $\mathcal{O}$ has a cokernel in C_X. Morphisms from $(X, \mathcal{O})$ to $(X', \mathcal{O}')$ are morphisms $X \xrightarrow{f} X'$ such that there exists an isomorphism $f^*(\mathcal{O}') \xrightarrow{\sim} \mathcal{O}$. Composition is defined in an obvious way. Objects of the category $|Cat|^o_{\mathbb{Z}}$ will be called $\mathbb{Z}$-'spaces'.

4.5.2.1. A reformulation. By 4.5.1, $\mathbb{Z}$-spaces are pairs $(X, \mathcal{O})$ such that $\mathcal{O}$ is an abelian group in the category C_X^{op} and the canonical functor

$$f_{\mathcal{O}*} : C_X \longrightarrow \Gamma_X\mathcal{O} - mod, \quad M \longmapsto C_X(\mathcal{O}, M), \tag{6}$$

has a left adjoint; or, equivalently, $f_{\mathcal{O}*}$ is a direct image functor of a continuous morphism.

4.5.2.2. Example. If C_X is an additive category with small coproducts and cokernels, then $(X, \mathcal{O})$ is a $\mathbb{Z}$-space for any $\mathcal{O} \in ObC_X$.

4.5.3. Affine $\mathbb{Z}$-spaces. We call a $\mathbb{Z}$-space $(X, \mathcal{O})$ *affine* if the canonical morphism $f_{\mathcal{O}} : X \longrightarrow \mathbf{Sp}(\Gamma_X\mathcal{O})$ is an isomorphism; i.e. the functor $f_{\mathcal{O}*}$ (see (6)) is a category equivalence. By a Mitchel's theorem, affine $\mathbb{Z}$-spaces are pairs $(X, \mathcal{O})$, where C_X is an abelian category with small coproducts, and $\mathcal{O}$ is a projective cogenerator of finite type. We denote by $\mathbf{Aff}_{\mathbb{Z}}$ the full subcategory of the category $|Cat|^o_{\mathbb{Z}}$ formed by affine $\mathbb{Z}$-spaces.

The functor $\mathbf{Sp} : Rings^{op} \longrightarrow |Cat|^o$, $R \longmapsto \mathbf{Sp}(R)$, gives rise to the functor

$$\mathbf{Sp}_{\mathbb{Z}} : Rings^{op} \longrightarrow |Cat|^o_{\mathbb{Z}}, \quad R \longmapsto (\mathbf{Sp}(R), R)$$

which takes values in the subcategory $\mathbf{Aff}_{\mathbb{Z}}$. We denote the image of the functor $\mathbf{Sp}_{\mathbb{Z}}$ by $\mathfrak{Aff}_{\mathbb{Z}}$. Thus, objects of the category $\mathfrak{Aff}_{\mathbb{Z}}$ are pairs $(\mathbf{Sp}(R), R)$ and morphisms from $(\mathbf{Sp}(R), R) \longrightarrow (\mathbf{Sp}(T), T)$ are morphisms $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp}(T)$ corresponding to unital ring morphisms $T \longrightarrow R$. The functor $\mathbf{Sp}_{\mathbb{Z}}$ induces an inclusion functor $\gamma_* : \mathfrak{Aff}_{\mathbb{Z}} \longrightarrow |Cat|^o_{\mathbb{Z}}$ which takes values in the subcategory of affine $\mathbb{Z}$-spaces.

4.5.4. Proposition. *The functor $\gamma_* : \mathfrak{Aff}_{\mathbb{Z}} \longrightarrow |Cat|^o_{\mathbb{Z}}$ is fully faithful and has a left adjoint. In particular, the functor γ_* induces an equivalence of $\mathfrak{Aff}_{\mathbb{Z}}$ and the category of affine $\mathbb{Z}$-spaces.*

Proof. Let f be a morphism $(X, \mathcal{O}) \longrightarrow (X', \mathcal{O}')$. A choice of an inverse image functor, f^*, of f and an isomorphism $\lambda : f^*(\mathcal{O}') \xrightarrow{\sim} \mathcal{O}$ determines a ring morphism $\psi_{f^*,\lambda} : \Gamma_{X'}\mathcal{O}' \longrightarrow \Gamma_X\mathcal{O}$. One can check that the corresponding morphism of categoric spectra, $\mathbf{Sp}(\Gamma_X\mathcal{O}) \longrightarrow \mathbf{Sp}(\Gamma_{X'}\mathcal{O}')$, does not depend on choices. Thus we have a functor $\gamma^* : |Cat|^o_{\mathbb{Z}} \longrightarrow \mathfrak{Aff}_{\mathbb{Z}}$. By 4.5.1, we have a natural morphism $\eta_\gamma : Id_{|Cat|^o_{\mathbb{Z}}} \longrightarrow \gamma_*\gamma^*$. And there is an isomorphism $\epsilon_\gamma : \gamma^*\gamma_* \xrightarrow{\sim} Id_{\mathfrak{Aff}_{\mathbb{Z}}}$. These are adjunction morphisms. Since ϵ_γ is an isomorphism, the functor γ_* is fully faithful. ∎

4.5.5. A description of the category $\mathfrak{Aff}_{\mathbb{Z}}$. Let $\mathfrak{A}ss$ denote the category whose objects are associative rings; and morphisms from a ring R to a ring S are equivalence classes of ring morphisms $R \longrightarrow S$ by the following equivalence relation: two ring morphisms $f, g : R \longrightarrow S$ are equivalent if they are conjugated, i.e. $g(-) = tf(-)t^{-1}$ for an invertible element t of S.

4.5.5.1. Proposition. *Two ring morphisms, $R \overset{\phi}{\underset{\psi}{\rightrightarrows}} S$, are conjugate iff the corresponding inverse image functors, $R - mod \overset{\phi^*}{\underset{\psi^*}{\rightrightarrows}} S - mod$, are isomorphic.*

Proof. (a) Suppose that ψ and ϕ are conjugate, i.e. there exists an invertible element, t, of S such that $\psi(r) = t\phi(r)t^{-1}$ for all $r \in R$. For any R-module $\mathcal{M} = (M, m)$, we have a commutative diagram

$$\begin{array}{ccc} S \otimes M & \xrightarrow{\cdot t} & S \otimes M \\ {\scriptstyle \gamma_\psi}\downarrow & & \downarrow{\scriptstyle \gamma_\phi} \\ S \otimes_{R,\psi} M & \xrightarrow{\lambda_t} & S \otimes_{R,\phi} M \end{array} \qquad (1)$$

Here $\cdot t$ denotes the S-module morphism $s \otimes z \longmapsto st \otimes z$ for all $s \in S,\ z \in M$; $\gamma_\psi,\ \gamma_\phi$ are canonical epimorphisms.

In fact, for any $s \in S,\ r \in R,\ z \in M$, $\gamma_\psi(s\psi(r) \otimes z) = \gamma_\psi(s \otimes r \cdot z)$, and $\cdot t(s \otimes r \cdot z) = st \otimes r \cdot z$.

On the other hand, $\cdot t(s\psi(r) \otimes z) = s\psi(r)t \otimes z = st\phi(r) \otimes z$, and $\gamma_\phi(st\phi(r) \otimes z) = \gamma_\phi(st \otimes r \cdot z)$. Since γ_ψ is by definition the cokernel of two maps

$$S \otimes_k R \otimes_k M \overset{\psi_l}{\underset{\psi_r}{\rightrightarrows}} S \otimes_k M, \quad s \otimes r \otimes z \overset{\psi_l}{\longmapsto} s\psi(r) \otimes z, \text{ and } s \otimes r \otimes z \overset{\psi_r}{\longmapsto} s \otimes r \cdot z,$$

it follows the existence of a (necessarily unique) morphism $\lambda_t : S \otimes_{R,\psi} M \longrightarrow S \otimes_{R,\phi} M$ such that the diagram (1) commutes; i.e. λ_t is given by $\gamma_\psi(s \otimes z) \longmapsto \gamma_\phi(st \otimes z)$.

(b) Conversely, suppose $\phi,\ \psi$ are unital ring morphisms such that there is a functorial isomorphism $u : \psi^* \xrightarrow{\sim} \phi^*$. Identifying both $\phi^*(R)$ and $\psi^*(R)$ with the left S-module S, we obtain, in particular, an S-module morphism $u(R) : S \longrightarrow S$. Since S is a ring with unit, $u(R)$ equals to $\cdot t : s \longmapsto st$ for some $t \in S$. Since u is a functor morphism, for any $r \in R$, $u(R) \circ \psi^*(\cdot r) = \phi^*(\cdot r) \circ u(R)$. This means that for any $s \in S$, $s\psi(r)t = st\phi(r)$, hence $\psi(r) = t\phi(r)t^{-1}$. ∎

4.5.5.2. Corollary. *The functor $\mathbf{Sp}_{\mathbb{Z}} : Rings^{op} \longrightarrow |Cat|^o_{\mathbb{Z}}, \quad R \longmapsto (\mathbf{Sp}(R), R)$ induces an isomorphism of categories $\mathfrak{A}ss^{op} \xrightarrow{\sim} \mathfrak{Aff}_{\mathbb{Z}}$.*

4.5.5.3. Corollary. *The functor $\mathbf{Sp}_{\mathbb{Z}} : Rings^{op} \longrightarrow |Cat|^o_{\mathbb{Z}}$ induces an equivalence of categories $\mathfrak{A}ss^{op} \longrightarrow \mathbf{Aff}_{\mathbb{Z}}$.*

Proof. This follows from 4.5.4 and 4.5.5.2. ∎

4.5.6. Remark. If S is a commutative ring, then for any ring R, the surjection $Rings(R, S) \longrightarrow \mathfrak{A}ss(R, S)$ is a bijective map. In particular, the full subcategory of $\mathfrak{A}ss$

formed by commutative rings is isomorphic to the category $CRings$ of commutative rings. Thus, the equivalence of categories $\mathfrak{A}ss^{op} \longrightarrow \mathbf{Aff}_{\mathbb{Z}}$ induces an equivalence between the category $CRings^{op}$ opposite to the category of commutative unital rings and the full subcategory $\mathbf{CAff}_{\mathbb{Z}}$ formed by affine $\mathbb{Z}$-spaces $(X, \mathcal{O})$ such that the global sections ring $\Gamma_X \mathcal{O}$ is commutative. This shows by passing that the category $\mathbf{CAff}_{\mathbb{Z}}$ of commutative affine $\mathbb{Z}$-spaces is equivalent to the category of commutative affine schemes in the usual sense.

4.6. Proposition. *Let $X \in Ob|Cat|^o$ be such that the category C_X has cokernels of pairs of morphisms. Then $(X, \mathcal{O})$ is a $\mathbb{Z}$-space for any object $\mathcal{O}$ of C_X such that there exists a coproduct of any small set of copies of $\mathcal{O}$. In particular, continuous morphisms $X \longrightarrow \mathbf{Sp}(R)$ are in one-to-one correspondence with isomorphism classes of right R-modules $(\mathcal{O}, \phi)$ in C_X such that $(X, \mathcal{O})$ is a $\mathbb{Z}$-space.*

In particular, if C_X is an abelian category with small coproducts, then morphisms $X \longrightarrow \mathbf{Sp}(R)$ are in one-to-one correspondence with isomorphism classes of right R-modules in C_X.

Proof. The assertion is a corollary of 4.5.4. ■

4.6.1. Example. Let $X = \mathbf{Sp}(S)$ for some associative ring S, i.e. C_X is the category $S - mod$ of left S-modules. By 4.6, continuous morphisms from $\mathbf{Sp}(S) \longrightarrow \mathbf{Sp}(R)$ are in one-to-one correspondence with isomorphism classes of right R-modules in the category $S - mod$. Notice that the category of right R-modules in $S - mod$ is isomorphic to the category of (S, R)-bimodules. If $\mathcal{O}$ is a (S, R)-bimodule corresponding to a morphism $f : \mathbf{Sp}(S) \longrightarrow \mathbf{Sp}(R)$, then $L \longmapsto Hom_S(\mathcal{O}, L)$ is a direct image functor of f. Therefore $N \longmapsto \mathcal{O} \otimes_R N$ is an inverse image functor of f. Thus we have recovered a classical fact already mentioned in 3.8.1.

4.7. Additive morphisms and continuous morphisms. For any objects X, Y of Cat^{op}, denote by $Cat^{op}_c(X, Y)$ the full subcategory of the category $Cat^{op}(X, Y)$ whose objects are continuous morphisms. If C_X, C_Y are (pre)additive categories, we call a morphism $X \xrightarrow{f} Y$ *additive* if its inverse image functor is additive. We denote by $Cat^{op}_a(X, Y)$ the full subcategory of the category $Cat^{op}(X, Y)$ whose objects are additive morphisms. Continuous morphisms are additive, i.e. $Cat^{op}_c(X, Y)$ is a (full) subcategory of $Cat^{op}_a(X, Y)$.

4.7.1. Proposition. *Let X be an object of Cat^{op} such that C_X is an abelian category with small coproducts. Then for any unital associative ring R, the inclusion functor $Cat^{op}_c(X, \mathbf{Sp}(R)) \longrightarrow Cat^{op}_a(X, \mathbf{Sp}(R))$ has a right adjoint.*

Proof. Since C_X is an abelian category with small coproducts (or, what is the same, colimits of small diagrams), it follows from 4.5.1 that for any object $\mathcal{O} \in ObC_X$, the functor

$$f_{\mathcal{O}*} = C_X(\mathcal{O}, -) : C_X \longrightarrow \Gamma_X \mathcal{O} - mod \tag{6}$$

is a direct image functor of a continuous morphism $f_{\mathcal{O}} : X \longrightarrow \mathbf{Sp}(\Gamma_X \mathcal{O})$. To every morphism $X \xrightarrow{f} \mathbf{Sp}(R)$, we assign the composition of the morphism $f_{\mathcal{O}}$, where $\mathcal{O} = f^*(R)$, and the morphism $\mathbf{Sp}(\Gamma_X \mathcal{O}) \longrightarrow \mathbf{Sp}(R)$ corresponding to the right R-module structure $\phi_f : R \longrightarrow \Gamma_X \mathcal{O}$. The assertion follows from 4.6. Details are left to the reader. ■

4.7.2. Corollary. *Let R, S be associative unital rings. The functor*

$$(S,R)-bimod \longrightarrow Cat^{op}(\mathbf{Sp}(S), \mathbf{Sp}(R)) \tag{7}$$

which assigns to each (S,R)-bimodule $\mathcal{M}$ the morphism $f_{\mathcal{M}}$ with the inverse image functor $\mathcal{M} \otimes_R - : R-mod \longrightarrow S-mod$ has a right adjoint.

Proof. The assertion follows from 4.7.1 and the fact that the functor (7) establishes an equivalence between the category (S,R)-bimodules and the category of continuous morphisms from $(\mathbf{Sp}(S)$ to $\mathbf{Sp}(R))$ (see 4.6.1). ∎

4.8. Grothendieck categories and flat localizations of spectra of rings. Recall that a *Grothendieck category* is an abelian category with small coproducts, exact inductive limits, and cogenerators.

Let C_X be a Grothendieck category. By 4.6, every object $\mathcal{O}$ of C_X defines a continuous morphism $f_{\mathcal{O}} : X \longrightarrow \mathbf{Sp}(\Gamma_X \mathcal{O})$ with a direct image functor $C_X(\mathcal{O}, -)$. Here $\Gamma_X \mathcal{O}$ denotes the ring $C_X(\mathcal{O}, \mathcal{O})^o$.

4.8.1. Proposition. *Let C_X be a Grothendieck category. A continuous morphism $X \xrightarrow{f} \mathbf{Sp}(R)$ is a flat localization iff the corresponding right R-module $(\mathcal{O}, \phi_f)$ (cf. 4.6) has the following properties:*

(a) $\mathcal{O}$ is a cogenerator of the category C_X;

(b) the ring morphism $\phi_f : R \longrightarrow \Gamma_X \mathcal{O}$ is flat (i.e. it makes $\Gamma_X \mathcal{O} \simeq f_* f^*(R)$ a flat right R-module), and the morphism $\Gamma_X \mathcal{O} \otimes_R \Gamma_X \mathcal{O} \longrightarrow \Gamma_X \mathcal{O}$ induced by the multiplication on $\Gamma_X \mathcal{O}$ is an isomorphism.

Proof. 1) Suppose the conditions (a) and (b) hold.

(a) If C_X is a Grothendieck category, then the morphism $f_{\mathcal{O}} : X \longrightarrow \mathbf{Sp}(\Gamma_X \mathcal{O})$ with the direct image functor given by $C_X(\mathcal{O}, -)$ is a localization iff $\mathcal{O}$ is a cogenerator of the category C_X.

This assertion follows from the argument of the Theorem by Gabriel and Popescu which describes Grothendieck categories as quotient categories (localizations) of categories of modules. See [BD, 6.25].

(b) The condition (b) means exactly that the morphism $\mathbf{Sp}\Gamma_X \mathcal{O} \longrightarrow \mathbf{Sp}(R)$ corresponding to the ring morphism ϕ_f is a localization.

Indeed, the (right) flatness of $\phi_f : R \longrightarrow \Gamma_X \mathcal{O}$ means that the corresponding inverse image functor, $\phi_f^* = \Gamma_X \mathcal{O} \otimes_R -$, is exact. The condition that $\Gamma_X \mathcal{O} \otimes_R \Gamma_X \mathcal{O} \longrightarrow \Gamma_X \mathcal{O}$ is an isomorphism implies that the adjunction morphism $\phi_f^* \phi_{f*} \longrightarrow Id_{\Gamma_X \mathcal{O}-mod}$ is an isomorphism. The latter is equivalent to that ϕ_{f*} is fully faithful, i.e. ϕ_f^* is a localization.

Thus, f, being a composition of flat localizations is a flat localization.

2) Suppose $X \xrightarrow{f} \mathbf{Sp}(R)$ is a localization.

(i) For any continuous morphism $X \xrightarrow{f} \mathbf{Sp}(R)$, the object $\mathcal{O} = f^*(R)$ is a cogenerator iff f_* is a faithful functor.

In fact, by definition, the object $\mathcal{O}$ is a cogenerator iff the functor $C_X(\mathcal{O}, -)$ is faithful. But, this functor is isomorphic to f_*.

A continuous morphism f is a localization iff its direct image functor, f_*, is fully faithful; in particular, $\mathcal{O}$ is a cogenerator.

(ii) Since $\mathcal{O}$ is a cogenerator, the morphism $f_{\mathcal{O}} : X \longrightarrow \mathbf{Sp}(\Gamma_X\mathcal{O})$ is a flat localization. Let $\bar{\phi}_f$ denote the morphism $\mathbf{Sp}(\Gamma_X\mathcal{O}) \longrightarrow \mathbf{Sp}(R)$ corresponding to the ring morphism ϕ_f. Since the composition $f = \bar{\phi}_f \circ f_{\mathcal{O}}$ is a localization and $f_{\mathcal{O}}$ is a localization, it follows from the universal property of localizations that $\bar{\phi}_f$ is a localization. The latter means that the adjunction morphism $\bar{\phi}^*\bar{\phi}_*(M) = \Gamma_X\mathcal{O} \otimes_R \phi_*(M) \longrightarrow M$ is an isomorphism for any $\Gamma_X\mathcal{O}$-module M. Taking $M = \Gamma_X\mathcal{O}$, we obtain that the natural morphism $\Gamma_X\mathcal{O} \otimes_R \Gamma_X\mathcal{O} \longrightarrow \Gamma_X\mathcal{O}$ is an isomorphism.

(iii) It remains to verify that $\Gamma_X\mathcal{O}$ is flat as a right R-module. This is a consequence of the following general fact:

4.8.1.1. Lemma. *Let $X \xrightarrow{g} Y \xrightarrow{h} Z$ be continuous localizations. Suppose that the category C_Z has finite limits and that inverse image functors of g and $h \circ g$ are left exact. Then h^* is left exact.*

Let $\mathcal{D} : \mathfrak{D} \longrightarrow C_Y$ be a finite diagram. Since h^* is a localization, the functor h_* is fully faithful. By [GZ, I.1.4], $\lim \mathcal{D}$ exists if $\lim h_*\mathcal{D}$ exists, and the natural morphism $h^*(\lim h_*\mathcal{D}) \longrightarrow \lim h^*h_*\mathcal{D} \simeq \lim \mathcal{D}$ is an isomorphism. Therefore

$$g^*(\lim \mathcal{D}) \simeq g^*h^*(\lim h_*\mathcal{D}) \simeq (hg)^*(\lim h_*\mathcal{D}).$$

Since the functor $(hg)^*$ is exact,

$$(hg)^*(\lim h_*\mathcal{D}) \simeq \lim((hg)^*h_*\mathcal{D}) \simeq \lim(g^*h^*h_*\mathcal{D}) \simeq \lim(g^*\mathcal{D})$$

(the last isomorphism is induced by the adjunction isomorphism $h^*h_* \longrightarrow Id_{C_Y}$). Hence the assertion. ∎ ∎

4.8.1.2. Note. The requirement in 4.8.1.1 that localizations are continuous can be dropped. The argument in this case is a little bit more involved. It is based on the fact that if $X \xrightarrow{f} Y$ is a localization and the category C_Y has finite limits, then the functor f^* is exact iff the class of morphisms $\Sigma_f = \{s \in HomC_Y \mid f^*(s) \text{ is an isomorphism}\}$ satisfies the right Ore conditions (see 3.1.1, or [GZ, I.3.1 and 1.3.4]).

4.8.2. Corollary. *Let C_X be a Grothendieck category. Any continuous morphism $X \xrightarrow{f} \mathbf{Sp}(R)$ such that f_* is a faithful functor is uniquely represented as the composition*

$$X \xrightarrow{\psi_f} \mathbf{Sp}(R') \xrightarrow{\bar{\phi}_f} \mathbf{Sp}(R), \tag{8}$$

where $\bar{\phi}_f$ is an affine morphism corresponding to a ring morphism $\phi_f : R \longrightarrow R'$, and ψ_f is a flat localization such that the adjunction morphism $R' \longrightarrow \psi_{f*}\psi_f^*(R')$ is an isomorphism.

The morphism f is a localization (resp. a flat localization) iff $\bar{\phi}_f$ is a localization (resp. a flat localization).

Proof. Let $\mathcal{O} = f^*(R)$. The functor f_* is faithful iff the object $\mathcal{O}$ is a cogenerator in C_X. By (the argument of) Gabriel-Popescu Theorem, this implies that the canonical morphism $f_{\mathcal{O}} : X \longrightarrow \mathbf{Sp}(\Gamma_X\mathcal{O})$ is a flat localization. We set $\psi_f = f_{\mathcal{O}}$ and $R' = \Gamma_X\mathcal{O} = C_X(\mathcal{O},\mathcal{O})^o$. The morphism $\phi_f : R \longrightarrow R'$ is the right R-module structure on $\mathcal{O}$ (which is the same as the adjunction morphism $R \longrightarrow f_*f^*(R)$ if the ring $C_X(\mathcal{O},\mathcal{O})^o$ is identified with $f_*f^*(R)$). It follows from 4.8.1 that if f is a localization (i.e. f_* is fully faithful), then $\bar{\phi}_f$ is a localization.

The condition $R' \simeq \psi_{f*}\psi_f^*(R')$ implies that the ring R' is isomorphic to $\Gamma_X\mathcal{O}$, where $\mathcal{O} = f^*(R)$. This implies uniqueness (up to isomorphism) of the decomposition (8). ∎

5. Monads, comonads, and continuous morphisms.

5.1. Monads and their categoric spectrum. Let Y be an object of $|Cat|^o$. A *monad on* Y is by definition a monad on the category C_Y, i.e. a pair (F,μ), where F is a functor $C_Y \longrightarrow C_Y$ and μ a morphism $F^2 \longrightarrow F$ (multiplication) such that $\mu \circ F\mu = \mu \circ \mu F$ and there exists a morphism $\eta : Id_{C_Y} \longrightarrow F$ uniquely determined by the equalities $\mu \circ F\eta = id_F = \mu \circ \eta F$ (a unit).

A morphism from a monad $\mathcal{F} = (F,\mu)$ to a monad $\mathcal{F}' = (F',\mu')$ is given by a functor morphism $F \longrightarrow F'$ such that $\varphi\circ\mu = \mu'\circ\varphi\odot\varphi$ and $\varphi\circ\eta = \eta'$. Here $\varphi\odot\varphi = F'\varphi\circ\varphi F$, and η, η' are units of the monads resp. $\mathcal{F}$ and $\mathcal{F}'$. The composition of morphisms is defined naturally, so that the map $\mathfrak{Mon}_Y \longrightarrow End(C_Y)$ forgetting monad structure, i.e. sending a monad morphism $(F,\mu) \xrightarrow{\varphi} (F',\mu')$ to the natural transformation $F \xrightarrow{\varphi} F'$, is a functor.

For an object Y of $|Cat|^o$, we denote by $\mathfrak{Mon}_Y$ the category of monads on Y.

Given a monad $\mathcal{F} = (F,\mu)$ on Y, we denote by $(\mathcal{F}/Y) - mod$, or simply by $\mathcal{F} - mod$, the category of $(\mathcal{F}/Y)$-modules. Its objects are pairs (M,ξ), where $M \in ObC_Y$ and ξ is a morphism $F(M) \longrightarrow M$ such that $\xi \circ F\xi = \xi \circ \mu(M)$ and $\xi \circ \eta(M) = id_M$. Morphisms from (M,ξ) to (M',ξ') are given by morphisms $g : M \longrightarrow M'$ such that $\xi' \circ Fg = g \circ \xi$.

We denote by $\mathbf{Sp}(\mathcal{F}/Y)$ the object of $|Cat|^o$ such that the corresponding category is the category $(\mathcal{F}/Y) - mod$ of $(\mathcal{F}/Y)$-modules. The object $\mathbf{Sp}(\mathcal{F}/Y)$ is regarded as the *spectrum of the monad* $\mathcal{F}$ *in* $|Cat|^o$.

The forgetful functor

$$(\mathcal{F}/Y) - mod \xrightarrow{f_*} C_Y, \quad (M,\xi) \longmapsto M,$$

is a right adjoint to the functor

$$C_Y \xrightarrow{f^*} (\mathcal{F}/Y) - mod, \quad L \longmapsto (F(L),\mu(L)),\ (L \xrightarrow{g} N) \longmapsto (f^*(L) \xrightarrow{F(g)} f^*(L')).$$

In other words, we have a canonical continuous morphism $\mathbf{Sp}(\mathcal{F}/Y) \xrightarrow{f} Y$.

5.1.1. Example. Let R, S be unital associative rings. Any unital ring morphism $\varphi : S \longrightarrow R$ defines a monad, $R_\varphi^\sim = (R_\varphi,\mu_\varphi)$, on $Y = \mathbf{Sp}(S)$. Here the functor R_φ is $M \longmapsto R\otimes_S M$, and the multiplication is induced by the multiplication on R. The canonical morphism $\mathbf{Sp}(R/\mathbf{Sp}(S)) \longrightarrow \mathbf{Sp}(S)$ has the pull-back functor $\phi_* : R - mod \longrightarrow S - mod$ as a direct image functor. Notice that the category $(R_\varphi^\sim/\mathbf{Sp}(S))$-modules is isomorphic to the category $R - mod$ of R-modules; in particular, $\mathbf{Sp}(R_\varphi^\sim/\mathbf{Sp}(S)) \simeq \mathbf{Sp}(R)$.

If $S = \mathbb{Z}$, i.e. $C_Y = \mathbb{Z}-mod$, the category $(R/\mathbf{Sp}\mathbb{Z})-mod$ coincides with the category $R-mod$ of left R-modules. In this case, consistantly with our previous notations, we write $\mathbf{Sp}(R)$ instead of $\mathbf{Sp}(R/\mathbf{Sp}\mathbb{Z})$.

5.1.2. Example. Any monoid morphism $\mathcal{M} \xrightarrow{\phi} cN$ defines a monad, $\mathcal{F} = (F, \mu)$, on $Y = \mathbf{Sp}(\mathcal{M}/\mathcal{E})$, where the functor F is $\mathcal{M} \boxtimes_{\mathcal{N}} -$. It maps any left $\mathcal{M}$-set (L, ξ) to the cokernel of the pair of morphisms $\mathcal{N} \times \mathcal{M} \times L \overset{\longrightarrow}{\longrightarrow} \mathcal{N} \times L$, where one arrow is $\mathcal{N} \times \xi$ and another is the composition of $\mathcal{N} \times \mathcal{M} \times L \xrightarrow{\mathcal{N}\times\phi\times L} \mathcal{N} \times \mathcal{N} \times L \xrightarrow{\nu\times L} \mathcal{N} \times L$. Here $\mathcal{N} \times \mathcal{N} \xrightarrow{\nu} \mathcal{N}$ is the multiplication on $\mathcal{N}$. The multiplication $F^2 \xrightarrow{\mu} F$ is induced by the multiplication on $\mathcal{N}$. The canonical morphism $\mathbf{Sp}(\mathcal{F}/\mathbf{Sp}(\mathcal{M}/\mathcal{E})) \longrightarrow \mathbf{Sp}(\mathcal{M}/\mathcal{E})$ has the pull-back functor $\mathcal{N} - sets \xrightarrow{\phi_*} \mathcal{M} - sets$ as a direct image functor.

5.1.3. Example: localizations of modules. Let R be an associative unital ring and $\mathfrak{F}$ a set of left ideals in R. Denote by $R-mod_{\mathfrak{F}}$ the full subcategory of $R-mod$ whose objects are R-modules M such that the canonical morphism

$$M \longrightarrow Hom_R(m, M), \quad z \longmapsto (r \mapsto r \cdot z) \text{ for all } r \in m \text{ and } z \in M, \tag{1}$$

is an isomorphism for all $m \in \mathfrak{F}$. The inclusion functor

$$R - mod_{\mathfrak{F}} \xrightarrow{\mathfrak{j}_{\mathfrak{F}*}} R - mod \tag{2}$$

preserves limits, hence it has a left adjoint, $\mathfrak{j}_{\mathfrak{F}}^*$. Since $\mathfrak{j}_{\mathfrak{F}*}$ is fully faithful, $\mathfrak{j}_{\mathfrak{F}}^*$ is a localization. The R-module $R_{\mathfrak{F}} = \mathfrak{j}_{\mathfrak{F}*}\mathfrak{j}_{\mathfrak{F}}^*(R)$ has a structure of a ring uniquely determined by the fact that the adjunction arrow, $R \xrightarrow{\eta_{\mathfrak{F}}} R_{\mathfrak{F}}$ is a ring morphism. There is a canonical functor morphism

$$R_{\mathfrak{F}} \otimes_R - \xrightarrow{\tau_{\mathfrak{F}}} \mathfrak{j}_{\mathfrak{F}*}\mathfrak{j}_{\mathfrak{F}}^*. \tag{3}$$

Suppose all ideals in $\mathfrak{F}$ are projective modules. Then the inclusion functor (2) is exact. This implies that the morphism

$$R_{\mathfrak{F}} \otimes_R M \xrightarrow{\tau_{\mathfrak{F}}(M)} \mathfrak{j}_{\mathfrak{F}*}\mathfrak{j}_{\mathfrak{F}}^*(M)$$

is an isomorphism for any module M of finite type.

If $\mathfrak{F}$ consists of projective ideals of finite type, then (2) is strictly exact, i.e. it has a right adjoint. In this case, the localization $R-mod \longrightarrow R-mod_{\mathfrak{F}}$ is an affine morphism, or, equivalently, the functor morphism (3) is an isomorphism. Thus the category $R-mod_{\mathfrak{F}}$ is equivalent to the category $R_{\mathfrak{F}} - mod$ of left $R_{\mathfrak{F}}$-modules.

5.1.3.1. Note. In general, the localization $\mathfrak{j}_{\mathfrak{F}}$ is not flat, i.e. the functor $\mathfrak{j}_{\mathfrak{F}}^*$ is not exact. Denote by $\mathfrak{F}^-$ the set of all left ideals of the ring R such that the canonical morphism $M \longrightarrow Hom_R(m, M)$ is an isomorphism for all $M \in ObR - mod_{\mathfrak{F}}$. Clearly $R - mod_{\mathfrak{F}^-} = R - mod_{\mathfrak{F}}$. It follows from results of Gabriel (cf. [Gab], or [BD, Ch. 6]) that the localization $\mathfrak{j}_{\mathfrak{F}}$ is flat iff $\mathfrak{F}^-$ is a *radical filter*; i.e. with any left ideal m, the set

$\mathfrak{F}^-$ contains left ideals $(m:r) = \{a \in R \mid ar \in m\}$ for all $r \in R$ and all left ideals n in R such that $(n:r) \in \mathfrak{F}^-$ for all $r \in m$. These conditions are equivalent to that the full subcategory $T_{\mathfrak{F}^-}$ of $R - mod$ whose objects are all R-modules M such that every element of M is annihilated by some ideal $m \in \mathfrak{F}^-$, is a Serre subcategory.

5.1.4. Curves. Let R be a ring of the homological dimension one, or, equivalently, every left ideal in R is projective. Then for any set of left ideals $\mathfrak{F}$, the inclusion functor (2) is exact. If, in addition, R is left noetherian, the functor (2) is strictly exact.

5.2. Morphisms of monads and morphisms of their categoric spectra. Let Y be an object of $|Cat|^o$ and $\mathcal{F}$, $\mathcal{F}'$ monads on Y. Any monad morphism $\mathcal{F} \xrightarrow{\varphi} \mathcal{F}'$ induces the 'pull-back' functor

$$(\mathcal{F}'/Y) - mod \xrightarrow{\varphi_*} (\mathcal{F}/Y) - mod, \quad (M, \xi) \longmapsto (M, \xi \circ \varphi(M)).$$

This correspondence defines a functor $\mathfrak{Mon}_Y^{op} \longrightarrow Cat/C_Y$ which takes values in the full subcategory of Cat/C_Y objects of which are functors $C_Z \longrightarrow C_Y$ having a left adjoint.

5.2.1. Reflexive pairs of arrows and weakly continuous functors and monads. Recall that a pair of arrows $M \overset{g_1}{\underset{g_2}{\rightrightarrows}} L$ in C_Y is called *reflexive*, if there exists a morphism $L \xrightarrow{h} M$ such that $g_1 \circ h = id_M = g_2 \circ h$.

We call a functor $C_Y \longrightarrow C_Z$ *weakly continous* if it preserves cokernels of reflexive pairs of arrows.

We call a monad $\mathcal{F} = (F, \mu)$ on Y *weakly continuous* if the functor $C_Y \xrightarrow{F} C_Y$ is weakly continuous. We denote by $\mathfrak{Mon}_Y^w$ the full subcategory of the category $\mathfrak{Mon}_Y$ whose objects are weakly continuous monads on Y.

5.2.2. Lemma. *Let $\mathcal{F}$, $\mathcal{F}'$ be monads on Y and φ a monad morphism $\mathcal{F} \longrightarrow \mathcal{F}'$. Suppose the category C_Y has cokernels of reflexive pairs of morphisms and the monad $\mathcal{F}'$ is weakly continuous. Then the functor φ_* has a left adjoint.*

In particular, the map $(\mathcal{F}/Y) \longmapsto \mathbf{Sp}(\mathcal{F}/Y)$, $\varphi \longmapsto [\varphi^*]$ is a functor,

$$\mathbf{Sp}_Y : \mathfrak{Mon}_Y^w \longrightarrow |Cat|^o, \tag{1}$$

which takes values in the subcategory $|Cat|^o_{cont}$ of $|Cat|^o$ formed by continuous morphisms.

Proof. The left adjoint, $(\mathcal{F}/Y) - mod \xrightarrow{\varphi^*} (\mathcal{F}'/Y) - mod$ assigns to each $(\mathcal{F}/Y)$-module $(M, F(M) \xrightarrow{\xi} M)$ the cokernel of the pair of arrows

$$F'F(M) \overset{\mu' \circ F'\varphi}{\underset{F'\xi}{\rightrightarrows}} F'(M). \tag{1}$$

Since by hypothesis F' preserves cokernels of reflexive pairs and both arrows (1) are $\mathcal{F}'$-module morphisms, there exists a unique $\mathcal{F}'$-module structure on the cokernel of (1). Details are left to the reader. ■

5.2.3. Note. Suppose that the category C_X has colimits of certain type $\mathfrak{D}$, and let $\mathcal{F} = (F, \mu)$ be a monad on X such that the functor F preserves colimits of this type. Then the category $(\mathcal{F}/X) - mod$ has colimits of this type.

In fact, for a diagram $\mathfrak{D} \xrightarrow{\mathcal{D}} (\mathcal{F}/X) - mod$, the colimit of the composition $f_* \circ \mathcal{D}$ (where f_* is the forgetful functor $(\mathcal{F}/X) - mod \longrightarrow C_X$) has a unique $\mathcal{F}$-module structure, $\xi_{\mathcal{D}}$. The $\mathcal{F}$-module $(colim(f_* \circ \mathcal{D}), \xi_{\mathcal{D}})$ is a colimit of the diagram $\mathcal{D}$.

In particular, if $\mathcal{F} = (F, \mu)$ is a weakly continuous monad on X, and the category C_X has cokernels of reflexive pairs of arrows, then the category $(\mathcal{F}/X) - mod$ has cokernels of reflexive pairs of arrows.

5.3. Comonads and their cospectrum. A *comonad on* Y is a monad on the dual object ('space') Y^o defined by $C_{Y^o} = C_Y^{op}$. In other words, a comonad on Y is a pair (G, δ), where G is a functor $C_Y \longrightarrow C_Y$ and δ a functor morphism $G \longrightarrow G^2$ (a comultiplication) such that $G\delta \circ \delta = \delta G \circ \delta$ and $G\epsilon \circ \delta = id_G = \epsilon G \circ \delta$ for a uniquely determined morphism $G \xrightarrow{\epsilon} Id_{C_Y}$ (a counit).

We denote the category of comonads on Y by $\mathfrak{CMon}_Y$. It is defined by the formula $\mathfrak{CMon}_Y = \mathfrak{Mon}_{Y^o}$.

Comodules over a comonad $\mathcal{G} = (G, \delta)$ are just modules over the dual monad on Y^o. In terms of Y, a $\mathcal{G}$-comodule is a pair (M, ξ), where $M \in ObC_Y$ and ξ a morphism $M \longrightarrow G(M)$ such that $\delta(M) \circ \xi = G\xi \circ \xi$ and $\epsilon(M) \circ \xi = id_M$. We denote the category of comodules over $\mathcal{G}$ by $(Y \backslash \mathcal{G}) - Comod$, or simply by $\mathcal{G} - Comod$.

We denote by $\mathbf{Sp^o}(Y \backslash \mathcal{G})$ the object of $|Cat|^o$ (or Cat^{op}) such that the corresponding category is $(Y \backslash \mathcal{G}) - Comod$. This definition can be rephrased as follows:

$$\mathbf{Sp^o}(Y \backslash \mathcal{G}) = \mathbf{Sp}(\mathcal{G}^o / Y^o)^o. \tag{1}$$

Here $\mathcal{G}^o$ is the monad (G^o, δ^o) on Y^o dual to the comonad $\mathcal{G}$.

We call $\mathbf{Sp^o}(Y \backslash \mathcal{G})$ the *cospectrum of the comonad* $\mathcal{G}$ *in* $|Cat|^o$.

By duality, there is a canonical continuous morphism $Y \xrightarrow{g} \mathbf{Sp^o}(Y \backslash \mathcal{G})$ with an inverse image functor

$$(Y \backslash \mathcal{G}) - Comod \xrightarrow{g^*} C_Y, \quad (M, \xi) \longmapsto M, \tag{2}$$

and having a direct image functor

$$C_Y \xrightarrow{g_*} (Y \backslash \mathcal{G}) - Comod, \quad L \longmapsto (G(L), \delta(L)). \tag{3}$$

5.3.1. Example. Let R be an associative unital ring and $\mathcal{H} = (H, \delta)$ a coalgebra in the monoidal category of R-bimodules. This means that H is an R-bimodule, δ an R-bimodule morphism $H \longrightarrow H \otimes_R H$ such that $\delta \otimes_R id_H \circ \delta = id_H \otimes_R \delta \circ \delta$, and there exists a (necessarily unique) R-bimodule morphism $\epsilon : H \otimes_R H \longrightarrow R$ such that $\lambda_r(H) \circ \epsilon \otimes_R id_H \circ \delta = id_H = \lambda_l(H) \circ id_H \otimes_R \epsilon \circ \delta$. Here $\lambda_l(H) : R \otimes_R H \longrightarrow H$ and $\lambda_r(H) : H \otimes_R \longrightarrow H$ are canonical isomorphisms. The coalgebra $\mathcal{H}$ induces a comonad on the category $R - mod$ of left R-modules tensoring by H over R, $L \longmapsto H \otimes_R L$, as a functor and with comultiplication $H \otimes_R - \longrightarrow H \otimes_R H \otimes_R -$ induced by the comultiplication δ.

The canonical morphism $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp^o}(\mathbf{Sp}(R)\backslash\mathcal{H})$ has the forgetful functor $(\mathbf{Sp}(R)\backslash\mathcal{H}) - Comod \longrightarrow R - mod$ as an inverse image functor.

5.3.2. Functoriality of the cospectrum. Let $\mathcal{G} = (G, \delta)$ and $\mathcal{G}' = (G', \delta')$ be comonads on Y and ψ a comonad morphism $\mathcal{G} \longrightarrow \mathcal{G}'$; i..e. ψ is a morphism of functors $G \longrightarrow G'$ such that $\delta' \circ \psi = \psi \odot \psi \circ \delta$ and $\epsilon' \circ \psi = \epsilon$. Here $\psi \odot \psi = G'\psi \circ \psi G$, and ϵ, ϵ' are counits of the comonads resp. $\mathcal{G}$ and $\mathcal{G}'$. The morphism ψ induces the 'pull-back' functor

$$(Y\backslash\mathcal{G}) - Comod \xrightarrow{\psi^*} (Y\backslash\mathcal{G}') - Comod, \quad (M, \xi) \longmapsto (M, \psi(M) \circ \xi), \tag{4}$$

which is regarded as an inverse image functor of a morphism

$$\mathbf{Sp^o}(\psi) : \mathbf{Sp^o}(Y\backslash\mathcal{G}') \longrightarrow \mathbf{Sp^o}(Y\backslash\mathcal{G}) \tag{5}$$

The map (4) defines a functor

$$\widetilde{\mathbf{Sp}}^o_Y : \mathfrak{CMon}_Y^{op} \longrightarrow (C_Y\backslash Cat^{op})_{\mathfrak{c}}, \tag{6}$$

where $(C_Y\backslash Cat^{op})_{\mathfrak{c}}$ denotes the full subcategory of the category $C_Y\backslash Cat^{op}$ whose objects are continuous morphisms.

