ON DONALDSON AND SEIBERG-WITTEN INVARIANTS

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ABSTRACT. In these lecture notes, we sketch the proof (due to Thomas Leness and the author) of Witten's conjecture that the Donaldson and Seiberg-Witten series coincide, modulo powers of degree $c+2$, with $c = -\frac{1}{4}(7\chi + 11\sigma)$, for four-manifolds obeying some mild conditions, where χ and σ are their Euler characteristic and signature. We use the moduli space of SO(3) monopoles as a cobordism between a link of the Donaldson moduli space of anti-self-dual SO(3) connections and links of the moduli spaces of Seiberg-Witten monopoles. Gluing techniques allow us to compute contributions from Seiberg-Witten moduli spaces lying in the first (or "one-bubble") level of the Uhlenbeck compactification of the moduli space of SO(3) monopoles.

1. INTRODUCTION

This article consists of a very lightly edited version of the author's notes for his lecture at the Mathematische Arbeitstagung 2001, hosted by the Max Planck Institut für Mathematik, Bonn. The severe time constraints imposed by the rapid production schedule for the Arbeitstagung proceedings guaranteed that these notes remain rough and informal, glossing over technical details and qualifications appropriate for more complete published accounts elsewhere [7], [8], [11]. Despite all their shortcomings, we hope that they provide a brief introduction to some of the author's joint work with Thomas Leness and which is reasonably clear, even if not necessarily precise in all respects.

1.1. **Witten's conjecture.** Two kinds of invariants can be used to explore the classification problem for compact, smooth 4-manifolds:

- *Donaldson invariants*, defined using an SO(3) Yang-Mills gauge theory (1986–1994).
- *Seiberg-Witten invariants*, defined using a U(1) monopole gauge theory (1994).

We shall restrict our attention throughout to the case of closed, oriented 4-manifolds with $b_1 = 0$ and odd $b_2^+ > 1$. The conjectured relationship between these gauge theory invariants is described below:

Conjecture 1.1. [25] A 4-manifold X has KM-simple type if and only if it has SW-simple type. If X has simple type, the KM and SW basic classes coincide, and the Donaldson and Seiberg-Witten series obey

$$\mathbf{D}_X^w(h) = 2^{2-c(X)} e^{\frac{1}{2}Q_X(h,h)} \mathbf{SW}_X^w(h), \quad h \in H_2(X;\mathbb{R}). \tag{1}$$

Here, the 4-manifold X has intersection form

$$Q_X : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z},$$

Euler characteristic χ, signature σ and

$$c(X) = -\frac{1}{4}(7\chi + 11\sigma).$$

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We shall recall the definitions of the Donaldson and Seiberg-Witten series shortly.

1.2. **Remarks on the problem.** Before proceeding to discuss our work on Witten's conjecture, it is interesting to compare the mathematician's and physicists' approaches to establishing (1).

Witten employs a certain $N = 2$ *supersymmetric quantum Yang-Mills theory*. He uses rescaling, $g_t = t^2 g$, of the Riemannian metric g on X and metric independence of the correlation functions to relate the Donaldson invariants ($t \to 0$) with the Seiberg-Witten invariants ($t \to \infty$).

The mathematical approach to a proof of Witten's formula proposed by Pidstrigatch and Tyurin [22] instead employs an SO(3) monopole gauge theory which generalizes both the instanton and U(1) monopole gauge theories. All three gauge theories are classical field theories and their solutions are invariant under metric rescaling, whereas Witten's quantum field theory is sensitive to metric rescaling.

Apparently, the SO(3) monopole gauge theory provides a purely classical field theory alternative to Witten's quantum field theory method. The problem of determining the relationship between these two approaches is surely an important one worth exploring further.

2. SO(3) monopoles

2.1. **Clifford modules and spin structures.** Given a Riemannian metric g on X, let $V \to X$ be a Hermitian bundle with Clifford map $\rho : T^*X \to \mathrm{End}_{\mathbb{C}}(V)$,

$$\rho(\alpha)^\dagger = -\rho(\alpha), \quad \text{and} \quad \rho(\alpha)^2 = -g(\alpha, \alpha), \quad \alpha \in \Omega^1(X, \mathbb{R}).$$

Then (ρ, V) defines a Clifford or $\mathbb{C}\ell(T^*X)$ module structure on V. If $W \to X$ is a complex-rank four Hermitian bundle, then $\mathfrak{s} = (\rho, W)$ is a *spin^c structure*, familiar from Seiberg-Witten theory [23]. If $V \to X$ is a complex-rank eight Hermitian bundle, then we call $\mathfrak{t} = (\rho, V)$ a *spin^u structure*.

2.2. **From spin^u structures to SO(3) bundles.** Given $\mathfrak{t} = (\rho, V)$ on X, one obtains

- An *SO(3) subbundle*,
$$\mathfrak{g}_{\mathfrak{t}} \subset \mathfrak{su}(V),$$
characterized as the span of the sections ξ of $\mathfrak{su}(V)$ such that $[\xi, \rho(\omega)] = 0$, for all $\omega \in \Omega^\bullet(X, \mathbb{R})$.
- A complex line bundle,
$$\det{}^{\frac{1}{2}}(V^+),$$
where $V = V^+ \oplus V^-$ and $V^\pm$ are the ∓ 1 eigenspaces of $\rho(\mathrm{vol})$ on V.
- Splittings, if $\Lambda^2(T^*X) = \Lambda^+ \oplus \Lambda^-$ and $\Lambda^\pm$ are the ± 1 eigenspaces of $*_g$ on $\Lambda^2(T^*X)$, and $\rho : \Lambda^\pm \cong \mathfrak{su}(W^\pm)$ are the usual isomorphisms of SO(3) bundles,
$$\mathfrak{su}(V^\pm) \cong \rho(\Lambda^\pm) \oplus \rho(\Lambda^\pm) \otimes \mathfrak{g}_{\mathfrak{t}} \oplus \mathfrak{g}_{\mathfrak{t}}.$$

Moreover, for any choice of spin^c structure $\mathfrak{s} = (\rho, W)$, one further obtains

- A complex-rank two Hermitian bundle,
$$E = \mathrm{Hom}_{\mathbb{C}\ell(T^*X)}(W, V).$$
- A Clifford module isomorphism,
$$V \cong W \otimes E.$$
- An isomorphism of SO(3) bundles
$$\mathfrak{su}(E) \cong \mathfrak{g}_{\mathfrak{t}}.$$

- An isomorphism of complex line bundles,

$$\det^{\frac{1}{2}}(V^+) \cong \det(W^+) \otimes \det(E).$$

2.3. **SO(3)-monopole equations.** We call a pair (A, Φ) an *SO(3) monopole* if

$$\begin{aligned} \mathrm{ad}^{-1}(F^+_{\hat{A}}) - \rho^{-1}(\Phi \otimes \Phi^*)_{00} &= 0, \\ D_A \Phi &= 0 \end{aligned}$$

where A is a spin connection on V, inducing a fixed connection $A|_{\det(V^+)} = 2A