5.3.2.1. Proposition. *The functor (6) is fully faithful and has a right adjoint.*

Proof. Let

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ f \nwarrow & & \nearrow g \\ & Y & \end{array}$$

be a morphism in $(C_Y\backslash Cat^{op})_{\mathfrak{c}}$ given by the commutative diagram

$$\begin{array}{ccc} C_Z & \xrightarrow{h^*} & C_X \\ f^* \searrow & & \swarrow g^* \\ & C_Y & \end{array}$$

of functors. Fix direct image functors of f and g and the corresponding adjunction arrows. Set

$$\varphi_h = f^*(f_*\epsilon_g \circ \eta_f h^* g_*) : g^*g_* \longrightarrow f^*f_*. \tag{7}$$

One can check that ϕ_h is a monad morphism $\mathcal{G}_g \longrightarrow \mathcal{G}_f$ and the map $\tilde{\Gamma}_Y : h \longmapsto \varphi_h$ is functorial. The composition $\tilde{\Gamma}_Y \circ \widetilde{\mathbf{Sp}}^o_Y$ is the identical functor which provides one of the adjunction arrows and shows that the functor $\widetilde{\mathbf{Sp}}^o_Y$ is fully faithful. We leave the other adjunction arrow to the reader (it is defined in 5.4). ■

Recall that pair of arrows $M \overset{g_1}{\underset{g_2}{\rightrightarrows}} L$ is called *coreflexive*, if there exists a morphism $L \xrightarrow{h} M$ such that $h \circ g_1 = id_M = h \circ g_2$.

5.3.2.2. Lemma. *Suppose that the category C_Y has kernels of coreflexive pairs of morphisms and the functor G preserves these kernels. Then the functor ψ^* has a right adjoint, i.e. the morphism (4) is continuous.*

Proof. The assertion is the dual version of 5.2.2. ■

5.4. Beck's theorem. Let $X \xrightarrow{f} Y$ be a continuous morphism in $|Cat|^o$ with inverse image functor f^*, direct image functor f_*, and adjunction morphisms

$$Id_{C_Y} \xrightarrow{\eta_f} f_*f^* \quad \text{and} \quad f^*f_* \xrightarrow{\epsilon_f} Id_{C_X}.$$

Let $\mathcal{G}_f$ denote the comonad (G_f, δ_f), where $G_f = f^*f_*$ and $\delta_f = f^*\eta_f f_*$. There is a commutative diagram

$$\begin{array}{ccc} C_Y & \xrightarrow{\tilde{f}^*} & (X\backslash\mathcal{G}_f) - Comod \\ & {\scriptstyle f^*}\searrow \quad \swarrow{\scriptstyle \breve{f}^*} & \\ & C_X & \end{array} \qquad (1^o)$$

Here $\tilde{f}^*$ is the canonical functor $C_Y \longrightarrow (X\backslash\mathcal{G}_f) - Comod$, $M \longmapsto (f^*(M), f^*\eta_f(M))$, and $\breve{f}^*$ is the forgetful functor $(X\backslash\mathcal{G}_f) - Comod \longrightarrow C_X$. The diagram (1^o) is regarded as the diagram of inverse image functors of the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\breve{f}} & \mathbf{Sp^o}(X\backslash\mathcal{G}_f) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \tilde{f}} & \\ & Y & \end{array} \qquad (2^o)$$

in $|Cat|^o$. The following statement is one of the versions of the Beck's theorem.

5.4.1. Theorem. *Let $X \xrightarrow{f} Y$ be a continuous morphism.*

(a) If the category C_X has kernels of coreflexive pairs of arrows, then the functor $\tilde{f}^*$ has a right adjoint, $\tilde{f}_*$, i.e. $\mathbf{Sp^o}(X\backslash\mathcal{G}_f) \xrightarrow{\tilde{f}} Y$ is a continuous morphism.

(b) If, in addition, f is weakly flat, i.e. the functor f^* preserves kernels of coreflexive pairs, then the adjunction arrow $\tilde{f}^*\tilde{f}_* \xrightarrow{\epsilon_{\tilde{f}}} Id_{(X\backslash\mathcal{G}_f)-Comod}$ is an isomorphism, i.e. $\hat{f}_*$ is a fully faithful functor, or, equivalently, $\tilde{f}^*$ is a localization.

(c) If, in addition to (a) and (b), f^* reflects isomorphisms, then the adjunction arrow $Id_{C_Y} \xrightarrow{\eta_{\tilde{f}}} \tilde{f}_*\tilde{f}^*$ is an isomorphism too, i.e. $\tilde{f}$ is an isomorphism.

Proof. See [MLM], IV.4.2, or [ML], VI.7. ■

We need also the dual version of the theorem 5.4.1. Let $\mathcal{F}_f$ denote the monad (F_f, μ_f), where $F_f = f_*f^*$ and $\mu_f = f_*\epsilon_f f^*$. There is a commutative diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\bar{f}_*} & (\mathcal{F}_f/Y) - mod \\ & {f_*}\searrow \quad \swarrow{\hat{f}_*} & \\ & C_Y & \end{array} \qquad (1)$$

Here $\bar{f}_*$ is the canonical functor

$$C_X \longrightarrow (\mathcal{F}_f/Y) - mod, \quad M \longmapsto (f_*(M), f_*\epsilon_f(M)),$$

$\hat{f}_*$ the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$.

5.4.2. Theorem. *Let $X \xrightarrow{f} Y$ be a continuous morphism.*

(a) If the category C_Y has cokernels of reflexive pairs of arrows, then the functor $\bar{f}_*$ has a left adjoint, $\bar{f}^*$; hence $\bar{f}_*$ is a direct image functor of a continuous morphism $\bar{X} \xrightarrow{f} \mathbf{Sp}(\mathcal{F}_f/Y)$.

(b) If, in addition, the functor f_* preserves cokernels of reflexive pairs, then the adjunction arrow $\bar{f}^*\bar{f}_* \longrightarrow Id_{C_X}$ is an isomorphism, i.e. $\bar{f}_*$ is a localization.

(c) If, in addition to (a) and (b), the functor f_* is conservative, then $\bar{f}_*$ is a category equivalence.

Proof. The theorem is dual (hence equivalent) to the theorem 5.4.1. ∎

If the condition (a) in 5.4.2 holds, then to the diagram (1), there corresponds a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & \mathbf{Sp}(\mathcal{F}_f/Y) \\ {\scriptstyle f}\searrow & & \swarrow{\scriptstyle \hat{f}} \\ & Y & \end{array} \qquad (2)$$

in $|Cat|^o$. If the condition (c) in 5.4 holds, the morphism $\bar{f}$ in (2) is an isomorphism.

Thus, given a continuous morphism $X \xrightarrow{f} Y$ such that the category C_Y has cokernels of reflexive pairs of arrows, we have a commutative diagram

$$\begin{array}{ccccc} & & X & \xrightarrow{\bar{f}} & \mathbf{Sp}(\mathcal{F}_f/Y) \\ & {\scriptstyle \check{f}}\swarrow & & \searrow^{f} \quad \swarrow{\scriptstyle \hat{f}} & \\ \mathbf{Sp^o}(X\backslash\mathcal{G}) & & \xrightarrow{\tilde{f}} & Y & \end{array} \qquad (3)$$

Notice that the diagrams (1) and (1^o) are uniquely defined by the data $(f^*, f_*, \epsilon_f, \eta_f)$, since the monad $\mathcal{F}_f$, the comonad $\mathcal{G}_f$, and the functors $\bar{f}_* : C_X \longrightarrow (F_f, Y) - mod$ in (1) and $\tilde{f}^* : C_Y \longrightarrow (X\backslash\mathcal{G}_f) - Comod$ are defined in terms of this data. Given the functor f^* (resp. f_*), the rest of the data, f_*, ϵ_f, η_f (resp. f^*, ϵ_f, η_f), is determined uniquely up to isomorphism. Thus, the monad $\mathcal{F}_f$ and the comonad $\mathcal{G}_f$ in the diagrams (1) and (1^o) are determined by f^* uniquely up to isomorphism.

5.5. Weakly flat and weakly affine morphisms. We call a functor *weakly continuous* (resp. *weakly flat*) if it preserves cokernels of reflexive pairs of arrows (resp. kernels of coreflexive pairs of arrows).

We call a monad (F, μ) *weakly continuous* if the functor F is weakly continuous, and a comonad (G, δ) *weakly flat* if the functor G is weakly flat.

Let $\mathfrak{Mon}^w_X$ denote the full subcategory of the category $\mathfrak{Mon}_X$ of monads on X spanned by weakly continuous monads, and let $\mathfrak{CMon}^w_X$ be the full subcategory of the category $\mathfrak{CMon}_X$ of comonads on X spanned by weakly flat comonads on X.

We call a continuous morphism $X \xrightarrow{f} Y$ *weakly affine* if its direct image functor is weakly continuous and the category C_X has cokernels of reflexive pairs of arrows. Let Aff_Y^w denote the full subcategory of the category $|Cat|^o/Y$ whose objects are weakly affine morphisms to Y.

We call a continuous morphism $X \xrightarrow{f} Y$ in *weakly flat* if its inverse image functor is weakly flat and the category C_Y has kernels of coreflexive pairs of arrows. We denote by $Flat_X^w$ the full subcategory of the category $X\backslash|Cat|^o$ whose objects are weakly flat morphisms from X.

5.5.1. Proposition. *(a) Suppose the category C_Y has cokernels of reflexive pairs of arrows. The map $(\mathcal{F}/Y) \longmapsto (\mathbf{Sp}(\mathcal{F}/Y) \to Y)$ defines a full functor*

$$\mathbf{Sp}_Y^w : (\mathfrak{Mon}_Y^w)^{op} \longrightarrow Aff_Y^w.$$

(b) Dually, if the category C_Y has kernels of coreflexive pairs of arrows, then the map $(X\backslash\mathcal{G}) \longmapsto (X \to \mathbf{Sp^o}(X\backslash\mathcal{G}))$ defines a full functor

$$\mathbf{Sp^o}_X^w : (\mathfrak{CMon}_X^w)^{op} \longrightarrow Flat_X^w.$$

Proof. These facts follow from Beck's Theorem and the following

5.5.2. Lemma. *Let $X \xrightarrow{f} Y$ be a continuous morphism with a direct image functor f_* and an inverse image functor f^*.*

(a) Suppose the morphism f is monadic and the category C_Y has colimits of a type $\mathfrak{S}$. Then f_* preserves colimits of the type $\mathfrak{S}$ iff the functor $F_f = f_*f^*$ has this property.

(b) Dually, if the morphism f is comonadic and the category C_X has limits of a certain type, then f^* preserves these limits iff the functor $G_f = f^*f_*$ does the same.

Recall that a continuous morphism $X \xrightarrow{f} Y$ is called *comonadic* if the induced morphism $\tilde{f} : \mathbf{Sp^o}(X\backslash\mathcal{G}_f) \longrightarrow Y$ is an isomorphism.

Dually, a continuous morphisms $X \xrightarrow{f} Y$ (in $|Cat|^o$, or in Cat^{op}) is called *monadic* if the associated morphism $\bar{X} \xrightarrow{f} \mathbf{Sp}(\mathcal{F}_f/Y)$ is an isomorphism.

The proof of the lemma and details of the proof of 5.5.1 are left to the reader. ■

5.5.3. Proposition. *Let*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & f \searrow \quad \swarrow g & \\ & Z & \end{array}$$

be a commutative diagram in $|Cat|^o$. Suppose C_Z has colimits of reflexive pairs of arrows. If f and g are weakly affine, then h is weakly affine.

Dually, if

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & f \nwarrow \quad \nearrow g & \\ & Z & \end{array}$$

is a commutative diagram in $|Cat|^o$ such that C_Z has kernels of coreflexive pairs of arrows and the morphisms f and g are weakly flat, then h is weakly flat.

Proof. Fix inverse and direct image functors of f and g together with adjunction morphisms. By hypothesis, the canonical morphisms $C_X \longrightarrow (\mathcal{F}_f/Z) - mod$ and $C_Y \longrightarrow (\mathcal{F}_g/Z) - mod$ are category equivalences. Here $\mathcal{F}_f = (f_* f^*, \mu_f)$ and $\mathcal{F}_g = (g_* g^*, \mu_g)$ are monads associated with resp. f and g. It follows from the dual version of 5.3.2.1 (or the argument of 10.7.1 in [R4]) that a choice of an inverse image functor h^* of the morphism h determines a monad morphism $\mathcal{F}_g \xrightarrow{\phi_h} \mathcal{F}_f$ such that the diagram

$$\begin{array}{ccc} C_Y & \xrightarrow{\sim} & (\mathcal{F}_g/Z) - mod \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle \phi_h^*} \\ C_X & \xrightarrow{\sim} & (\mathcal{F}_f/Z) - mod \end{array}$$

quasi-commutes. Here ϕ_h^* is the inverse image functor associated with the monad morphism ϕ_h (a left adjoint to the pull-back functor). The pull-back functor, ϕ_{h_*}, is, evidently, conservative and weakly continuous. The latter follows from the fact that the monads $\mathcal{F}_f$ and $\mathcal{F}_g$ are weakly continuous. ■

6. Continuous monads and affine morphisms. Duality. A functor F is called *continuous* if it has a right adjoint. A monad $\mathcal{F} = (F, \mu)$ on Y (i.e. on the category C_Y) is called *continuous* if the functor F is continuous.

Dually, a comonad $\mathcal{G} = (G, \delta)$ on Y is called *cocontinuous* if the functor G has a left adjoint. In other words, a cocontinuous comonad on Y is the same as a continuous monad on Y^o and vice versa.

It follows that a continuous monad is weakly continuous, because the the functor F preserves all colimits. Dually, a cocontinuous monad is weakly flat.

6.1. Duality. Let $\mathcal{F} = (F, \mu)$ be a continuous monad on Y; i.e. the functor F has a right adjoint, $F^\wedge$. The multiplication $F^2 \xrightarrow{\mu} F$ induces a morphism $F^\wedge \xrightarrow{\delta} (F^\wedge)^2$ which is a comonad structure on $F^\wedge$ with the counit $F^\wedge \xrightarrow{\epsilon} Id_{C_Y}$ induced by the unit $Id_{C_Y} \xrightarrow{\eta} F$ of the monad $\mathcal{F}$. Thus, we have a comonad, $\mathcal{F}^\wedge = (F^\wedge, \mu^\wedge)$ dual to the monad $\mathcal{F}$. The map which assigns to any morphism $F(L) \longrightarrow L$, $L \in ObC_Y$, the dual morphism $L \longrightarrow F^\wedge(L)$ induces an isomorphism of categories

$$\Phi : (\mathcal{F}/Y) - mod \xrightarrow{\sim} (Y \backslash \mathcal{F}^\wedge) - Comod \qquad (1)$$

such that the diagram

$$\begin{array}{ccc} (\mathcal{F}/Y) - mod & \xrightarrow{\Phi} & (Y \backslash \mathcal{F}^\wedge) - Comod \\ & {\scriptstyle \hat{f}_*} \searrow \quad \swarrow {\scriptstyle \breve{f}^*} & \\ & C_Y & \end{array} \qquad (2)$$

commutes. Here $\breve{f}^*$ denotes the functor forgetting $\mathcal{F}^\wedge$-comodule structure.

It follows from the construction that $\mathcal{F}^\wedge$ is a cocontinuous comonad on Y determined by the monad $\mathcal{F}$ uniquely up to isomorphism.

Conversely, to any cocontinuous comonad, $\mathcal{G} = (G, \delta)$, on Y, there corresponds a continuous monad $\mathcal{G}^\vee = (G^\vee, \delta^\vee)$, where $G^\vee$ is a left adjoint to G. The monad $\mathcal{G}^\vee$ is determined by $\mathcal{G}$ uniquely up to isomorphism, and we have a comonad and monad isomorphisms, respectively

$$\mathcal{G} \xrightarrow{\sim} (\mathcal{G}^\vee)^\wedge \quad \text{and} \quad \mathcal{F} \xrightarrow{\sim} (\mathcal{F}^\wedge)^\vee.$$

6.2. Proposition. *A monad $\mathcal{F} = (F, \mu)$ on Y is continuous iff the canonical morphism $\mathbf{Sp}(\mathcal{F}/Y) \xrightarrow{\hat{f}} Y$ is affine. Dually, a comonad $\mathcal{G} = (G, \delta)$ on Y is cocontinuous iff the canonical morphism $Y \longrightarrow \mathbf{Sp^o}(Y\backslash\mathcal{G})$ is coaffine.*

Proof. A canonical direct image functor of $\hat{f}$ is the forgetful functor

$$(\mathcal{F}/Y) - mod \xrightarrow{\hat{f}_*} C_Y, \ (M, \xi) \longmapsto M.$$

Since the functor $\hat{f}_*$ is, evidently, conservative, the morphism $\hat{f}$ is affine iff $\hat{f}_*$ has a right adjoint.

(a) If $\hat{f}^!$ is a right adjoint to $\hat{f}_*$, then the functor $F^\wedge = \hat{f}_*\hat{f}^!$ is a right adjoint to $F = \hat{f}_*\hat{f}^*$. Here $\hat{f}^*$ denotes the functor $L \longmapsto (F(L), \mu(L))$.

(b) Conversely, suppose $\mathcal{F} = (F, \mu)$ is a continuous monad on Y; i.e. the functor F has a right adjoint, $F^\wedge$. The functor $\breve{f}^*$ in the diagram (2) has a right adjoint, $\breve{f}_*$, which maps every object M of C_Y to the $(Y\backslash\mathcal{F}^\wedge)$-comodule $(\mathcal{F}^\wedge(M), M \xrightarrow{\delta(M)} (F^\wedge)^2(M))$. It follows from the commutativity of (2) that the functor $\hat{f}^! = \Phi^{-1} \circ \breve{f}_* : C_Y \longrightarrow \mathcal{F} - mod$ is a right adjoint to the forgetful functor $\mathcal{F} - mod \xrightarrow{\hat{f}_*} C_Y$. Since $\hat{f}_*$ is, obviously, conservative, it is a direct image functor of an affine morphism $\mathbf{Sp}(\mathcal{F}/Y) \longrightarrow Y$. ∎

6.2.1. Corollary. *Suppose that the category C_Y has cokernels of reflexive pairs of arrows. A continuous morphism $X \xrightarrow{f} Y$ in $|Cat|^o$ is affine iff its direct image functor $C_X \xrightarrow{f_*} C_Y$ is the composition of a category equivalence*

$$C_X \longrightarrow (\mathcal{F}_f/Y) - mod$$

for a continuous monad $\mathcal{F}_f$ in C_Y and the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$. The monad $\mathcal{F}_f$ is determined by f uniquely up to isomorphism.

Proof. The conditions of the Beck's theorem are fullfiled if f is affine, hence f_* is the composition of an equivalence $C_X \longrightarrow (\mathcal{F}_f/Y) - mod$ for a monad $\mathcal{F}_f = (f_*f^*, \mu_f)$ in C_Y and the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$ (see (1)). The functor $F_f = f_*f^*$ has a right adjoint $f_*f^!$, where $f^!$ is a right adjoint to f_*. The rest follows from 6.2. ∎

6.3. Proposition. *Suppose X is an object of $|Cat|^o$ such that the category C_X has kernels of reflexive pairs of arrows. Let $\mathcal{F}_f = (F_f, \mu_f)$ and $\mathcal{F}_g = (F_g, \mu_g)$ be continuous monads on X. Then for any monad morphism $\mathcal{F}_f \xrightarrow{\varphi} \mathcal{F}_g$, the corresponding morphism*

$$\mathbf{Sp}(\varphi) : \mathbf{Sp}(\mathcal{F}_f/X) \longrightarrow \mathbf{Sp}(\mathcal{F}_g/X)$$

is affine.

Proof. The morphism φ induces a dual comonad morphism $\mathcal{F}_g^\wedge \xrightarrow{\hat{\varphi}} \mathcal{F}_f^\wedge$ such that the diagram

$$\begin{array}{ccc} (\mathcal{F}_g/X) - mod & \xrightarrow{\varphi_*} & (\mathcal{F}_f/X) - mod \\ \Phi_{\mathcal{F}_g} \downarrow & & \downarrow \Phi_{\mathcal{F}_f} \\ (X\backslash\mathcal{F}_g^\wedge) - Comod & \xrightarrow{\hat{\varphi}^*} & (X\backslash\mathcal{F}_f^\wedge) - Comod \end{array} \qquad (3)$$

commutes. Here $\Phi_{\mathcal{F}_f}$ and $\Phi_{\mathcal{F}_g}$ are the canonical category isomorphisms (cf. 6.1). Since the category C_X has kernels of coreflexive pairs of arrows and the functor $F_g^\wedge$ preserves limits (in particular, it preserves kernels of pairs of arrows), the functor $\hat{\varphi}^*$ has a right adjoint (cf. 5.3.2.2), hence φ_* has a right adjoint. Since the functor φ_* is conservative, the morphism $\mathbf{Sp}(\varphi)$ is affine. ■

6.4. Proposition. *Let*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ f \searrow & & \swarrow g \\ & Z & \end{array}$$

be a commutative diagram in $|Cat|^o$. Suppose the category C_Z has cokernels of coreflexive pairs of arrows. If f and g are affine, then h is affine.

Proof. Fix inverse and direct image functors of f and g together with adjunction morphisms. By the Beck's theorem, the canonical morphisms $C_X \longrightarrow (\mathcal{F}_f/Z) - mod$ and $C_Y \longrightarrow (\mathcal{F}_g/Z) - mod$ are category equivalences. Here $\mathcal{F}_f = (f_*f^*, \mu_f)$ and $\mathcal{F}_g = (g_*g^*, \mu_g)$ are monads associated with resp. f and g. By 5.5.3 (or 10.7.1 in [R4]), a choice of an inverse image functor h^* of the morphism h determines a monad morphism $\mathcal{F}_g \xrightarrow{\phi_h} \mathcal{F}_f$ such that the diagram

$$\begin{array}{ccc} C_Y & \xrightarrow{\sim} & (\mathcal{F}_g/Z) - mod \\ h^* \downarrow & & \downarrow \phi_h^* \\ C_X & \xrightarrow{\sim} & (\mathcal{F}_f/Z) - mod \end{array}$$

quasi-commutes. By 6.3, since the monads $\mathcal{F}_g$ and $\mathcal{F}_f$ are continuous, the direct image functor ϕ_{h*} (the pull-back by the morphism ϕ_h) has a right adjoint, $\phi_h^!$. ■

For $Z \in Ob|Cat|^o$, denote by Aff_Z the full subcategory of $|Cat|^o/Z$ whose objects are affine morphisms. Let $|Cat|^o_{aff}$ be the subcategory of $|Cat|^o$ formed by affine morphisms.

6.4.1. Proposition. *Suppose that the category C_Z has cokernels of reflexive pairs of arrows. Then the natural embedding $|Cat|^o_{aff}/Z \longrightarrow Aff_Z$ is an isomorphism of categories.*

Proof. The assertion is a corollary of 6.4. ■

6.5. Proposition. *Let $X \xrightarrow{f} Y$ be an affine morphism in $|Cat|^o$. If the category C_Y is additive (resp. abelian, resp. abelian with small coproducts, resp. a Grothendieck category), then the category C_X has the same property.*

Proof. By 6.2, the category C_X is equivalent to the category $(\mathcal{F}_f/Y)-mod$ of (F_f/Y)-modules for a continuous monad $\mathcal{F} = (F_f, \mu_f)$ on Y. Since the functor F_f has a right adjoint and the category C_Y is additive, F_f is additive and preserves colimits of arbitrary small diagrams. This implies that for any diagram $D \xrightarrow{\mathfrak{D}} \mathcal{F}-mod$, the object $\mathrm{colim}(f_*\circ\mathfrak{D})$ (where f_* is the forgetful functor $(\mathcal{F}/Y)-mod \longrightarrow C_Y$) has a unique $(\mathcal{F}_f/Y)$-module structure $\xi_{\mathfrak{D}}$ such that all morphisms $f_*\mathfrak{D}(x) \longrightarrow \mathrm{colim}(f_*\circ\mathfrak{D})$ are $(\mathcal{F}/Y)$-module morphisms $\mathfrak{D}(x) \longrightarrow (\mathrm{colim}(f_*\circ\mathfrak{D}), \xi_{\mathfrak{D}})$. This implies the assertion. Details are left to the reader. ∎

6.6. Affine morphisms to $\mathbf{Sp}(R)$.

6.6.1. Proposition. *Let R be an associative unital ring. A continuous morphism $X \xrightarrow{f} \mathbf{Sp}(R)$ in $|Cat|^o$ is affine iff its direct image functor, $C_X \xrightarrow{f_*} R-mod$, is the composition of an equivalence of categories $C_X \longrightarrow R_f-mod$ for an associative unital ring R_f and the pull-back functor $R_f-mod \xrightarrow{\phi_*} R-mod = C_Y$ for a ring morphism $R \xrightarrow{\phi} R_f$ determined by f uniquely up to isomorphism.*

Proof. (i) If $X \xrightarrow{f} Y$ is such a morphism in $|Cat|^o$ that $C_X \xrightarrow{f_*} C_Y$ is an equivalence of categories, then f is, obviously, affine. The morphism $S-mod \longrightarrow R-mod$ corresponding to a ring morphism $R \longrightarrow S$ is affine by 3.8. Finally, the composition of affine morphisms is affine.

(ii) Conversely, suppose that $X \xrightarrow{f} \mathbf{Sp}(R)$ is an affine morphism. Then the functor $f_*f^* : R-mod \longrightarrow R-mod$ has a right adjoint, hence it is isomorphic to the functor $R_f\otimes_R - : L \longmapsto R_f\otimes_R L$ for some R-bimodule R_f. The monad structure on f_*f^* induces an associative ring structure, $R_f\otimes_R R_f \xrightarrow{m_f} R_f$, on R_f; and the adjunction morphism $Id_{R-mod} \xrightarrow{\eta_f} f_*f^*$ corresponds to a ring morphism $R \xrightarrow{\phi} R_f$ so that the diagrams of functor morphisms

$$\begin{array}{ccc} Id_{R-mod} & \xrightarrow{\sim} & R\otimes_R - \\ \eta_f \downarrow & & \downarrow \phi\otimes_R \\ f_*f^* & \xrightarrow{\sim} & R_f\otimes_R - \end{array} \quad \text{and} \quad \begin{array}{ccc} (f_*f^*)^2 & \xrightarrow{\sim} & R_f\otimes_R R_f\otimes_R - \\ \mu_f \downarrow & & \downarrow m_f \\ f_*f^* & \xrightarrow{\sim} & R_f\otimes_R - \end{array} \qquad (2)$$

commute. Thus we have a commutative diagram

$$\begin{array}{ccc} (\mathcal{F}_f/\mathbf{Sp}(R))-mod & \xrightarrow{\sim} & R_f-mod \\ \hat{f}_* \searrow & & \swarrow \phi_* \\ & R-mod & \end{array} \qquad (3)$$

in which the horizontal arrow is an isomorphism of categories. Combining with the commutative diagram (1), we obtain a commutative diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\sim} & R_f-mod \\ f_* \searrow & & \swarrow \phi_* \\ & R-mod & \end{array} \qquad (4)$$

in which the horizontal arrow is an equivalence of categories.

Notice that $R_f = f_*f^*(R)$. Therefore, the ring morphism $R \xrightarrow{\phi} R_f$ is defined uniquely up to isomorphism by a choice of the functor f^*. ■

6.6.2. A comparison of two descriptions. Let $X \xrightarrow{f} \mathbf{Sp}(R)$ be an affine morphism. Being continuous, the morphism f is determined uniquely up to isomorphism by the object $\mathcal{O} = f^*(R)$, and a right R-module structure $R \longrightarrow \Gamma_X\mathcal{O} = C_X(\mathcal{O},\mathcal{O})^o$ (cf. 4.1). By 4.5, we have a commutative diagram of direct image functors of continuous morphisms

$$\begin{array}{ccc} C_X & \xrightarrow{f_{\mathcal{O}^*}} & \Gamma_X\mathcal{O} - mod \\ f_* \searrow & & \swarrow \bar{\phi}_{f*} \\ & R - mod & \end{array} \qquad (1)$$

Here $\bar{\phi}_{f*}$ is the pull-back by the ring morphism $R \xrightarrow{\phi_f} \Gamma_X\mathcal{O}$ defining a right R-module structure on $\mathcal{O}$. The morphism $f_{\mathcal{O}*}$ has an inverse image functor $f_{\mathcal{O}}^*$ which maps the left module $\Gamma_X\mathcal{O}$ to $\mathcal{O}$. The adjunction morphism $\Gamma_X\mathcal{O} \longrightarrow f_{\mathcal{O}*}f_{\mathcal{O}}^*(\Gamma_X\mathcal{O})$ is an isomorphism. Since morphisms f and $\mathbf{Sp}\Gamma_X\mathcal{O} \xrightarrow{\bar{\phi}_f} \mathbf{Sp}(R)$ are affine, the morphism $f_{\mathcal{O}}$ is affine too (cf. 6.3). In particular, $f_{\mathcal{O}*}f_{\mathcal{O}}^*$ has a right adjoint, hence it preserves colimits. Since $\Gamma_X\mathcal{O}$ is a generator of the category $\Gamma_X\mathcal{O} - mod$, the isomorphness of the adjunction arrow $\Gamma_X\mathcal{O} \longrightarrow f_{\mathcal{O}*}f_{\mathcal{O}}^*(\Gamma_X\mathcal{O})$ implies that $M \longrightarrow f_{\mathcal{O}*}f_{\mathcal{O}}^*(M)$ is an isomorphism for any $\Gamma_X\mathcal{O}$-module M. This means that the functor $f_{\mathcal{O}}^*$ is fully faithful, hence $f_{\mathcal{O}*}$ is a localization. Since by condition $f_{\mathcal{O}*}$ is conservative, it is a category equivalence.

This shows that C_X is naturally equivalent to the category of $\Gamma_X\mathcal{O}$-modules. Thus the ring morphism $R \longrightarrow R_f$ in 6.6.1 is isomorphic to the ring morphism $R \longrightarrow \Gamma_X\mathcal{O}$ defining a right R-module structure on the object $\mathcal{O}$.

This observation was made before (in [R], Ch.7) using a slightly different argument.

7. Flat descent.

7.1. Continuous, flat comonads. A comonad $\mathcal{G} = (G, \delta)$ on X is called
— *continuous* if the functor G has a right adjoint,
— *flat* if the functor G preserves finite limits,
— *weakly flat* if the functor G preserves kernels of coreflexive pairs of arrows,
— *conservative* if the functor G is conservative.

7.2. Proposition. *Let $X \xrightarrow{f} Y$ be a continuous morphism, and let C_X have kernels of coreflexive pairs of morphisms. The morphism $X \xrightarrow{f} Y$ in $|Cat|^o$ is affine, flat (resp. weakly flat), and conservative iff its inverse image functor $C_Y \xrightarrow{f^*} C_X$ is the composition of an equivalence $C_Y \longrightarrow (X\backslash\mathcal{G}_f) - Comod$ for a continuous flat (resp. weakly flat) conservative comonad $\mathcal{G}_f$ on X and the forgetful functor $(X\backslash\mathcal{G}_f) - Comod \longrightarrow C_X$. The monad $\mathcal{G}_f$ is determined by f uniquely up to isomorphism.*

Proof. The conditions of the Beck's theorem are fullfiled if f is weakly flat and conservative, hence f^* is the composition of an equivalence $C_Y \longrightarrow (X\backslash\mathcal{G}_f) - Comod$ for a comonad $\mathcal{G}_f = (f^*f_*, \delta_f)$ on X and the forgetful functor $(X\backslash\mathcal{G}_f) - Comod \longrightarrow C_X$ (see

(1)). If f is affine, then the functor $G_f = f^*f_*$ has a right adjoint $f^!f_*$, where $f^!$ is a right adjoint to f_*.

Let now $\mathcal{G} = (G, \delta)$ be a continuous comonad on X and f^* the forgetful functor $(X\backslash\mathcal{G}) - Comod \longrightarrow C_X$. The functor f_* which assigns to each object M of C_X the $\mathcal{G}$-comodule $(G(M), \delta(M))$ is right adjoint to f^*: the canonical adjunction arrows are

$$\epsilon_f : f^*f_* = G \longrightarrow Id_{C_X} \quad \text{and} \quad \eta_f : Id_{C_Y} \longrightarrow f_*f^*,$$

where $C_Y = \mathcal{G} - Comod$, ϵ_f is the counit of the monad $\mathcal{G}$ and

$$\eta_f(M, \xi) = \xi : (M, \xi) \longrightarrow f_*f^*(M, \xi) = (G(M), \delta(M))$$

for any $\mathcal{G}$-comodule $(M, M \xrightarrow{\xi} G(M))$.

Let $G^!$ be a right adjoint to G, and let $GG^! \xrightarrow{\epsilon'} Id_{C_X}$ and $Id_{C_X} \xrightarrow{\eta'} G^!G$ be adjunction arrows. Let $f^!$ denote the functor

$$C_Y = (X\backslash\mathcal{G}) - Comod \longrightarrow C_X, \ (M, \xi) \longmapsto G^!(M).$$

Since $f^!f_* = G^!G$, the adjunction arrow η' is a morphism $Id_{C_X} \longrightarrow f^!f_*$. The composition $f_*f^!$ assigns to each $\mathcal{G}$-comodule (M, ξ) the $\mathcal{G}$-comodule $f_*G^!(M) = (GG^!(M), \delta G^!(M))$. One can check that the adjunction arrow $\epsilon'(M) : GG^!(M) \longrightarrow M$ is a $\mathcal{G}$-comodule morphism $f_*G^!(M) \longrightarrow (M, \xi)$, i.e. the diagram

$$\begin{array}{ccc} GG^!(M) & \xrightarrow{\epsilon'(M)} & M \\ {\scriptstyle \delta G^!(M)} \downarrow & & \downarrow {\scriptstyle \xi} \\ G^2G^!(M) & \xrightarrow{G\epsilon'(M)} & G(M) \end{array} \qquad (1)$$

commutes. This implies that $\epsilon' f^*$ and η' are adjunction morphisms, hence the assertion. ■

7.2.1. Corollary. *Let a morphism $X \xrightarrow{f} Y$ be affine, weakly flat, and conservative. If the category C_X is additive (resp. abelian, resp. abelian with small coproducts, resp. a Grothendieck category), then the category C_Y has the same property, and the morphism f is flat.*

Proof. Under the hypothesis, the category C_Y is equivalent to the category $(X\backslash\mathcal{G}_f)$-comodules for a continuous comonad $\mathcal{G}_f = (G_f, \delta_f)$ on X. Since the functor $C_X \xrightarrow{G_f} C_X$ has a right adjoint, it is additive and preserves small colimits. Since $G_f = f^*f_*$, the functor f_* preserves all small limits, and the functor f^* preserves kernels of coreflexive pairs of arrows, the functor G_f preserves kernels of coreflexive pairs of arrows too. For additive categories (more generally, for categories with coproducts and a zero object) functors which preserve kernels of coreflexive pairs of arrows preserve kernels of any pairs of arrows. Thus G_f preserves kernels of any pairs of arrows and, being additive, products (which coincide with coproducts), hence G_f reserves limits of any finite diagrams. This implies that the category $(X\backslash\mathcal{G}_f) - Comod$ has limits of finite diagrams which are preserved (and reflected) by the forgetful functor $(X\backslash\mathcal{G}_f) - Comod \longrightarrow C_X$. This implies the additivity of $(X\backslash\mathcal{G}_f) - Comod$. The rest follows from the compatibility of G_f with arbitrary small colimits (cf. the argument of 6.5). ■

7.3. Affine, flat morphisms from $\mathbf{Sp}(R)$. If R is an associative ring and $\mathcal{G}$ a comonad on $\mathbf{Sp}(R)$, we shall write for convenience $(R\backslash\mathcal{G})$ instead of $(\mathbf{Sp}(R)\backslash\mathcal{G})$.

7.3.1. Proposition. *A continuous morphism $\mathbf{Sp}(R) \xrightarrow{f} X$ in $|Cat|^o$ is flat, conservative, and affine iff its inverse image functor, $C_X \xrightarrow{f^*} R - mod$, is the composition of an equivalence of categories $C_X \longrightarrow (R\backslash\mathcal{H}_f) - Comod$ for a coalgebra $\mathcal{H}_f = (H_f, \delta_f)$ in the category of R-bimodules such that H_f is a flat right R-module, and the forgetful functor $(R\backslash\mathcal{H}_f) - mod \longrightarrow R - mod$.*

Proof. Let $\mathbf{Sp}(R) \xrightarrow{f} X$ be a flat, conservative, and affine morphism with an inverse image functor f^*. By 7.2, the functor $C_X \xrightarrow{f^*} R - mod$ is the composition of a category equivalence $C_X \longrightarrow (R\backslash\mathcal{G}_f) - Comod$ for a comonad $\mathcal{G}_f = (G_f, \delta_f)$ on $\mathbf{Sp}(R)$ and the forgetful functor $(R\backslash\mathcal{G}_f) - mod \longrightarrow R - mod$. Since the comonad $\mathcal{G}_f$ is continuous, the functor G_f is isomorphic to the functor $H_f \otimes_R -$ for an R-bimodule H_f (equal to $G_f(R)$). The comultiplication $G_f \xrightarrow{\delta_f} G_f^2$ induces a comultiplication $H_f \longrightarrow H_f \otimes_R G_f$ (see the argument of 6.6.1).

Conversely, let $\mathcal{H} = (H, \delta)$ be a coalgebra in the category of R-bimodules, and let f^* denote the forgetful functor

$$(R\backslash\mathcal{H}) - Comod \longrightarrow R - mod, \quad (M, M \to H(M)) \longmapsto M. \tag{1}$$

The functor f^* has a right adjoint,

$$L \longmapsto \mathcal{H} \otimes_R L = (H \otimes_R L, \delta \otimes_R L) \tag{2}$$

(see the argument of 7.2). The comonad $\mathcal{H}\otimes_R$ is continuous, since the functor $H \otimes_R -$ has a right adjoint, $Hom_R(H, -)$.

The functor f^* being flat is equivalent to the flatness of H as a right R-module. The assertion follows now from 7.2. ■

7.3.2. Corollary. *Let $(C, \mathcal{O})$ be a ringed category, and let $X \xrightarrow{f} Y$ be a morphism of presheaves of sets on C such that its inverse image functor, $Qcoh_Y \xrightarrow{f^*} Qcoh_X$, is flat, conservative, and its direct image functor is conservative and has a right adjoint. Suppose X is representable. Then there exists a coalgebra $\mathcal{H}$ in the category of $\mathcal{O}(X)$-bimodules such that f^* is the composition of a category equivalence $Qcoh_Y \xrightarrow{\sim} (\mathcal{O}(X)\backslash\mathcal{H}) - Comod$ and the forgetful functor $(\mathcal{O}(X)\backslash\mathcal{H}) - Comod \longrightarrow \mathcal{O}(X) - mod$.*

Proof. The assertion follows from 7.3.1. ■

7.3.3. Example: semiseparated schemes and algebraic spaces. Let $\mathcal{X}$ be a scheme, or an algebraic space. Recall that an affine cover $\{U_i \xrightarrow{u_i} \mathcal{X} \mid i \in J\}$ of $\mathcal{X}$ is called *semiseparated* if each morphism $U_i \xrightarrow{u_i} \mathcal{X}$ is affine. A scheme (or an algebraic space) is called *semiseparated* if it has a semiseparated cover. Evidently, every separated algebraic space (or scheme) is semiseparated.

If $\{U_i \to \mathcal{X} \mid i \in J\}$ is a semiseparated cover of $\mathcal{X}$, the corresponding morphism

$$\mathcal{U} = \coprod_{i \in J} U_i \xrightarrow{\ \pi\ } \mathcal{X}$$

is affine which implies that the space of relations, $\mathcal{R} = \coprod_{i,j \in J} U_i \times_{\mathcal{X}} U_j \simeq \mathcal{U} \times_{\mathcal{X}} \mathcal{U}$ is affine too. Since morphisms u_i are étale, their inverse image functors, u_i^* are flat and the family $\{u_i^* | \ i \in J\}$ is conservative. The latter means exactly that an inverse image functor π^* of the morphism π is flat and conservative. It follows by construction, that the inverse images of projections $\mathcal{R} \rightrightarrows \mathcal{U}$ are flat and conservative (equivalently, faithfully flat). And they are affine, since both $\mathcal{R}$ and $\mathcal{U}$ are affine.

8. Generalities on finiteness conditions and smooth and étale morphisms. Fix a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$. In what follows, $\mathfrak{A}$ is a category of 'local', or 'affine', objects, $\mathfrak{B}$ is a category of spaces, and $\mathfrak{F}$ assigns to local objects corresponding spaces. In a standard commutative prototype, $\mathfrak{B}$ is the category of locally ringed topological spaces, otherwise called *geometric spaces*, $\mathfrak{A}$ is the category opposite to the category $CRings$ of commutative unital rings, and the functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ assigns to every commutative ring R the affine scheme $(SpecR, \mathcal{O}_R)$.

8.1. Locally finitely presentable objects and morphisms. Given a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$, we call an object X of $\mathfrak{B}$ *of* $(\mathfrak{A}, \mathfrak{F})$*-finite type* (resp. $(\mathfrak{A}, \mathfrak{F})$*-finitely presentable*), if for any filtered projective system $\mathcal{D} \xrightarrow{\mathfrak{D}} \mathfrak{A}$ such that there exists $\lim(\mathfrak{F} \circ \mathfrak{D}$, the canonical map

$$\mathrm{colim}\ \mathfrak{B}(\mathfrak{F} \circ \mathfrak{D}, X) \longrightarrow \mathfrak{B}(\lim(\mathfrak{F} \circ \mathfrak{D}), X) \qquad (1)$$

is injective (resp. bijective).

We call a morphism $X \xrightarrow{f} Y$ of $\mathfrak{B}$ *of* $(\mathfrak{A}, \mathfrak{F})$*-finite type* (resp. $(\mathfrak{A}, \mathfrak{F})$*-finitely presentable*), if for any filtered projective system $\mathcal{D} \xrightarrow{\mathfrak{D}} \mathfrak{A}/Y$, the canonical morphism

$$\mathrm{colim}\ \mathfrak{B}/Y(\mathfrak{F}_Y \circ \mathfrak{D}, (X, f)) \longrightarrow \mathfrak{B}/Y(\lim(\mathfrak{F}_Y \circ \mathfrak{D}), (X, f)) \qquad (2)$$

is injective (resp. bijective), provided $\lim(\mathfrak{F}_Y \circ \mathfrak{D})$ exists. Here $\mathfrak{F}_Y$ denotes the canonical functor $\mathfrak{F}/Y \longrightarrow \mathfrak{B}/Y$.

It follows from these definitions that if the category $\mathfrak{B}$ has a final object, $\bullet$, then an object X of $\mathfrak{B}$ is of $(\mathfrak{A}, \mathfrak{F})$-finite type (resp. $\mathfrak{A}$-finitely presentable) iff the unique morphism $X \longrightarrow \bullet$ is of $(\mathfrak{A}, \mathfrak{F})$-finite type (resp. $(\mathfrak{A}, \mathfrak{F})$-finitely presentable).

8.1.1. Proposition. *Let $\Sigma^1_{\mathfrak{A}}$ (resp. $\Sigma^0_{\mathfrak{A}}$) denote the class of all $(\mathfrak{A}, \mathfrak{F})$-finitely presentable morphisms (resp. morphisms of $(\mathfrak{A}, \mathfrak{F})$-finite type) of the category $\mathfrak{B}$.*

(a) Both $\Sigma^0_{\mathfrak{A}}$ and $\Sigma^1_{\mathfrak{A}}$ are closed under compositions and contain all isomorphisms.

(b) If the morphism f in the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

belongs to $\Sigma^i_{\mathfrak{A}}$, then f' belongs to $\Sigma^i_{\mathfrak{A}}$, $i = 0, 1$.

(c) Suppose that $X \xrightarrow{f} Y$ and $Z \xrightarrow{h} W$ are morphisms over an object S which belong to $\Sigma^i_{\mathfrak{A}}$. If $X \times_S Z$ and $Y \times_S W$ exist, then the morphism $X \times_S Z \xrightarrow{f \times_S h} Y \times_S W$ belongs to $\Sigma^i_{\mathfrak{A}}$, $i = 0, 1$.

(d) If the composition $g \circ f$ of two morphisms is $(\mathfrak{A}, \mathfrak{F})$-finitely presentable and g is of $(\mathfrak{A}, \mathfrak{F})$-finite type, then f is $(\mathfrak{A}, \mathfrak{F})$-finitely presentable.

Proof. The argument is similar to that of [KR2, 5.12.2]. ■

8.2. Formally smooth, formally unramified and formally étale morphisms. Fix a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ and a family, $\mathfrak{M}$, of morphisms of $\mathfrak{A}$ containing all identical morphisms.

8.2.1. Definitions. (i) We call a morphism $X \xrightarrow{f} Y$ of $\mathfrak{B}$ *formally $\mathfrak{M}$-smooth* if any commutative diagram

$$\begin{array}{ccc} \mathfrak{F}(T) & \xrightarrow{g} & X \\ {\scriptstyle \mathfrak{F}(\phi)}\downarrow & & \downarrow{\scriptstyle f} \\ \mathfrak{F}(S) & \xrightarrow{g'} & Y \end{array} \qquad (1)$$

such that $\phi \in \mathfrak{M}$ extends to a commutative diagram

$$\begin{array}{ccc} \mathfrak{F}(T) & \xrightarrow{g'} & X \\ {\scriptstyle \mathfrak{F}(\phi)}\downarrow & \nearrow_{\gamma} & \downarrow{\scriptstyle f} \\ \mathfrak{F}(S) & \xrightarrow{g} & Y \end{array} \qquad (2)$$

(ii) We call $X \xrightarrow{f} Y$ *formally $\mathfrak{M}$-unramified* if for any commutative diagram (1) such that $\phi \in \mathfrak{M}$, there exists at most one morphism $S \xrightarrow{\gamma} X$ such that the diagram (2) commutes.

(iii) We call $X \xrightarrow{f} Y$ *formally $\mathfrak{M}$-étale* if it is both formally $\mathfrak{M}$-smooth and formally $\mathfrak{M}$-unramified.

We denote by $\mathfrak{M}_{fsm}$ (resp. $\mathfrak{M}_{fnr}$, resp. $\mathfrak{M}_{fet}$) the class of all formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale) morphisms.

8.2.2. Proposition. *(a) Each monomorphism is formally unramified and each isomorphism is formally étale.*

(b) Composition of formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale) morphisms is formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale).

(c) Let $X \xrightarrow{f} Y$, $Y \xrightarrow{h} Z$ be morphisms of $\mathfrak{B}$.

(i) If $h \circ f$ is formally $\mathfrak{M}$-unramified, then f is formally $\mathfrak{M}$-unramified.

(ii) Suppose h is formally $\mathfrak{M}$-unramified. If $X \xrightarrow{h \circ f} Z$ is formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-étale), then f is formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-étale).

(d) Let $X \xrightarrow{\xi} S \xleftarrow{\xi'} X'$ and $Y \xrightarrow{\nu} S \xleftarrow{\nu'} Y'$ be morphisms such that there exist $X \times_S X'$ and $Y \times_S Y'$. Let $(X, \xi) \xrightarrow{f} (Y, \nu)$ and $(X', \xi') \xrightarrow{f'} (Y', \nu)$ be morphisms of objects over S. The morphisms f, f' are formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale) iff the morphism $f \times_S f' : X \times_S X' \longrightarrow Y \times_S Y'$ has the respective property.

(e) Let $X \xrightarrow{f} S \xleftarrow{h} Y$ be such a diagram that there exists a fiber product $X \times_S Y$. If f is formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally étale), then the canonical projection $X \times_S Y \xrightarrow{f'} Y$ is formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale).

Proof. See [KR2, 6.6]. ∎

8.3. Smooth, unramified, and étale morphisms. We call a morphism $X \xrightarrow{f} Y$ $\mathfrak{M}$-*smooth* (resp. $\mathfrak{M}$-*étale*, resp. $\mathfrak{M}$-*unramified*) if it is $(\mathfrak{A}, \mathfrak{F})$-finitely presentable and formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale).

We denote by $\mathfrak{M}_{sm}$ (resp. $\mathfrak{M}_{nr}$, resp. $\mathfrak{M}_{et}$) the family of all $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-unramified, resp. $\mathfrak{M}$-étale) morphisms.

We call a morphism $X \xrightarrow{f} Y$ $\mathfrak{M}$-*open immersion* if it is an $\mathfrak{M}$-smooth monomorphism.

8.3.1. Proposition. *(a) Each monomorphism is $\mathfrak{M}$-unramified and each isomorphism is $\mathfrak{M}$-open immersion.*

(b) Composition of $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-unramified, resp. $\mathfrak{M}$-étale) morphisms is $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-unramified, resp. $\mathfrak{M}$-étale).

(c) Let $X \xrightarrow{f} Y$, $Y \xrightarrow{h} Z$ be morphisms of $\mathfrak{B}$.

(i) If $g \circ f$ is formally $\mathfrak{M}$-unramified and g is of $\mathfrak{M}$-finite type, then f is $\mathfrak{M}$-unramified.

(ii) Suppose g is $\mathfrak{M}$-unramified. If $X \xrightarrow{g \circ f} Z$ is $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-étale), then f is $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-étale).

(d) Let $X \xrightarrow{\xi} S \xleftarrow{\xi'} X'$ and $Y \xrightarrow{\nu} S \xleftarrow{\nu'} Y'$ be morphisms such that there exist $X \times_S X'$ and $Y \times_S Y'$. Let $(X, \xi) \xrightarrow{f} (Y, \nu)$ and $(X', \xi') \xrightarrow{f'} (Y', \nu)$ be morphisms of objects over S. The morphisms f, f' are $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-unramified, resp. $\mathfrak{M}$-étale) iff $f \times_S f' : X \times_S X' \longrightarrow Y \times_S Y'$ has the respective property.

(e) Let $X \xrightarrow{f} S \xleftarrow{h} Y$ be such a diagram that there exists a fiber product $X \times_S Y$. If f is $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-unramified, resp. étale), then the projection $X \times_S Y \xrightarrow{f'} Y$ is $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-unramified, resp. $\mathfrak{M}$-étale).

Proof. The assertion follows from 8.2.2 and 8.1.1. ∎

8.4. Standard examples.

8.4.1. Let $\mathfrak{A}$ be the category $CRings^{op}$, as in 8.4.1. Let $\mathfrak{B}$ be the category $\mathcal{E}sp$ of *spaces* in the sense of Grothendieck (and [DG]); i.e. $\mathfrak{B}$ is the category of sheaves of sets on $\mathfrak{A}$ for the flat topology. In other words, objects of $\mathfrak{B}$ are functors $CRings \longrightarrow Sets$ which preserve finite products, and for any faithfully flat ring morphism $R \longrightarrow T$, the diagram

$$X(R) \longrightarrow X(T) \rightrightarrows X(T \otimes_R T) \qquad (1)$$

is exact. The functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ is the Yoneda functor which maps every object R of $\mathfrak{A}$ to the functor $\mathfrak{A}(-, R) = CRings(R, -)$ represented by R (here we identify objects of $\mathfrak{A}$ with the corresponding objects of $CRings$).

Then $(\mathfrak{A}, \mathfrak{F})$-finitely presentable morphisms (resp. morphims of $(\mathfrak{A}, \mathfrak{F})$-finite type) are precisely locally finitely presentable morphisms (resp. morphisms of locally finite type).

We take as $\mathfrak{M}$ the family of all morphisms of $\mathfrak{A}$ such that the corresponding ring morphism is an epimorphism with a nilpotent kernel. The formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale) morphisms are formally smooth (resp. formally unramified, resp. formally étale) in the usual sense. Therefore, $\mathfrak{M}$-smooth, $\mathfrak{M}$-unramified, $\mathfrak{M}$-étale morphisms are resp. smooth, unramified and étale. And $\mathfrak{M}$-open immersions are precisely open immersions in the conventional sense.

8.4.2. Let $\mathfrak{A}$ be the opposite category to the category Alg_k of associative unital k-algebras, Let $\mathfrak{B}$ be the category of presheaves of sets on $\mathfrak{A}$, i.e. functors $Alg_k \longrightarrow Sets$, which are *local* in the following sense: they preserve finite products, and for any faithfully flat k-algebra morphism $R \longrightarrow T$, the diagram

$$X(R) \longrightarrow X(T) \rightrightarrows X(T \star_R T) \qquad (1)$$

is exact. Here $\star_R$ denote the 'star'-product of rings over R (which is a traditional name for a push-forward of associative rings). We denote this category by $\mathcal{E}sp_{\mathcal{NC}}$ and call its objects 'noncommutative spaces', or simply 'spaces'. The functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ is the Yoneda embedding, $R \longmapsto \mathfrak{A}(-, R) = Alg_k(R, -)$ (here we identify objects of $\mathfrak{A}$ with the corresponding k-algebras).

It follows that $(\mathfrak{A}, \mathfrak{F})$-finitely presentable morphisms (resp. morphims of $(\mathfrak{A}, \mathfrak{F})$-finite type) are precisely locally finitely presentable morphisms (resp. morphisms of locally finite type) in the sense of [KR, 5.12.1].

We take as $\mathfrak{M}$ the family of all morphisms of $\mathfrak{A}$ such that the corresponding k-algebra morphism is an epimorphism with a nilpotent kernel. Then formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale) morphisms are formally smooth (resp. formally unramified, resp. formally étale) in the sense of [KR2, 5.8].

8.5. Smooth and étale morphisms of 'spaces'. Open immersions.

8.5.1. Cosubspaces and closed immersions. Let X be a 'space', that is an object of $|Cat|^o$. We call Y a *cosubspace* of X if C_Y is a full subcategory of the category C_X

which is closed under finite limits and colimits taken in C_X and, in addition, the following condition holds:

If $M \rightrightarrows N$ is a pair of arrows such that $M \in ObC_Y$ (resp. $N \in ObC_Y$) and the kernel (resp. cokernel) of the pair $M \rightrightarrows N$ exists in C_X, then this kernel (resp. cokernel) belongs to the subcategory C_Y. In particular, C_Y is strictly full (i.e. it contains with any of its objects all objects of C_X isomorphic to this object).

We call a continuous morphism $U \xrightarrow{u} X$ a *closed immersion* if its direct image functor, u_*, induces an equivalence between C_U and C_Y for some cosubspace Y of X. In particular, u_* is a fully faithful.

8.5.1.1. Topologizing subcategories. Suppose C_X is an abelian category, and C_Y is a subcategory of C_X. Then Y is a cosubspace of X iff C_Y is a *topologizing* subcategory; i.e. C_Y is closed by direct sums and subquotients (taken in C_X).

8.5.2. Thickennings. We call a closed immersion $U \xrightarrow{u} T$ a *thickenning* (and say that T is a thickenning of U), if the smallest saturated multiplicative system in $HomC_T$ containing $u_*(HomC_U)$ coincides with $HomC_T$. The latter condition means precisely that if $Y \xrightarrow{q} X$ is an exact localization such that the composition q^*u_* maps all arrows to isomorphisms, then C_Y is a groupoid.

8.5.2.1. Abelian case. Suppose that the category C_T is abelian. Then a continuous morphism $U \xrightarrow{u} T$ is a thickenning iff its direct image functor, u_*, induces an equivalence between C_U and a topologizing subcategory of C_T, and the smallest thick subcategory of C_T containing $u_*(C_U)$ coincides with C_T.

8.5.3. Formally smooth, formally unramified, and formally étale morphisms. Fix a 'space' S, and consider the category $|Cat|^o_S$ and its full subcategory Aff_S. Recall that $|Cat|^o_S$ is a full subcategory of $|Cat|^o/S$ whose objects are pair (X, f), where $X \xrightarrow{f} S$ is a continuous morphism; and Aff_S is the full subcategory of $|Cat|^o/S$ whose objects are pairs (Y, g), where $Y \xrightarrow{g} S$ is an affine morphism.

Let $X \xrightarrow{f} Y$ be a morphism in $|Cat|^o/S$. We call the morphism f *formally smooth* (resp. *formally unramified*) if for every commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{g'} & X \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & Y \end{array}$$

of S-'spaces' such that U and T belong to Aff_S and $U \xrightarrow{u} T$ is a thickening, there exists (resp. at most one) morphism $T \xrightarrow{\gamma} X$ such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{g'} & X \\ {\scriptstyle u}\big\downarrow & \nearrow_{\gamma} & \big\downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & Y \end{array}$$

commutes. The morphism $X \xrightarrow{f} Y$ is called formally étale if it is both formally smooth and formally unramified.

We call an S-'space' (X, ξ) *formally smooth* (resp. *formally unramified*, resp. *formally étale*), if the morphism $(X, \xi) \xrightarrow{\xi} (S, id_S)$ is formally smooth (resp. formally unramified, resp. formally étale).

8.5.3.1. Proposition. *(a) The class of formally smooth (resp. formally unramified, resp. formally étale) morphisms is closed under composition and contains all isomorphisms.*

(b) If $h \circ f$ is formally unramified, then f is formally unramified.

(c) Suppose a morphism h is formally unramified. If $X \xrightarrow{h \circ f} Z$ is formally smooth (resp. formally étale), then f is formally smooth (resp. formally étale).

(d) Let $X \xrightarrow{\xi} T \xleftarrow{\xi'} X'$ and $Y \xrightarrow{\nu} T \xleftarrow{\nu'} Y'$ be morphisms such that there exist $X \times_T X'$ and $Y \times_T Y'$. Let $(X, \xi) \xrightarrow{f} (Y, \nu)$ and $(X', \xi') \xrightarrow{f'} (Y', \nu)$ be morphisms of objects over T. The morphisms f, f' are formally smooth (resp. formally unramified, resp. formally étale) iff the morphism $f \times_T f' : X \times_T X' \longrightarrow Y \times_T Y'$ has the respective property.

(e) Let $X \xrightarrow{f} T \xleftarrow{h} Y$ be such a diagram that there exists a fiber product $X \times_T Y$. If f is formally smooth (resp. formally unramified, resp. formally étale), then the canonical projection $X \times_T Y \xrightarrow{f'} Y$ is formally smooth (resp. formally unramified, resp. formally étale).

Proof. The assertion follows from [KR2 6.6]. ∎

8.5.5. Locally finitely presentable S-'spaces' and morphisms of S-'spaces'. Let S be a 'space'. We take as $\mathfrak{B}$ the category $|Cat|_S^o$. If $\mathfrak{A}$ is the subcategory Aff_S of affine S-'spaces', then we shall call $\mathfrak{A}$-presentable S-'spaces' (resp. morphisms of S-'spaces') *locally presentable*. Similarly, we call (morphisms of) S-'spaces' of $\mathfrak{A}$-finite type *locally of finite type*.

8.5.6. Smooth, unramified, and étale S-'spaces' and morphisms of S-'spaces'. Open immersions. We call a morphism $X \xrightarrow{f} Y$ of S-'spaces' *smooth* (resp. *étale*, resp. *unramified*) if it is locally finitely presentable and formally smooth (resp. formally étale, resp. formally unramified).

We call a morphism $X \xrightarrow{f} Y$ of S-'spaces' an *open immersion* if it is a smooth monomorphism.

An S-'space' (X, ξ) is called smooth (resp. unramified, resp. étale) iff it is formally smooth (resp. formally unramified, resp. formally étale) and locally finitely presentable.

Since (S, id_S) is a final object in $|Cat|_S^o$, an S-'space' (X, ξ) is smooth (resp. unramified, resp. étale) if the unique morphism $(X, \xi) \longrightarrow (S, id_S)$ is smooth (resp. unramified, resp. étale).

9. Locally affine 'spaces' and schemes.

9.1. Generalities on glueing. Fix a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$.

We assume that the category $\mathfrak{B}$ is endowed with a quasi-pretopology, τ. The latter is a function which assigns to each object X of $\mathfrak{B}$ a family, τ_X, of covers of X. An element of τ_X is set of arrows $\{U_i \overset{u_i}{\to} X \mid i \in J\}$. We assume that any isomorphism forms a cover, and the composition of covers is a cover.

A cover of the form $\{\mathfrak{F}(U_i) \overset{u_i}{\to} X \mid i \in J\}$ of an object X is called a $(\mathfrak{A}, \tau)$-*cover*, or simply $\mathfrak{A}$-*cover*, if τ is fixed.

An object X of $\mathfrak{B}$ is called *locally* $(\mathfrak{A}, \tau)$-*affine* (or locally $\mathfrak{A}$-affine, if no ambiguity arises) if it has an $(\mathfrak{A}, \tau)$-cover.

We denote by $Sp_{\mathfrak{A},\tau}$ the full subcategory of the category $\mathfrak{B}$ whose objects are locally $(\mathfrak{A}, \tau)$-affine.

9.1.1. Quasi-finite locally $\mathfrak{A}$-affine objects. Given a quasi-pretopology τ on $\mathfrak{B}$, let $\tau_{\mathfrak{f}}$ denote the quasi-pretopology formed by all finite covers of τ. We call an object X of $\mathfrak{B}$ *quasi-finite locally* $(\mathfrak{A}, \tau)$-*affine* if it is locally $(\mathfrak{A}, \tau_{\mathfrak{f}})$-affine.

9.1.2. 2-$\mathfrak{A}$-covers and 2-locally $\mathfrak{A}$-affine objects. An $(\mathfrak{A}, \tau)$-cover $\{\mathfrak{F}(U_i) \overset{u_i}{\to} X \mid i \in J\}$ will be called a *2-*$(\mathfrak{A}, \tau)$-*cover* (or *2-*$\mathfrak{A}$-*cover*), if for any $i, j \in J$, there exists a set of morphisms $(U_i, u_i) \longleftarrow (U_{ij}^{\nu}, u_{ij}^{\nu}) \longrightarrow (U_j, u_j)$, $\nu \in J_{ij}$ in the category $\mathfrak{F}/X$ such that the corresponding set of morphisms $\{\mathfrak{F}(U_{ij}^{\nu}) \longrightarrow \mathfrak{F}(U_i) \times_X \mathfrak{F}(U_j) \mid \nu \in J_{ij}\}$ is a cover for any $i, j \in J$. We call the diagram

$$(U_i, u_i) \longleftarrow (U_{ij}^{\nu}, u_{ij}^{\nu}) \longrightarrow (U_j, u_j), \ \nu \in J_{ij}, \ i, j \in J, \tag{1}$$

(in the category $\mathfrak{F}/X$) a *diagram of relations* of the 2-cover $\mathfrak{U} = \{\mathfrak{F}(U_i) \overset{u_i}{\to} X \mid i \in J\}$.

We call an object X of $\mathfrak{B}$ *2-locally* $\mathfrak{A}$-*affine* if it has a 2-locally $\mathfrak{A}$-affine cover.

9.1.3. Weakly semiseparated covers. We call an $\mathfrak{A}$-cover $\{\mathfrak{F}(U_i) \overset{u_i}{\to} X \mid i \in J\}$ *weakly semiseparated* if for any $i, j \in J$, there exists a diagram

$$(U_i, u_i) \longleftarrow (U_{ij}, u_{ij}) \longrightarrow (U_j, u_j)$$

in $\mathfrak{F}/X$ such that the square

$$\begin{array}{ccc} \mathfrak{F}(U_{ij}) & \longrightarrow & \mathfrak{F}(U_j) \\ \downarrow & & \downarrow \\ \mathfrak{F}(U_i) & \longrightarrow & X \end{array}$$

is cartesian; in particular, the object $\mathfrak{F}(U_i) \times_X \mathfrak{F}(U_j)$ is isomorphic to an object of the form $\mathfrak{F}(U_{ij})$. It follows that any weakly semiseparated $\mathfrak{A}$-cover is a 2-$\mathfrak{A}$-cover.

We say that an object X of $\mathfrak{B}$ is $\mathfrak{A}$-*weakly semiseparated* if it has a weakly semiseparated $\mathfrak{A}$-cover.

9.2. $\mathfrak{A}$-Representable morphisms and covers, and locally $\mathfrak{A}$-representable objects. If $\mathcal{E}$ is a subcategory of $\mathfrak{B}$ such that $Ob\mathcal{E} = Ob\mathfrak{B}$, we denote by $\tau^{\mathcal{E}}$ the quasi-pretopology on $\mathfrak{B}$ formed by all covers $\{U_i \overset{u_i}{\to} X \mid i \in J\}$ in τ such that all morphisms u_i belong to $\mathcal{E}$. Given a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$, we have a natural choice of the subcategory $\mathcal{E}$, which is the subcategory of $\mathfrak{A}$-*representable morphisms* described below.

We call a morphism $X \xrightarrow{f} Y$ of the category $\mathfrak{B}$ $\mathfrak{A}$-*representable* if for any morphism $\mathfrak{F}(V) \xrightarrow{g} Y$, there exist morphisms $\mathfrak{F}(W) \xrightarrow{\tilde{g}} X$ and $W \xrightarrow{v} V$ such that

$$\begin{array}{ccc}
\mathfrak{F}(W) & \xrightarrow{\tilde{g}} & X \\
{\scriptstyle \mathfrak{F}(v)}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
\mathfrak{F}(V) & \xrightarrow{g} & Y
\end{array}$$

is a cartesian square; in particular, it commutes.

9.2.1. Lemma. *(a) Every isomorphism in $\mathfrak{B}$ is $\mathfrak{A}$-representable.*
(b) The composition of $\mathfrak{A}$-representable morphisms is $\mathfrak{A}$-representable.
(c) If in a cartesian square

$$\begin{array}{ccc}
X' & \longrightarrow & X \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
Y' & \longrightarrow & Y
\end{array}$$

the morphism $X \xrightarrow{f} Y$ is $\mathfrak{A}$-representable, then the morphism $X' \xrightarrow{f'} Y'$ is $\mathfrak{A}$-representable.

Proof. The assertion (a) is obvious.

The assertions (b) and (c) follow from the general nonsense fact that the composition of cartesian squares is a cartesian square: if in the commutative diagram

$$\begin{array}{ccccc}
X'' & \longrightarrow & X' & \longrightarrow & X \\
{\scriptstyle f''}\big\downarrow & & {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
Y'' & \longrightarrow & Y' & \longrightarrow & Y
\end{array}$$

both squares are cartesian, then the square

$$\begin{array}{ccc}
X'' & \longrightarrow & X \\
{\scriptstyle f''}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
Y'' & \longrightarrow & Y
\end{array}$$

is cartesian. Details are left to the reader. ■

9.2.2. Representable covers. We call a cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ $\mathfrak{A}$-*representable* if each morphism u_i of the cover is $\mathfrak{A}$-representable. We denote by $\tau^{\mathfrak{A}}$ the function which assigns to each object X of the category $\mathfrak{B}$ the set, $\tau_X^{\mathfrak{A}}$, of $\mathfrak{A}$-representable covers of X.

9.2.2.1. Lemma. *The function $\tau^{\mathfrak{A}}$ is a quasi-pretopology on $\mathfrak{B}$. If τ is a pretopology, then $\tau^{\mathfrak{A}}$ is a pretopology.*

Proof. The assertion is a corollary of 9.2.1. ■

9.2.3. Locally $\mathfrak{A}$-representable objects. Evidently, every representable $\mathfrak{A}$-cover is weakly semiseparated (cf. 9.1.3.). In particular, it is a 2-$\mathfrak{A}$-cover.

We say that an object X of $\mathfrak{B}$ is *locally $\mathfrak{A}$-representable* if it has a representable $\mathfrak{A}$-cover. Thus, every locally $\mathfrak{A}$-representable object is locally $\mathfrak{A}$-affine.

9.3. Coinduced pretopology. Let $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ be a functor, and let $\mathfrak{T}$ be a quasi-pretopology on $\mathfrak{A}$. The coinduced quasi-pretopology, $\mathfrak{T}^{\mathfrak{F}}$, on $\mathfrak{B}$ is defined as follows: a set of arrows $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is a cover of X iff for any morphism $\mathfrak{F}(V) \xrightarrow{g} X$, there exists a cover $\{V_j \xrightarrow{v_j} V \mid j \in I\} \in \mathfrak{T}_V$ such that for every $j \in I$, the morphism $g \circ \mathfrak{F}(v_j) : \mathfrak{F}(V_j) \longrightarrow X$ factors through u_i for some $i \in J$ (cf. [R4, 4.4]).

9.3.1. Proposition. *Suppose $\mathfrak{B}$ is a category with fiber products. Then the coinduced quasi-pretopology $\mathfrak{T}^{\mathfrak{F}}$ on $\mathfrak{B}$ is a pretopology.*

Proof. Let $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a cover in $\mathfrak{T}^{\mathfrak{F}}$ and $Y \xrightarrow{g} X$ an arbitrary morphism. The claim (equivalent to the proposition) is that the set of arrows $\{U_i \times_X Y \xrightarrow{\bar{u}_i} Y \mid i \in J\}$ is a cover in $\mathfrak{T}^{\mathfrak{F}}$.

In fact, let $\mathfrak{F}(V) \xrightarrow{v} Y$ be an arbitrary morphism. Since $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is a cover, there exists a cover $\{V_j \xrightarrow{v_i} V \mid j \in I\}$ in $\mathfrak{T}$ such that for any $j \in I$, there exists $i_j \in J$ and a morphism $\mathfrak{F}(V_j) \xrightarrow{\bar{v}_j} U_{i_j}$ which make the diagram

$$\begin{array}{ccc} \mathfrak{F}(V_j) & \xrightarrow{\bar{v}_j} & U_{i_j} \\ {\scriptstyle \mathfrak{F}(v_j)} \downarrow & & \downarrow {\scriptstyle u_{i_j}} \\ \mathfrak{F}(V) & \xrightarrow{g \circ v} & X \end{array} \qquad (1)$$

commute. The commutativity of (1) implies the existence if a unique morphism $\mathfrak{F}(V_j) \xrightarrow{v'_j} U_{i_j} \times_X Y$ such that the diagram

$$\begin{array}{ccc} \mathfrak{F}(V_j) & \xrightarrow{\bar{v}'_j} & U_{i_j} \times_X Y \\ {\scriptstyle \mathfrak{F}(v_j)} \downarrow & & \downarrow {\scriptstyle u_{i_j}} \\ \mathfrak{F}(V) & \xrightarrow{v} & Y \end{array} \qquad (2)$$

commutes and the morphism $\mathfrak{F}(V_j) \xrightarrow{\bar{v}_j} U_{i_j}$ is the composition of $\mathfrak{F}(V_j) \xrightarrow{v'_j} U_{i_j} \times_X Y$ and the canonical projection $U_{i_j} \times_X Y \longrightarrow U_{i_j}$. This shows that $\{U_i \times_X Y \xrightarrow{\bar{u}_i} Y \mid i \in J\}$ is a cover in $\mathfrak{T}^{\mathfrak{F}}$. ∎

9.3.2. Note. Let $\mathfrak{A}$ be a category with fiber products and $\mathfrak{T}$ a quasi-pretopology on $\mathfrak{A}$. Taking $\mathfrak{F} = Id_{\mathfrak{A}}$, we obtain the coinduced pretopology, $\mathfrak{T}^g$, on $\mathfrak{B} = \mathfrak{A}$. The pretopology $\mathfrak{T}^g$ is the finest pretopology among those pretopologies on $\mathfrak{A}$ which are coarser than $\mathfrak{T}$.

9.4. Standard commutative examples.

9.4.1. Geometric spaces and schemes. Let $\mathfrak{B}$ be the category of locally ringed topological spaces which we call otherwise *geometric spaces*, $\mathfrak{A}$ the category opposite to the category $CRings$ of commutative unital rings, $\mathfrak{F}$ the functor $\mathfrak{A} \longrightarrow \mathfrak{B}$ which assigns to every commutative ring its spectrum. The pretopology on $\mathfrak{B}$ is the standard *Zariski* pretopology given by families of open immersions covering the underlying space: a set $\{(U_i, \mathcal{O}_{U_i}) \xrightarrow{u_i} (X, \mathcal{O}_X) \mid i \in J\}$ of open immersions is a cover iff $\bigcup_{i \in J} U_i = X$.

Then locally $\mathfrak{A}$-affine objects of $\mathfrak{B}$ are arbitrary schemes.

9.4.1.1. Semiseparated schemes. Locally $\mathfrak{A}$-representable objects of $\mathfrak{B}$ are precisely *semiseparated schemes.* Recall that a scheme $\mathcal{X} = (X, \mathcal{O})$ is called *semiseparated* if it has an affine cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ such that each morphism $U_i \xrightarrow{u_i} X$ is representable. Clearly, every semiseparated scheme is weakly separated.

9.4.2. Quasi-finite $\mathfrak{A}$-objects. Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ are same as in 9.4.1. Then quasi-finite $\mathfrak{A}$-objects (i.e. locally $(\mathfrak{A}, \tau_{3_\mathfrak{f}})$-affine objects, where $\tau_{3_\mathfrak{f}}$ is the subpretopology of τ_3 formed by finite covers, cf. 9.1.1) are exactly quasi-compact schemes.

Notice that 2-locally $(\mathfrak{A}, \tau_{3_\mathfrak{f}})$-affine objects are quasi-compact quasi-separated schemes.

9.4.3. Spaces as sheaves of sets. Let $\mathfrak{A}$ be the category $CRings^{op}$, as in 9.2.1. Let $\mathfrak{B}$ be category $\mathcal{E}sp$ of sheaves of sets on $CRings^{op}$ for the **fpqc** topology, and let $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ be the Yoneda embedding: $R \longmapsto CRings(R, -)$ (see 8.4.1).

Zariski covers in $CRings^{op} = \mathfrak{A}$ are given by sets of morphisms $\{R \longrightarrow R_i \mid i \in J\}$ such that R_i is a localization of R at an element of R (that is at the multiplicative set generated by this element), and $\bigcup_{i \in J} Spec(R_i) = Spec(R)$. Zariski covers form a (Zariski) pretopology, $\mathfrak{T}_3$. We define *Zariski* pretopology on $\mathfrak{B} = \mathcal{E}sp$ as the pretopology coinduced by $\mathfrak{T}_3$ (cf. 9.3).

Locally affine $\mathfrak{A}$-objects in this setting are schemes in the sense of [DG], that is schemes realized as functors $CRings \longrightarrow Sets$. The functor $\mathcal{S}$ which assigns to each geometric space $\mathbf{X} = (X, \mathcal{O})$ the functor $R \longmapsto Hom((SpecR, \mathcal{O}_R), \mathbf{X})$ establishes an equivalence between *geometric* schemes and *functorial* schemes.

Representable morphisms in $\mathfrak{B}$ are corepresentable functors $CRings \longrightarrow Sets$. The functor $\mathcal{S}$ induces an equivalence of the category of semiseparated schemes (cf. 9.4.1.1) and the category of locally $\mathfrak{A}$-representable objects of $\mathfrak{B}$.

Replacing the Zariski pretopology $\mathfrak{T}_3$ by its finite version, $\mathfrak{T}_{3_\mathfrak{f}}$, we obtain a full subcategory of the category of locally $\mathfrak{A}$-affine objects formed by quasi-finite locally $\mathfrak{A}$-affine objects. The functor $\mathcal{S}$ induces an equivalence of this category and the category of quasi-compact geometric schemes. The functor $\mathcal{S}$ induces an equivalence of the category of 2-locally $(\mathfrak{A}, \mathfrak{T}_{3_\mathfrak{f}})$-affine objects and the category of quasi-compact, quasi-separated geometric schemes.

9.5. Standard noncommutative examples. We take as $\mathfrak{A}$ the category $\mathbf{Aff_k} = Alg_k^{op}$ opposite to the category of associative unital k-algebras, together with one of the canonical quasi-pretopologies defined below.

9.5.1. Canonical quasi-pretopologies on $\mathbf{Aff_k} = \mathbf{Alg_k^{op}}$. We call the image in $\mathbf{Aff_k}$ of a set of k-algebra morphisms $\{R \longrightarrow R_i \mid i \in J\}$ an **fpqc** *cover* if all morphisms $R \longrightarrow R_i$ are flat (i.e. R_i is a flat right R-module), and there is a finite subset I of J

such that the family of functors $\{R_i \otimes_R \ \mid i \in I\}$ is conservative. The composition of **fpqc** covers is an **fpqc** cover, and any faithfully flat ring morphism (in particular, any ring isomorphism) $R \longrightarrow S$ forms an **fpqc** cover. Thus, **fpqc** covers form a quasi-pretopology which we denote by $\tau_{\mathbf{fpqc}}$.

We call an **fpqc** cover $\{\mathbf{Spec}R_i \longrightarrow \mathbf{Spec}R \mid i \in J\}$ an **lqc** *cover* if the corresponding ring morphisms $R \longrightarrow R_i$ are localizations, or, equivalently, the corresponding 'restriction of scalars' functor $R_i - mod \longrightarrow R - mod$ is full (hence fully faithful). It follows from [R4, 2.6.3.1] that **lqc** covers form pretopology on **Aff** which we denote by $\tau_{\mathbf{lqc}}$.

We call an **fpqc** cover $\{R \longrightarrow R_i \mid i \in J\}$ an **fppf** *cover* if it consists of finitely presentable morphisms. We denote the **fppf** quasi-pretopology by $\tau_{\mathbf{fppf}}$.

A set of algebra morphisms $\{R \longrightarrow R_i \mid i \in J\}$ defines a *Zariski cover* if it consists of finitely presentable localizations and the family of functors $\{R_i \otimes_R \mid i \in J\}$ is conservative. Zariski covers form a pretopology which we denote by $\tau_{\mathfrak{Z}}$ and call it the *Zariski pretopology.*

9.5.2. Noncommutative schemes as presheaves of sets. Let $\mathfrak{B}$ be the category $\mathcal{E}sp_{\mathcal{NC}}$ of sheaves of sets on $\mathbf{Aff}_k$ for **fpqc** quasi-pretopology. In other words, objects of $\mathfrak{B}$ are functors **Rings** $\longrightarrow$ *Sets* which preserve finite products, and for any faithfully flat ring morphism $R \longrightarrow T$, the diagram

$$X(R) \longrightarrow X(T) \rightrightarrows X(T \star_R T) \tag{1}$$

is exact. The functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ is the Yoneda embedding, $R \longmapsto Alg_k(R, -)$ (see 8.4.2).

Let $\tau_{\mathfrak{z}}^{\mathbf{o}}$ denote the pretopology on $\mathfrak{B}$ coinduced by the Zariski pretopology $\tau_{\mathfrak{z}}$ via the functor $\mathfrak{F}$. We define *schemes* as locally $(\mathfrak{A}, \tau_{\mathfrak{z}}^{\mathbf{o}})$-affine objects of $\mathfrak{B}$.

9.5.3. Remark on fpqc-locally affine spaces. Let the category $\mathfrak{B}$ and the functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ be the same as in 9.5.2. But, we take the **fpqc** quasi-pretopology on $\mathfrak{A} = Aff_k$ instead of the Zariski pretopology. Let $\tau_{\mathbf{fpqc}}^{\mathbf{o}}$ denote the pretopology coinduced on $\mathfrak{B}$ by the **fpqc** quasi-pretopology on $\mathfrak{A} = A^{op}$. Applying the formalism of 9.1, we obtain *locally* $(\mathfrak{A}, \tau_{\mathbf{fpqc}}^{\mathbf{o}})$*-affine spaces.*

This approach, however, is less satisfactory in the case of general **fpqc** covers, than in the case of Zariski covers. The reason is that **fpqc** covers do not form a pretopology; hence the operation of coinduction decimates the original quasi-pretopology on Aff_k. Fortunately, this inconvenience is easily avoided by defining **fpqc** quasi-pretopology directly on the category $\mathfrak{B}$.

9.6. Flat quasi-pretopologies in $|Cat|^o$. Let $\mathfrak{B} = |Cat|^o$. We call a set of morphisms $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ in $|Cat|^o$ a *weakly flat cover* if all u_i are weakly flat and the set of their inverse image functors, $\{u_i^* \mid i \in J\}$, is conservative. This defines a *weakly flat* quasi-pretopology, τ^w, on the category $|Cat|^o$.

9.6.1. Finiteness conditions. We call a weakly flat cover an **fpqc** cover, if it contains a finite subcover. We denote the corresponding quasi-pretopology by $\tau_{\mathbf{fpqc}}$.

Let $\mathfrak{E}$ be a set of types of diagrams. We denote by $\tau_{\mathbf{fpqc}}^{\mathfrak{E}}$ the quasi-pretopology defined as follows: $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ belongs to $\tau_{\mathbf{fpqc}}^{\mathfrak{E}}$ iff it is a weakly flat **fpqc** cover such that all direct image functors, u_{i*} preserve colimits from $\mathfrak{E}$.

We denote by $\tau^{af}_{\mathbf{fpqc}}$ the quasi-pretopology formed by weakly flat **fpqc** covers which consist of affine morphisms. We denote by τ^{w}_{1} the quasi-pretopology generated by weakly flat covers which consist of one morphism.

Finally, we denote by τ^{af}_{1} the quasi-pretopology generated by weakly flat covers which consist of one *affine* morphism.

9.6.2. Semiseparated covers. We call a weakly flat cover $\{U_i \overset{u_i}{\to} X \mid i \in J\}$ *semiseparated* if all morphisms u_i are affine. We denote the corresponding quasi-pretopology by τ^{af}.

9.6.3. Proposition. *Let* $\mathfrak{U} = \{U_i \overset{u_i}{\to} X \mid i \in J\}$ *be a weakly flat cover and* $\mathcal{U} = \coprod_{i\in J} U_i \xrightarrow{u} X$ *the canonical morphism corresponding to the cover* $\mathfrak{U}$.

(a) If the category C_X *has products of* J *objects, then the morphism* $\mathcal{U} \xrightarrow{u} X$ *is weakly flat and conservative.*

(b) If the category C_X *is additive and the cover* $\mathfrak{U}$ *is finite and semiseparated (i.e. every morphism* u_i *is affine), then the morphism* $\mathcal{U} \xrightarrow{u} X$ *is affine.*

Proof. The family of inverse image functors $\mathfrak{U} = \{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ is conservative iff the corresponding functor

$$C_X \xrightarrow{u^*} \prod_{i\in J} C_{U_i} = C_{\mathcal{U}}, \quad x \longmapsto (u_i^*(x)|i \in J),$$

is conservative. Similarly, the functors u_i^* preserve kernels of coreflexive pairs of arrows for all $i \in J$ iff the functor u^* has the same property.

(a) Suppose the category C_X has products of J objects. Then the functor $u_* : (a_i|i \in J) \longmapsto \prod_{i\in J} u_{i*}(a_i)$ is a right adjoint to the functor u^*.

(b) If every direct image functor u_{i*} is conservative, then the functor u_* is conservative. If the category C_X is additive and the cover $\mathfrak{U}$ is finite (i.e. J is finite), then $u_*(a_i|i \in J) = \coprod_{i\in J} u_{i*}(a_i)$ for any object $(a_i|i \in J)$ of the category $C_{\mathcal{U}}$, and for any $x \in ObC_X$, we have:

$$C_X\big(u_*(a_i|i \in J), x\big) = C_X\big(\coprod_{i\in J} u_{i*}(a_i), x\big) \simeq \prod_{i\in J} C_X\big(u_{i*}(a_i), x\big) \simeq$$

$$\prod_{i\in J} C_X(a_i, u_i^!(x)) = C_{\mathcal{U}}\big((a_i|i \in J), (u_i^!(x)|i \in J)\big).$$

Here $u_i^!$ is a right adjoint to the direct image functor u_{i*}. This shows that the functor

$$C_X \xrightarrow{u^!} C_{\mathcal{U}}, \quad x \longmapsto (u_i^!(x)|i \in J),$$

is a right adjoint to the functor u_*. ■

9.7. Locally affine morphisms. Relative schemes. Fix a 'space' S, and consider the category $\mathfrak{B} = |Cat|^o_S$. Recall that $|Cat|^o_S$ is a full subcategory of $|Cat|^o/S$ whose objects are pairs (X, f), where $X \xrightarrow{f} S$ is a continuous morphism.

We assume that the category C_S has cokernels of reflexive pairs of arrows. There are two extreme choices of the category of 'local' (or 'affine') objects $\mathfrak{A}$. The largest (in a certain sense) choice is the category $\mathfrak{A} = Aff_S^w$ of weakly affine morphisms to S. The other extremity is the category Aff_S of affine morphisms to S (as in 8.5.3). Intermediate choices are categories $Aff_S^{\mathfrak{S}}$, where $\mathfrak{S}$ is a set of types of diagrams: objects of $Aff_S^{\mathfrak{S}}$ are pairs (X, f), where f is a continuous morphism $X \longrightarrow S$ such that f_* is weakly affine and preserves colimits of diagrams of $\mathfrak{S}$.

In each of these cases, the functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ is the inclusion functor.

Any quasi-pretopology on $|Cat|^o$ induces a quasi–pretopology on $|Cat|^o/S$. In particular, each of the canonical quasi-pretopologies on $|Cat|^o$ defined in 9.7 induces a canonical quasi-pretopology on $\mathfrak{B} = |Cat|^o_S$. Thus, we have the quasi-pretopology $\tau_{\mathbf{fpqc}}$ given by finite weakly flat covers and its versions, $\tau^{\mathfrak{E}}_{\mathbf{fpqc}}$ and $\tau^{af}_{\mathbf{fpqc}}$ (cf. 9.6.1).

9.7.1. Locally affine S-'spaces'. We call an object $(X, X \xrightarrow{f} S)$ of the category $\mathfrak{B} = |Cat|^o_S$ a *locally affine S-'space'*, if it is a locally $\mathfrak{A}$-affine object of $|Cat|^o_S$ for the quasi-pretopology $\tau_{\mathbf{fpqc}}$, and $\mathfrak{A} = Aff_S$.

An object $(X, X \xrightarrow{f} S)$ of $|Cat|^o_S$ is called a *semiseparated locally affine S-'space'*, if it is a locally $\mathfrak{A}$-affine object of $|Cat|^o_S$ with respect to the quasi-pretopology $\tau^{af}_{\mathbf{fpqc}}$ and $\mathfrak{A} = Aff_S$. In other words, $(X, X \xrightarrow{f} S)$ has a finite weakly flat Aff_S-affine cover which consists of affine morphisms.

Restricting to **fpqc** covers $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ such that all morphisms u_i are flat localizations, we obtain quasi-pretopologies $\tau_{\mathbf{lpqc}}$, $\tau^{\mathfrak{E}}_{\mathbf{lpqc}}$, and $\tau^{af}_{\mathbf{lpqc}}$.

9.7.2. Relative schemes. We call a conservative set $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ of morphisms of $|Cat|^o_S$ a *Zariski cover*, if all morphisms u_i are locally finitely presentable localizations. Zariski covers form a quasi-pretopology, $\tau_{\mathfrak{Z}}$.

We call an object $(X, X \xrightarrow{f} S)$ of the category $|Cat|^o_S$ an *S-scheme*, if it is a locally $\mathfrak{A}$-affine object of $|Cat|^o_S$ with respect to the quasi-pretopology $\tau_{\mathfrak{Z}}$ and $\mathfrak{A} = Aff_S$.

Taking Zariski covers which are **fpqc** covers, we obtain the corresponding versions of Zariski quasi-pretopology: $\tau_{\mathfrak{Z}}$, $\tau^{\mathfrak{E}}_{\mathfrak{Z}}$, and $\tau^{af}_{\mathfrak{Z}}$.

We call an object $(X, X \xrightarrow{f} S)$ of $|Cat|^o/S$ a *semiseparated S-scheme*, if it is a locally $\mathfrak{A}$-affine object of $|Cat|^o/S$ with respect to the quasi-pretopology $\tau^{af}_{\mathfrak{Z}}$ and $\mathfrak{A} = Aff_S$. In other words, $(X, X \xrightarrow{f} S)$ has a finite weakly flat Aff_S-affine Zariski cover which consists of affine localizations.

9.8. Locally S-affine 'spaces' and schemes. Fix an object S of the category $\mathfrak{B} = |Cat|^o$. One can take as $\mathfrak{A}$ the category Aff_S^w, or Aff_S, or $Aff_S^{\mathfrak{S}}$ for some set $\mathfrak{S}$ of types of diagrams. This time, $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ is the natural functor which maps an object $(X, X \to S)$ of the category $\mathfrak{A}$ to X. We call an object X of $\mathfrak{B}$ *S-locally affine 'space'* if it is a locally $(\mathfrak{A}, \mathfrak{F})$-affine object with respect to the quasi-pretopology $\tau_{\mathbf{fpqc}}$.

We call a conservative set $\{\mathfrak{F}(U_i) \xrightarrow{u_i} X \mid i \in J\}$ of morphisms of $|Cat|^o_S$ a *Zariski cover*, if all morphisms u_i are $(\mathfrak{A}, \mathfrak{F})$-finitely presentable localizations. Zariski covers form a quasi-pretopology which we denote by $\tau_{\mathfrak{Z}}$, as in the relative case sketched in 9.7.2.

We call an object $(X, X \xrightarrow{f} S)$ of the category $|Cat|^o_S$ an S-*scheme*, if it is a locally $\mathfrak{A}$-affine object of $|Cat|^o_S$ with respect to the quasi-pretopology τ_3 and $\mathfrak{A} = Aff_S$.

9.8.1. Locally affine $\mathbb{Z}$-spaces and $\mathbb{Z}$-schemes. Let $S = \mathbf{Sp}\mathbb{Z}$ (i.e. C_S is the category of abelian groups). We call an object X of $\mathfrak{B}$ *locally affine* $\mathbb{Z}$-*space* (resp. a $\mathbb{Z}$-*scheme*) if it is a locally $\mathfrak{A}$-affine object with respect to the quasi-pretopology $\tau_{\mathbf{fpqc}}$ (resp. the Zariski quasi-pretopology τ_3).

9.8.2. Note. Let $\mathfrak{A}$ be the category $\mathbf{Rings}^{op}$, and let $\mathfrak{F}$ be the functor $\mathbf{Sp}$ which assigns to each associative unital ring its categoric spectrum. Then the category of locally $(\mathfrak{A}, \mathfrak{F})$-affine 'spaces' is isomorphic to the category of locally affine $\mathbb{Z}$-'spaces' defined in 9.8.1. Similarly, the category of $(\mathfrak{A}, \mathfrak{F})$-schemes (defined in an obvious way) is isomorphic to the category of $\mathbb{Z}$-schemes.

9.9. The structure of locally affine 'spaces'. Fix a 'space' S. Let $\mathfrak{A}$ be the category Aff_S of affine morphisms to S, or the category $Aff_S^{\mathfrak{S}}$ for some set of diagram types $\mathfrak{S}$, and let $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B} = |Cat|^o$ be the forgerful functor $(X, X \to S) \longmapsto X$.

Suppose that C_S has finite products and cokernels of reflexive pairs of arrows. Let $\{\mathfrak{F}(U_i, \bar{u}_i) = U_i \xrightarrow{u_i} X \mid i \in J\}$ be a finite $\mathfrak{A}$-cover of X, and let $U = \coprod_{i \in J} U_i \xrightarrow{u} X$ be the corresponding morphism.

If C_X has finite products, then u is a continuous morphism (see 9.6.3), hence u is weakly flat and conservative. By Beck's theorem, X is isomorphic to $\mathbf{Sp^o}(U\backslash\mathcal{G}_u)$; i.e. the category C_X is equivalent to the category of $(U\backslash\mathcal{G}_u)$-comodules.

10. 'Spaces' of morphisms and flat descent.

10.1. 'Spaces' of morphisms. For any two objects X, Y of the category $|Cat|^o$, we denote by $\mathcal{H}om(X, Y)$ the object of $|Cat|^o$ such that $C_{\mathcal{H}om(X,Y)}$ is the category $Cat^{op}(X, Y)$ of (inverse image) functors from C_Y to C_X.

Let $\bullet$ be the standard initial object of the category $|Cat|^o$, i.e. $C_\bullet$ is the category with one morphism. For any object X of $|Cat|^o$, there are natural isomorphisms:

$$\mathcal{H}om(\bullet, X) \simeq \bullet \quad \text{and} \quad \mathcal{H}om(X, \bullet) \simeq X.$$

As it is observed in 2.2, $|Cat|^o(X, \bullet) \simeq |X|$, where $|X|$ is the set of isomorphism classes of objects of C_X. In particular, $|Cat|^o(\mathcal{H}om(X, Y), \bullet) \simeq |Cat|^o(X, Y)$.

Let X, Y, Z be objects of $|Cat|^o$. The composition of functors (1-morphisms of the category Cat)

$$Cat(C_X, C_Y) \times Cat(C_Y, C_Z) \longrightarrow Cat(C_X, C_Z)$$

can be rewritten as

$$C_{\mathcal{H}om(Y,X)} \times C_{\mathcal{H}om(Y,X)} = C_{\mathcal{H}om(Y,X) \coprod \mathcal{H}om(Z,Y)} \longrightarrow C_{\mathcal{H}om(Z,X)}$$

and interpreted as a inverse image functor of a morphism

$$\mathcal{H}om(Z, X) \longrightarrow \mathcal{H}om(Y, Z) \coprod \mathcal{H}om(X, Y) \qquad (1)$$

Applying to (1) the functor $|Cat|^o(-,\bullet)$, we obtain the composition in $|Cat|^o$:

$$|Cat|^o(Y,X) \times |Cat|^o(Z,Y) \longrightarrow |Cat|^o(Z,X).$$

10.1.1. Remark. One can view Cat as a monoidal category with the monoidal structure given by the product of categories. The unit object is the category $C_\bullet$, and for any two objects, C_X, C_Y, of the category Cat, the category $\mathcal{H}om(X,Y) = \mathcal{H}om(C_Y, C_X)$ of functors $C_Y \longrightarrow C_X$ is the *inner hom* from C_Y to C_X. Thus, we have an enriched monoidal category. Formally dualizing this structure, we obtain the one described above, i.e. the category $|Cat|^o$ enriched by 'spaces' of morphisms $\mathcal{H}om(X,Y)$ for any pair of objects X, Y.

10.1.2. Morphisms of S-'spaces'. Fix an object S of the category $|Cat|^o$ and consider the category $|Cat|^o/S$ of objects over S. For any two objects, $\mathcal{X} = (X,f)$, $\mathcal{Y} = (Y,g)$ of $|Cat|^o/S$, we denote by $\mathcal{H}om_S(\mathcal{X},\mathcal{Y})$ the object of $|Cat|^o$ such that $C_{\mathcal{H}om_S(\mathcal{X},\mathcal{Y})}$ is the full subcategory of the category $C_{\mathcal{H}om(X,Y)} = \mathcal{H}om(C_Y, C_X)$ formed by those functors $\phi^* : C_Y \longrightarrow C_X$ for which the diagram

$$\begin{tikzcd} C_Y \arrow[rr, "\phi^*"] & & C_X \\ & C_S \arrow[ul, "g^*"] \arrow[ur, "f^*"'] & \end{tikzcd}$$

quasi commutes. If ϕ^* is an object of $C_{\mathcal{H}om_S(\mathcal{X},\mathcal{Y})}$ and ψ^* is an object of $C_{\mathcal{H}om_S(\mathcal{Y},\mathcal{Z})}$, then $\phi^*\psi^*$ is an object of $C_{\mathcal{H}om_S(\mathcal{X},\mathcal{Z})}$. Thus, we have the enriched category $|Cat|^o/S$ of S-'spaces' such that the forgetful functor $|Cat|^o/S \longrightarrow |Cat|^o$ defines a morphism of enriched categories.

10.2. Certain enriched categories of morphisms. For any two objects X and Y of the category $|Cat|^o$, we denote by $\mathcal{H}om^w(X,Y)$, $\mathcal{H}om_w(X,Y)$, and $\mathcal{H}om^c(X,Y)$ objects of $|Cat|^o$ defined as follows:

$C_{\mathcal{H}om^w(X,Y)}$ is the full subcategory of $C_{\mathcal{H}om(X,Y)}$ formed by *weakly right exact* functors $C_Y \longrightarrow C_X$, i.e. functors which preserve cokernels of reflexive pairs of arrows;

$C_{\mathcal{H}om_w(X,Y)}$ is the full subcategory of $C_{\mathcal{H}om(X,Y)}$ formed by *weakly left exact* functors, i.e. functors $C_Y \longrightarrow C_X$ which preserve kernels of coreflexive pairs of arrows.

$C_{\mathcal{H}om^c(X,Y)}$ is the full subcategory of $C_{\mathcal{H}om(X,Y)}$ whose objects are continuous functors $C_Y \longrightarrow C_X$. In particular, $C_{\mathcal{H}om^c(X,Y)}$ is a full subcategory of the category $C_{\mathcal{H}om^w(X,Y)}$.

Continuous morphisms play a special role, as the following example shows.

10.2.1. Example. For any unital associative rings R, S, there is a natural isomorphism $Hom^c(\mathbf{Sp}(R), \mathbf{Sp}(S)) \simeq \mathbf{Sp}(S \otimes R^o)$.

In fact, the category of continuous functors from $S - mod = C_{\mathbf{Sp}(S)}$ to $R - mod = C_{\mathbf{Sp}(R)}$ is equivalent to the category of (R,S)-bimodules, which, in turn, is isomorphic to the category of $R \otimes S^o$-modules.

10.2.1.1. Note. Suppose that S and R are commutative rings. Then $S \otimes R^o = S \otimes R$ is a commutative ring. Thus, if U and V are spectra of commutative rings, then $Hom^c(U,V)$ is the spectrum of a commutative ring.

10.2.2. Generalizations of $\mathcal{H}om^w$ and $\mathcal{H}om_w$. Let $\mathfrak{S}$ and $\mathfrak{T}$ be sets of types of diagrams. For any two objects, X, Y of $|Cat|^o$, we denote by $\mathcal{H}om^{\mathfrak{T}}(X,Y)$ (resp. by $\mathcal{H}om_{\mathfrak{S}}(X,Y)$) the object of $|Cat|^o$ such that $C_{\mathcal{H}om^{\mathfrak{T}}(X,Y)}$ (resp. $C_{\mathcal{H}om_{\mathfrak{S}}(X,Y)}$) is a full subcategory of $C_{\mathcal{H}om(X,Y)}$ whose objects are all functors $C_Y \longrightarrow C_X$ which preserve colimits of diagrams from $\mathfrak{T}$ (resp. limits of diagrams from $\mathfrak{S}$). Besides $\mathcal{H}om^w$ and $\mathcal{H}om_w$, the cases of special interest are when $\mathfrak{S}$ (or $\mathfrak{T}$, or both) consist of all finite diagrams, or $\mathfrak{S}$ consists of countable diagrams.

We denote by $\mathcal{H}om^{\mathfrak{T}}_{\mathfrak{S}}(X,Y)$ the intersection (or, ruther, cointersection) of $\mathcal{H}om_{\mathfrak{S}}(X,Y)$ and $\mathcal{H}om^{\mathfrak{T}}(X,Y)$ which is the object of $|Cat|^o$ defined by

$$C_{\mathcal{H}om^{\mathfrak{T}}_{\mathfrak{S}}(X,Y)} = C_{\mathcal{H}om_{\mathfrak{S}}(X,Y)} \bigcap C_{\mathcal{H}om^{\mathfrak{T}}(X,Y)}$$

i.e. objects of $C_{\mathcal{H}om^{\mathfrak{T}}_{\mathfrak{S}}(X,Y)}$ are functors which preserves limits from $\mathfrak{S}$ and colimits from $\mathfrak{T}$. In particular, objects of $C_{\mathcal{H}om^w_w(X,Y)}$ are functors $C_Y \longrightarrow C_X$ which are weakly right and left exact.

10.2.3. Functoriality of these $\mathcal{H}om$-s. The maps

$$(X,Y) \longmapsto \mathcal{H}om^c(X,Y) \quad \text{and} \quad (X,Y) \longmapsto \mathcal{H}om^{\mathfrak{T}}_{\mathfrak{S}}(X,Y),$$

in particular, the maps $(X,Y) \longmapsto \mathcal{H}om^w(X,Y)$ and $(X,Y) \longmapsto \mathcal{H}om_w(X,Y)$, are functorial and stable under the composition (1), hence they define enriched subcategories of the enriched category of morphisms.

10.3. Flat descent of morphisms. Any pair $U \xrightarrow{u} X$ and $V \xrightarrow{v} Y$ of morphisms such that v is continuous induces a morphism

$$(u|v) : \mathcal{H}om(U,V) \longrightarrow \mathcal{H}om(X,Y) \tag{1}$$

having an inverse image functor $(u^*|v_*)$ defined by

$$C_{\mathcal{H}om(X,Y)} \longrightarrow C_{\mathcal{H}om(U,V)}, \quad \phi^* \longmapsto u^*\phi^*v_*. \tag{2}$$

10.3.1. Lemma. *(a) If $U \xrightarrow{u} X$ and $V \xrightarrow{v} Y$ are continuous morphisms, then $(u|v)$ is continuous.*

(b) If u^* and v_* are weakly continuous (i.e. they preserve cokernels of reflexive pairs of arrows), then the morphism $(u|v)$ induces a morphism

$$(u|v)_w : \mathcal{H}om^w(U,V) \longrightarrow \mathcal{H}om^w(X,Y).$$

(c) If the morphisms u, v are continuous and v_* has a right adjoint (e.g. v is affine), then the morphism $(u|v)$ induces a morphism

$$(u|v)_c : \mathcal{H}om^c(U,V) \longrightarrow \mathcal{H}om^c(X,Y)$$

which is continuous if u_* has a right adjoint.

Proof. (a) If u has a direct image, u_*, then the functor

$$(u_*|v^*) : C_{\mathcal{H}om(U,V)} \longrightarrow C_{\mathcal{H}om(X,Y)}, \quad \psi^* \longmapsto u_*\phi^*v^*, \tag{3}$$

is a direct image functor of the morphism $(u|v)$.

In fact,

$$(u_*|v^*) \circ (u^*|v_*) = (u_*u^*|v_*v^*) : \phi^* \longmapsto u_*u^*\phi^*v_*v^*,$$

and

$$(u^*|v_*) \circ (u_*|v^*) = (u^*u_*|v^*v_*) : \psi^* \longmapsto u^*u_*\psi^*v^*v_*.$$

Adjunction arrow $\eta_u,\ \eta_v$ (resp. $\epsilon_u,\ \epsilon_v$) determine adjunction arrows

$$\eta_{(u|v)} : Id_{C_{\mathcal{H}om(X,Y)}} \longrightarrow (u|v)_* \circ (u|v)^* = (u_*u^*|v_*v^*)$$

and

$$\epsilon_{(u|v)} : (u|v)^*(u|v)_* = (u^*u_*|v^*v_*) \longrightarrow Id_{C_{\mathcal{H}om(U,V)}}.$$

(b) If the functors u^*, v_* and $\phi^* : C_Y \longrightarrow C_X$ preserve kernels of coreflexive pairs of arrows, then their composition, $u^*\phi^*v_* = (u|v)^*(\phi^*)$ has the same property.

(c) If each of the functors u^*, v_* and ϕ^* has a right adjoint, then the functor $(u|v)^*(\phi^*) = u^*\phi^*v_*$ has a right adjoint. Similarly, if the functors u_* and ψ^* have right adjoints, then $(u|v)_*(\psi^*) = u_*\psi^*v^*$ has a right adjoint. ■

10.3.2. Lemma. *Let $U \xrightarrow{u} X$ be a continuous morphism with an inverse image u^* and a direct image u_*. Suppose the category C_X has kernels of coreflexive pairs of arrows and the functor u^* preserves these kernels. Then the morphism u (i.e. its inverse image u^*) is conservative iff the canonical diagram*

$$Id_{C_X} \xrightarrow{\eta_u} F_u \overset{F_u\eta_u}{\underset{\eta_u F_u}{\rightrightarrows}} F_u^2 \tag{4}$$

is exact. Here $F_u = u_*u^*$ and η_u is an adjunction morphism.

Proof. Since the pair of arrows

$$F_u \overset{F_u\eta_u}{\underset{\eta_u F_u}{\rightrightarrows}} F_u^2 \tag{5}$$

is coreflexive, and by hypothesis, the category C_X has kernels of coreflexive arrows, the pair (5) has a kernel, K_η, and the morphism $\eta : Id_{C_X} \longrightarrow F_u$ is uniquely decomposed into $Id_{C_X} \xrightarrow{s_u} K_\eta \xrightarrow{\kappa_u} F_u$, where κ_u is the canonical morphism. Notice that the diagram

$$u^* \xrightarrow{u^*\eta_u} u^*F_u \overset{u^*F_u\eta_u}{\underset{u^*\eta_u F_u}{\rightrightarrows}} u^*F_u^2 \tag{6}$$

is exact without any additional conditions on u (except for u being continuous). Since u^* preserves kernels of coreflexive pairs of arrows, the exactness of (6) is equivalent to that $u^*(s_u)$ is an isomorphism. Thus, if u^* is conservative, then s_u is an isomorphism, i.e. the diagram (4) is exact.

Conversely, suppose the diagram (4) is exact. Let $s : L \longrightarrow M$ be a morphism such that $u^*(s)$ is an isomorphism. Then the vertical arrows $F_u(s)$ and $F_u^2(s)$ in the commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\eta_u(L)} & F_u(L) & \overset{F_u\eta_u(L)}{\underset{\eta_uF_u(L)}{\rightrightarrows}} & F_u^2(L) \\ s\downarrow & & F_u(s)\downarrow & & \downarrow F_u^2(s) \\ M & \xrightarrow{\eta_u(M)} & F_u(M) & \overset{F_u\eta_u(M)}{\underset{\eta_uF_u(M)}{\rightrightarrows}} & F_u^2(M) \end{array} \qquad (7)$$

are isomorphisms. Since the horizontal subdiagrams in (7) are exact, this implies that s is an isomorphism. ∎

10.3.3. Note. We shall also refer to 10.3.2 when the dual assertion is needed:

Let $U \xrightarrow{u} X$ be a continuous morphism with an inverse image u^* and a direct image u_*. Suppose the category C_U has cokernels of reflexive pairs of arrows and the functor u_* preserves these cokernels. Then the functor u_* is conservative iff the canonical diagram

$$G_u^2 \overset{G_u\epsilon_u}{\underset{\epsilon_uG_u}{\rightrightarrows}} G_u \xrightarrow{\epsilon_u} Id_{C_U} \qquad (8)$$

is exact. Here $G_u = u^*u_*$ and ϵ_u an adjunction morphism.

10.3.4. Proposition. *Let $U \xrightarrow{u} X$ and $V \xrightarrow{v} Y$ be continuous morphisms.*

(a) If u is weakly coflat (i.e. u_* preserves cokernels of reflexive pairs of arrows), then the morphism

$$(u|v) : \mathcal{H}om(U,V) \longrightarrow \mathcal{H}om(X,Y)$$

has the same property.

(b) Suppose both u and v are weakly affine (i.e. they are weakly coflat and their direct image functors are conservative); and let the category C_X have cokernels of reflexive pairs of arrows. Then the morphism

$$(u|v)_w : \mathcal{H}om^w(U,V) \longrightarrow \mathcal{H}om^w(X,Y)$$

is weakly affine.

(c) Suppose that u and v are weakly flat and conservative, and the category C_U has limits of coreflexive pairs of arrows. Then the morphism

$$(u|v)^w : \mathcal{H}om_w(U,V) \longrightarrow \mathcal{H}om_w(X,Y)$$

is weakly flat and conservative.

Proof. (a) The functor

$$(u|v)_* : C_{\mathcal{H}om(U,V)} \longrightarrow C_{\mathcal{H}om(X,Y)}, \quad \phi^* \longmapsto u_*\phi^*v^*,$$

preserves all colimits the functor u_* preserves, because the functor $\phi^* \longmapsto \phi^*v^*$ preserves all (small) colimits and limits.

(b) Let $G_v = v^*v_*$, and let $\epsilon_v : G_v \longrightarrow Id_{C_V}$ be an adjunction morphism. If v_* preserves cokernels of reflexive pairs of arrows and is conservative, then, by 10.3.3, the diagram

$$G_v^2 \underset{\epsilon_v G_v}{\overset{G_v\epsilon_v}{\rightrightarrows}} G_v \xrightarrow{\epsilon_v} Id_{C_V} \tag{9}$$

is exact. If $\phi^* : C_Y \longrightarrow C_X$ is an object of $C_{\mathcal{H}om^w(X,Y)}$, i.e. ϕ^* preserves cokernels of reflexive pairs of arrows, the diagram $\phi^*(9)$ is exact. If, in addition, the morphism u_* preserves cokernels of reflexive pairs of arrows and is conservative, hence the diagram $G_u^2 \underset{\epsilon_u G_u}{\overset{G_u\epsilon_u}{\rightrightarrows}} G_u \xrightarrow{\epsilon_u} Id_{C_U}$ is exact, than the diagram

$$G_u^2\phi^*G_v^2 = G_{(u|v)}^2(\phi^*) \underset{\epsilon_{(u|v)}G_{(u|v)}}{\overset{G_{(u|v)}\epsilon_{(u|v)}}{\rightrightarrows}} G_{(u|v)}(\phi^*) = G_u\phi^*G_v \xrightarrow{\epsilon_{(u|v)}} \phi^* \tag{10}$$

is exact. If the category C_X has a certain type of colimits, the category $C_{\mathcal{H}om(X,Y)}$ and its subcategory $C_{\mathcal{H}om^w(X,Y)}$ have the same type of colimits. In particular, $C_{\mathcal{H}om^w(X,Y)}$ has cokernels of reflexive pairs of arrows, because C_X has them.

Thus, the morphism $(u|v)_w : \mathcal{H}om^w(U,V) \longrightarrow \mathcal{H}om^w(X,Y)$ satisfies the conditions of 9.3.3, hence its direct image functor is conservative.

(c) The assertion (c) is dual (therefore equivalent) to the assertion (b). ■

10.3.5. Corollary. *Let $U \xrightarrow{u} X$ and $V \xrightarrow{v} Y$ be continuous morphisms.*

(a) Suppose both u and v are weakly affine; and let the category C_X have cokernels of reflexive pairs of arrows. Then the morphism

$$(u|v)_w : \mathcal{H}om^w(U,V) \longrightarrow \mathcal{H}om^w(X,Y)$$

is monadic, i.e. it is isomorphic to the canonical morphism

$$\mathbf{Sp}(\mathcal{F}_{(u|v)_w}/\mathcal{H}om^w(X,Y)) \longrightarrow \mathcal{H}om^w(X,Y).$$

Here $\mathcal{F}_{(u|v)^w}$ is the monad on $\mathcal{H}om^w(X,Y)$ associated with the (choice of an inverse and a direct image functors and adjunction arrows of the) morphism $(u|v)$.

(b) Suppose that u and v are weakly flat and conservative, and the category C_U has limits of coreflexive pairs of arrows. Then the morphism

$$(u|v)^w : \mathcal{H}om_w(U,V) \longrightarrow \mathcal{H}om_w(X,Y)$$

is comonadic, i.e. it is isomorphic to the canonical morphism

$$\mathcal{H}om_w(U,V) \longrightarrow \mathbf{Sp^o}(\mathcal{H}om_w(U,V)\backslash\mathcal{G}_{(u|v)^w}).$$

Here $\mathcal{G}_{(u|v)^w}$ *is the comonad on* $\mathcal{H}om_w(U,V)$ *associated with the morphism* $(u|v)$*.*

Proof. (a) By 10.3.4(b), the morphism $(u|v)_w$ satisfies the conditions of the Beck's theorem 5.4.2.

(b) Similarly, by 10.3.4(c), the morphism $(u|v)^w$ satisfies the conditions of the (dual) Beck's theorem 5.4.1.

The assertions follow from the respective versions of the Beck's theorem. ■

Denote by $\mathcal{H}om^{wfl}(X,Y)$ the object (in $|Cat|^o$) of *weakly flat morphisms*: objects of the category of quasi-coherent modules on $\mathcal{H}om^{wfl}(X,Y)$ are functors $\phi^* : C_Y \longrightarrow C_X$ which have a right adjoint and preserve kernels of coreflexive pairs of arrows.

10.3.6. Proposition. *Let* $U \xrightarrow{u} X$ *and* $V \xrightarrow{v} Y$ *be affine weakly flat morphisms. Then a direct image functor of the morphism*

$$(u|v)_c : \mathcal{H}om^c(U,V) \longrightarrow \mathcal{H}om^c(X,Y)$$

is conservative and preserves colimits of small diagrams.

The morphism $(u|v)_c$ *induces a weakly flat conservative morphism*

$$(u|v)_{wfl} : \mathcal{H}om^{wfl}(U,V) \longrightarrow \mathcal{H}om^{wfl}(X,Y)$$

whose direct image functor preserves colimits of small diagrams.

Proof. Since u and v are affine, in particular u_* and v_* have right adjoints, it follows from 10.3.1(c) that $(u|v)_c$ is a continuous morphism. By 10.3.4, a direct image functor of the morphism $(u|v)_c$ is conservative. ■

10.3.7. Corollary. *Let* $U \xrightarrow{u} X$ *and* $V \xrightarrow{v} Y$ *be affine weakly flat morphisms. Then*

(a) The morphism

$$(u|v)_c : \mathcal{H}om^c(U,V) \longrightarrow \mathcal{H}om^c(X,Y)$$

is isomorphic to the canonical morphism

$$\mathbf{Sp}(\mathcal{F}_{(u|v)_c}/\mathcal{H}om^c(X,Y)) \longrightarrow \mathcal{H}om^c(X,Y),$$

where $\mathcal{F}_{(u|v)_c} = (F_{(u|v)_c}, \mu_(u|v))$ *is a monad associated with* $(u|v)_c$*. The functor* $F_{(u|v)_c}$ *preserves colimits.*

(b) The morphism $(u|v)_c$ *is the composition of the standard morphism*

$$\mathcal{H}om^c(U,V) \longrightarrow \mathbf{Sp^o}(\mathcal{H}om^c(U,V)\backslash\mathcal{G}_{(u|v)_c})$$

and the canonical morphism

$$\mathbf{Sp^o}(\mathcal{H}om^c(U,V)\backslash\mathcal{G}_{(u|v)_c}) \longrightarrow \mathcal{H}om^c(X,Y). \tag{11}$$

The morphism (11) is a localization.

(c) The morphism

$$(u|v)_{wfl} : \mathcal{H}om^{wfl}(U,V) \longrightarrow \mathcal{H}om^{wfl}(X,Y)$$

is isomorphic to the canonical morphism

$$\mathcal{H}om^{wfl}(U,V) \longrightarrow \mathbf{Sp^o}(\mathcal{H}om^{wfl}(U,V)\backslash\mathcal{G}_{(u|v)_{wfl}}),$$

where $\mathcal{G}_{(u|v)_{wfl}} = (G_{(u|v)_{wfl}}, \delta_{(u|v)})$ *is a comonad associated with* $(u|v)_{wfl}$*. The functor* $G_{(u|v)_{wfl}}$ *preserves colimits.*

Proof. The assertion is a consequence of 10.3.6 and the Beck's theorem 5.4.1. ∎

10.3.7.1. A description of the morphisms in 10.3.7. Since the morphisms $U \xrightarrow{u} X$ and $V \xrightarrow{v} Y$ are affine and weakly flat, X and Y can be identified with resp. $\mathbf{Sp^o}(U\backslash\mathcal{G}_u)$ and $\mathbf{Sp^o}(V\backslash\mathcal{G}_v)$. One can define a (fully faithful) direct image functor of the morphism (11) as follows.

Let $\mathcal{M} = (M,\zeta)$ be a $\mathcal{G}_{(u|v)}$-comodule; and let $\mathcal{L} = (L,\xi_L)$ be a $\mathcal{G}_v$-comodule (i.e. an object of C_Y). Notice that the $\mathcal{G}_{(u|v)}$-comodule structure on M is the composition of a right coaction, $\zeta^v : M \longrightarrow MG_v$, and a left coaction, $\zeta_u : M \longrightarrow G_uM$. In other words, $\mathcal{M}$ is naturally regarded as a $((U\backslash\mathcal{G}_u),(V\backslash\mathcal{G}_v))$-bicomodule (M,ζ_u,ζ^v).

Thus, we have a coreflexive pair of morphisms

$$M(L) \underset{\zeta^v(L)}{\overset{M(\xi_L)}{\rightrightarrows}} MG_v(L). \tag{12}$$

The comodule structure $\zeta_u : M \longrightarrow G_uM$ induces $\mathcal{G}_u$-comodule structure on $M(L)$ and $MG_v(L)$, so that (12) becomes a diagram of $\mathcal{G}_u$-comudules. Since the functor $G_u = u^*u_*$ preserves kernels of coreflexive pairs of arrows, the kernel $\widetilde{\mathcal{M}}(\mathcal{L})$ of the (coreflexive) pair (12) has a (necessarily unique) $\mathcal{G}_u$-comodule structure, $\xi_{\widetilde{\mathcal{M}}(\mathcal{L})} : \widetilde{\mathcal{M}}(\mathcal{L}) \longrightarrow G_u\widetilde{\mathcal{M}}(\mathcal{L})$, such that the canonical morphism $\widetilde{\mathcal{M}}(\mathcal{L}) \longrightarrow \mathcal{M}(\mathcal{L})$ is a $\mathcal{G}_u$-comodule morphism.

The correspondence $(\mathcal{M},\mathcal{L}) \longmapsto (\widetilde{\mathcal{M}}(\mathcal{L}),\xi_{\mathcal{M}(\mathcal{L})})$ is functorial in $\mathcal{M}$ and $\mathcal{L}$. The functor which assigns to each $(\mathcal{H}om^c(U,V)\backslash\mathcal{G}_{(u|v)})$-comodule $\mathcal{M}$ the functor

$$(Y\backslash\mathcal{G}_v) - Comod \longrightarrow (X\backslash\mathcal{G}_u) - Comod, \quad \mathcal{L} \longmapsto (\widetilde{\mathcal{M}}(\mathcal{L}),\xi_{\widetilde{\mathcal{M}}(\mathcal{L})})$$

is a direct image functor of the morphism (11).

10.3.7.2. The composition of morphisms. Let $U \xrightarrow{u} X$, $V \xrightarrow{v} Y$ and $W \xrightarrow{w} Z$ be affine weakly flat morphisms which allows to assume that $X = \mathbf{Sp^o}(U\backslash\mathcal{G}_u)$,

$Y = \mathbf{Sp^o}(V\backslash\mathcal{G}_v)$, and $Z = \mathbf{Sp^o}(W\backslash\mathcal{G}_w)$. Let $\mathcal{M} = (M, \zeta_u, \zeta^v)$ be a $((U\backslash\mathcal{G}_u), (V\backslash\mathcal{G}_v))$-bicomodule and $\mathcal{N} = (N, \zeta_v, \zeta^w)$ a $((V\backslash\mathcal{G}_v), (W\backslash\mathcal{G}_w))$-bicomodule. Consider the diagram

$$MN \underset{\zeta_v}{\overset{\zeta^v}{\rightrightarrows}} MG_vN.$$

This diagram is coreflexive, hence by hypothesis it has a kernel, $M \odot N$. The coaction $\zeta^w : N \longrightarrow NG_w$ induces a right $(W\backslash\mathcal{G}_w)$-comodule structure, $M \odot \zeta^w$ on $M \odot N$. Since the functor $G_u = u^*u_*$ is weakly flat, by a standard argument, the left $(U\backslash\mathcal{G}_u)$-comodule structure $\zeta_u : M \longrightarrow G_uM$ induces a left $(U\backslash\mathcal{G}_u)$-comodule structure, $\zeta \odot N$, on $M \odot N$. We denote by $\mathcal{M} \odot \mathcal{N}$ the $((U\backslash\mathcal{G}_u), (W\backslash\mathcal{G}_w))$-bicomodule $(M \odot N, \zeta_u \odot N, M \odot \zeta^w)$.

The functor

$$\phi^*_{\mathcal{M}\odot\mathcal{N}} : (W\backslash\mathcal{G}_w) - Comod \longrightarrow (U\backslash\mathcal{G}_u) - Comod$$

corresponding to $\mathcal{M} \odot \mathcal{N}$ is isomorphic to the composition $\phi^*_{\mathcal{M}} \circ \phi^*_{\mathcal{N}}$ of the functors corresponding to the resp. bicomodules $\mathcal{N}$ and $\mathcal{M}$. This follows from general facts on commuting limits and from the fact that the limits of functors are computed object-wise (cf. [GZ], Glossary).

Complementary facts.

C1. Continuous morphisms to the categoric spectrum of a monoid. Let $\mathcal{M}$ be a monoid. Denote by $\mathbf{Sp}(\mathcal{M}/\mathcal{E})$ the object of $|Cat|^o$ corresponding to the category $\mathcal{M} - Sets$ of $\mathcal{M}$-sets. We call $\mathbf{Sp}(\mathcal{M}/\mathcal{E})$ the *categoric spectrum of the monoid* $\mathcal{M}$.

C1.1. Proposition. *Let* $X \xrightarrow{f} \mathbf{Sp}(\mathcal{M}/\mathcal{E})$ *be a continuous morphism. Then*

(a) The morphism f *is determined by the object* $\mathcal{O} = f^*(\mathcal{M})$ *uniquely up to isomorphism.*

(b) There exists a coproduct of any set of copies of $\mathcal{O}$.

(c) The object $\mathcal{O}$ *has a structure of an* $\mathcal{M}^o$*-module, i.e. there is a monoid morphism* $\mathcal{M} \xrightarrow{\phi_f} C_X(\mathcal{O}, \mathcal{O})^o$.

Proof. These facts follow from the canonical isomorphisms

$$Hom_X(f^*(\mathcal{M}), -) \simeq Hom_{\mathcal{M}}(\mathcal{M}, f_*(-)) \simeq f_*(-).$$

Since f^* has a right adjoint, it preserves colimits, in particular coproducts. Notice that the category of $\mathcal{M}$-sets has small coproducts and cokernels of pairs of arrows, hence it has arbitrary small colimits. The rest of the argument follows the same lines as the argument of 4.1. Details are left to the reader. ∎

For any object $\mathcal{O}$ of the category C_X, we denote by $\Gamma_X \mathcal{O}$ the monoid $C_X(\mathcal{O}, \mathcal{O})^o$.

We denote by $|Cat|^o(\mathcal{M}/\mathcal{E})$ the category of *right* $(\mathcal{M}/\mathcal{E})$*-modules*. Its objects are triples $(X, \mathcal{O}_X, \phi)$, where X is an object of $|Cat|^o$, $(\mathcal{O}_X, \phi)$ is a right $\mathcal{M}$-module in the category C_X, i.e. ϕ is a monoid morphism $\mathcal{M} \longrightarrow C_X(\mathcal{O}_X, \mathcal{O}_X)^o$ (cf. C1.1(c)). Morphisms from $(X, \mathcal{O}_X, \phi)$ to $(Y, \mathcal{O}_Y, \psi)$ are given by morphisms $X \xrightarrow{f} Y$ such that there exists an isomorphism $\lambda : f^*(\mathcal{O}_Y) \xrightarrow{\sim} \mathcal{O}_X$ making the diagram

$$\begin{array}{ccc} \mathcal{M}^o & \xrightarrow{\phi} & C_X(\mathcal{O}_X, \mathcal{O}_X) \\ \psi \downarrow & & \downarrow \mathfrak{c}_\lambda \\ C_Y(\mathcal{O}_Y, \mathcal{O}_Y) & \xrightarrow{f^*_{\mathcal{O}_Y, \mathcal{O}_Y}} & C_X(f^*(\mathcal{O}_Y), f^*(\mathcal{O}_Y)) \end{array} \qquad (2^{bis})$$

commute. Here $\mathfrak{c}_\lambda$ denotes the conjugation by λ, $h \longmapsto \lambda^{-1} \circ h \circ \lambda$.

Let $|Cat|^o(\mathcal{M}/\mathcal{E})_\mathfrak{c}$ denote the full subcategory of the category $|Cat|^o(\mathcal{M}/\mathcal{E})$ whose objects are triples $(X, \mathcal{O}_X, \phi)$ such that the following conditions hold:

(a) There exists a coproduct of any set of copies of $\mathcal{O}_X$.

(b) Let Φ_ϕ denote the functor from the subcategory $\mathfrak{L}_{\mathcal{M}}$ of free $(\mathcal{M}/\mathcal{E})$-sets to C_X which is uniquely defined by the action ϕ (thanks to (b) above). The image by Φ_ϕ of any pair of arrows $X_1 \rightrightarrows X_0$ has a cokernel.

Let $Cat^{op}(\mathcal{M}/\mathcal{E})_\mathfrak{c}$ denote the preimage of $|Cat|^o(\mathcal{M}/\mathcal{E})_\mathfrak{c}$ in $Cat^{op}(\mathcal{M}/\mathcal{E})$. On the other hand, let $(|Cat|^o/\mathbf{Sp}(\mathcal{M}/\mathcal{E})_\mathfrak{c}$ denote the full subcategory of the category $|Cat|^o/\mathbf{Sp}(\mathcal{M}/\mathcal{E})$ whose objects are continuous morphisms to $\mathbf{Sp}(\mathcal{M}/\mathcal{E})$, and let $(Cat^{op}/\mathbf{Sp}(\mathcal{M}/\mathcal{E}))_\mathfrak{c}$ denote its preimage in $Cat^{op}/\mathbf{Sp}(\mathcal{M}/\mathcal{E})$.

C1.2. Proposition. *The functor (3) induces an equivalence of the fibered categories*

$$\begin{pmatrix} (Cat^{op}/\mathbf{Sp}(\mathcal{M}/\mathcal{E}))_{\mathfrak{c}} \\ \big\downarrow \\ (|Cat|^o/\mathbf{Sp}(\mathcal{M}/\mathcal{E}))_{\mathfrak{c}} \end{pmatrix} \longrightarrow \begin{pmatrix} (Cat^{op}(\mathcal{M}/\mathcal{E})_{\mathfrak{c}} \\ \big\downarrow \\ |Cat|^o(\mathcal{M}/\mathcal{E})_{\mathfrak{c}} \end{pmatrix}. \qquad (4^{bis})$$

Proof. The argument is similar to that of 4.4.2. ■

C1.3. $\mathcal{E}$-Spaces. Denote by $|Cat|^o_{\mathcal{E}}$ the category whose objects are pairs $(X, \mathcal{O})$, where $X \in Ob|Cat|^o$ and $\mathcal{O}$ is an object of C_X such that there exist coproducts of small sets of copies of $\mathcal{O}$ and any pair of arrows $\mathcal{O}^{\oplus I} \rightrightarrows \mathcal{O}^{\oplus J}$ between coproducts of copies of $\mathcal{O}$ has a cokernel in C_X (equivalently, every small diagram $\mathfrak{D} : \mathcal{D} \longrightarrow C_X$ which maps all objects to $\mathcal{O}$ has a colimit in C_X). Morphisms from $(X, \mathcal{O})$ to $(X', \mathcal{O}')$ are morphisms $X \xrightarrow{f} X'$ such that there exists an isomorphism $f^*(\mathcal{O}') \xrightarrow{\sim} \mathcal{O}$. Composition is defined in an obvious way. Objects of the category $|Cat|^o_{\mathcal{E}}$ will be called $\mathcal{E}$-'spaces'.

It follows that $\mathcal{E}$-'spaces' are precisely those pairs $(X, \mathcal{O})$ for which the functor

$$f_{\mathcal{O}*} : C_X \longrightarrow \Gamma_X\mathcal{O} - Sets, \;\; M \longmapsto C_X(\mathcal{O}, M),$$

is a direct image functor of a continuous morphism $f_{\mathcal{O}} : X \longrightarrow \mathbf{Sp}(\Gamma_X\mathcal{O}/\mathcal{E})$; i.e. $f_{\mathcal{O}*}$ has a left adjoint.

C1.4. Affine $\mathcal{E}$-spaces. We call a $\mathcal{E}$-space $(X, \mathcal{O})$ *affine* if the canonical morphism $f_{\mathcal{O}} : X \longrightarrow \mathbf{Sp}(\Gamma_X\mathcal{O})$ is an isomorphism; i.e. the functor $f_{\mathcal{O}*}$ (see (6)) is a category equivalence. We denote by $\mathbf{Aff}_{\mathcal{E}}$ the full subcategory of the category $|Cat|^o_{\mathcal{E}}$ objects of which are affine $\mathcal{E}$-spaces.

Let $\mathfrak{Aff}_{\mathcal{E}}$ denote the subcategory of the category $|Cat|^o_{\mathcal{E}}$ whose objects are pairs $(\mathbf{Sp}(\mathcal{M}/\mathcal{E}), \mathcal{M})$. Morphisms from $(\mathbf{Sp}(\mathcal{M}/\mathcal{E}), \mathcal{M})$ to $(\mathbf{Sp}(\mathcal{M}'/\mathcal{E}), \mathcal{M}')$ are morphisms $\mathbf{Sp}(\mathcal{M}/\mathcal{E}) \longrightarrow \mathbf{Sp}(\mathcal{M}'/\mathcal{E})$ corresponding to monoid morphisms $\mathcal{M}' \longrightarrow \mathcal{M}$. There is an inclusion functor $\mathfrak{Aff}_{\mathcal{E}} \xrightarrow{\gamma_*} |Cat|^o_{\mathcal{E}}$ which takes values in the subcategory of affine $\mathcal{E}$-spaces.

C1.4.1. Proposition. *The functor $\mathfrak{Aff}_{\mathcal{E}} \xrightarrow{\gamma_*} |Cat|^o_{\mathcal{E}}$ is fully faithful and has a left adjoint. In particular, γ_* induces an equivalence of $\mathfrak{Aff}_{\mathcal{E}}$ and the category of affine $\mathcal{E}$-spaces.*

Proof. The argument is similar to that of 4.5.4 and is left to the reader. ■

C2. Continuous morphisms, monads, and localizations. Let $X \xrightarrow{q} Y$ be a localization with an inverse image functor $C_Y \xrightarrow{q^*} C_X$. Let Σ_q denote the class of all morphisms s in C_Y such that $q^*(s)$ is invertible. A functor $C_Y \xrightarrow{F} C_Y$ is compatible with the localization q iff $F(\Sigma_q) \subseteq \Sigma_q$. In this case, there exists a unique functor $C_X \xrightarrow{\bar{F}} C_X$ such that $q^* \circ F = \bar{F} \circ q^*$.

C2.1. Proposition. *Let $X \xrightarrow{q} Y$ be a continuous localization.*

(a) A functor $C_Y \xrightarrow{F} C_Y$ is compatible with q iff the canonical morphism

$$q^* \circ F \longrightarrow q^* \circ F \circ q_* q^* \qquad (1)$$

is an isomorphism.

(b) Suppose $C_Y \xrightarrow{F} C_Y$ is compatible with the localization q, and let $\bar{F}$ be the functor $C_X \longrightarrow C_X$ such that $q^* \circ F = \bar{F} \circ q^*$.

(i) If C_Y has colimits of certain type, then C_X has colimits of this type. If F preserves colimits of this type, then the functor $\bar{F}$ has the same property.

(ii) If C_Y has limits of certain type, then C_X has limits of this type. If F and q^* preserve limits of this type (e.g. finite limits), then the functor $\bar{F}$ has the same property.

(iii) If F has a right adjoint, then $\bar{F}$ has a right adjoint.

Proof. (a) The assertion follows from C2.4.1.1 applied to $f^* = q^* \circ F$.

(b) (i) Let $\mathcal{D} : D \longrightarrow C_X$ be a small diagram such that there exists $colim(q_*\mathcal{D})$. Then there exists the colimit of $\mathcal{D}$ and $colim(\mathcal{D}) = q^* colim(q_*\mathcal{D})$. Suppose the functor F preserves the colimit of $q_*\mathcal{D}$. Since $\bar{F} \simeq q^*Fq_*$, we have:

$$\bar{F}(colim(\mathcal{D})) \simeq q^*Fq_*q^*(colim(q_*\mathcal{D})) \simeq q^*F(colim(q_*\mathcal{D})) \simeq q^*(colim(Fq_*\mathcal{D})$$

(the second isomorphism here is due to the isomorphism $q^*F \xrightarrow{\sim} q^*Fq_*q^*$). Since q^* has a right adjoint, it preserves colimits. Therefore

$$q^*(colim(Fq_*\mathcal{D}) \simeq colim(q^*Fq_*\mathcal{D}) \simeq colim(\bar{F}\mathcal{D})$$

whence the assertion.

(ii) Let $\mathcal{D} : D \longrightarrow C_X$ be a small diagram such that there exists $\lim(q_*\mathcal{D})$. Then, by [GZ, I.1.4], there exists the limit of $\mathcal{D}$ and $\lim(\mathcal{D}) = q^*(\lim(q_*\mathcal{D}))$. Let the functor F preserve the colimit of $q_*\mathcal{D}$. As in (i), we have:

$$\bar{F}(\lim(\mathcal{D})) \simeq q^*Fq_*q^*(\lim(q_*\mathcal{D})) \simeq q^*F(\lim(q_*\mathcal{D})) \simeq q^*(\lim(Fq_*\mathcal{D})$$

If q^* preserves limit of $Fq_*\mathcal{D}$, we continue as follows:

$$q^*(\lim(Fq_*\mathcal{D}) \simeq (\lim(q^*Fq_*\mathcal{D}) \simeq \lim(\bar{F}\mathcal{D}).$$

(iii) Let $F^!$ be a right adjoint to F. Set $\bar{F}^! = q^*F^!q_*$. By (a), $\bar{F} \simeq q^*Fq_*$. Thus we have morphisms

$$\bar{F}\bar{F}^! \xrightarrow{\sim} (q^*Fq_*q^*)F^!q_* \xrightarrow{\sim} q^*FF^!q_* \xrightarrow{q^*\epsilon_F q_*} q^*q_* \xrightarrow{\epsilon_q} Id_{C_X}. \tag{3}$$

and

$$Id_{C_X} \xrightarrow{\sim} q^*q_* \xrightarrow{q^*\eta_F q_*} q^*F^!Fq_* \xrightarrow{q^*F^!\eta_q Fq_*} q^*F^!q_*q^*Fq_* \xrightarrow{\sim} \bar{F}^!\bar{F}. \tag{4}$$

The compositions of the sequence of morphisms resp. (3) and (4) are adjunction arrows. ∎

C2.2. Proposition. *Let $X \xrightarrow{q} Y$ be a localization and $\mathcal{F} = (F, \mu)$ a monad on Y such that the endofunctor F is compatible with q. Then the monad $\mathcal{F}$ induces a monad, $\bar{\mathcal{F}} = (\bar{F}, \bar{\mu})$, on X defined uniquely up to isomorphism.*

(i) If $\mathcal{F}$ is continuous (i.e. F has a right adjoint), then the monad $\bar{\mathcal{F}}$ is continuous.

(ii) If C_Y has colimits of certain type, then C_X has colimits of this type. If F preserves colimits of this type, then $\bar{F}$ has the same property.

(iii) If C_Y has limits of certain type, then C_X has limits of this type. If F and q^* preserve limits of this type, then $\bar{F}$ has the same property.

Proof. Fix an inverse image, q^*, of the localization q. Let $\bar{F}$ be a unique endofunctor $\bar{F} : C_X \longrightarrow C_X$ such that $q^* \circ F = \bar{F} \circ q^*$. Then $q^* \circ F^2 = \bar{F}^2 \circ q^*$, and, by the universal property of localizations, there exists a unique morphism $\bar{\mu} : \bar{F}^2 \longrightarrow \bar{F}$ such that $q^*\mu = \bar{\mu}q^*$. We leave to the reader verifying that $\bar{\mu}$ is a monad structure on $\bar{F}$.

The assertions (i), (ii), (iii) follow from the corresponding assertions of C2.1. ■

C2.2.1. Remark. The same assertion holds for comonads. In fact, the first part is obtained by dualization. The parts (i) and (ii) are statements about endofunctors.

C3. Cones of non-unital monads and rings.

C3.1. Non-unital monads. Let X be a 'space' such that C_X is an additive category, and let $\mathbb{F}_+ = (F_+, \mu)$ be a non-unital additive monad on X; i.e. F_+ is an additive functor $C_X \longrightarrow C_X$ and μ is a functor morphism $F_+^2 \longrightarrow F_+$ such that $\mu \circ F_+\mu = \mu \circ \mu F_+$. Let $\mathbb{F}_+ - mod_1$ denote the category of non-unital $\mathbb{F}_+$-modules. Its objects are pairs (M, ξ), where $M \in ObC_X$ and ξ a morphism $F_+(M) \longrightarrow M$ such that $\xi \circ \mu(M) = \xi \circ F_+\xi$. A morphism $(M, \xi) \longrightarrow (M', \xi')$ is given by a morphism $M \xrightarrow{f} M'$ such that $\xi' \circ F_+(f) = f \circ \xi$. Composition is defined naturally, so that map which assigns to each $\mathbb{F}_+$-module (M, ξ) the object M and to every $\mathbb{F}_+$-module morphism $(M, \xi) \xrightarrow{f} (M', \xi')$ the morphism $M \xrightarrow{f} M'$ is a functor, $\mathbb{F}_+ - mod_1 \xrightarrow{\mathfrak{f}_*} C_X$. This functor has a canonical left adjoint, $\mathfrak{f}^*$, which maps every object N of C_X to the $\mathbb{F}_+$-module $(N \oplus F(N), \xi_N)$, where the action $F_+(N \oplus F_+(N)) = F_+(N) \oplus F_+^2(N) \xrightarrow{\xi_N} N \oplus F_+(N)$ is the composition of the morphism $F_+(N) \oplus F_+^2(N) \xrightarrow{(id_{F_+(N)}, \mu(N)} F_+(N)$ and the embedding $F_+(N) \longrightarrow N \oplus F_+(N)$.

Thus, $\mathfrak{f}_*\mathfrak{f}^* = Id_{C_X} \oplus F$. We denote $Id_{C_X} \oplus F_+$ by F and the monad corresponding to the pair of adjoint functors $\mathfrak{f}_*, \mathfrak{f}^*$ by $\mathbb{F} = (F, \mu_1)$. It is easy to see that the category $\mathbb{F}_+ - mod_1$ of non-unital $\mathbb{F}_+$-modules is isomorphic to the category $\mathbb{F} - mod$ of unital $\mathbb{F}$-modules.

There is a natural embedding $C_X \longrightarrow \mathbb{F}_+ - mod$ which assigns to each object M of C_X the $\mathbb{F}_+$-module $(M, 0)$. We denote the image of C_X in $\mathbb{F}_+ - mod$ (i.e. the full subcategory generated by trivial modules) by $\mathcal{T}_{\mathbb{F}_+}$.

C3.2. The cone of a non-unital monad. Suppose that the category C_X is abelian. We denote by $C_{\mathbf{Cone}(\mathbb{F}_+/X)}$ the quotient, $\mathbb{F}_+ - mod_1/\mathcal{T}_{\mathbb{F}_+}^-$, of the category $\mathbb{F}_+ - mod_1$ by the smallest Serre subcategory containing $\mathcal{T}_{\mathbb{F}_+}$. This defines a 'space' $\mathbf{Cone}(\mathbb{F}_+/X)$.

C3.2.1. Proposition. *If $\mathbb{F}_+$ is a unital monad, then $C_{\mathbf{Cone}(\mathbb{F}_+/X)}$ is naturally equivalent to the category $\mathbb{F}_+ - mod$ of unital $\mathbb{F}_+$-modules, i.e. the 'space'* $\mathbf{Cone}(\mathbb{F}_+/X)$ *is isomorphic to* $\mathbf{Sp}(\mathbb{F}_+/X)$.

Proof. If the monad $\mathbb{F}_+ = (F_+, \mu)$ is unital with the unit element $Id \xrightarrow{e} F_+$, then there is a monad epimorphism $F = Id \oplus F_+ \xrightarrow{\gamma} F_+$ defined by (e, id_{F_+}). The corresponding pull-back functor, γ_*, is the inclusion functor $\mathbb{F}_+ - mod$ into $\mathbb{F}_+ - mod_1$. Its left adjoint, γ^* assigns to each object (M, ξ) of $\mathbb{F}_+ - mod_1$ the cokernel of the pair of morphisms $M \underset{\xi e(M)}{\overset{id_M}{\rightrightarrows}} M$. Since the functor γ_* is fully faithful, its left adjoint γ^* is an exact localization, and the kernel of γ^* coincides with the subcategory $\mathcal{T}_{\mathbb{F}_+}$. Therefore, $\mathcal{T}_{\mathbb{F}_+}$ is, in this case, a Serre subcategory, i.e. $\mathcal{T}_{\mathbb{F}_+} = \mathcal{T}_{\mathbb{F}_+}^-$, whence the assertion. ■

C3.2.2. Corollary. *If $\mathbb{F}_+$ is a unital monad, then $\mathbf{Sp}(\mathbb{F}/X) \simeq \mathbf{Sp}(\mathbb{F}_+/X) \coprod X$.*

Proof. If $\mathbb{F}_+$ is a unital monad, then (by the argument of C3.2.1) $\mathbb{F} \simeq \mathbb{F}_+ \prod Id_X$, where Id_X denotes the identical monad (Id_{C_X}, id). This implies that the category $\mathbb{F} - mod$ of $\mathbb{F}$-modules is equivalent to the product $\mathbb{F}_+ - mod \prod C_X$, hence the assertion. ■

C3.3. The cone of an associative ring. Let $X = \mathbf{Sp}(R)$, where R_0 is a unital associative ring, and let R_+ be an R_0*-ring.* The latter means that R_0 is an associative ring, not unital in general, in the category of R_0-bimodules; i.e. the mutiplication in R_+ is given by an R_0-bimodule morphism $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$ satisfying the associativity condition. The R_0-ring R_+ defines a non-unital monad $\mathbb{F}_+ = (F_+, \mu)$ on X, where F_+ is the endofunctor $R_+ \otimes_{R_0} -$ on $C_X = R_0 - mod$ and $F_+^2 \xrightarrow{\mu} F_+$ is induced by the multiplication $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$. The category $\mathbb{F}_+ - mod_1$ of non-unital $\mathbb{F}_+$-modules is the category of unital R_0-modules endowed with a non-unital R_+-module structure compatible with the action of R_0 on the module and on R_+. We write $R_+ - mod_1$ instead of $\mathbb{F}_+ - mod_1$ and $\mathcal{T}_{R_+}$ instead of $\mathcal{T}_{\mathbb{F}_+}$. By definition $\mathcal{T}_{\mathbb{F}_+}$ is the full subcategory of $R_+ - mod_1$ spanned by modules with zero action.

The associated *augmented* monad $\mathbb{F}$ (cf. C3.1) is isomorphic to the monad associated with the unital R_0-ring $R = R_0 \oplus R_+$ which we call the *augmented* R_0*-ring* corresponding to R_+. The category $R_+ - mod_1$ is isomorphic to the category $R - mod$ of unital R-modules.

We shall write $\mathbf{Cone}(R_+/R_0)$, or simply $\mathbf{Cone}(R_+)$, instead of $\mathbf{Cone}(\mathbb{F}_+/\mathbf{Sp}(R_0))$.

We shall identify $R_+ - mod_1$ with $R - mod$ whenever it is convenient by some reason. Thus, $\mathcal{T}_{R_+}$ is viewed as the full subcategory of $R - mod$ whose objects are modules annihilated by the *irrelevant* ideal R_+; and we write $C_{\mathbf{Cone}(R_+)} = R - mod/\mathcal{T}_{R_+}^-$, where $\mathcal{T}_{R_+}^-$ is the smallest Serre subcategory of the category $R - mod$ containing $\mathcal{T}_{R_+}$ (getting back the definition of a cone in 1.6). The localization functor $R - mod \xrightarrow{u^*} R - mod/\mathcal{T}_{R_+}^-$ is an inverse image functor of a morphism of 'spaces' $\mathbf{Cone}(R_+) \xrightarrow{u} \mathbf{Sp}(R)$. The functor u^* has a (necessarily fully faithful) right adjoint, i.e. the morphism u is continuous. The composition of the morphism u with the natural affine morphism $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp}(R_0)$ is a continuous morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Sp}(R_0)$. Its direct image functor is (regarded as) the *global sections functor.*

C3.3.1. Proposition. *If R_+ is a unital ring, then $\mathbf{Cone}(R_+/R_0) \simeq \mathbf{Sp}(R_+)$ and $\mathbf{Sp}(R) \simeq \mathbf{Sp}(R_+) \coprod \mathbf{Sp}(R_0)$.*

Proof. The assertion follows from C3.2.1 and C3.2.2. ■

C3.3.2. Lemma. *Let $\mathcal{J}$ be a two-sided ideal in the ring R contained in R_+ (i.e. a two-sided ideal in R_+ which is an R_0-bimodule). Let $T_{R_+|\mathcal{J}}$ denote the full subcategory of*

$R - mod$ *whose objects are* R*-modules annihilated by* $\mathcal{J}$*; and let* $T^-_{R|\mathcal{J}}$ *be the Serre subcategory spanned by* $T_{R|\mathcal{J}}$*. The quotient category* $R - mod/T^-_{R|\mathcal{J}}$ *is equivalent to* $C_{\mathbf{Cone}(\mathcal{J})}$*.*

Proof. The embedding $\mathcal{J} \hookrightarrow R$ induces a unital ring morphism $\widetilde{\mathcal{J}} \xrightarrow{\iota} R$, where $\widetilde{\mathcal{J}}$ is the ring $R_0 \oplus \mathcal{J}$ with natural multiplication. The pull-back functor $R - mod \xrightarrow{\iota_*} \widetilde{\mathcal{J}} - mod$ induces a functor from the subcategory $T_{R|\mathcal{J}}$ to the subcategory $T_{\mathcal{J}}$. Since the functor ι_* is exact (in a strong sense, that is it preserves small limits and colimits), it maps the Serre subcategory $T^-_{R|\mathcal{J}}$ to the Serre subcategory $T^-_{\mathcal{J}}$. Thus we have a commutative diagram

$$\begin{array}{ccc} R - mod & \xrightarrow{\iota_*} & \widetilde{\mathcal{J}} - mod \\ \uparrow & & \uparrow \\ T^-_{R|\mathcal{J}} & \longrightarrow & T^-_{\mathcal{J}} \end{array} \tag{1}$$

of exact functors. Therefore the functor ι_* induces a functor

$$R - mod/T^-_{R|\mathcal{J}} \longrightarrow \widetilde{\mathcal{J}} - mod/T^-_{\mathcal{J}}. \tag{2}$$

The functor (2) is a category equivalence. In fact, let $\widetilde{\mathcal{J}} - mod \xrightarrow{\Psi} R - mod$ be the functor which assigns to every $\widetilde{\mathcal{J}}$-module M the R-module $\mathcal{J}M$. The cokernel of the embedding $\mathcal{J}M \hookrightarrow M$ belongs to $T_{\mathcal{J}}$, hence the localization $\widetilde{\mathcal{J}} - mod \longrightarrow \widetilde{\mathcal{J}} - mod/T^-_{\mathcal{J}}$ maps this embedding to an isomorphism. We assign to each object M of $\widetilde{\mathcal{J}} - mod$ the composition of the functor Ψ and the localization $R - mod \longrightarrow R - mod/T^-_{R|\mathcal{J}}$. It follows that this functor factors through the localization $\widetilde{\mathcal{J}} - mod \longrightarrow \widetilde{\mathcal{J}} - mod/T^-_{\mathcal{J}}$, i.e. it defines (uniquely) a functor $\widetilde{\mathcal{J}} - mod/T^-_{\mathcal{J}} \xrightarrow{\Phi} R - mod/T^-_{R|\mathcal{J}}$. The functor Φ is a quasi-inverse to the functor (2). ■

C3.3.3. Example: quasi-affine schemes. Quasi-affine schemes are defined (in [EGA II, 5.1.1]) as open *quasi-compact* subschemes ofaffine schemes. Open subschemes of $\mathbf{Spec}A$ are in one-to-onecorrespondence with the radical ideals in A. Quasi-compactness ofan open set defined by an ideal $\mathcal{J}$ means that $\mathcal{J}$ is the radicalof its finitely generated subideal (this holds in noncommutative case too, see [R, I.5.6]. One can show that the category ofquasi-coherent sheaves on the open subscheme of $\mathbf{Spec}A$ defined by the ideal $\mathcal{J}$ is equivalent to the quotient category$A - mod/T^-_{A|\mathcal{J}}$. By C3.3.2, the latter category is equivalent tothe category $C_{\mathbf{Cone}(\mathcal{J})} = \widetilde{\mathcal{J}} - mod/T^-_{\mathcal{J}}$ of modules on the cone $\mathbf{Cone}(\mathcal{J})$ of the (non-unital) R_0-ring $\mathcal{J}$.

C3.3.4. Functoriality. Let $R_0 - Rings$ denote the category of (not necessarily unital) R_0-rings. A morphism of such rings, $R_+ \xrightarrow{\phi} S_+$, is an R_0-bimodule morphism compatible with multiplication. The morphism ϕ induces the pull-back functor

$$S_+ - mod_1 \xrightarrow{\phi_*} R_+ - mod_1$$

which maps the subcategory T_{S_+} of trivial S_+-modules to the category T_{R_+} of trivial R_+-modules. Since ϕ_* is exact and preserves small colimits, it maps the Serre subcategory

$T^-_{S_+}$ spanned by T_{S_+} to the Serre subcategory $T^-_{R_+}$ spanned by T_{R_+}. Therefore, ϕ_* induces a unique functor

$$S_+ - mod_1/T^-_{S_+} \xrightarrow{\bar{\phi}_*} R_+ - mod_1/T^-_{R_+} \qquad (3)$$

such that the diagram

$$\begin{array}{ccc} S_+ - mod_1/T^-_{S_+} & \xrightarrow{\bar{\phi}_*} & R_+ - mod_1/T^-_{R_+} \\ q^*_S \uparrow & & \uparrow q^*_R \\ S_+ - mod_1 & \xrightarrow{\phi_*} & R_+ - mod_1 \end{array} \qquad (4)$$

commutes. In general, the functor $\bar{\phi}_*$ does not have a left adjoint, hence it cannot be interpreted as a direct image functor of a continuous morphism.

C3.3.4.1. The category $R_0 - Rings_1$. We denote by $R_0 - Rings_1$ the subcategory of $R_0 - Rings$ formed by R_0-ring morphisms $R_+ \xrightarrow{\phi} S_+$ whose inverse image functor, $R_+ - mod_1 \xrightarrow{\phi^*} S_+ - mod_1$, is *compatible* with the localizations at resp. $T^-_{R_+}$ and $T^-_{S_+}$. The compatibility means that there exists a functor $C_{\mathbf{Cone}(R_+)} \xrightarrow{\bar{\phi}^*} C_{\mathbf{Cone}(S_+)}$ such that the diagram

$$\begin{array}{ccc} C_{\mathbf{Cone}(R_+)} = R_+ - mod_1/T^-_{R_+} & \xrightarrow{\bar{\phi}^*} & S_+ - mod_1/T^-_{S_+} = C_{\mathbf{Cone}(S_+)} \\ q^*_R \uparrow & & \uparrow q^*_S \\ R_+ - mod_1 & \xrightarrow{\phi^*} & S_+ - mod_+ \end{array} \qquad (5)$$

commutes. Thanks to the universal property of localizations, the functor $\bar{\phi}$ is uniquely determined by the commutativity of (5).

Evidently, all ring isomorphisms belong to $R_0 - Rings_1$. It follows from the universal property of localizations that the composition of morphisms of $R_0 - Rings_1$ belongs to $R_0 - Rings_1$; i.e. $R_0 - Rings_1$ is, indeed, a subcategory of the category $R_0 - Rings$. The map $R_+ \longmapsto \mathbf{Cone}(R_+)$ extends to a functor $R_0 - Rings_1^{op} \longrightarrow |Cat|^o$ which we denote by **Cone**.

C3.3.4.2. Remarks. (a) For any morphism $R_+ \xrightarrow{\varphi} S_+$ of $R_0 - Rings$, the functor

$$\bar{\varphi}^* = q^*_S \varphi^*_1 q_{R*} : C_{\mathbf{Cone}(R_+)} \longrightarrow C_{\mathbf{Cone}(S_+)} \qquad (6)$$

might be regarded as an inverse image functor of a morphism $\mathbf{Cone}(S_+) \xrightarrow{\bar{\varphi}} \mathbf{Cone}(R_+)$. Notice, however, that the map $\varphi \longmapsto \bar{\varphi}$ is not functorial, unless morphisms are picked from the subcategory $R_0 - Rings_1$.

(b) For any morphism $R_+ \xrightarrow{\varphi} S_+$ of $R_0 - Rings_1$, the corresponding morphism $\mathbf{Cone}(S_+) \xrightarrow{\bar{\varphi}} \mathbf{Cone}(R_+)$ is continuous.

This follows from the fact that the functor $C_{\mathbf{Cone}(R_+)} \xrightarrow{q_{R^*}} R_+ - mod_1$ has a right adjoint and from the formula (6).

C3.3.4.3. Proposition. *Let S_+ be an R_0-ring, e a central idempotent element in S_+ (i.e. $e^2 = e$). Then $R_+ = \{r \in S \mid re = er = r\}$ is an R_0-subring in S_+, and the inclusion $R \hookrightarrow S$ is a morphism of $R_0 - Rings_1$.*

Proof is left to the reader. ■

C3.3.5. Remark. For any R_0-ring R_+, we have a canonical morphism (Zariski open immersion) $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Sp}(R)$, $R = R_0 \oplus R_+$, which depends functorially on R_+ (a functor from $R_0 - Rings_1^{op}$). This morphism can be regarded as a noncommutative analogue of the Stone compactification of a locally compact space. If the ring R_+ is unital, then $\mathbf{Sp}(R)$ is the disjoint union of $\mathbf{Sp}(R_+)$ and $\mathbf{Sp}(R_0)$ (see C3.3).

C3.3.6. Hopf actions and cross-products. Let R_0 be an associative unital k-algebra. We call an R_0-ring R_+ an *$(R_0|k)$-ring* if the R_0-ring structure makes R_+ a k-algebra, i.e. $\lambda r = r\lambda$ for all $r \in R_+$ and $\lambda \in k$. Let $\mathcal{H} = (\delta, H, \mu)$ be a k-bialgebra. Here $H \xrightarrow{\delta} H \otimes_k H \xrightarrow{\mu} H$ are resp. comultiplication and multiplication. Recall that a Hopf action of $\mathcal{H}$ on a k-algebra R_+ is a unital $\mathcal{H}$-module structure on R such that the multiplication $R_+ \otimes_k R_+ \longrightarrow R_+$ is an $\mathcal{H}$-module morphism. We assume that $\mathcal{H}$ acts trivially on R_0. Then the cross-product $R_+\#\mathcal{H}$ is an $(R_0|k)$-ring.

The Hopf action of $\mathcal{H}$ on R_+ induces an endofunctor, $\widetilde{H}$, on the category $R_+ - mod_1$. This endofunctor assigns to any (non-unital) R_+-module $\mathcal{M} = (M, R_+ \otimes_k M \xrightarrow{\xi} M)$ the R_+-module $\mathcal{H} \otimes_k \mathcal{M} = (H \otimes_k M, \xi_{\mathcal{H}})$, where the action $\xi_{\mathcal{H}}$ is the composition of

$$R_+ \otimes_k H \otimes_k M \xrightarrow{\sim} H \otimes_k R_+ \otimes_k M \longrightarrow H \otimes_k H \otimes_k R_+ \otimes_k M \longrightarrow H \otimes_k R_+ \otimes_k M \longrightarrow H \otimes_k M.$$

Here the second arrow is induced by the comultiplication δ, the third arrow by the action τ, and the fourth arrow by the R-module structure ξ. The multiplication $H \otimes_k H \xrightarrow{\mu} H$ induces a monad structure, $\widetilde{H}^2 \xrightarrow{\widetilde{\mu}} \widetilde{H}$, on $\widetilde{H}$. One can see that the category $R_+\#\mathcal{H} - mod_1$ is isomorphic to the category $\widetilde{\mathcal{H}} - mod$, where $\widetilde{\mathcal{H}}$ denotes the monad $(\widetilde{H}, \widetilde{\mu})$. This follows from the observation that the functor $\widetilde{H}$ is isomorphic to $R\#\mathcal{H} \otimes_R -$, where $R = R_0 \oplus R_+$ is the augmented R_0-ring. This observation implies, on the other hand, that the functor $\widetilde{H}$ is continuous (i.e. it has a right adjoint) and that there is a natural isomorphism between the category $R_+\#\mathcal{H} - mod_1$ and the category $(\widetilde{\mathcal{H}}/R) - mod$ of modules over the monad $\widetilde{\mathcal{H}}$. Here we write $(\widetilde{\mathcal{H}}/R)$ instead of $(\widetilde{\mathcal{H}}/\mathbf{Sp}(R))$ and identify $R_+ - mod$ with $R - mod$. Thus, we have a natural isomorphism $\mathbf{Sp}(\widetilde{\mathcal{H}}/R) \xrightarrow{\sim} \mathbf{Sp}(R\#\mathcal{H})$ such that the diagram

$$\begin{tikzcd}[column sep=tiny] \mathbf{Sp}(\widetilde{\mathcal{H}}/R) \arrow[rr, "\sim"] \arrow[dr] & & \mathbf{Sp}(R\#\mathcal{H}) \arrow[dl] \\ & \mathbf{Sp}(R) & \end{tikzcd} \qquad (7)$$

commutes.

The following assertion provides another family of morphisms of $R_0 - Rings_1$.

C3.3.6.1. Proposition. *Let $H \otimes_k R_+ \xrightarrow{\tau} R_+$ be a Hopf action of an k-bialgebra $\mathcal{H} = (\delta, H, \mu)$ on a $(R_0|k)$-ring R_+. Suppose the functor $H \otimes_k -$ is flat. Then the monad $\widetilde{\mathcal{H}}$ on $\mathbf{Sp}(R)$ induces a monad $\bar{\mathcal{H}}$ on $\mathbf{Cone}(R_+)$ such that there is a canonical commutative diagram*

$$\begin{array}{ccc} \mathbf{Sp}(\bar{\mathcal{H}}/\mathbf{Cone}(R_+)) & \xrightarrow{\sim} & \mathbf{Cone}(R_+\#\mathcal{H}) \\ & \searrow \quad \swarrow & \\ & \mathbf{Cone}(R_+) & \end{array} \qquad (8)$$

of affine morphisms.

In particular, the canonical morphism $R_+ \longrightarrow R_+\#\mathcal{H}$ belongs to $R_0 - Rings_1$.

Proof. It follows that the functor $\widetilde{H}$ maps the subcategory T_{R_+} to itself. Since the functor $H \otimes_k - : k - mod \longrightarrow k - mod$ is flat (i.e. it is exact and preserves colimits of small diagrams), the functor $R_+ - mod_1 \xrightarrow{\widetilde{H}} R_+ - mod_1$ is flat too. Therefore, the Serre subcategory $T^-_{R_+}$ is stable under $\widetilde{H}$, and the functor $\widetilde{H}$ induces a continuous functor $R_+ - mod_1/T^-_{R_+} = C_{\mathbf{Cone}(R_+)} \xrightarrow{\bar{H}} C_{\mathbf{Cone}(R_+)}$. By C2.2, the multiplication $\widetilde{H}^2 \xrightarrow{\widetilde{\mu}} \widetilde{H}$ induces a multiplication $\bar{H}^2 \xrightarrow{\bar{\mu}} \bar{H}$. The isomorphism of categories

$$R_+\#\mathcal{H} - mod_1 \xrightarrow{\sim} (\widetilde{\mathcal{H}}/R) - mod$$

mentioned above induces an isomorphism $C_{\mathbf{Cone}(R\#\mathcal{H})} \xrightarrow{\sim} (\bar{\mathcal{H}}/\mathbf{Cone}(R)) - mod$, regarded as an inverse image functor of an isomorphism $\mathbf{Sp}(\bar{\mathcal{H}}/\mathbf{Cone}(R)) \xrightarrow{\sim} \mathbf{Cone}(R\#\mathcal{H})$ such that the diagram (8) commutes. The monad $(\widetilde{\mathcal{H}}/R)$ is continuous (i.e. the functor $\widetilde{H}$ has a right adjoint), because $\widetilde{H}$ is isomorphic to the (obviously) continuous functor $R\#\mathcal{H} \otimes_R -$. By C2.2(i), this implies that the monad $\bar{\mathcal{H}}$ on $\mathbf{Cone}(R)$ is continuous. By 6.2, the latter means precisely that the natural morphism $\mathbf{Sp}(\bar{\mathcal{H}}/\mathbf{Cone}(R)) \longrightarrow \mathbf{Cone}(R)$ is affine. Therefore, by the commutativity of (8), the morphism $\mathbf{Cone}(R\#\mathcal{H}) \longrightarrow \mathbf{Cone}(R)$ is affine. This shows, in particular, the canonical morphism $R_+ \longrightarrow R_+\#\mathcal{H}$ belongs to $R_0 - Rings_1$. ∎

C4. Noncommutative projective spectra.

C4.1. $\mathbf{Proj}_{\mathcal{G}}$. Fix a monoid $\mathcal{G}$. Let $\mathbb{F}_+ = (F_+, \mu)$ be a $\mathcal{G}$-graded (non-unital in general) monad on X. Let $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1$ denote the category of $\mathcal{G}$-graded non-unital $\mathbb{F}_+$-modules and preserving gradings morphisms. Let

$$gr_{\mathcal{G}}\mathbb{F}_+ - mod_1 \xrightarrow{\pi^*} \mathbb{F}_+ - mod_1 \qquad (1)$$

be the functor forgetting the grading. We denote by $gr_{\mathcal{G}}\mathcal{T}_{\mathbb{F}_+}$ the preimage of the subcategory $\mathcal{T}_{\mathbb{F}_+}$ in $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1$. Let $C_{\mathbf{Proj}_{\mathcal{G}}}(\mathbb{F}_+)$ be the quotient category $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1/gr_{\mathcal{G}}\mathcal{T}^-_{\mathbb{F}_+}$. This defines a 'space' $\mathbf{Proj}_{\mathcal{G}}(\mathbb{F}_+) = \mathbf{Proj}_{\mathcal{G}}(\mathbb{F}_+/X)$.

C4.2. Actions. Let $\mathcal{G}$ be a monoid. An action of $\mathcal{G}$ on a 'space' X is a monoidal functor $\mathcal{G} \xrightarrow{\mathfrak{L}} \widetilde{End}(C_X)$. Here $\mathcal{G}$ is viewed as a discrete monoidal category and $\widetilde{End}(C_X)$

denote the (strict) monoidal category of endofunctors $C_X \longrightarrow C_X$; i.e. $\widetilde{End}(C_X) = (End(C_X), \circ)$.

C4.2.1. Examples. (a) Let $\mathbb{F}_+$ be a (non-unital in general) $\mathcal{G}$-graded monad on a 'space' X; and let $C_Y = gr_{\mathcal{G}}\mathbb{F}_+ - mod_1$. For any $\mathcal{G}$-graded $\mathbb{F}_+$-module $N = \oplus_{\nu \in \mathcal{G}} N_\nu$ and any $\gamma \in \mathcal{G}$, we denote by $N[\gamma]$ the $\mathcal{G}$-graded $\mathbb{F}_+$-module defined by $N[\gamma]_\sigma = N_{\sigma\gamma}$. This defines a *strict* action of $\mathcal{G}$ on the 'space' Y. Here *strict* means that the monoidal functor $\mathcal{G} \xrightarrow{\mathfrak{L}} \widetilde{End}(C_X)$ is strict, that is $N[\gamma_1\gamma_2] = (N[\gamma_2])[\gamma_1]$ for all N.

(b) The action of $\mathcal{G}$ on the 'space' Y in (a) (i.e. on the category $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1$), induces an action of $\mathcal{G}$ on $\mathbf{Proj}_{\mathcal{G}}(\mathbb{F}_+)$.

C4.3. Proposition. *Let $\mathbb{T}$ be a $\mathcal{G}$-stable, topologizing subcategory of $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1$, and let $\widetilde{\mathbb{T}}$ denote the image of $\mathbb{T}$ in $\mathbb{F}_+ - mod_1$. Then ${\pi^*}^{-1}(\widetilde{\mathbb{T}}^-) \subseteq \mathbb{T}^-$.*

If the 'space' X has the property (sup) and the functor F_+ preserves supremums of subobjects, then ${\pi^*}^{-1}(\widetilde{\mathbb{T}}^-) = \mathbb{T}^-$.

Proof. (a) Since the functor (1) is exact, the preimage, ${\pi^*}^{-1}(\widetilde{\mathbb{T}}^-)$, of the Serre subcategory $\widetilde{\mathbb{T}}^-$ is a Serre subcategory of the category $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1$. The inclusion ${\pi^*}^{-1}(\widetilde{\mathbb{T}}^-) \subseteq \mathbb{T}^-$ is equivalent to that every nonzero object of ${\pi^*}^{-1}(\widetilde{\mathbb{T}}^-)$ has a nonzero subobject which belongs to $\mathbb{T}$; or, what is the same, for any nonzero object, N, of ${\pi^*}^{-1}(\widetilde{\mathbb{T}}^-)$, there exists a nonzero morphism $L \xrightarrow{g} N$, with $L \in Ob\mathbb{T}$. We can and will assume that L is generated by one of its homogeneous components. Then $\mathbb{F}_+ - mod_1(L, N)$ is a $\mathcal{G}$-graded $\mathbb{Z}$-module, and some of homogeneous components of the morphism g are nonzero. Replacing the module L by the module $L[\gamma]$ for an appropriate $\gamma \in \mathcal{G}$, we can assume that the homogeneous component of g of zero degree is nonzero. Thus, there exists a nonzero morphism $L[\gamma] \longrightarrow N$ of graded $\mathbb{F}_+$-modules. Since the subcategory $\mathbb{T}$ is stable under the action of $\mathcal{G}$, the object $L[\gamma]$ belongs to $\mathbb{T}$.

(b) Since X has the property (sup) and F preserves supremums, both categories, $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1$ and $\mathbb{F}_+ - mod_1$ possess this property too. Therefore, every object, M, of $\mathbb{T}^-$ has a filtration, $\{M_i \mid i \geq 0\}$ such that $M_i = sup(M_j | j < i)$, if i is a limit ordinal, and M_{i+1}/M_i belongs to $\mathbb{T}$. But, this implies that M is an object of $\widetilde{\mathbb{T}}^-$; i.e. we have the inverse inclusion, $\mathbb{T}^- \subseteq {\pi^*}^{-1}(\widetilde{\mathbb{T}}^-)$. ■

C4.3.1. Corollary. *(a) ${\pi^*}^{-1}(\mathcal{T}^-_{\mathbb{F}_+}) \subseteq gr_{\mathcal{G}}\mathcal{T}^-_{\mathbb{F}_+}$.*

(b) If the 'space' X has the property (sup) and the functor F_+ preserves supremums of subobjects, then $gr_{\mathcal{G}}\mathcal{T}^-_{\mathbb{F}_+} = {\pi^*}^{-1}(\mathcal{T}^-_{\mathbb{F}_+})$.

Proof. Set $\mathbb{T} = gr_{\mathcal{G}}\mathcal{T}_{\mathbb{F}_+}$. Then $\widetilde{\mathbb{T}}^-$ coincides with $\mathcal{T}^-_{\mathbb{F}_+}$, hence the assertion. ■

C4.3.2. Corollary. *Suppose that X has the property (sup) and F_+ preserves supremums of subobjects. Then the forgetful functor (1) induces a faithful exact functor*

$$C_{\mathbf{Proj}_{\mathcal{G}}(\mathbb{F}_+)} \xrightarrow{\mathfrak{p}^*} C_{\mathbf{Cone}(\mathbb{F}_+)} \tag{2}$$

Proof. By C4.3.1(b), $gr_{\mathcal{G}}\mathcal{T}^-_{\mathbb{F}_+} = \pi^{*^{-1}}(\mathcal{T}^-_{\mathbb{F}_+})$, where $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1 \xrightarrow{\pi^*} \mathbb{F}_+ - mod_1$ is the forgetful functor. The functor $\pi^{*^{-1}}$ induces a faithful functor between quotient categories

$$gr_{\mathcal{G}}\mathbb{F}_+ - mod_1/gr_{\mathcal{G}}\mathcal{T}^-_{\mathbb{F}_+} \longrightarrow \mathbb{F}_+ - mod_1/\mathcal{T}^-_{\mathbb{F}_+}.$$

This functor is exact because the inclusion functor $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1 \longrightarrow \mathbb{F}_+ - mod_1$ is exact. Hence the assertion. ∎

The functor (2) is regarded as an inverse image functor of a morphism ('projection') $\mathbf{Cone}(\mathbb{F}_+) \xrightarrow{\mathfrak{p}} \mathbf{Proj}_{\mathcal{G}}(\mathbb{F}_+)$.

C4.3.3. The Proj of an associative ring. Let R_0 be an associative unital ring and $\mathcal{G}$ a monoid. Let R_+ be a $\mathcal{G}$-graded R_0-ring, which is, by definition, a $\mathcal{G}$-graded ring in the category of R_0-bimodules. Then we have the category $gr_{\mathcal{G}}R_+ - mod_1$ of $\mathcal{G}$-graded R_+-modules and its subcategory $gr_{\mathcal{G}}\mathcal{T}_{R_+} = \mathcal{T}_{R_+} \cap gr_{\mathcal{G}}R_+ - mod_1$. We obtain the 'space' $\mathbf{Proj}_{\mathcal{G}}(R_+) = \mathbf{Proj}_{\mathcal{G}}(R)$ defined by

$$C_{\mathbf{Proj}_{\mathcal{G}}(R_+)} = gr_{\mathcal{G}}R_+ - mod_1/gr_{\mathcal{G}}\mathcal{T}^-_{R_+}.$$

Since the conditions of C4.3.1(b) hold, $gr_{\mathcal{G}}\mathcal{T}^-_{R_+} = gr_{\mathcal{G}}R_+ - mod_1 \cap \mathcal{T}^-_{R_+}$, and, therefore, we have a canonical projection

$$\mathbf{Cone}(R_+) \xrightarrow{\mathfrak{p}} \mathbf{Proj}_{\mathcal{G}}(R_+).$$

Taking $X = \mathbf{Sp}(R_0)$ (i.e. $C_X = R_0 - mod$), we can identify R_+ with the monad $\mathbb{F}_+ = (F_+, \mu)$, where $F_+ = R_+ \otimes_{R_0} -$ and $F_+^2 \xrightarrow{\mu} F_+$ is determined by the multiplication $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$. If R_+ is $\mathcal{G}$-graded, then the monad $\mathbb{F}_+$ is $\mathcal{G}$-graded. We have natural isomorphisms $\mathbf{Cone}(R_+) \simeq \mathbf{Cone}(\mathbb{F}_+)$ and $\mathbf{Proj}_{\mathcal{G}}(R_+) \simeq \mathbf{Proj}_{\mathcal{G}}(\mathbb{F}_+)$.

We have recovered the construction 1.7 illustrated by examples 1.8, 1.9, and 1.10. One can approach to these examples from a different side, via Hopf actions.

C5. Hopf actions. Let $\mathcal{G}$ be a monoid and R_0 an associative unital k-algebra. For a $\mathcal{G}$-graded $(R_0|k)$-ring R_+, we denote by $gr_{\mathcal{G}}R_+ - mod_1$ the category of non-unital $\mathcal{G}$-graded R_+-modules. Let $\mathcal{H} = (\delta, H, \mu)$ be a $\mathcal{G}$-graded k-bialgebra with comultiplication δ and multiplication μ; and let $\mathcal{H} \otimes_k R_+ \xrightarrow{\tau} R_+$ is a Hopf action compatible with grading. Recall that a Hopf action of $\mathcal{H}$ on a k-algebra R_+ is a unital $\mathcal{H}$-module structure on R such that the multiplication $R_+ \otimes_k R_+ \longrightarrow R_+$ is an $\mathcal{H}$-module morphism. We assume that $\mathcal{H}$ acts trivially on R_0. Then the cross-product $R_+ \# \mathcal{H}$ is a $\mathcal{G}$-graded $(R_0|k)$-ring.

The Hopf action of $\mathcal{H}$ on R_+ induces an endofunctor, $H_{\mathcal{G}}$, on the category $gr_{\mathcal{G}}R_+ - mod_1$ which assigns to any (non-unital) $\mathcal{G}$-graded R_+-module $\mathcal{M} = (M, R_+ \otimes_k M \xrightarrow{\xi} M)$ the $\mathcal{G}$-graded R_+-module $\mathcal{H} \otimes_k \mathcal{M} = (H \otimes_k M, \xi_{\mathcal{H}})$, where the action $\xi_{\mathcal{H}}$ is same as in the non-graded case (cf. C3.3.6). The multiplication $H \otimes_k H \xrightarrow{\mu} H$ gives rise to a monad $\mathcal{H}_{\mathcal{G}} = (H_{\mathcal{G}}, \mu_{\mathcal{G}})$ (like in C3.3.6); and the category $gr_{\mathcal{G}}R_+ \# \mathcal{H} - mod_1$ is isomorphic to the category $\mathcal{H}_{\mathcal{G}} - mod$. By an argument similar to that of C3.3.6, the monad $\mathcal{H}_{\mathcal{G}}$ is continuous (i.e. the functor $H_{\mathcal{G}}$ has a right adjoint) which is equivalent to that the forgetful functor $\mathcal{H}_{\mathcal{G}} - mod \longrightarrow gr_{\mathcal{G}}R_+ - mod_1$ is a direct image functor of an affine morphism.

C5.1. Proposition. *Let $H \otimes_k R_+ \xrightarrow{\tau} R_+$ be a Hopf action of an k-bialgebra $\mathcal{H} = (\delta, H, \mu)$ on a $\mathcal{G}$-graded $(R_0|k)$-ring R_+. Suppose the functor $H \otimes_k -$ is flat. Then the monad $\widetilde{\mathcal{H}}$ on $\mathbf{Sp}(R)$ induces a monad $\bar{\mathcal{H}}$ on $\mathbf{Proj}_{\mathcal{G}}(R_+)$ such that there is a canonical commutative diagram*

$$\begin{array}{ccc} \mathbf{Sp}(\bar{\mathcal{H}}/\mathbf{Proj}_{\mathcal{G}}(R_+)) & \xrightarrow{\sim} & \mathbf{Proj}_{\mathcal{G}}(R_+\#\mathcal{H}) \\ & \searrow \quad \swarrow & \\ & \mathbf{Proj}_{\mathcal{G}}(R_+) & \end{array} \tag{8}$$

of affine morphisms.

Proof. The argument is similar to that of C3.3.6.1. Details are left to the reader. ■

C5.2. Example. Let G be a connected reductive algebraic group over an algebraicly closed field k of zero characteristic. Fix a Borel subgroup B, a maximal unipotent subgroup U, and a maximal torus H chosen in a compatible way: H and U are subgroups of B, and $B = HU$). Let R be the algebra of regular functions on the homogeneous space G/U (called after I. M. Gelfand the 'base affine space'). The algebra R is the direct sum of all simple finite dimensional modules, each appears once; i.e. $R = \oplus_{\lambda \geq 0} R_\lambda$, where λ runs through nonnegative integral weights. Then $R_0 = k$, and $R_+ = \oplus_{\lambda > 0} R_\lambda$ is a $\mathcal{G}$-graded k-algebra. Here $\mathcal{G}$ is the group of intergral weights of the group G.

The category $C_{\mathbf{Cone}(R_+)}$ is equivalent to the category of quasi-coherent sheaves on the base affine space G/U. The category $C_{\mathbf{Proj}_{\mathcal{G}}(R_+)}$ is equivalent to the category of quasi-coherent sheaves on the flag variety G/B. We refer for details to [LR4].

C5.2.1. Note. If the group G is simply connected, this construction can be given in terms of the Lie algebra $\mathfrak{g}$ of G and its Cartan subalgebra $\mathfrak{h}$, as it is done in 1.9.

C5.2.2. D-modules. By construction, there is a Hopf action on R of the universal enveloping (Hopf) algebra $U(\mathfrak{g})$. Consider instead of R the crossed product $R_+\#U(\mathfrak{g})$.

The universal enveloping algebra, $\mathcal{H}$, of the Cartan subalgebra, $\mathfrak{h}$, acts on the algebra R according the decomposition $R = \oplus_{\lambda \geq 0} R_\lambda$: each R_λ is a one-dimensional representation of $\mathcal{H}$ with the weight λ tensored by the vector space R_λ. This is a Hopf action commuting with the action of $U(\mathfrak{g})$, hence it determines to a Hopf action of $\widetilde{U}(\mathfrak{g}) = U(\mathfrak{g}) \otimes_k \mathcal{H}$ on R_+.

The category $C_{\mathbf{Cone}(R_+\#\widetilde{U}(\mathfrak{g}))}$ is equivalent to the category of D-modules on the base affine space G/U.

The category $C_{\mathbf{Proj}_{\mathcal{G}}(R_+\#\widetilde{U}(\mathfrak{g}))}$ is equivalent to the category of D-modules on the flag variety G/B.

We can express these facts saying that the category of D-modules on the base affine space G/U is the category of quasi-coherent sheaves on the noncommutative quasi-affine 'space' $\mathbf{Cone}(R_+\#\widetilde{U}(\mathfrak{g}))$ and the category of D-modules on the flag variety G/B is the category of quasi-coherent sheaves on the noncommutative 'space' $\mathbf{Proj}_{\mathcal{G}}(R_+\#\widetilde{U}(\mathfrak{g}))$. Both are semiseparated (actually, separated) noncommutative schemes.

C5.3. Example: quantum affine base space and quantum flag variety. Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of a semisimple Lie algebra, $\mathfrak{g}$, and let $\mathcal{H}$ be its maximal torus (this time canonical). We define R and R_+ as in C5.2; i.e. $R = \oplus_{\lambda \geq 0} R_\lambda$ and $R_+ = \oplus_{\lambda > 0} R_\lambda$, where R_λ is the simple $U_q(\mathfrak{g})$-module with the highest weight λ. The multiplication is given by choosing projections $R_\lambda \otimes R_\mu \longrightarrow R_{\lambda+\mu}$ for different λ and μ in an appropriate way (see [LR4] for details).

We define the *quantum base affine space of* $\mathfrak{g}$ as the 'space' $\mathbf{Cone}(R_+)$ and the *quantum flag variety of* $\mathfrak{g}$ as the 'space' $\mathbf{Proj}_{\mathcal{G}}(R_+)$.

C5.3.1. D-modules on the quantum base affine space and the quantum flag variety. Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of a semisimple Lie algebra, $\mathfrak{g}$, and let $\mathcal{H}$ be its maximal a torus. Let $R = \oplus_{\lambda \geq 0} R_\lambda$ be the algebra of functions on the quantum base affine space, and $R_+ = \oplus_{\lambda \geq 0} R_\lambda$ the *quantum base affine space of* $\mathfrak{g}$ which is by difinition the spectrum $\mathbf{Cone}(R_+)$ of the algebra R_+ (see C5.2).

The maximal torus $\mathcal{H}$ acts on R_+, and this action commutes with the action of $U_q(\mathfrak{g})$. Thus R_+ has a structure of a $\widetilde{U}_q(\mathfrak{g})$-module, where $\widetilde{U}_q(\mathfrak{g}) = U_q(\mathfrak{g}) \otimes_k \mathcal{H}$. By C2.2, $\widetilde{U}_q(\mathfrak{g})$ induces a continuous monad, $U_q^{\sim}(\mathfrak{g})$, on $\mathbf{Cone}(R_+)$. And we have the commutative diagram

$$\begin{array}{ccc} \mathbf{Sp}(U_q^{\sim}(\mathfrak{g})/\mathbf{Cone}(R_+)) & \xrightarrow{\ \sim\ } & \mathbf{Cone}(R_+\#\widetilde{U}_q(\mathfrak{g})) \\ & \searrow \quad \swarrow & \\ & \mathbf{Cone}(R_+) & \end{array}$$

The action of $\widetilde{U}_q(\mathfrak{g})$ on R_+ respects $\mathcal{G}$-grading, hence it induces a continuous monad, $\bar{U}_q(\mathfrak{g})$, on $\mathbf{Proj}_{\mathcal{G}}(R_+)$; and we have a commutative diagram

$$\begin{array}{ccc} \mathbf{Sp}(\bar{U}_q(\mathfrak{g})/\mathbf{Proj}_{\mathcal{G}}(R_+)) & \xrightarrow{\ \sim\ } & \mathbf{Proj}_{\mathcal{G}}(R_+\#\widetilde{U}_q(\mathfrak{g})) \\ & \searrow \quad \swarrow & \\ & \mathbf{Proj}_{\mathcal{G}}(R_+) & \end{array}$$

It is shown in [LR4] that the monad $\bar{U}_q(\mathfrak{g})$ on $\mathbf{Proj}_{\mathcal{G}}(R_+)$ is compatible with the affine localizations

$$\mathbf{Sp}((S_w^{-1}R)_0) \longrightarrow \mathbf{Proj}_{\mathcal{G}}(R_+), \quad w \in W, \tag{1}$$

of the quantum flag variety described in 1.10.1. Similarly, the monad $U_q^{\sim}(\mathfrak{g})$ on $\mathbf{Cone}(R_+)$ is compatible with the affine localizations

$$\mathbf{Sp}(S_w^{-1}R) \longrightarrow \mathbf{Cone}(R_+), \quad w \in W, \tag{2}$$

of the quantum base affine 'space' (see 1.10.1).

Applying C2.2, we obtain that the affine cover (1) of the quantized flag variety, $\mathbf{Proj}_{\mathcal{G}}(R_+)$, induces an affine cover of $\mathbf{Proj}_{\mathcal{G}}(R_+\#\widetilde{U}_q(\mathfrak{g}))$.

Similarly, the affine cover (1) of the quantum base affine space $\mathbf{Cone}(R_+)$ induces an affine cover of the 'space' $\mathbf{Cone}(R_+\#\widetilde{U}_q(\mathfrak{g}))$.

Moreover, it follows from C2.2 that in both cases the morphisms of affine cover are affine localizations. Therefore, $\mathbf{Proj}_{\mathcal{G}}(R_+\#\widetilde{U}_q(\mathfrak{g}))$ and $\mathbf{Cone}(R_+\#\widetilde{U}_q(\mathfrak{g}))$ are examples of semiseparated noncommutative D-schemes.

C6. The category of coalgebras and the category of flat, affine covers.

C6.1. The category of coalgebras. Let R be an associative, unital ring, $\mathcal{H} = (H, \delta)$ a coalgebra in the category of R-bimodules, $\phi : R \longrightarrow S$ a ring morphism. The comultiplication δ induces a comultiplication, $\delta_\phi : H_\phi \longrightarrow H_\phi \otimes_S H_\phi$, on the S-bimodule $H_\phi = S \otimes_R H \otimes_R S$. In fact, $H_\phi \otimes_S H_\phi \simeq S \otimes_R H \otimes_R S \otimes_R H \otimes_R S$, and the comultiplication δ_ϕ is determined by the composition of $\delta : H \longrightarrow H \otimes_R H$ and $H \otimes_R H \longrightarrow H \otimes_R S \otimes_R H$ given by $a \otimes b \longmapsto a \otimes 1 \otimes b$.

We denote by $\mathfrak{Coalg}$ the *category of coalgebras*: its objects are pairs $(R\backslash\mathcal{H})$, where R is an associative ring and $\mathcal{H} = (H, \delta)$ is a coalgebra in the category of R-bimodules. Morphisms from $(R\backslash\mathcal{H})$ to $(R'\backslash\mathcal{H}')$ are pairs (ϕ, λ_ϕ), where ϕ is a ring morphism $R \longrightarrow R'$, λ_ϕ is a morphism of coalgebras $\mathcal{H}_\phi \longrightarrow \mathcal{H}'$ in the category of coalgebras in R-bimodules. The composition is defined in an obvious way.

We denote by $\mathfrak{Coalg}_{\mathfrak{fl}}$ the full subcategory of $\mathfrak{Coalg}$ whose objects are $(R\backslash\mathcal{H})$, $\mathcal{H} = (H, \delta)$, such that H is flat as a right R-module.

There is a forgetful functor

$$\Phi^* : \mathfrak{Coalg} \longrightarrow Rings, \quad (R\backslash\mathcal{H}) \longmapsto R, \ (\phi, \lambda_\phi) \longmapsto \phi \tag{1}$$

which is right adjoint to the functor

$$\Phi_* : Rings \longrightarrow \mathfrak{Coalg}, \quad R \longmapsto (R\backslash R), \ \psi \longmapsto (\psi, \psi). \tag{2}$$

The adjunction morphisms are $\epsilon_\Phi = id : \Phi^*\Phi_* \longrightarrow Id_{Rings}$ and $\eta_\Phi : Id_{\mathfrak{Coalg}} \longrightarrow \Phi_*\Phi^*$; the latter morphism assigns to each object $(R\backslash\mathcal{H})$ of $\mathfrak{Coalg}$ the morphism $(id_R, \mathfrak{e}_\mathcal{H})$, where $\mathfrak{e}_\mathcal{H}$ denotes the counit $\mathcal{H} \longrightarrow R$ of the coalgebra $\mathcal{H}$ (which is a coalgebra morphism). Since the first adjunction arrow is an isomorphism, the functor Φ_* is fully faithful. In other words, the pair of functors $\mathfrak{Coalg} \rightleftarrows Rings$ is a Q^o-category.

C6.2. The category $\mathfrak{ACov}$ of affine covers. On the other hand, consider the category $\mathfrak{ACov}$ of *affine covers* whose objects are flat, conservative, affine morphisms $\mathbf{Sp}(R) \longrightarrow X$. Morphisms from $\mathbf{Sp}(R) \xrightarrow{\phi} X$ to $\mathbf{Sp}(S) \xrightarrow{\psi} Y$ are commutative diagrams

$$\begin{array}{ccc} \mathbf{Sp}(R) & \xrightarrow{f} & \mathbf{Sp}(S) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X & \xrightarrow{h} & Y \end{array} \tag{3}$$

such that the morphism $\mathbf{Sp}(R) \xrightarrow{f} \mathbf{Sp}(S)$ is affine.

For each object $(R\backslash\mathcal{H})$ of the category $\mathfrak{Coalg}$, the forgetful functor $(R\backslash\mathcal{H})-Comod \longrightarrow R-mod$ is an inverse image functor of an affine morphism $\mathbf{Sp}(R) \xrightarrow{f} \mathbf{Sp}(R\backslash\mathcal{H})$. The morphism f is flat iff H is flat as a right R-module (see 7.3.1). Any morphism $(\phi, \lambda_\phi) : (R\backslash\mathcal{H}) \longrightarrow (S\backslash\mathcal{G})$ induces a functor

$$|\phi, \lambda_\phi|^* : (R\backslash\mathcal{H}) - Comod \longrightarrow (S\backslash\mathcal{G}) - Comod, \tag{4}$$

$$(M, M \xrightarrow{\xi} H \otimes_R M) \longmapsto (\phi^*(M), \phi^*(M) \to G \otimes_S \phi^*(M)),$$

where $\phi^*(M) = S \otimes_R M$, and the morphism $\phi^*(M) \to G \otimes_S \phi^*(M))$ is the composition of

$$\phi^*(\xi) : \phi^*(M) \longrightarrow \phi^*(H \otimes_R M) = S \otimes_R H \otimes_R M,$$

$$S \otimes_R H \otimes_R M \longrightarrow S \otimes_R H \otimes_R S \otimes_R M,$$

and

$$\lambda_\phi \otimes_S M : S \otimes_R H \otimes_R S \otimes_R M \longrightarrow G \otimes_S \phi^*(M) = G \otimes_R M.$$

The functor $|\phi, \lambda_\phi)|^*$ is (regarded as) an inverse image functor of a morphism $|\phi, \lambda_\phi| : \mathbf{Sp}(S\backslash\mathcal{G}) \longrightarrow \mathbf{Sp}(R\backslash\mathcal{H})$. The map $(\phi, \lambda_\phi) \longmapsto |\phi, \lambda_\phi|^*$ extends to a pseudo-functor $\mathfrak{Coalg} \longrightarrow Cat$ which induces a functor $\mathfrak{Coalg}^{op} \longrightarrow |Cat|^o$. It follows that the diagram

$$\begin{array}{ccc} R - mod & \xrightarrow{\phi^*} & S - mod \\ \uparrow & & \uparrow \\ (R\backslash\mathcal{H}) - Comod & \xrightarrow{|\phi, \lambda_\phi|^*} & (S\backslash\mathcal{G}) - Comod \end{array} \qquad (5)$$

commutes. In particular, the corresponding diagram in $|Cat|^o$,

$$\begin{array}{ccc} \mathbf{Sp}(S) & \xrightarrow{|\phi|} & \mathbf{Sp}(R) \\ \downarrow & & \downarrow \\ \mathbf{Sp}(S\backslash\mathcal{G}) & \xrightarrow{|\phi, \lambda_\phi|} & \mathbf{Sp}(R\backslash\mathcal{H}) \end{array} \qquad (6)$$

commutes. Thus we have a contravariant functor

$$\mathfrak{Coalg}^{op} \longrightarrow \mathfrak{ACov} \qquad (7)$$

from the category of coalgebras to the category of affine covers.

C6.3. Proposition. *The image of the functor (7) is equivalent to the category of affine covers.*

Proof. Let

$$\begin{array}{ccc} \mathbf{Sp}(S) & \xrightarrow{f} & \mathbf{Sp}(R) \\ \phi \downarrow & & \downarrow \psi \\ X & \xrightarrow{h} & Y \end{array} \qquad (8)$$

be a morphism of covers, i.e. the morphisms ϕ, ψ, f are affine and, in addition, ϕ and ψ are flat and conservative.

Choosing inverse image functors of each of the morphisms in the diagram, we obtain a quasi-commutative diagram

$$\begin{array}{ccc} R-mod & \xrightarrow{f^*} & S-mod \\ \phi^* \uparrow & & \uparrow \psi^* \\ C_Y & \xrightarrow{h^*} & C_X \end{array} \qquad (9)$$

(a) Since f is affine, its direct image functor, f_* is the composition of and equivalence of the categories (Morita equivalence) $S-mod \longrightarrow S'-mod$ and the pull-back functor $S'-mod \longrightarrow R-mod$ by a ring morphism $\hat{f} : R \longrightarrow S'$ (cf. 6.6.1). Thus the inverse image functor f^* of f is the composition of $\hat{f}^* = S' \otimes_R - : R-mod \longrightarrow S'-mod$ and an equivalence $S'-mod \longrightarrow S-mod$. Replacing S by S', we shall assume that the functor f^* in (9) is $S \otimes_R -$ for a ring morphism $R \xrightarrow{\widehat{f}} S$.

(b) The inverse image functor h^* defines a canonical functor morphism

$$(\psi f)^*(\psi f)_* \xrightarrow{\lambda_h} \phi^*\phi_*. \qquad (10)$$

Denote for convenience the composition ψf by g. The quasi-commutativity of (9) means that there is an isomorphism $\mathfrak{c} = \mathfrak{c}_{\phi,h} : \phi^* h^* \xrightarrow{\sim} g^*$. This isomorphism induces a morphism $h^* \xrightarrow{f\mathfrak{c}'} \phi_* g^*$. We define λ_h as the composition of the following morphisms:

$$g^* g_* \xrightarrow{\mathfrak{c}^{-1} g_*} \phi^* h^* g_* \xrightarrow{\phi^* \mathfrak{c}' g_*} \phi^*\phi_* g^* g_* \xrightarrow{\phi^*\phi_*} \phi^*\phi_*. \qquad (11)$$

(c) The morphism $g^* g_* \xrightarrow{\lambda_h} \phi^*\phi_*$ is a comonad morphism.

The proof of this fact follows from the argument of Proposition 1.0.7.1 in [R4], where the siminar fact is proven for the dual situation. One needs only to switch to dual categories and functors.

(d) Since the functors ϕ^* and ψ^* are flat and conservative, the diagram (9) can be included into the diagram

$$\begin{array}{ccc} R-mod & \xrightarrow{f^*} & S-mod \\ \phi^* \uparrow & & \uparrow \psi^* \\ C_Y & \xrightarrow{h^*} & C_X \\ \uparrow & & \uparrow \\ (R\backslash\mathcal{G}_\phi)-Comod & \xrightarrow{\mathfrak{h}^*} & (S\backslash\mathcal{G}_\psi)-Comod \end{array} \qquad (12)$$

Here $\mathcal{G}_\phi$ is the comonad $(\phi^*\phi_*, \delta_\phi)$ and $\mathcal{G}_\psi = (\psi^*\psi_*, \delta_\psi)$, lower vertical arrows are canonical category equivalences given by Beck's theorem. The functor $\mathfrak{h}^*$ is induced by the monad morphism λ_h defined by (11) as follows. To any $\mathcal{G}_\psi$-comodule $\mathcal{L} = (L, L \xrightarrow{\xi} \psi^*\psi_*(L))$, the functor $\mathfrak{h}^*$ assigns the $\mathcal{G}_\phi$-comodule $(f^*(L), \xi')$, where the $\mathcal{G}_\phi$-coalgebra structure, $\xi' : f^*(L) \longrightarrow \phi^*\phi_* f^*(L)$, is the composition of

$$f^*(L) \xrightarrow{f^*(\xi)} f^*\psi^*\psi_*(L) \xrightarrow{f^*\psi^*\psi_*\eta_f} f^*\psi^*\psi_* f_* f^*(L) \xrightarrow{\sim} g^* g_* f^*(L) \xrightarrow{\lambda_h f^*(L)} \phi^*\phi_* f^*(L).$$

The diagram

$$\begin{array}{ccc} R-mod & \xrightarrow{f^*} & S-mod \\ \phi_1^* \uparrow & & \uparrow \psi_1^* \\ (R\backslash\mathcal{G}_\phi)-Comod & \xrightarrow{\mathfrak{h}^*} & (S\backslash\mathcal{G}_\psi)-Comod \end{array} \qquad (13)$$

obtained from (12) by composing the vertical arrows commutes. Here ψ_1^* and ϕ_1^* are functors forgeting the respective comodule structures.

(e) It follows from (the argument of) 7.3.1 that the comonad $\mathcal{G}_\phi$ is isomorphic to the comonad $\mathcal{H}_\phi \otimes_R -$ for a coalgebra $\mathcal{H}_\phi = (H_\phi, \delta_\phi)$ in the category of R-bimodules determined uniquely up to isomorphism (see 7.3.1(2)). Therefore the category $(R\backslash\mathcal{G}_\phi) - Comod$ is naturally isomorphic to the category $(R\backslash\mathcal{H}_\phi) - Comod$ of $(R\backslash\mathcal{H}_\phi)$-comodules. Similarly, $(S\backslash\mathcal{G}_\psi) - Comod \simeq (S\backslash\mathcal{H}_\psi) - Comod$ for a coalgebra $\mathcal{H}_\psi$ in the category of S-bimodules. The morphism λ_h (defined by (11)) together with the ring morphism $R \xrightarrow{\hat{f}} S$ (cf. (a)) define a coalgebra morphism $(\hat{f}, \lambda_h') : \mathcal{H}_\phi \longrightarrow \mathcal{H}_\psi$ such that the diagram

$$\begin{array}{ccc} (R\backslash\mathcal{G}_\phi)-Comod & \xrightarrow{\mathfrak{h}^*} & (S\backslash\mathcal{G}_\psi)-Comod \\ \uparrow & & \uparrow \\ (R\backslash\mathcal{H}_\phi)-Comod & \xrightarrow{(\hat{f},\lambda_h')^*} & (S\backslash\mathcal{H}_\psi)-Comod \end{array} \qquad (14)$$

commutes. Combining (13) and (14), we obtain a commutative diagram

$$\begin{array}{ccc} R-mod & \xrightarrow{f^*} & S-mod \\ \uparrow & & \uparrow \\ (R\backslash\mathcal{H}_\phi)-Comod & \xrightarrow{(\hat{f},\lambda_h')^*} & (S\backslash\mathcal{H}_\psi)-Comod \end{array} \qquad (15)$$

in which the vertical arrows are functors forgetting the comodule structrure. This implies the assertion. ■

C6.4. Corollary. *Let*

$$\begin{array}{ccc} \mathbf{Sp}(R) & \xrightarrow{f} & \mathbf{Sp}(S) \\ \phi \downarrow & & \downarrow \psi \\ X & \xrightarrow{h} & Y \end{array} \qquad (3)$$

be a morphism of affine flat covers. Then the morphism $X \xrightarrow{h} Y$ has a direct image functor.

Proof. Thanks to C6.3, it suffices to show that for any coalgebra morphism

$$(\phi, \lambda_\phi) : (R\backslash\mathcal{H}) \longrightarrow (S\backslash\mathcal{G})$$

the functor

$$|\phi, \lambda_\phi|^* : (R\backslash\mathcal{H}) - Comod \longrightarrow (S\backslash\mathcal{G}) - Comod$$

has a right adjoint. This follows from 5.3.2.2. ■

C6.5. Proposition. *Let* $\mathbf{Sp}(R) \xrightarrow{\phi} X$ *and* $\mathbf{Sp}(S) \xrightarrow{\psi} Y$ *be affine flat covers. Then* X *is isomorphic to* Y *iff there exist affine flat covers* $\mathbf{Sp}(R) \xleftarrow{f} \mathbf{Sp}(T) \xrightarrow{g} \mathbf{Sp}(S)$ *such that the associated coalgebras* $\mathcal{H}_{\phi f}$ *and* $\mathcal{H}_{\psi g}$ *in the category of* T*-bimodules are isomorphic.*

Proof. (a) For any two affine flat covers, $\mathbf{Sp}(R) \xrightarrow{\phi} X$ and $\mathbf{Sp}(T) \xrightarrow{f} \mathbf{Sp}(R)$, the categories $\mathcal{H}_\phi - Comod$ and $\mathcal{H}_{\phi f} - Comod$ are equivalent, or, what is the same, $\mathbf{Sp}(R\backslash\mathcal{H}_\phi) \simeq \mathbf{Sp}(T, \mathcal{H}_{\phi f})$.

In fact, the composition of affine flat covers is an affine flat cover. Therefore, by Beck's theorem, the category C_X is equivalent both to $\mathcal{H}_\phi - Comod$ and $\mathcal{H}_{\phi f} - Comod$.

(b) Let

$$X \xleftarrow{\phi} \mathbf{Sp}(R) \xleftarrow{f} \mathbf{Sp}(T) \xrightarrow{g} \mathbf{Sp}(S) \xrightarrow{\psi} Y$$

be affine flat covers such that the coalgebras $\mathcal{H}_{\phi f}$ and $\mathcal{H}_{\psi g}$ are isomorphic. By (a), the category C_X is equivalent to $(T\backslash\mathcal{H}_{\phi f}) - Comod$, the category C_Y is equivalent to $(T\backslash\mathcal{H}_{\psi g}) - Comod$, and the isomorphism of the coalgebras induces an isomorphism of the categories of comodules. Hence C_X is equivalent to C_Y.

(c) Let $\mathbf{Sp}(R) \xrightarrow{\phi} X$ and $\mathbf{Sp}(S) \xrightarrow{\psi} Y$ be affine covers and α an isomorphism $Y \xrightarrow{\sim} X$. We claim that there exist affine covers $\mathbf{Sp}(R) \xleftarrow{f} \mathbf{Sp}(T) \xrightarrow{g} \mathbf{Sp}(S)$ such that the diagram

$$\begin{tikzcd} & \mathbf{Sp}(T) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathbf{Sp}(R) \arrow[d, "\phi"'] & & \mathbf{Sp}(S) \arrow[d, "\psi"] \\ X & & Y \arrow[ll, "\alpha"'] \end{tikzcd} \tag{17}$$

commutes. Replacing ψ by $\alpha\psi$, we assume that α is the identical isomorphism $X \longrightarrow X$.

The category C_X is equivalent to the category $(R\backslash\mathcal{H}_\phi) - Comod$ for a coalgebra $\mathcal{H}_\phi = (H_\phi, \delta_\phi)$ in the category of R-bimodules such that H_ϕ is flat as a right R-module. This implies that C_X is a Grothendieck category. In particular, it has arbitrary (small) colimits.

Let $\mathcal{F}_\phi = (F_\phi, \mu_\phi) = (\phi_*\phi^*, \mu_\phi)$ and $\mathcal{F}_\psi = (F_\psi, \mu_\psi) = (\psi_*\psi^*, \mu_\psi)$ be monads associated with resp. ϕ and ψ. By 2.6.2.3 in [R4], there exists a free product, $\mathcal{F}_\phi \star \mathcal{F}_\psi$, of the monads $\mathcal{F}_\phi$ and $\mathcal{F}_\psi$. The canonical monad morphisms $\mathcal{F}_\phi \longrightarrow \mathcal{F}_\phi \star \mathcal{F}_\psi \longleftarrow \mathcal{F}_\psi$ induce a commutative diagram

$$\begin{tikzcd} (\mathcal{F}_\phi \star \mathcal{F}_\psi/X) - mod \arrow[r] \arrow[d] & (\mathcal{F}_\phi/X) - mod \arrow[d] \\ (\mathcal{F}_\psi/X) - mod \arrow[r] & C_X \end{tikzcd} \tag{18}$$

of the corresponding direct image functors. Since the monads $\mathcal{F}_\phi$ and $\mathcal{F}_\psi$ are continuous, their coproduct, $\mathcal{F}_\phi \star \mathcal{F}_\psi$, is continuous too (by 2.6.2.3 in [R4]). This implies that all morphisms in the diagram (18) are affine. By Beck's theorem, the category $(\mathcal{F}_\phi/X) - mod$ is equivalent to the category $R - mod$ and $(\mathcal{F}_\psi/X) - mod$ is equivalent to the category $S - mod$. Applying the Beck's theorem to the (direct image of the) affine morphism $(\mathcal{F}_\phi \star \mathcal{F}_\psi/X) - mod \longrightarrow R - mod$, we obtain the commutative diagram

$$\begin{array}{ccc} T - mod & \longrightarrow & (\mathcal{F}_\phi \star \mathcal{F}_\psi/X) - mod \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow \\ R - mod & \longrightarrow & (\mathcal{F}_\phi/X) - mod \end{array} \qquad (19)$$

of (direct image) functors in which the horizontal arrows are category equivalences and the ring, the morphism f_* is the pull-back of a ring morphism $f : R \longrightarrow T$ which is defined uniquely up to isomorphism. Combining (18) and (19) and using an equivalence of $(\mathcal{F}_\psi/X) - mod$ to $S - mod$, we obtain a quasi-commutative diagram

$$\begin{array}{ccc} T - mod & \xrightarrow{g_*} & S - mod \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle \psi_*} \\ R - mod & \xrightarrow{\phi_*} & C_X \end{array} \qquad (20)$$

Since the inverse image functors $(\mathcal{F}_\phi/X) - mod \longleftarrow C_X \longrightarrow (\mathcal{F}_\psi/X) - mod$ are flat and conservative, the inverse image functors

$$(\mathcal{F}_\phi/X) - mod \longrightarrow (\mathcal{F}_\phi \star \mathcal{F}_\psi/X) - mod \longleftarrow (\mathcal{F}_\psi/X) - mod$$

have the same property (by 2.6.4.1, or 4.3.2 in [R4]). Therefore f_* and g_* in (19) are direct image functors of affine flat covers. ∎

C6.5.1. The dual description. Let $\mathbf{Sp}(R) \xrightarrow{f} X$ be an affine morphism. Denote by $f^\wedge$ the *dual* morphism $X \longrightarrow \mathbf{Sp}(R)$ with an inverse image functor f_* (cf. 3.4). Since f is affine, the morphism $f^\wedge$ is continuous: $f^\wedge_* = f^!$. By 4.1, the morphism $f^\wedge$ (hence the morphism f) is uniquely defined by the right R-module $(\mathcal{O}_*, \hat{\phi})$, where $\mathcal{O}_* = f^{\wedge *}(R) = f_*(R)$ and $\hat{\phi}$ is the canonical ring morphism $R \longrightarrow C_X(\mathcal{O}_*, \mathcal{O}_*)^o = C_X(f_*(R), f_*(R))^o$. The functor $f^\wedge_* = f^!$ is the composition of the functor

$$C_X(\mathcal{O}_*, -) : C_X \longrightarrow C_X(\mathcal{O}_*, \mathcal{O}_*)^o - mod$$

and the pull-back

$$\hat{\phi}_* : C_X(\mathcal{O}_*, \mathcal{O}_*)^o - mod \longrightarrow R - mod$$

by the ring morphism $\hat{\phi}$. Let $X = \mathbf{Sp}(R\backslash\mathcal{H})$, where $\mathcal{H} = (H, \delta)$ is a coalgebra in the category of R-bimodules, and let f be the standard morphism $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp}(R\backslash\mathcal{H})$ having as an inverse image functor the forgetful functor $(R\backslash\mathcal{H}) - Comod \longrightarrow R - mod$ (cf. 7.3.1(1)) and as a direct image functor the functor $L \longmapsto \mathcal{H} \otimes_R L$ (cf. 7.3.1(2)). Then the object $\mathcal{O}_*$ coincides with the $\mathcal{H}$-comodule $\mathcal{H} = (H, \delta)$, and the functor $\widetilde{f}_* = f^!$ is isomorphic to

$$(R\backslash\mathcal{H}) - Comod \longrightarrow R - mod, \quad L \longmapsto Hom_R(H, L) \tag{1}$$

C6.6. Affine covers over the spectrum of a ring.

C6.6.1. Affine covers over $\mathbf{Sp}(S)$. Fix an associative ring S and denote by $\mathfrak{ACov}/\mathbf{Sp}(S)$ the category of affine covers over $\mathbf{Sp}(S)$. Its objects are diagrams

$$\mathbf{Sp}(R) \xrightarrow{\phi} X \xrightarrow{f} \mathbf{Sp}(S)$$

such that ϕ is affine, flat, and conservative, and the composition $f \circ \phi$ is affine. Morphisms from $\mathbf{Sp}(R) \xrightarrow{\phi} X \xrightarrow{f} \mathbf{Sp}(S)$ to $\mathbf{Sp}(T) \xrightarrow{\psi} Y \xrightarrow{g} \mathbf{Sp}(S)$ are commutative diagrams

$$\begin{tikzcd} \mathbf{Sp}(R) \arrow[r, "\gamma"] \arrow[d, "\phi"'] & \mathbf{Sp}(T) \arrow[d, "\psi"] \\ X \arrow[r, "h"] \arrow[dr, "f"'] & Y \arrow[d, "g"] \\ & \mathbf{Sp}(S) \end{tikzcd} \tag{1}$$

The composition is defined in an obvious way. Notice that, since the compositions $f \circ \phi$ and $g \circ \psi$ in (1) are affine, the morphism γ is affine too (see 6.4).

C6.6.2. Coalgebras over a ring. For convenience, we denote by $S\backslash\mathfrak{Coalg}$ the category $(S\backslash S)\backslash\mathfrak{Coalg}$ of coalgebras over the coalgebra $(S\backslash S)$. It follows from definitions that the category $S\backslash\mathfrak{Coalg}$ admits the following description. Objects of $S\backslash\mathfrak{Coalg}$ are triples $((R\backslash\mathcal{H}), \phi, \tau)$, where $(R\backslash\mathcal{H})$ is an object of the category $\mathfrak{Coalg}$ (i.e. $\mathcal{H} = (H, \delta)$ is a coalgebra in the category of R-bimodules), ϕ is a ring morphism $S \longrightarrow R$, and τ is a coalgebra morphism $(R\backslash R \otimes_S R) \longrightarrow (R\backslash\mathcal{H})$. In particular, $\epsilon \circ \tau : R \otimes_S R \longrightarrow R$ is the morphism induced by the multiplication, μ_R, on R (cf. C6.1). Here ϵ denotes the counit, $H \longrightarrow R$, of the coalgebra $\mathcal{H}$.

For any ring S, the map which assigns to every object $((R\backslash\mathcal{H}), S \xrightarrow{\phi} R, \tau,)$ of the category $S\backslash\mathfrak{Coalg}$ the diagram

$$\mathbf{Sp}(R) = \mathbf{Sp^o}(R\backslash R) \xrightarrow{\mathbf{Sp^o}(\epsilon)} \mathbf{Sp^o}(R\backslash\mathcal{H}) \xrightarrow{\mathbf{Sp^o}(\phi,\tau)} \mathbf{Sp^o}(S\backslash S) = \mathbf{Sp}(S) \tag{2}$$

extends naturally to a functor

$$(S\backslash\mathfrak{Coalg})^{op} \longrightarrow \mathfrak{ACov}/\mathbf{Sp}(S) \tag{3}$$

C6.6.3. Proposition. *The image of the functor (3) is equivalent to the category $\mathfrak{ACov}/\mathbf{Sp}(S)$.*

Proof. Let $\mathbf{Sp}(R) \xrightarrow{\phi} X \xrightarrow{f} \mathbf{Sp}(S)$ be an affine cover over $\mathbf{Sp}(S)$. Replacing it by an isomorphic cover, we can assume that $X = \mathbf{Sp^o}(R\backslash\mathcal{H})$ for a coalgebra $(R\backslash\mathcal{H})$, the inverse image ϕ^* of ϕ is the forgetful functor $(R\backslash\mathcal{H}) - Comod \longrightarrow R - mod$, and the composition $f \circ \phi$ corresponds to a ring morphism $S \longrightarrow R$. The latter means that $(f \circ \phi)^*(S) \simeq \phi^*(f^*(S)) \simeq R$, where the isomorphisms are (R, S)-bimodule isomorphisms. We can assume that $\phi^*(f^*(S)) = R$. Since ϕ^* is the functor forgetting the coaction, the $(R\backslash\mathcal{H})$-comodule $f^*(S)$ is of the form (R, γ) for some coaction $\gamma : R \longrightarrow H \otimes_R R \simeq H$. The fact that the R-module morphism γ is an H-comodule structure and an (R, S)-bimodule morphism implies that its composition with the natural isomorphism $H \otimes_R R \xrightarrow{\sim} H$ induces a coalgebra morphism $(R\backslash R \otimes_S R) \longrightarrow (R\backslash\mathcal{H})$. Thus we have assigned to an affine cover over $\mathbf{Sp}(S)$ a coalgebra over S which the functor (3) maps to an affine cover that is isomorphic to the affine cover we have started with. ∎

C6.6.4. Remark. Proposition C6.6.3 can be also deduced directly from C6.3 as follows. Since $\mathbb{Z}$ is the initial object of the category *Rings* of unital rings, the fully faithful functor

$$\Phi_* : Rings \longrightarrow \mathfrak{Coalg}, \quad R \longmapsto (R\backslash R),$$

induces a fully faithful functor $\Phi_*^+ : Rings \longrightarrow \mathbb{Z}\backslash\mathfrak{Coalg}$ which is right adjoint to the forgetful functor

$$\Phi^{+*} : \mathbb{Z}\backslash\mathfrak{Coalg} \longrightarrow Rings, \quad (R\backslash\mathcal{H}, R \otimes_{\mathbb{Z}} R \xrightarrow{\tau} H) \longmapsto R.$$

(cf. C6.1). Moreover, for any ring S the natural functor

$$(S\backslash S)\backslash(\mathbb{Z}\backslash\mathfrak{Coalg}) \longrightarrow (S\backslash S)\backslash\mathfrak{Coalg}$$

is an isomorphism of categories.

C7. Affine flat covers and descent of flat morphisms. If $U = \mathbf{Sp}(R)$ and $V = \mathbf{Sp}(S)$, then the category $C_{\mathcal{H}om^c(U,V)}$ is equivalent to the category of (R, S)-bimodules, or, equivalently, $\mathcal{H}om^c(U, V)$ is isomorphic to $\mathbf{Sp}(R \otimes S^o)$ (cf. 9.2.1).

The object $\mathcal{H}om^{wfl}(U, V)$ corresponds to the full subcategory of the category of (R, S)-bimodules whose objects are bimodules which are flat as right S-modules. We shall write $\mathcal{H}om^{fl}(U, V)$ instead of $\mathcal{H}om^{wfl}(U, V)$.

Notice that since the categories C_X and C_Y are abelian, weakly flat morphisms are just flat. By this reason, we shall write $\mathcal{H}om^{fl}(X, Y)$ instead of $\mathcal{H}om^{wfl}(X, Y)$.

Let $U \xrightarrow{u} X$ and $V \xrightarrow{v} Y$ be affine flat covers (i.e. affine, flat, conservative morphisms); hence $X \simeq \mathbf{Sp^o}(R\backslash\mathcal{H}_u)$ and $Y \simeq \mathbf{Sp^o}(S\backslash\mathcal{H}_v)$, where $\mathcal{H} = (H_u, \delta_u)$ is a coalgebra in the category of R-bimodules and $\mathcal{H}_v = (H_v, \delta_v)$ is a coalgebra in the category of S-bimodules. The coalgebras $\mathcal{H}_u$, $\mathcal{H}_v$ determine a coalgebra $\mathcal{H}_{(u|v)} = (H_{(u|v)}, \delta_{(u|v)})$ in the category of $R \otimes S^o$-bimodules which is naturally identified with $\mathcal{H}_u \otimes \mathcal{H}_v^o$. The latter follows from the fact that the tensoring by $H_{(u|v)}$ maps the bimodule $R \otimes S^o$ to $H_u \otimes H_v$ (i.e. $H_{(u|v)} = H_u \otimes H_v$), and the comultiplication $\delta_{(u|v)}$ on $H_u \otimes H_v$ is the one induced by the comultiplications (δ_u, δ_v).

C7.1. Proposition. *Let* $\mathbf{Sp}(R) \xrightarrow{u} X$ *and* $\mathbf{Sp}(S) \xrightarrow{v} Y$ *be affine flat covers. Then* *(a) The pair (u,v) determines an affine morphism*

$$\mathbf{Sp}(R\otimes S^o) \longrightarrow \mathcal{H}om^c(X,Y). \tag{1}$$

which is the decomposition of

$$\mathbf{Sp}(R\otimes S^o) \longrightarrow \mathbf{Sp^o}(R\otimes S^o\backslash\mathcal{H}_u\otimes\mathcal{H}_v^o) \tag{2}$$

and

$$\mathbf{Sp^o}(R\otimes S^o\backslash\mathcal{H}_u\otimes\mathcal{H}_v^o) \longrightarrow \mathcal{H}om^c(X,Y) \tag{3}$$

(b) The morphism (3) is an exact localization (i.e. its inverse image functor is an exact localization). Equivalently, an inverse image functor of (3) is exact and its direct image functor is fully faithful.

(c) The morphism (1) (or (3)) induces an equivalence of the category $C_{\mathcal{H}om^{fl}(X,Y)}$ of flat morphisms $X\longrightarrow Y$ and the full subcategory of the category $(R\otimes S^o\backslash\mathcal{H}_u\otimes\mathcal{H}_v^o)-Comod$ whose objects are comodules (M,ξ) such that the $R\otimes S^o$-module M is flat as a right S-module.

Proof. The assertion follows from 9.7 and the Beck's theorem 5.4.1. ∎

C7.2. Morphisms corresponding to bicomodules. Under the conditions of C7.1, we can identify X with $\mathbf{Sp^o}(R\backslash\mathcal{H}_u)$ and Y with $\mathbf{Sp^o}(S\backslash\mathcal{H}_v)$.

One can define a (fully faithful) direct image functor of the morphism (11) as follows.

Let $\mathcal{M}$ be a $\mathcal{H}_{(u|v)}$-comodule which is convenient to regard as an $(\mathcal{H}_u,\mathcal{H}_v)$-bicomodule, $\mathcal{M}=(M, H_u\otimes_S M \overset{\zeta_u}{\leftarrow} M \overset{\zeta_v}{\rightarrow} M\otimes_S \mathcal{H}_v)$. It follows from 10.7.1 that a direct image functor of the morphism (3) assigns to $\mathcal{M}$ a functor

$$C_Y=(S\backslash\mathcal{H}_v)-Comod \longrightarrow (R\backslash\mathcal{H}_u)-Comod = C_X$$

which maps each $\mathcal{H}_v$-comodule, $\mathcal{L}=(L,\xi_L)$ to a kernel of the pair of morphisms

$$M\otimes_S L \underset{\zeta_v\otimes L}{\overset{M\otimes_S\xi_L}{\rightrightarrows}} M\otimes_S H_v\otimes_S L. \tag{4}$$

with a comodule structure induced by the coaction $\zeta_u : M\longrightarrow H_u\otimes_S M$.

C7.2.1. Composition of morphisms. Let $u:\mathbf{Sp}(R)\longrightarrow X$, $v:\mathbf{Sp}(S)\longrightarrow Y$ and $w:\mathbf{Sp}(T)\longrightarrow Z$ be affine flat morphisms which allows to assume that $X=\mathbf{Sp^o}(R\backslash\mathcal{H}_u)$, $Y=\mathbf{Sp^o}(S\backslash\mathcal{H}_v)$, and $Z=\mathbf{Sp^o}(T\backslash\mathcal{H}_w)$. Let $\mathcal{M}=(M,\zeta_u,\zeta^v)$ be a $((R\backslash\mathcal{H}_u),(S\backslash\mathcal{H}_v))$-bicomodule and $\mathcal{N}=(N,\zeta_v,\zeta^w)$ a $((S\backslash\mathcal{H}_v),(T\backslash\mathcal{H}_w))$-bicomodule.

Denote by $M\boxtimes_{(S\backslash\mathcal{H}_v)}N$, or simply by $M\boxtimes N$, the kernel of the diagram

$$M\otimes_S N \underset{\zeta_v}{\overset{\zeta^v}{\rightrightarrows}} M\otimes_S H_v\otimes_S N.$$

The right coaction $\zeta^w : N \longrightarrow N \otimes_S H_w$ induces a right $(T\backslash\mathcal{H}_w)$-comodule structure, and the left $(R\backslash\mathcal{H}_u)$-comodule structure $\zeta_u : M \longrightarrow H_u \otimes_S M$ induces a left $(R\backslash\mathcal{H}_u)$-comodule structure on $M \boxtimes N$. Thus we obtain a $((R\backslash\mathcal{H}_u),(T\backslash\mathcal{H}_w))$-bicomodule $\mathcal{M}\boxtimes\mathcal{N} = (M \boxtimes N, \zeta_u \boxtimes N, M \boxtimes \zeta^w)$.

There is a natural morphism from the composition $\phi^*_{\mathcal{M}} \circ \phi^*_{\mathcal{N}}$ of the functors corresponding to the resp. bicomodules $\mathcal{N}$ and $\mathcal{M}$ to the functor

$$\phi^*_{\mathcal{M}\boxtimes\mathcal{N}} : (T\backslash\mathcal{H}_w) - Comod \longrightarrow (R\backslash\mathcal{H}_u) - Comod$$

corresponding to the $(R\backslash\mathcal{H}_u),(T\backslash\mathcal{H}_w))$-bicomodule $\mathcal{M}\boxtimes\mathcal{N}$. This morphism is an isomorphism if M is flat as a right S-module.

C7.3. The bicategory of affine flat covers. We define the bicategory of affine flat covers as follows. Its objects are affine flat covers $\mathbf{Sp}(R) \xrightarrow{u} X$. The category of morphisms from $\mathbf{Sp}(R) \xrightarrow{u} X$ to $\mathbf{Sp}(S) \xrightarrow{v} Y$ is the category of $((R\backslash\mathcal{H}_u),(S\backslash\mathcal{H}_v))$-bicomodules. The composition is given by the tensor product ($\boxtimes$, see C7.2.1) of bicomodules.

C7.3.1. Proposition. *The category of morphisms from a flat cover $\mathbf{Sp}(R) \xrightarrow{u} X$ to a flat cover $\mathbf{Sp}(S) \xrightarrow{v} Y$ is determined by the objects X and Y uniquely up to equivalence.*

Proof. Let $\mathbf{Sp}(R) \xrightarrow{u} X$ and $\mathbf{Sp}(R)' \xrightarrow{u'} X$ be two affine flat covers of X. By C6.5, there exists affine flat covers $\mathbf{Sp}(R) \xleftarrow{p} \mathbf{Sp}(T) \xrightarrow{p'} \mathbf{Sp}(R)'$ such that $u \circ p = u' \circ p'$. Thus, it suffices to show that the category of morphisms from $\mathbf{Sp}(R) \xrightarrow{u} X$ to $\mathbf{Sp}(S) \xrightarrow{v} Y$ is equivalent to the category of morphisms from $\mathbf{Sp}(T) \xrightarrow{up} X$ to $\mathbf{Sp}(S) \xrightarrow{v} Y$.

The functor which assigns to every $((R\backslash\mathcal{H}_u),(S\backslash\mathcal{H}_v))$-bicomodule $\mathcal{M} = (M,\zeta_u,\zeta_v)$ the $((T\backslash\mathcal{H}_{up}),(S\backslash\mathcal{H}_v))$-bicomodule $T \otimes_S \mathcal{M} = (M,\zeta_{up},\zeta_v)$ is a category equivalence. This follows from 7.4.5, or directly from the Beck's theorem (see the part (a) of the argument of 7.4.5). ∎

C7.4. Descent of morphisms over $\mathbf{Sp}(T)$. Fix an object Z of the category $|Cat|^o$. For any two objects $\mathbf{X} = (X, X \xrightarrow{f} Z)$, $\mathbf{Y} = (Y, Y \xrightarrow{f} Z)$ of the category $|Cat|^o/Z$, we denote by $\mathcal{H}om_Z(\mathbf{X},\mathbf{Y})$ (resp. $\mathcal{H}om^c_Z(\mathbf{X},\mathbf{Y})$, resp. $\mathcal{H}om^{fl}_Z(\mathbf{X},\mathbf{Y})$) the object of $|Cat|^o$ corresponding to the category of functors $\phi^* : C_Y \longrightarrow C_X$ (resp. having a right adjoint, resp. exact and having a right adjoint) such that the diagram

$$\begin{tikzcd} C_Y \arrow[rr, "\phi^*"] & & C_X \\ & C_Z \arrow[ul, "g^*"] \arrow[ur, "f^*"'] & \end{tikzcd} \tag{5}$$

quasi-commutes.

Suppose that $Z = \mathbf{Sp}(T)$ for an associative unital ring T; and let $u : \mathbf{Sp}(R) \longrightarrow X$ and $v : \mathbf{Sp}(S) \longrightarrow Y$ be affine flat covers such that the compositions $\mathbf{Sp}(R) \xrightarrow{fu} \mathbf{Sp}(T) \xrightarrow{gv} \mathbf{Sp}(S)$ are affine. Thanks to 7.6.3 (see also 7.6.2), we can assume that the compositions $f \circ u$, $g \circ v$ correspond to ring morphisms resp. $T \longrightarrow R$ and $T \longrightarrow S$, and $X = \mathbf{Sp^o}(R\backslash\mathcal{H}_u)$

and $Y = \mathbf{Sp^o}(S\backslash\mathcal{H}_v)$ for coalgebras $(R\backslash\mathcal{H}_u)$ and $(S\backslash\mathcal{H}_v)$ in resp. the category of R- and S-bimodules. The morphisms $X \longrightarrow \mathbf{Sp}(T)$ and $Y \longrightarrow \mathbf{Sp}(T)$ are described by resp.

$$(T \xrightarrow{\phi_f} R, (R\backslash R \otimes_T R) \xrightarrow{\tau_u} (R\backslash\mathcal{H}_u)) \quad \text{and} \quad (T \xrightarrow{\phi_f} S, (S\backslash S \otimes_T S) \xrightarrow{\tau_u} (S\backslash\mathcal{H}_v)),$$

(cf. C6.6.2). The quasi-commutativity of the diagram (5) implies in particular that $\phi^*(\mathcal{O}_Y) \simeq \mathcal{O}_X$. Here $\mathcal{O}_X = f^*(\mathcal{O}_T) = f^*(T)$, and $\mathcal{O}_Y = g^*(\mathcal{O}_T) = g^*(T)$. The morphisms f and g are continuous (this follows from 7.6.3). Therefore, by 3.4.1, the object $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) determines the morphism f (resp. g) uniquely up to isomorphism. This, in turn, implies that the diagram (5) quasi-commutes if and only if $\phi^*(\mathcal{O}_Y) \simeq \mathcal{O}_X$. The object $\mathcal{O}_X = f^*(\mathcal{O}_T) = f^*(T)$ is (isomorphic to) the $(R\backslash\mathcal{H}_u)$-comodule $(R, R \xrightarrow{\tau'_u} H_u)$, and $\mathcal{O}_Y = g^*(\mathcal{O}_T) = g^*(T)$ is (isomorphic to) the $(S\backslash\mathcal{H}_v)$-comodule $(S, S \xrightarrow{\tau'_v} H_v)$. The coactions τ'_u and τ'_v induce the coalgebra morphisms $(R\backslash R \otimes_T R) \xrightarrow{\tau_u} (R\backslash\mathcal{H}_u)$ and $(S\backslash S \otimes_T S) \xrightarrow{\tau_v} (S\backslash\mathcal{H}_v)$ which appear above (cf. C6.6.2).

Suppose the functor $\phi^* : C_Y \longrightarrow C_X$ is given by an $((R\backslash\mathcal{H}_u), (S\backslash\mathcal{H}_v))$-bicomodule $\mathcal{M} = (M, H_u \otimes_R M \xleftarrow{\zeta_u} M \xrightarrow{\zeta_v} M \otimes_S H_v)$. Then by C7.2 applied to the $S\backslash\mathcal{H}_v$)-comodule $\mathcal{O}_Y = (S, \tau_v)$, the $(R\backslash\mathcal{H}_u)$-comodule $\phi^*(\mathcal{O}_Y)$ is a kernel of the pair of morphisms

$$M \underset{\zeta_v}{\overset{M\otimes_S \tau_v}{\rightrightarrows}} M \otimes_S H_v. \tag{6}$$

with a comodule structure induced by the coaction $\zeta_u : M \longrightarrow H_u \otimes_S M$.

Therefore, the diagram (5) is quasi-commutative iff the kernel of (6) is isomorphic to $u^*(\mathcal{O}_X) = R$ and the $(R\backslash\mathcal{H}_u)$-comodule structure induced on R by $M \xrightarrow{\zeta_u} H_u \otimes_R M$ is isomorphic to $\tau_u : R \longrightarrow H_u$.

The argument above proves the following

C7.4.1. Proposition. *An $((R\backslash\mathcal{H}_u), (S\backslash\mathcal{H}_v))$-bicomodule $\mathcal{M} = (M, \zeta_u, \zeta_v)$ determines a continuous morphism from $\mathbf{X} = (X, X \xrightarrow{f} \mathbf{Sp}(T))$ to $\mathbf{Y} = (Y, Y \xrightarrow{g} \mathbf{Sp}(T))$ iff there exists an $(R\backslash\mathcal{H}_u)$-comodule morphism $\lambda : (R, \tau_u) \longrightarrow (M, \zeta_u)$ such that the diagram*

$$R \xrightarrow{\lambda} M \underset{\zeta_v}{\overset{M\otimes_S \tau_v}{\rightrightarrows}} M \otimes_S H_v \tag{7}$$

is exact.

C7.4.2. Corollary. *The category of flat morphisms from $\mathbf{X} = (X, X \xrightarrow{f} \mathbf{Sp}(T))$ to $\mathbf{Y} = (Y, Y \xrightarrow{g} \mathbf{Sp}(T))$ is equivalent to the category whose objects are pairs $(\mathcal{M}, \lambda)$, where $\mathcal{M} = (M, \zeta_u, \zeta_v)$ is an $((R\backslash\mathcal{H}_u), (S\backslash\mathcal{H}_v))$-bicomodule such that M is flat as a right S-module, and λ an $(R\backslash\mathcal{H}_u)$-comodule morphism $(R, \tau_u) \longrightarrow (M, \zeta_u)$ such that the diagram*

$$R \xrightarrow{\lambda} M \underset{\zeta_v}{\overset{M\otimes_S \tau_v}{\rightrightarrows}} M \otimes_S H_v \tag{7}$$

is exact. Morphisms $(\mathcal{M}, \lambda) \longrightarrow (\mathcal{M}', \lambda')$ *are* $((R\backslash\mathcal{H}_u), (S\backslash\mathcal{H}_v))$*-bicomodule morphisms* $\psi : \mathcal{M} \longrightarrow \mathcal{M}'$ *which make the diagram*

$$\begin{tikzcd} (M, \zeta_u) \arrow[rr, "\psi"] & & (M', \zeta'_u) \\ & (R, \tau_u) \arrow[ul, "\lambda"] \arrow[ur, "\lambda'"'] & \end{tikzcd} \tag{8}$$

commute.

Proof. The fact follows from C7.1(c) and C7.4.1. Details are left to the reader. ∎

C8. Flat descent and relations.

C8.1. Proposition. *(a) Let* $\mathfrak{X} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \mathfrak{Y} \xrightarrow{f} X$ *be an exact diagram in* $|Cat|^o$*. Then*

(i) The morphism f *(i.e. its inverse image functor* f^**) is conservative.*

(ii) If the functors $C_{\mathfrak{Y}} \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} C_{\mathfrak{X}}$ *preserve limits (resp. colimits) of a certain type, then* f^* *has the same property.*

(b) Let $\mathfrak{X} \xleftarrow{p} \mathfrak{Z} \xrightarrow{q} \mathfrak{Y}$ *be a diagram in* $|Cat|^o$*. Suppose the functors* p^* *and* q^* *preserve limits (resp. colimits) of a certain type, then the inverse image functors of the canonical coprojections*

$$\mathfrak{X} \xrightarrow{\pi_p} \mathfrak{X} \coprod_{p,q} \mathfrak{Y} \xleftarrow{\pi_q} \mathfrak{Y}$$

have the same property.

Proof. (a) By 2.2, the category C_X is the kernel of the pair $C_{\mathfrak{Y}} \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} C_{\mathfrak{X}}$ of inverse image functors of p_1 and p_2. This means that the category C_X can be described as follows: its objects are pairs (M, ϕ), where $M \in ObC_{\mathfrak{Y}}$ and ϕ is an isomorphism $p_1^*(M) \xrightarrow{\sim} p_2^*(M)$. A morphism from (M_1, ϕ_1) to (M_2, ϕ_2) is given by a morphism $M_1 \xrightarrow{g} M_2$ such that the diagram

$$\begin{tikzcd} p_1^*(M_1) \arrow[r, "p_1^*(g)"] \arrow[d, "\phi_1"'] & p_1^*(M_1) \arrow[d, "\phi_2"] \\ p_2^*(M_1) \arrow[r, "p_2^*(g)"] & p_2^*(M_1) \end{tikzcd} \tag{1}$$

commutes. A natural inverse image functor, $C_X \xrightarrow{f^*} C_{\mathfrak{Y}}$, of the morphism f assigns to every object (M, ϕ) of C_X the object M.

(i) It follows from the construction that f^* is conservative.

(ii) If p_1^*, p_2^* preserve a certain type of limits (resp. colimits), it follows from the description of the category C_X that the functor $(M, \phi) \overset{f^*}{\longmapsto} M$ preserves the same type of limits (resp. colimits).

(b) The category $C_{\mathfrak{X}\coprod_{p,q}\mathfrak{Y}}$ is described as follows. Its objects are triples (M,ϕ,L), where $M \in ObC_{\mathfrak{X}}$, $L \in ObC_{\mathfrak{Y}}$, and ϕ is an isomorphism $p^*(M) \xrightarrow{\sim} q^*(L)$. Morphisms $(M,\phi,L) \longrightarrow (M',\phi',L')$ are pairs of morphisms $(M \xrightarrow{u} M', L \xrightarrow{v} L')$ such that the diagram

$$\begin{array}{ccc} p^*(M) & \xrightarrow{\ \phi\ } & q^*(L) \\ {\scriptstyle p^*(u)}\downarrow & & \downarrow{\scriptstyle q^*(v)} \\ p^*(M') & \xrightarrow{\ \phi'\ } & q^*(L') \end{array}$$

commutes. Composition is defined in an obvious way. A natural inverse image functor of the projection π_p (resp. π_q) maps an object (M,ϕ,L) to M (resp. to L) and a morphism $(M,\phi,L) \xrightarrow{(u,v)} (M',\phi',L')$ to $M \xrightarrow{u} M'$ (resp. to $L \xrightarrow{v} L'$).

It follows from this description (as in (ii) above) that if the functors p^* and q^* preserve (co)limits of certain type, then the functors π_p^* and π_q^* have the same property. ∎

C8.1.1. Corollary. *Let $\mathfrak{F} = (\mathcal{F} \xrightarrow{\pi} \mathcal{E})$ be a fibered category.*

(a) Let $\mathfrak{R} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathfrak{U} \xrightarrow{f} \mathfrak{X}$ be an exact diagram of presheaves of sets on the category $\mathcal{E}$. Then

(i) The morphism f is conservative.

(ii) If the functors $Qcoh(\mathfrak{F}/\mathfrak{R}) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} Qcoh(\mathfrak{F}/\mathfrak{U})$ preserve a certain type of limits (resp. colimits), then f^* has the same property.

(b) Let $X \xleftarrow{p} Z \xrightarrow{q} Y$ be a diagram of presheaves of sets on the category $\mathcal{E}$. If the functors p^* and q^* preserve limits (resp. colimits) of a certain type, then the inverse image functors of the canonical coprojections

$$X \xrightarrow{\pi_p} X\coprod_{p,q} Y \xleftarrow{\pi_q} Y$$

have the same property.

Proof. By [KR3, 11.1.5.2(b)], the category $Qcoh(\mathfrak{F}/\mathfrak{X})$ of quasi-coherent modules on $\mathfrak{X}$ is the kernel of the pair $Qcoh(\mathfrak{F}/\mathcal{G}) \rightrightarrows Qcoh(\mathfrak{F}/\mathfrak{R})$ of inverse image functors of p_1 and p_2, and the category $Qcoh(\mathfrak{F}/X\coprod_{p,q} Y)$ is the pull-back of the inverse image functors $Qcoh(\mathfrak{F}/X) \xrightarrow{p^*} Qcoh(\mathfrak{F}/Z) \xleftarrow{q^*} Qcoh(\mathfrak{F}/Y)$. The assertion follows now from C8.1. ∎

C8.1.1.1. Proposition. *Let $\mathfrak{F} = (\mathcal{F} \xrightarrow{\pi} \mathcal{E})$ be a fibered category, and let $\mathfrak{T}$ be a topology on $\mathcal{E}$ which is coarser than the topology of effective descent.*

(a) Let $\mathfrak{R} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathcal{G} \xrightarrow{f} X$ be an exact diagram of sheaves of sets on $(\mathcal{E},\mathfrak{T})$. Then

(i) The morphism f is conservative.

(ii) If the functors $Qcoh(\mathfrak{F}/\mathfrak{R}) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} Qcoh(\mathfrak{F}/\mathcal{G})$ preserve a certain type of limits (resp. colimits), then f^* has the same property.

(b) Let

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{p} & \mathfrak{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi_p} \\ \mathfrak{Y} & \xrightarrow{\pi_q} & \mathfrak{X}\coprod_{p,q}\mathfrak{Y} \end{array}$$

be a cartesian coproduct of sheaves of sets on $(\mathcal{E}, \mathfrak{T})$*. If the functors* p^*, q^* *preserve limits (resp. colimits) of a certain type, then the functors* π_p^*, π_q^* *have the same property.*

Proof. Let X' be the cokernel of the pair $\mathfrak{R} \rightrightarrows \mathcal{G}$ in the category $\mathcal{E}^\wedge$ of presheaves on $\mathcal{E}$. Since the topology$\mathfrak{T}$ is coarser than the topology of effective descent and X is asheaf associated with the presheaf X', the unique presheaf morphism $X' \longrightarrow X$ induces an equivalence of categories $Qcoh(\mathfrak{F}/X) \xrightarrow{\sim} Qcoh(\mathfrak{F}/X')$ (see [KR3, 2.6]. The assertion follows now from C8.1.1. ∎

C8.2. Proposition. *Let* $\mathfrak{X} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathfrak{Y} \xrightarrow{f} X$ *be an exact diagram in* $|Cat|^o$ *such that*
(i) inverse image functors of p_1 *and* p_2 *preserve kernels of coreflexive pairs of arrows;*
(ii) the morphism f *is continuous;*
(iii) The category $C_{\mathfrak{Y}}$ *has kernels of coreflexive pairs of arrows.*

Then the morphism f *is comonadic, i.e.* $\mathfrak{Y} \xrightarrow{f} X$ *is isomorphic to the canonical morphism* $\mathfrak{Y} \longrightarrow \mathbf{Sp^o}(\mathfrak{Y}\backslash\mathcal{H}_f)$*, where* $\mathcal{H}_f$ *is a comonad associated with (a choice of inverse and direct image functors and adjunction morphisms of) the morphism* f*.*

Proof. By C8.1, the morphism f is weakly flat, i.e. its inverse image functor preserves kernels of coreflexive pairs of arrows. The assertion now follows from the Beck's theorem (see 5.4.1). ∎

C8.2.1. Remark. Let

$$\mathfrak{X} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathfrak{Y} \xrightarrow{f} X$$

be an exact diagram in $|Cat|^o$. Suppose the morphisms p_1, p_2 in the diagram are continuous. Then their inverse image functors resp. p_1^* and p_2^* preserve colimits of any small diagram. By C8.1(ii), this implies that f^* preserves colimits of any small diagram. Recall that if the category C_X has colimits of small diagrams, f is continuous iff the following conditions hold (cf. [BD]):

(a) f^* preserves colimits of small diagrams;
(b) for any $W \in ObC_{\mathfrak{Y}}$, there exists a morphism $f^*(V) \longrightarrow W$ for some $V \in ObC_X$.

Notice that the condition (b) is fulfilled if the category C_X has an initial object (for instance, C_X is (pre)additive). In fact, every functor having a right adjoint maps an initial object to initial object. In particular, each of the inverse image functors, p_1^* and p_2^*, maps an initial object, $\bullet$, of C_X to an initial object of $C_{\mathfrak{Y}}$. Since all initial objects are isomorphic, there is an isomorphism $\phi : p_1^*(\bullet) \xrightarrow{\sim} p_1^*(\bullet)$. In other words, $(\bullet, \phi)$ is an (initial) object of the category C_X which f^* maps to $\bullet$, an initial object of $C_{\mathfrak{Y}}$, hence the condition (b).

C8.3. Example: quasi-compact quasi-separated schemes. Let $\mathcal{X}$ be a quasi-compact, quasi-separated scheme or algebraic space. Then $\mathcal{X}$ has a finite affine cover

$\{U_i \longrightarrow \mathcal{X} \mid i \in J\}$ and for any $i, j \in J$, the intersection $U_i \bigcap U_j$ has a finite affine cover $\{U_{ij}^k \longrightarrow U_i \bigcap U_j \mid k \in J_{ij}\}$. Set $\mathcal{U} = \coprod_{i \in J} U_i$ and $\mathcal{R} = \coprod_{i,j \in J} \coprod_{k \in J_{ij}} U_{ij}^k$. Then we have an exact diagram of schemes (resp. algebraic spaces)

$$\mathcal{R} \rightrightarrows \mathcal{U} \longrightarrow \mathcal{X} \qquad (1)$$

Since $\mathcal{R}$ and $\mathcal{U}$ are affine schemes, the projections $\mathcal{R} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \mathcal{U}$ are affine morphisms. By construction, inverse image functors of p_1 and p_2 are conservative and exact, as well as the functor π^*. As any morphism to a quasi-separated quasi-compact scheme, π has a direct image functor (this, by the way, follows from the argument of C8.2.1) which makes π (or ruther the corresponding morphism of categories of quasi-coherent sheaves) satisfy the conditions of Beck's theorem; hence $Qcoh_X$ is equivalent to the category of $\mathcal{G}_\pi$-comodules, where $\mathcal{G}_\pi = (\pi^*\pi_*, \delta_\pi)$ is the comonad associated with the continuous morphism π. This, however, does not imply that the morphism $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ is affine.

The morphism π being not affine is equivalent to any of the following conditions:
(a) the direct image functor π_* is not right exact;
(b) the comonad $\mathcal{G}_\pi$ is not continuous;
(c) the functor $G_\pi = \pi^*\pi_*$ is not exact;
(d) the functor $F_\pi = \pi_*\pi^*$ is not exact.

This follows from the equivalence ot the following conditions on a scheme morphism π:
(i) π is affine (in the conventional sense);
(ii) its direct image functor is exact (Serre's criterion);
(iii) π induces an affine morphism of categories of quasi-coherent sheaves.

Notice, however, that the flatness of the morphism π (i.e. the exactness of π^*) is equivalent to that the functor G_π is left exact.

Thus, for any quasi-compact quasi-separated scheme or algebraic space X, the object $|Qcoh_X|$ of $|Cat|^o$ corresponding to the category of quasi-coherent sheaves on X is isomorphic to the cospectrum of a left exact comonad on $\mathbf{Sp}(R)$ for some (commutative) ring R which is not, in general, given by a coalgebra in the category of R-bimodules.

C9. Monads, comonads, and relations. For any object X of the category $|Cat|^o$, we denote by $\mathfrak{Mon}_X$ the category of monads on X and by $\mathfrak{CMon}_X$ the category of comonads on X. We have two functors

$$\mathbf{Sp}_X : \mathfrak{Mon}_X^{op} \longrightarrow |Cat|^o/X, \quad (\mathcal{F}/X) \longmapsto (\mathbf{Sp}(\mathcal{F}/X) \longrightarrow X), \qquad (1)$$

and

$$\mathbf{Sp}_X^{\mathbf{o}} : \mathfrak{CMon}_X^{op} \longrightarrow X\backslash|Cat|^o, \quad (X\backslash\mathcal{G}) \longmapsto (X \longrightarrow \mathbf{Sp}^{\mathbf{o}}(X\backslash\mathcal{G})). \qquad (2)$$

C9.1. Base change. Let $X \xrightarrow{f} Y$ be a continuous morphism with an inverse image functor f^*, a direct image functor f_* and adjunction arrows $Id_{C_X} \xrightarrow{\eta_f} f_*f^*$ and $f^*f_* \xrightarrow{\epsilon_f} Id_{C_Y}$. Let $\mathcal{G} = (G, \delta)$ be a comonad on Y with counit ε. The comultiplication δ

and the adjunction arrow η_f induce a comultiplication δ^f on $G^f = f^*Gf_*$ defined as the composition

$$G^f = f^*Gf_* \xrightarrow{f^*\delta f_*} f^*G^2f_* \xrightarrow{f^*G\eta_f Gf_*} (f^*Gf_*)^2 = (G^f)^2. \tag{1}$$

The counit ε and the adjunction arrow ϵ_f determine the counit, ε^f, of (G^f, δ^f) given by the composition

$$G^f = f^*Gf_* \xrightarrow{f^*\varepsilon f_*} f^*f_* \xrightarrow{\epsilon_f} Id_{C_X}. \tag{2}$$

C9.1.1. Lemma. *Let $X \xrightarrow{f} Y$ be a continuous morphism with inverse and direct image functors resp. f^* and f_* and adjunction arrows $Id_{C_X} \xrightarrow{\eta_f} f_*f^*$ and $f^*f_* \xrightarrow{\epsilon_f} Id_{C_Y}$. Let $\mathcal{G} = (G, \delta)$ be a comonad on Y with counit ε.*

(a) For any comonad $\mathcal{G} = (G, \delta)$ on Y having the counit ε, the pair $\mathcal{G}^f = (G^f, \delta^f)$ is a comonad on X with the counit ε^f.

(b) The correspondence $\mathcal{G} \longmapsto \mathcal{G}^f$ extends naturally to a functor $\mathfrak{CMon}_Y \longrightarrow \mathfrak{CMon}_X$.

(c) For every comonad $\mathcal{G} = (G, \delta)$ on Y, the inverse image functor f^* induces a functor

$$f^*_{\mathcal{G}} : (Y\backslash\mathcal{G}) - Comod \longrightarrow (X\backslash\mathcal{G}^f) - Comod \tag{3}$$

which can be regarded as an inverse image functor of a morphism

$$f_{\mathcal{G}} : \mathbf{Sp^o}(X\backslash\mathcal{G}^f) \longrightarrow \mathbf{Sp^o}(Y\backslash\mathcal{G}). \tag{4}$$

(d) If the category C_X has kernels of coreflexive pairs of arrows, then the functor (3) has a right adjoint, i.e. the morphism (4) is continuous.

(e) If the comonad $\mathcal{G} = (G, \delta)$ and the morphism f are weakly flat (i.e. the functors G and f^* preserve kernels of coreflexive pairs of arrows), then $\mathcal{G}^f$ is weakly flat too.

Proof. (a) The comonad $\mathcal{G}$ can be regarded as the one obtained from an adjoint pair of functors, i.e. $G = g^*g_*$, $\delta = g^*\eta_g g_*$, where η_g is an adjunction arrow, and the counit ε is a complementary adjunction morphism ϵ_g (see 5.3). Then the $\mathcal{G}^f$ becomes the comonad corresponding to the pair of adjoint functors f^*g^*, g_*f_* and the adjunction morphisms $g_*\eta_f g^* \circ \eta_g$ and $\epsilon_f \circ f^*\epsilon_g f_*$.

(b) The functoriality of the map $\mathcal{G} \longmapsto \mathcal{G}^f$ is evident.

(c) The functor (3) assigns to every $(Y\backslash\mathcal{G})$-comodule (L, ξ) the $(X\backslash\mathcal{G}^f)$-comodule $(f^*(L), \xi^f)$, where the coaction $\xi^f : f^*(L) \longrightarrow G^f(f^*(L) = f^*Gf_*f^*(L)$ is the composition $G\eta_f(L) \circ f^*(\xi)$.

(d) The comonad $\mathcal{G}$ corresponds to an adjoint pair of functors, $g^* \dashv g_*$, regarded as resp. inverse and direct image functors of a continuous morphism $Y \xrightarrow{g} Z = \mathbf{Sp^o}(Y\backslash\mathcal{G})$. By (a), $\mathcal{G}^f$ is the comonad corresponding to the adjoint pair f^*g^*, g_*f_* of resp. inverse and direct image functors of the continuous morphism $gX \xrightarrow{f} Z$. By 5.4, the inverse image functor f^*g^* of gf decomposes canonically into

$$C_Z = (Y\backslash\mathcal{G}) - Comod \xrightarrow{f^*_{\mathcal{G}}} (X\backslash\mathcal{G}_{gf}) - Comod \longrightarrow C_Y.$$

Since C_X has kernels of coreflexive pairs of arrows, it follows from 5.4.1(a) (Beck's theorem) that the functor $f^*_{\mathcal{G}}$ in this decomposition has a right adjoint.

(e) If the functors f^* and G preserve limits of certain type, the functor $G^f = f^*Gf_*$ does the same, since f_* preserves limits of any small diagram. ∎

C9.2. Morphisms $(f\backslash\lambda)$**.** Let X, Y be objects of $|Cat|^o$; and let $\mathcal{G}$ and $\mathcal{H}$ be comonads resp. on X and Y. A morphism $(Y\backslash\mathcal{H}) \longrightarrow (X\backslash\mathcal{G})$ is a pair $(f\backslash\lambda)$, where f is a continuous morphism $X \longrightarrow Y$, λ is a comonad morphism $(X\backslash\mathcal{H}^f) \longrightarrow (X\backslash\mathcal{G})$. It follows from this definition that the morphism $(f\backslash\lambda)$ is decomposed into a morphism of underlying objects and a morphism of comonads:

$$\begin{tikzcd}[column sep=small] (Y\backslash\mathcal{H}) \arrow[rr, "(f\backslash\lambda)"] \arrow[dr, "(f\backslash\mathcal{H}^f)"'] & & (X\backslash\mathcal{G}) \\ & (X\backslash\mathcal{H}^f) \arrow[ur, "(X\backslash\lambda)"'] & \end{tikzcd} \qquad (5)$$

Here $(X\backslash\lambda) = (id_X\backslash\lambda)$ and $(f\backslash\mathcal{H}^f) = (f\backslash id_{\mathcal{H}^f})$. Any morphism $(f\backslash\lambda) : (X\backslash\mathcal{G}) \longrightarrow (Y\backslash\mathcal{H})$ induces a functor

$$|f\backslash\lambda|^* : (Y\backslash\mathcal{H}) - Comod \longrightarrow (X\backslash\mathcal{G}) - Comod \qquad (6)$$

which is the composition of the functor

$$f^*_{\mathcal{H}} : (Y\backslash\mathcal{H}) - Comod \longrightarrow (X\backslash\mathcal{H}^f) - Comod \qquad (7)$$

(cf. C9.1) and the inverse image

$$\lambda^* : (X\backslash\mathcal{H}^f) - Comod \longrightarrow (X\backslash\mathcal{G}) - Comod \qquad (8)$$

of the morphism λ (see (5) above and 5.3.2). The functor $|f\backslash\lambda|^*$ is regarded as an inverse image functor of a morphism

$$\mathbf{Sp^o}(f\backslash\lambda) : \mathbf{Sp^o}(X\backslash\mathcal{G}) \longrightarrow \mathbf{Sp^o}(Y\backslash\mathcal{H}). \qquad (9)$$

We shall use also a short-hand notation $|f\backslash\lambda|$ instead of $\mathbf{Sp^o}(f\backslash\lambda)$.

C9.2.1. Proposition. *Let $\mathcal{G} = (G, \delta_{\mathcal{G}})$ and $\mathcal{H} = (H, \delta_{\mathcal{H}})$ be comonads on resp. Y and X, and let $(f\backslash\lambda) : (X\backslash\mathcal{G}) \longrightarrow (Y\backslash\mathcal{H})$ be a comonad morphism. If the category C_X has kernels of coreflexive pairs of arrows and the functors f^* and H preserve these kernels, then the morphism $\mathbf{Sp^o}(f\backslash\lambda)$ is continuous, i.e. the functor $|f\backslash\lambda|^*$ has a right adjoint.*

Proof. By definition, the functor $|f\backslash\lambda|^*$ is the composition of the functors (7) and (8). By C9.1(d), the functor (7) has a right adjoint if C_X has kernels of coreflexive pairs of arrows. By 5.3.2.2, the functor (8) has a right adjoint if the category C_X has kernels of coreflexive pairs of arrows and the functor H^f preserves these kernels. The latter condition holds if H and f have this property. ∎

C9.3. Proposition. *Let*

$$(Y\backslash\mathcal{H}) \overset{(p\backslash\lambda)}{\underset{(q\backslash\gamma)}{\rightrightarrows}} (X\backslash\mathcal{G})$$

be comonad morphisms such that

(i) The pull-back, $(X\backslash\mathcal{H}^p \times_{\lambda,\gamma} \mathcal{H}^q)$*, of comonad morphisms*

$$(X\backslash\mathcal{H}^p) \xrightarrow{(X\backslash\lambda)} (X\backslash\mathcal{G}) \xleftarrow{(X\backslash\gamma)} (X\backslash\mathcal{H}^q) \tag{3}$$

exists.

(ii) The category C_Y *has kernels of coreflexive pairs of arrows, andthe morphisms* $X \overset{p}{\underset{q}{\rightrightarrows}} Y$ *are weakly flat and conservative.*

Then the pushforward of

$$\mathbf{Sp^o}(Y\backslash\mathcal{H}) \xleftarrow{\mathbf{Sp^o}(q\backslash\gamma)} \mathbf{Sp^o}(X\backslash\mathcal{G}) \xrightarrow{\mathbf{Sp^o}(p\backslash\lambda)} \mathbf{Sp^o}(Y\backslash\mathcal{H})$$

is naturally isomorphic to $\mathbf{Sp^o}(X\backslash\mathcal{H}^p \times_{\lambda,\gamma} \mathcal{H}^q)$.

Proof. The pair of morphisms $(p\backslash\lambda)$, $(q\backslash\gamma)$ is described by the diagram

$$\begin{tikzcd}
(X\backslash\mathcal{H}^p) \arrow[r, "(X\lambda)"] & (X\backslash\mathcal{G}) & (X\backslash\mathcal{H}^q) \arrow[l, "(X\backslash\gamma)"'] \\
(Y\backslash\mathcal{H}) \arrow[u, "(p\backslash\mathcal{H}^p)"] & & (Y\backslash\mathcal{H}) \arrow[u, "(q\backslash\mathcal{H}^q)"']
\end{tikzcd} \tag{4}$$

If the condition (a) holds, i.e. a pull-back of the pair of comonad morphisms λ, γ exists, we complete (4) to a commutative diagram

$$\begin{tikzcd}
 & (X\backslash\mathcal{G}) & (X\backslash\mathcal{H}^q) \arrow[l, "(X\backslash\gamma)"'] & (Y\backslash\mathcal{H}) \arrow[l, "(q\backslash\mathcal{H}^q)"'] \\
(Y\backslash\mathcal{H}) \arrow[r, "(p\backslash\mathcal{H}^p)"] & (X\backslash\mathcal{H}^p) \arrow[u, "(X\backslash\lambda)"] & (X\backslash\mathcal{H}^p \times_{\lambda,\gamma} \mathcal{H}^q) \arrow[l, "(X\backslash\pi_p)"'] \arrow[u, "(X\backslash\pi_q)"']
\end{tikzcd} \tag{5}$$

Applying the functor $\mathbf{Sp^o}$ to (5), we obtain the diagram

$$\begin{tikzcd}
 & \mathbf{Sp^o}(X\backslash\mathcal{G}) \arrow[r, "\mathbf{Sp^o}(X\backslash\gamma)"] \arrow[d, "\mathbf{Sp^o}X\backslash\lambda)"'] & \mathbf{Sp^o}(X\backslash\mathcal{H}^q) \arrow[d, "\mathbf{Sp^o}(X\backslash\pi_q)"] & \mathbf{Sp^o}(Y\backslash\mathcal{H}) \arrow[l, "\mathbf{Sp^o}(q\backslash\mathcal{H}^q)"'] \\
\mathbf{Sp^o}(Y\backslash\mathcal{H}) \arrow[r, "\mathbf{Sp^o}(p\backslash\mathcal{H}^p)"] & \mathbf{Sp^o}(X\backslash\mathcal{H}^p) \arrow[r, "\mathbf{Sp^o}(X\backslash\pi_p)"] & \mathbf{Sp^o}(X\backslash\mathcal{H}^p \times_{\lambda,\gamma} \mathcal{H}^q) &
\end{tikzcd} \tag{6}$$

By condition (b) and Beck's theorem, both $\mathbf{Sp^o}(p\backslash\mathcal{H}^p)$ and $\mathbf{Sp^o}(q\backslash\mathcal{H}^q)$ are isomorphisms, hence the assertion. ∎

C9.3.1. Corollary. *Under the conditions of C9.3, the cokernel of*

$$\mathbf{Sp^o}(Y\backslash\mathcal{H}) \overset{\mathbf{Sp^o}(p\backslash\lambda)}{\underset{\mathbf{Sp^o}(q\backslash\gamma)}{\rightrightarrows}} \mathbf{Sp^o}(X\backslash\mathcal{G})$$

is naturally isomorphic to the cokernel of

$$\mathbf{Sp^o}(X\backslash\mathcal{H}^p) \rightrightarrows \mathbf{Sp^o}(X\backslash\mathcal{H}^p \times_{\lambda,\gamma} \mathcal{H}^q),$$

where one arrow is $\mathbf{Sp^o}(X\backslash\pi_p)$ *and another arrow is the composition of*

$$\mathbf{Sp^o}(X\backslash\pi_q) : \mathbf{Sp^o}(X\backslash\mathcal{H}^q) \longrightarrow \mathbf{Sp^o}(X\backslash\mathcal{H}^p \times_{\lambda,\gamma} \mathcal{H}^q)$$

and the isomorphisms

$$\mathbf{Sp^o}(X\backslash\mathcal{H}^p) \xrightarrow{Sp^o(p\backslash\pi_p)^{-1}} \mathbf{Sp^o}(Y\backslash\mathcal{H}) \quad \textit{and} \quad \mathbf{Sp^o}(Y\backslash\mathcal{H}) \xrightarrow{Sp^o(q\backslash\pi_q)} \mathbf{Sp^o}(X\backslash\mathcal{H}^q)$$

(see the diagram (6)).

Proof. The assertion follows from the diagram (6) and the general nonsense fact about the connection of the push-forward of a pair of arrows $x \xleftarrow{\alpha} y \xrightarrow{\beta} x$ and the cokernel of this pair. ■

C9.4. The case of affine covers and relations. Let

$$\mathbf{Sp}(R) \overset{p}{\underset{q}{\rightrightarrows}} \mathbf{Sp}\mathfrak{A} \xrightarrow{f} X \tag{1}$$

be an exact diagram in $|Cat|^o$ such that all morphisms are faithfully flat. To the pair of morphisms $p,\ q$, there corresponds the pair of morphisms

$$(R\backslash\mathcal{H}^p) \xrightarrow{(R\backslash\mu_p)} (R\backslash R) \xleftarrow{(R\backslash\mu_q)} (R\backslash\mathcal{H}^q) \tag{2}$$

(cf. C9.3(a)), where $\mathcal{H}^p$ is $(R \otimes_{(\mathfrak{A},p)} R, \delta_p)$ and $\mathcal{H}_q = (R \otimes_{(\mathfrak{A},q)} R, \delta_q)$, and morphisms μ_p and μ_q are induced by the multiplication $R \otimes R \longrightarrow R$. The pull-back of (2) is the coalgebra $(R\backslash(H_{p,q}, \delta_{p,q})$. Here $H_{p,q} = (R \otimes R)/(I_p \bigcap I_q)$, where I_p (resp. I_q) is the kernel of the epimorphism $R \otimes R \longrightarrow R \otimes_{(\mathfrak{A},p)} R$ (resp. $R \otimes R \longrightarrow R \otimes_{(\mathfrak{A},q)} R$); and the comultiplication $\delta_{p,q}$ is uniquely determined by the condition that the natural epimorphism $R \otimes R \longrightarrow H_{p,q}$ is a coalgebra morphism $(R\backslash(R \otimes R, \delta_R)) \longrightarrow (R\backslash(H_{p,q}, \delta_{p,q}))$.

Set $\mathcal{H}_{p,q} = (H_{p,q}, \delta_{p,q})$. It follows from the construction that the square

$$\begin{array}{ccc} (R\backslash\mathcal{H}_{p,q}) & \longrightarrow & (R\backslash\mathcal{H}^p) \\ \big\downarrow & & \big\downarrow \\ (R\backslash\mathcal{H}^q) & \longrightarrow & (R\backslash R) \end{array} \tag{3}$$

commutes. This implies that there is a natural morphism $\mathbf{Sp}\mathfrak{A} \longrightarrow \mathbf{Sp^o}(R\backslash\mathcal{H}_{p,q})$ which equalizes the pair of arrows $\mathbf{Sp}(R) \overset{p}{\underset{q}{\rightrightarrows}} \mathbf{Sp}\mathfrak{A}$. This morphism is the composition of the morphism $\mathbf{Sp^o}(R\backslash\mathcal{H}^p) \longrightarrow \mathbf{Sp^o}(R\backslash\mathcal{H}_{p,q})$ induced by the coalgebra morphism $(R\backslash\mathcal{H}_{p,q}) \longrightarrow (R\backslash\mathcal{H}^p)$ and the isomorphism $\mathbf{Sp}\mathfrak{A} \longrightarrow \mathbf{Sp^o}(R\backslash\mathcal{H}^p)$.

It is not true in general that the functor $\mathbf{Sp^o}$ transforms (3) into a cocartesian square, or, equivalently, that $\mathbf{Sp^o}(R\backslash\mathcal{H}_{p,q})$ is a cokernel of the pair of arrows $\mathbf{Sp}(R) \overset{p}{\underset{q}{\rightrightarrows}} \mathbf{Sp}\mathfrak{A}$.

C9.4.1. Proposition. *The square (3) is cocartesian, or, equivalently, the diagram*

$$\mathbf{Sp}(R) \overset{p}{\underset{q}{\rightrightarrows}} \mathbf{Sp}\mathfrak{A} \longrightarrow \mathbf{Sp^o}(R\backslash\mathcal{H}_{p,q}) \tag{4}$$

is exact, iff $H_{p,q}$ is flat as a right R-module.

Proof. The fact follows from C9.3 (and holds in a more general situation). Details are left to the reader. ■

C9.4.2. Remark. The realization of the cokernel of the pair $\mathbf{Sp}(R) \overset{p}{\underset{q}{\rightrightarrows}} \mathbf{Sp}\mathfrak{A}$ as $\mathbf{Sp^o}(R\backslash\mathcal{H}_{p,q})$ (under the conditions of C9.4.1) might be not the best choice. For instance, suppose that the object X in the exact sequence (1) is also affine, i.e. $X = \mathbf{Sp}(S)$ for some ring S. In this case, we can assume (after replacing some of the rings in (1) by Morita equivalent rings) that the diagram (1) is obtained by applying the functor $\mathbf{Sp}$ to the exact diagram of rings $S \xrightarrow{\phi} \mathfrak{A} \overset{\mathfrak{p}}{\underset{\mathfrak{q}}{\rightrightarrows}} R$. It follows (from Beck's theorem) that $\mathbf{Sp}(S) \simeq \mathbf{Sp^o}(\mathfrak{A}\backslash\mathfrak{A} \otimes_S \mathfrak{A})$.

C9.5. Quasi-coherent modules on a Grassmannian. Let $R \xrightarrow{\phi} S$ be a k-algebra morphism. Let M be a left R-module, and let $L,\ N$ be projective left R-modules of finite type. We have canonical isomorphisms

$$Hom_S(S \otimes_R M, S \otimes_R L) \simeq Hom_R(M, S \otimes_R L) \simeq Hom_{R^e}(M \otimes_k L^\vee_R, S) \tag{1}$$

(cf. the argument of [KR3, C9.1.3]). Here $R^e = R \otimes_k R^{op}$. In particular, the composition,

$$Hom_S(S \otimes_R M, S \otimes_R L) \times Hom_S(S \otimes_R L, S \otimes_R N) \xrightarrow{c^S_{M,L,N}} Hom_S(S \otimes_R M, S \otimes_R N),$$

determines a unique map

$$Hom_{R^e}(M \otimes_k L^\vee_R, S) \times Hom_{R^e}(L \otimes_k N^\vee_R, S) \xrightarrow{\lambda_{M,L,N}} Hom_{R^e}(M \otimes_k N^\vee_R, S)$$

The map $\lambda_{M,L,N}$ assigns to any pair of R^e-module morphisms, $M \otimes_k L^\vee_R \xrightarrow{u} S$ and $L \otimes_k N^\vee_R \xrightarrow{v} S$, the composition

$$M \otimes_k N^\vee_R \longrightarrow M \otimes_k L^\vee_R \otimes_R L \otimes_k N^\vee_R \xrightarrow{u \otimes_R v} S \otimes_R S \longrightarrow S$$

Here the first arrow is induced by the canonical map $k \longrightarrow L_R^\vee \otimes_R L$ (– the composition of the morphism $k \longrightarrow Hom_R(L, L)$ sending the unit of k to the identity morphism, id_L, and the isomorphism $Hom_R(L, L) \longrightarrow L_R^\vee \otimes_R L$) and the last arrow is induced by the multiplication on S.

The relation $u \circ v = id_{S \otimes_R L}$ (defining the functor $G_{_{M,L}}$) is expressed by the commutative diagram

$$\begin{array}{ccccc} L \otimes_k M_R^\vee \otimes_R M \otimes_k L_R^\vee & & \xrightarrow{\quad v \otimes_R u \quad} & & S \otimes_R S \\ \lambda_{L,M,L} \Big\uparrow & & & & \Big\downarrow \\ L \otimes L_R^\vee & \xrightarrow{ev_L} & R & \xrightarrow{\phi} & S \end{array} \tag{2}$$

Here ev_L denotes the evaluation morphism, $a \otimes \lambda \longmapsto \langle \lambda, a \rangle$.

Taking as S the R-ring $\mathbf{T}(L \otimes_k M_R^\vee \oplus M \otimes_k L_R^\vee)$ representing the functor

$$R\backslash Alg_k \longrightarrow Sets, \quad U \longmapsto Hom_U(U \otimes_R L, U \otimes_R M) \times Hom_U(U \otimes_R M, U \otimes_R L),$$

and as u, v canonical R-bimodule morphisms resp. $L \otimes_k M_R^\vee \longrightarrow U$ and $M \otimes_k L_R^\vee \longrightarrow U$, one might discribe an R-ring, $\mathcal{G} = \mathcal{G}_{_{M,L}}$, representing the functor $G_{_{M,L}} : R\backslash Alg_k \longrightarrow Sets$ (cf. C7.1) as the colimit of the (non-commutative) diagram (2).

The relations $u_j \circ (v_i \circ u_i) = u_i$, $i, j = 1, 2$, defining (together with relations $u_i \circ v_i = id_{S \otimes_R L}$, $i = 1, 2$) the functor $\mathfrak{R}_{_{M,L}}$ are expressed by the commutative diagrams

$$\begin{array}{ccc} M \otimes_k L_R^\vee \otimes_R L \otimes_k M_R^\vee \otimes_R M \otimes_k L_R^\vee & \xrightarrow{u_i \otimes_R v_i \otimes_R u_j} & S \otimes_R S \otimes_R S \\ \Big\uparrow & & \Big\downarrow \\ M \otimes_k M_R^\vee \otimes_R M \otimes_k L_R^\vee & \xrightarrow{(v_i \circ u_i) \otimes_R u_j} & S \otimes_R S \\ \Big\uparrow & & \Big\downarrow id \\ R \otimes_R M \otimes_k L_R^\vee & \xrightarrow{\phi \otimes_R u_j} & S \otimes_R S \end{array} \tag{3}$$

Fix an R-ring $\mathcal{G} = \mathcal{G}_{M,L}$ corepresenting the functor $G_{_{M,L}} : R\backslash Alg_k \longrightarrow Sets$. And let $\mathcal{G} \otimes_R L \xrightarrow{v} \mathcal{G} \otimes_R M \xrightarrow{u} \mathcal{G} \otimes_R L$ be a canonical splitting — the image of $id_{\mathcal{G}}$ under the isomorphism $Alg(\mathcal{G}, \mathcal{G}) \xrightarrow{\sim} G_{_{M,L}}(\mathcal{G})$. Set $p = v \circ u : \mathcal{G} \otimes_R M \longrightarrow \mathcal{G} \otimes_R M$. Denote by H a colimit of the diagram

$$\begin{array}{ccc} M \otimes_k M_R^\vee \otimes_R M \otimes_k L_R^\vee & \xrightarrow{p \otimes_R u} & \mathcal{G} \otimes_R \mathcal{G} \\ \Big\uparrow & & \Big\downarrow id \\ M \otimes_k L_R^\vee \simeq R \otimes_R M \otimes_k L_R^\vee & \xrightarrow{\phi \otimes_R u} & \mathcal{G} \otimes_R \mathcal{G} \end{array} \tag{4}$$

Note that H is a quotient of the bimodule $\mathcal{G} \otimes_R \mathcal{G}$. The comultiplication, δ, on H is uniquely defined by the condition: the canonical $\mathcal{G}$-bimodules epimorphism $\mathcal{G} \otimes_R \mathcal{G} \longrightarrow H$ is a coalgebra morphism.

The category of quasi-coherent sheaves on the locally affine space $Gr^+_{M,L}$ is equivalent to the category $(H,\delta) - comod$ of (H,δ)-comodules.

C9.5.1. Example: the zero dimensional projective space. Let $R = k$. The zero dimensional projective space is by definition $\mathbb{P}_{k^1} = Gr^+_{k^1,k^1}$. The algebra $\mathcal{G}$ representing the functor G_{k^1,k^1} is the quotient of the free k-algebra $k\langle x,y\rangle$ in two variables by the two-sided ideal generated by $xy-1$: $\mathcal{G} = k\langle x,y\rangle/(xy-1)$. Denote by e the element $1-yx$ of $\mathcal{G}$. It follows that e is an idempotent, $e^2 = e$. The coalgebra corresponding to the projection $\pi : \mathbf{Spec}(A) \longrightarrow \mathbb{P}_{k^1}$ is (H,δ), where the bimodule H enters into the exact sequence

$$0 \longrightarrow \mathcal{G}\otimes e\mathcal{G} \xrightarrow{\ i\ } \mathcal{G}\otimes\mathcal{G} \longrightarrow H \longrightarrow 0 \tag{1}$$

Here $\otimes = \otimes_k$, and i is defined by $a\otimes eb \longmapsto ax\otimes eb$. The comultiplication δ is uniquely determined by the commutativity of the diagram

$$\begin{array}{ccc} H & \xrightarrow{\ \delta\ } & H\otimes_{\mathcal{G}} H \\ \uparrow & & \uparrow \\ \mathcal{G}\otimes\mathcal{G} & \xrightarrow{\ \delta^{\sim}\ } & \mathcal{G}\otimes\mathcal{G}\otimes\mathcal{G} \end{array}$$

where $\delta^{\sim}$ sends $a\otimes b$ to $a\otimes 1\otimes b$. The counit $\epsilon : H \longrightarrow \mathcal{G}$ is induced by the multiplication $\mathcal{G}\otimes\mathcal{G} \longrightarrow \mathcal{G}$ in the algebra $\mathcal{G}$.

C10. Miscellaneous.

C10.1. Connections. Let $X \xrightarrow{f} Y$ be a continuous morphism. Fix inverse and direct image functors of f and adjunction arrows, $Id_{C_Y} \xrightarrow{\eta_f} f_*f^*$ and $f^*f_* \xrightarrow{\epsilon_f} Id_{C_X}$. Let $\mathcal{G}_f$ be the comonad associated with this data: $\mathcal{G}_f = (G_f, \delta_f)$, where $G_f = f^*f_*$ and $\delta_f = f^*\eta_f f_*$.

A $(X\backslash\mathcal{G}_f)$-connection is a pair (M,ρ), where M is an object of C_X and ρ a morphism $M \longrightarrow G_f(M)$ such that $\epsilon_f(M)\circ\rho = id_M$. A morphism from an f-connection (M,ρ) to an f-connection (M',ρ') is a morphism $g : M \longrightarrow M'$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ g\ } & M' \\ \rho\big\downarrow & & \big\downarrow\rho' \\ G_f(M) & \xrightarrow{\ G_f(g)\ } & G_f(M') \end{array}$$

commutes. Composition is defined in a natural way. We denote the category of $\mathcal{G}_f$-connections by $\mathfrak{Conn}(X\backslash\mathcal{G}_f)$.

A $\mathcal{G}_f$-connection (M,ρ) is called *integrable* if it is an $(X\backslash\mathcal{G}_f)$-comodule. Thus, the category $(X\backslash\mathcal{G}) - Comod$ is a full subcategory of the category $\mathfrak{Conn}(X\backslash\mathcal{G}_f)$ formed by connections and morphisms between them.

C10.1.1. Connections and monadic morphisms. Suppose that the morphism $X \xrightarrow{f} Y$ is monadic, i.e. the arrow $\bar{f}$ in the diagram 5.4(3) is an isomorphism. Then we can assume (for our immediate purposes) that $X = \mathbf{Sp}(\mathcal{F}_f/Y)$ and take the standard inverse and direct image functors of the canonical morphism $f : \mathbf{Sp}(\mathcal{F}_f/Y) \longrightarrow Y$. The functor $G_f : (\mathcal{F}_f/Y) - mod \longrightarrow (\mathcal{F}_f/Y) - mod$ assigns to each $(\mathcal{F}_f/Y)$-module $\mathcal{M} = (M, \xi)$ the $(\mathcal{F}_f/Y)$-module $(F_f(M), \mu_f(M))$. The comultiplication $\delta_f : G_f \longrightarrow G_f^2$ is $F_f\eta_f$, where η_f is the unit of the monad $\mathcal{F}_f$; the counit $\epsilon_f : G_f \longrightarrow Id_{(\mathcal{F}_f/Y)-mod}$ is defined by $\epsilon_f(M, \xi) = \xi : F_f(M) \longrightarrow M$.

A $(X\backslash\mathcal{G}_f)$-connection is a pair $(\mathcal{M}, \rho)$, where $\mathcal{M} = (M, \xi)$ is an $(\mathcal{F}_f/Y)$-module and ρ an $(\mathcal{F}_f/Y)$-module morphism $(M, \xi) \longrightarrow (F_f(M), \mu_f(M))$ such that $\xi \circ \rho = id_M$. In other words, ρ is a morphism $M \longrightarrow F_f(M)$ satisfying the equations

$$\mu_f(M) \circ F_f(\rho) = \rho \circ \xi, \quad \xi \circ \rho = id_M. \tag{1}$$

The connection $(\mathcal{M}, \rho)$ is integrable iff $F_f\eta_f(M) \circ \rho = F_f(\rho) \circ \rho$. It follows from (1) that (M, ξ) is the kernel of the pair of arrows

$$F_f\eta_f(M), F_f(\rho) : F_f(M) \rightrightarrows F_f^2(M) \tag{2}$$

and $\rho : (M, \xi) \longrightarrow \mathcal{F}_f(M) = (F_f(M), \mu_f(M))$ is a universal arrow. The pair

$$\eta_f(M), \rho : M \rightrightarrows F_f(M) \tag{3}$$

is coreflexive, since $\xi \circ F_f\eta_f(M) = id_M = \xi \circ \rho$. Suppose C_Y has kernels of coreflexive pairs, and let L be a kernel of the pair (3). If the functor F_f preserves the kernels of coreflexive pairs, then $\mathcal{F}_f(L) = (F_f(L), \mu_f(L))$ is a kernel of the pair (2) which implies that (M, ξ) is isomorphic to $\mathcal{F}_f(L)$.

C10.2. Weakly quasi-affine morphisms. We call a continuous morphism $X \xrightarrow{f} Y$ *weakly quasi-affine* if the canonical diagram

$$(G_f)^2 \underset{G_f\epsilon_f}{\overset{\epsilon_f G_f}{\rightrightarrows}} G_f \xrightarrow{\epsilon_f} Id_{C_X} \tag{1}$$

is exact. Here $G_f = f^*f_*$ and ϵ_f is an adjunction arrow.

C10.2.1. Proposition. *Let $Y \in Ob|Cat|^o$ be such that the category C_Y has cokernels of reflexive pairs of arrows. The following conditions on a continuous morphism $X \xrightarrow{f} Y$ are equivalent:*

(a) The canonical morphism $\bar{X} \xrightarrow{f} \mathbf{Sp}(\mathcal{F}_f/Y)$ (cf. 5.4.2) is a localization.

(b) The morphism f is weakly quasi-affine.

Proof. Since the category C_Y has cokernels of reflexive pairs of arrows, the functor $\bar{f}_* : C_Y \longrightarrow (\mathcal{F}_f/Y) - mod,\ L \longmapsto (f_*(L), f_*\epsilon_f(L))$, has a left adjoint, $\bar{f}^*$, which assigns to each $(\mathcal{F}_f/Y)$-module (M, ξ) a cokernel of the pair of arrows

$$f^*f_*f^*(M) \underset{f^*(\xi)}{\overset{\epsilon_f f^*(M)}{\rightrightarrows}} f^*(M). \tag{2}$$

(b)⇔(a). The exactness of the diagram (1) means precisely that the adjunction morphism $\epsilon_{\bar{f}} : \bar{f}^* \bar{f}_* \longrightarrow Id_{C_X}$ is an isomorphism. The latter is equivalent to that $\bar{f}_*$ is a fully faithful functor, i.e. $\bar{f}^*$ is a localization. ∎

C10.2.2. Remark. If a morphism $X \xrightarrow{f} Y$ is weakly quasi-affine, than an adjunction arrow $\epsilon_f : f^* f_* \longrightarrow Id_{C_X}$ is a strict epimorphism. Notice that ϵ_f is an epimorphism iff the functor f_* is faithful. The exactness of the diagram (1) implies that f_* is conservative. On the other hand, if the functor f_* reflects cokernels of reflexive pairs of arrows, then the diagram (1) is exact, i.e. f is weakly quasi-affine. This follows from the observation that the pair of morphisms $(f^* f_*)^2 \overset{\epsilon_f G_f}{\underset{G_f \epsilon_f}{\rightrightarrows}} f^* f_*$ is reflexive, and the diagram $f_*(1)$,

$$f_*(f^* f_*)^2 \overset{f_* \epsilon_f G_f}{\underset{f_* G_f \epsilon_f}{\rightrightarrows}} f_* f^* f_* \xrightarrow{f_* \epsilon_f} f_*,$$

is exact for any pair of adjoint functors f_*, f^* and an adjunction morphism ϵ_f.

By Beck's theorem, a morphism $X \xrightarrow{f} Y$ satisfying the equivalent conditions of C10.2.1 is *monadic* if its direct image functor f_* preserves cokernels of reflexive pairs of arrows.

C10.2.3. Weakly quasi-affine morphisms to the spectrum of a ring. Let R be an associative unital ring and f a continuous morphism $X \longrightarrow \mathbf{Sp}(R)$ with an inverse image functor f^* and a direct image f_*; and let $\mathcal{O} = f^*(R)$. Consider the decomposition

$$\begin{array}{ccc} C_X & \xrightarrow{\hat{f}_*} & \Gamma_X \mathcal{O} - mod \\ f_* \searrow & & \swarrow \phi_* \\ & R - mod & \end{array} \qquad (3)$$

Here $\Gamma_X \mathcal{O}$ denotes the ring $C_X(\mathcal{O}, \mathcal{O})^o$, ϕ_* is the pull-back functor by the ring morphism $\phi : R \longrightarrow \Gamma_X \mathcal{O}$ defining a right R-module structure on $\mathcal{O}$, and $\hat{f}_* = C_X(\mathcal{O}, -)$ (see 4.5).

C10.2.3.1. Proposition. *Let C_X be a Grothendieck category and R an associative unital ring.*

(a) A continuous morphism $X \xrightarrow{f} \mathbf{Sp}(R)$ is weakly quasi-affine iff its direct image functor is faithful.

(b) Every weakly quasi-affine morphism $X \longrightarrow \mathbf{Sp}(R)$ is a composition of an affine morphism and a flat localization.

Proof. By C10.2.2, direct image functor of a weakly quasi-affine morphism is faithful.

Conversely, if f_* is a faithful functor and C_X is a Grothendieck category, then, by 4.8.2, the canonical morphism $X \longrightarrow \mathbf{Sp}(\Gamma_X \mathcal{O})$ is a flat localization. Thus the first arrow in the canonical decomposition $X \longrightarrow \mathbf{Sp}(\Gamma_X \mathcal{O}) \longrightarrow \mathbf{Sp}(R)$ of the morphism f is a flat localization and the second one an affine morphism. ∎

C10.2.3.2. Two decompositions of a continuous morphism to the spectrum of a ring. Let $X \xrightarrow{f} \mathbf{Sp}(R)$ be continuous morphism to the spectrum of a unital associative ring. We have two canonical decompositions of the morphism f incorporated in the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f_{\mathcal{O}}} & \mathbf{Sp}(\Gamma_X\mathcal{O}) \\ {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle \phi_f} \\ \mathbf{Sp}(\mathcal{F}_f/R) & \xrightarrow{\hat{f}} & \mathbf{Sp}(R) \end{array} \tag{4}$$

where $\mathcal{F}_f$ is the monad (f_*f^*, μ_f) associated with f. Let $\Gamma_X\mathcal{O}^\sim$ denote the monad on $\mathbf{Sp}(R)$ (i.e. on $R-mod$) determined by the ring morphism ϕ_f. Then we have a natural isomorphism $\mathbf{Sp}(\Gamma_X\mathcal{O}^\sim/R) \xrightarrow{\sim} \mathbf{Sp}(\Gamma_X\mathcal{O})$ which makes the diagram

$$\begin{array}{ccc} \mathbf{Sp}(\Gamma_X\mathcal{O}^\sim/R) & \xrightarrow{\sim} & \mathbf{Sp}(\Gamma_X\mathcal{O}) \\ & \searrow \quad \swarrow{\scriptstyle \phi_f} & \\ & \mathbf{Sp}(R) & \end{array} \tag{5}$$

commute. The monad $\Gamma_X\mathcal{O}^\sim$ is exactly the continuous monad associated with the monad $\mathcal{F}_f$ (cf. C2.1). Thus, there exists a canonical monad morphism $\Gamma_X\mathcal{O}^\sim \longrightarrow \mathcal{F}_f$ to which there corresponds a morphism of their spectra,

$$\mathbf{Sp}(\mathcal{F}_f/R) \longrightarrow \mathbf{Sp}(\Gamma_X\mathcal{O}^\sim/R). \tag{6}$$

Adjoining the morphism (6) (or ruther the composition of (6) with the isomorphism $\mathbf{Sp}(\Gamma_X\mathcal{O}^\sim/R) \xrightarrow{\sim} \mathbf{Sp}(\Gamma_X\mathcal{O})$), we obtain a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f_{\mathcal{O}}} & \mathbf{Sp}(\Gamma_X\mathcal{O}) \\ {\scriptstyle \bar{f}}\downarrow & {\scriptstyle \lambda_f}\nearrow & \downarrow{\scriptstyle \phi_f} \\ \mathbf{Sp}(\mathcal{F}_f/R) & \xrightarrow{\hat{f}} & \mathbf{Sp}(R) \end{array} \tag{7}$$

If f is weakly quasi-affine, then, by C10.2.1, the morphism $\bar{f}$ is a localization. If, in addition, C_X is a Grothendieck category, then the morphism $f_{\mathcal{O}}$ is a flat localization.

The canonical morphism $\lambda_f : \mathbf{Sp}(\mathcal{F}_f/R) \longrightarrow \mathbf{Sp}(\Gamma_X\mathcal{O})$ is an isomorphism iff the morphism f is affine. In this case, $f_{\mathcal{O}} : X \longrightarrow \mathbf{Sp}(\Gamma_X\mathcal{O})$ is an isomorphism.

C10.2.4. Relatively ample morphisms. Let $X \xrightarrow{f} Y$ be a continuous morphism. We call a continuous morphism $\phi : U \longrightarrow X$ *f-ample* if the canonical diagram

$$\Big(G^2_{f\phi} \underset{G_{f\phi}\epsilon_{f\phi}}{\overset{\epsilon_{f\phi}G_{f\phi}}{\rightrightarrows}} G_{f\phi} \xrightarrow{\epsilon_{f\phi}} Id_{C_X}\Big)\phi^* \tag{1}$$

(where $G_{f\phi} = (f\phi)^*(f\phi)_*$ and $\epsilon_{f\phi}$ is an adjunction arrow) is exact.

It follows that id_X is f-ample iff f is weakly quasi-affine.

Denote by $U_{f\phi}$ the object of $|Cat|^o$ such that $C_{U_{f\phi}}$ is the full subcategory of C_U whose objects are all $M \in ObC_U$ for which the diagram

$$G^2_{f\phi}(M) \underset{G_{f\phi}\epsilon_{f\phi}}{\overset{\epsilon_{f\phi}G_{f\phi}}{\rightrightarrows}} G_{f\phi}(M) \xrightarrow{\epsilon_{f\phi}} M \qquad (2)$$

is exact. The condition 'ϕ is f-ample' implies that ϕ induces a continuous morphism $\varphi : U_{f\phi} \longrightarrow X$ such that $f\varphi$ is weakly quasi-affine.

C10.2.4.1. Note. The exactness of the diagram (1) implies that the functor $f_*\phi_*\phi^*$ is faithful and conservative.

C10.2.4.2. Example. Let $X \xrightarrow{f} Y$ be a continuous morphism, and let $\theta : X \longrightarrow X$ be a continuous endomorphism. Suppose C_X has countable coproducts. Then $F_\theta = \oplus_{n\geq 0}\theta^{*n}$ has a natural structure of a monad on X. Let $U = \mathbf{Sp}(F_\theta/X)$, and let $\phi = \phi_\theta$ be a canonical morphism $U \longrightarrow X$. Thus $\phi_*\phi^* = F_\theta$. If f_* preserves countable coproducts, $f_*\phi_*\phi^* \simeq \oplus_{n\geq 0} f_*\theta^{*n}$. In this case, if the morphism ϕ is f-ample, than the family $\{f_*\theta^{*n} \mid n \geq 0\}$ is conservative.

C10.2.4.2.1. Example. Let $X = (X, \mathcal{O}_X)$ be a scheme and $X \xrightarrow{f} Y$ a quasi-compact scheme morphism. Let $\mathcal{L}$ be an invertible sheaf on X. Set $\theta = \mathcal{L} \otimes_{\mathcal{O}_X} -$. The sheaf $\mathcal{L}$ is f-ample in the usual sense iff the morphism $\phi = \phi_\theta$ constructed in C10.2.4.2 is f-ample.

C10.3. Continuous, affine, and flat morphisms in a fibered category. Let $\mathfrak{P}$ be a subcategory of $|Cat|^o$, and let $p : \mathcal{F} \longrightarrow \mathcal{B}$ be a fibered category. We say that a morphism f of $\mathcal{B}$ belongs to $\mathfrak{P}$ if its image by by the canonical functor $\Psi_p : \mathcal{B} \longrightarrow |Cat|^o$ belongs to $\mathfrak{P}$. In particular, we call a morphism f of $\mathcal{B}$ *continuous* (resp. *affine*, resp. *flat*, resp. *faithful*) if its image by $\Psi_p : \mathcal{B} \longrightarrow |Cat|^o$ is continuous (resp. affine, resp. flat, resp. faithful).

C10.3.1. Continuous, affine, and flat morphisms in the fibered category of modules on affine schemes. Consider the fibered category $\mathfrak{M}(\mathcal{A}, \mathcal{O})$ over the category $\mathcal{A} = \mathbf{Aff}_k$ of affine k-schemes (cf. [KR3, 11.4.1]). All morphisms of Aff_k are affine. In fact, any morphism $f : \mathbf{Spec}R \longrightarrow \mathbf{Spec}S$ is given by an algebra morphism $S \longrightarrow R$. By 3.8, the corresponding inverse image functor f^* is $R\otimes_S : S-mod \longrightarrow R-mod$, the direct image functor f_* is the 'pull-back' functor $R-mod \longrightarrow S-mod$, and a right adjoint to f_* is given by $f^! : L \longmapsto Hom_S(R, L)$.

A morphism $f : \mathbf{Spec}R \longrightarrow \mathbf{Spec}T$ is flat (resp. faithful) iff the corresponding algebra morphism $T \longrightarrow R$ turns R into a flat right T-module, that is the functor $R \otimes_T -$ is exact (resp. faithful).

C10.3.2. Proposition. *Let X, Y be presheaves of sets on $\mathbf{Aff}_k$ and let $X \xrightarrow{f} Y$ be a morphism of presheaves. Suppose Y is affine (i.e. representable). Then X is affine iff the morphism $|Qcoh_X| \longrightarrow |Qcoh_Y|$ in $|Cat|^o$ induced by f is affine.*

Proof. Let Y be isomorphic to $\mathbf{Spec}R$ for some k-algebra R. By [KR3, 11.1.5.1], the category $Qcoh_Y$ is equivalent to the category $R-mod$.

(a) If X is affine, i.e. $X \simeq \mathbf{Spec}S$ for a k-algebra S, then $Qcoh_X$ is equivalent to the category $S - mod$ and we have a quasi-commutative diagram

$$\begin{array}{ccc} S - mod & \xrightarrow{\sim} & Qcoh_X \\ \phi^* \uparrow & & \uparrow f^* \\ R - mod & \xrightarrow{\sim} & Qcoh_X \end{array} \qquad (1)$$

where ϕ^* is the inverse image functor corresponding to a k-algebra morphism $R \xrightarrow{\phi} S$; i.e. $\phi^* = S \otimes_R -$. Since ϕ^* is an inverse image of an affine morphism (by the argument above or 3.8), f is affine.

(b) The converse assertion follows from 6.6.1. ∎

Let $\mathfrak{A}ss_k$ be the category whose objects are associative k-algebras. Morphisms from a k-algebra R to a k-algebra S are equivalence classes of algebra morphisms $R \longrightarrow S$ by the following equivalence relation: two algebra morphisms, $f, g : R \longrightarrow S$ are equivalent if they are conjugated, i.e. $g(-)t = tf(-)$ for an invertible element t of S.

Note that the restriction of the natural functor $Ass_k \longrightarrow \mathfrak{A}ss_k$ to the subcategory of commutative algebras is a strict fully faithful functor ('strict' means that it is injective on objects).

C10.3.3. Proposition. *The canonical functor $\mathbf{Aff}_k \longrightarrow |Cat|^o$ induces a faithful functor from the category $\mathfrak{A}ss_k^{op}$ to the subcategory $|Cat|^o_{aff}$ of $|Cat|^o$ formed by affine morphisms.*

Proof. The assertion is a corollary of [KR3, 5.10.1] and C10.3.2. ∎

Denote by $\mathfrak{B}\mathfrak{A}ss$ the category whose objects are associative unital rings and morphisms from R to S isomorphism classes of (S, R)-bimodules. The composition is induced by tensoring bimodules. Notice that there is a natural embedding $\mathfrak{A}ss \hookrightarrow \mathfrak{B}\mathfrak{A}ss$. Denote by $\mathfrak{M}\mathfrak{A}ss$ class of morphisms of the form $f \circ m$, where m is the isomorphism class of an invertible bimodule (Morita equivalence) and f belongs to $Hom(\mathfrak{A}ss)$.

C10.3.4. Proposition. *(a) $\mathfrak{M}\mathfrak{A}ss$ is a subcategory of $\mathfrak{B}\mathfrak{A}ss$.*

(b) The natural functor $\mathfrak{B}\mathfrak{A}ss^{op} \longrightarrow |Cat|^o_{\mathfrak{c}}$ is fully faithful.

(c) The restriction of the functor (b) to the subcategory $\mathfrak{M}\mathfrak{A}ss^{op}$ induces an equivalence $\mathfrak{M}\mathfrak{A}ss^{op} \longrightarrow (|Cat|^o/\mathbf{Sp}\mathbb{Z})_{aff}$.

Proof. The assertion follows from C10.3.3 and 6.4.1. ∎

C10.4. Additive monads and continuous monads. For any $X \in Ob|Cat|^o$ such that the category C_X is (pre)additive, denote by $\mathfrak{Mon}^a_X$ the category of *additive* monads on X, i.e. monads (F, μ) on C_X with an additive functor F. We denote by $\mathfrak{Mon}^c_X$ the category of continuous monads on X.

C10.4.1. Proposition. *Let R be an associative unital ring and $X = \mathbf{Sp}(R)$. The inclusion functor $\mathfrak{j}^* : \mathfrak{Mon}^c_X \hookrightarrow \mathfrak{Mon}^a_X$ has a left adjoint.*

Proof. Let $\mathcal{F} = (F, \mu)$ be an additive monad on $C_{\mathbf{Sp}(R)}$, and let $f : \mathbf{Sp}(\mathcal{F}/R) \longrightarrow \mathbf{Sp}(R)$ denote the canonical morphism with inverse image functor $V \longmapsto (F(V), \mu(V))$. In particular, the object $\mathcal{O} = f^*(R)$ is $(F(R), \mu(R))$, and $C_{\mathbf{Sp}(\mathcal{F}/R)}(\mathcal{O}, \mathcal{O}) \simeq f_*f^*(R) = F(R)$. The latter means that the R-bimodule $F(R)$ has a structure of a ring (in the category of R-bimodules) such that the adjunction arrow $R \longrightarrow F(R)$ is a ring morphism. Let $f_{\mathcal{O}*}$ denote the functor $C_{\mathbf{Sp}(\mathcal{F}/R)}(\mathcal{O}, -)$ and $f_{\mathcal{O}}^*$ its left adjoint. The functor $f_{\mathcal{O}*}f_{\mathcal{O}}^*$ is isomorphic to the functor $F_c = F(R) \otimes_R - : R - mod \longrightarrow R - mod$. Thus we obtain a monad $\mathfrak{F}_c$ on $R - mod$. The map $\mathfrak{F} \longmapsto \mathfrak{F}_c$ is functorial. We denote this functor by $\mathfrak{j}_*$.

Since the functor F_c preserves colimits of small diagrams, the natural isomorphism $F_c(R) \xrightarrow{\sim} F(R)$ extends canonically to the functor morphism $\varphi_F : F_c \longrightarrow F$ such that $\varphi_F(M)$ is an isomorphism for any projective R-module M of finite type.

In fact, the isomorphism $F_c(R) \xrightarrow{\sim} F(R)$ yields an isomorphism $\varphi_F(L) : F_c(L) \longrightarrow F(L)$ for any free R-module L of finite rank functorial in L. Since every free module of infinite rank is a colimit of its free submodules of finite rank, φ_F extends to all free modules. Since every R-module is a cokernel of a pair of R-module morphisms $L_1 \rightrightarrows L_0$ with L_0, L_1 free R-modules, and F_c preserves cokernels, φ_F extends to all R-modules in a unique way (thanks to universal property of colimits).

One can see that φ_F is a monad morphism. The function $\varphi : F \longmapsto \varphi_F$ defines a functor morphism $\mathfrak{j}^*\mathfrak{j}_* \longrightarrow Id_{\mathfrak{Mon}_X^a}$ which is an adjunction morphism. The other adjunction morphism, $Id_{\mathfrak{Mon}_X^c} \longrightarrow \mathfrak{j}_*\mathfrak{j}^*$ is identical. ∎

Let C_X be an additive category. Let $End_a(C_X)$ denote the category of additive functors $C_X \longrightarrow C_X$ and $End_c(C_X)$ its full subcategory formed by continuous endofunctors.

C10.4.2. Proposition. *Let C_X be an additive category with small colimits. Then*
(a) The inclusion functor $End_c(C_X) \longrightarrow End_a(C_X)$ has a left adjoint.
(b) The inclusion functor $\mathfrak{j}^* : \mathfrak{Mon}_X^c \hookrightarrow \mathfrak{Mon}_X^a$ has a left adjoint.

Proof. (a) Let F be a functor $C_X \longrightarrow C_X$. Consider the category $End_c(C_X)/F$ whose objects are pairs (G, ξ), where G is a continuous functor $C_X \longrightarrow C_X$ and ξ is a functor morphism, morphisms $(G, \xi) \longrightarrow (G', \xi')$ are functor morpisms $\psi : G \longrightarrow G'$ such that $\xi = \xi' \circ \psi$. Let $\mathcal{D}_F$ denote the diagram

$$End_c(C_X)/F \longrightarrow End(C_X), \ (G, \xi) \longmapsto G.$$

We denote a colimit of the diagram $\mathcal{D}_F$ by F_c^*. The functor F_c^*, being a colimit of a diagram of continuous functors, preserves colimits of small diagrams. Since for any $x, y \in ObC_X$, there exist morphisms $F_c^*(x) \longrightarrow y$, i.e. the category F_c^*/y is non-empty, the assignment to every $y \in ObC_X$ a colimit of the functor $F_c^*/y \longrightarrow C_X$, $(x, F_c^*(x) \xrightarrow{\xi} y) \longmapsto x$, determines a right adjoint, F_{c*}, to F_c^*.

It follows from the construction that there is a canonical morphism $\epsilon_{\mathfrak{j}}(F) : F_c^* \longrightarrow F$ which depends functorially on F. If F is continuous, then $\epsilon_{\mathfrak{j}}(F)$ can be chosen to be an identical morphism. This shows that $\mathfrak{j}^* : F \longmapsto F_c$ is a left adjoint to the inclusion functor $\mathfrak{j}_* : End_c(C_X) \hookrightarrow End_a(C_X)$ and $\epsilon_{\mathfrak{j}}$ is an adjunction morphism. The other adjunction morphism is identical.

(b) If $\mathcal{F} = (F, \mu)$ is a monad on C_X, then there is a unique monad structure, $\mu_c : (F_c^*)^2 \longrightarrow F_c^*$ such that the adjunction morphism $F_c^* \longrightarrow F$ is a monad morphism. It follows from (a) that the map $\mathcal{F} = (F, \mu) \longmapsto \mathcal{F}_c = (F_c^*, \mu_c)$ extends to a functor which is left adjoint to the inclusion functor $\mathfrak{j}^* : \mathfrak{Mon}_X^c \hookrightarrow \mathfrak{Mon}_X^a$. Details are left to the reader. ∎

C10.4.3. Proposition. *Suppose the category C_X has limits and colimits of small diagrams. Then the inclusion functor $\mathfrak{j}^* : \mathfrak{Mon}^c_X \hookrightarrow \mathfrak{Mon}_X$ has a left adjoint.*

Proof. Let $\mathcal{F} = (F, \mu)$ be a monad on X. Since Id_{C_X} is a continuous functor and there is a morphism $\eta : Id_{C_X} \longrightarrow F$ (the unit of $\mathcal{F}$), the category $End_c(C_X)/F$ (see the part (a) of the argument of C10.4.2) is non-empty. Let $\mathcal{D}_F$ denote the standard functor $End_c(C_X)/F \longrightarrow End(C_X)$. For any two objects, x, y of the category C_X, we have isomorphisms:

$$C_X(F_c(x), y) \simeq \lim C_X(\mathcal{D}_F(x), y) \simeq \lim C_X(x, \mathcal{D}_F^\vee(y)).$$

Here $\mathcal{D}_F^\vee$ denotes the diagram $End_c(C_X)/F \longrightarrow End(C_X)$ which assigns to any object (G^*, ξ) of $End_c(C_X)/F$ a right adjoint, G_*, to G^* and to any morphism $(G^*, \xi) \longrightarrow (H^*, \nu)$ the corresponding morphism $H_* \longrightarrow G_*$ of right adjoint functors. If the category C_X has limits, there exists a limit of $\mathcal{D}_F^\vee(y)$. Choosing this limit for each y, we define a functor which is a right adjoint to F_c. ∎

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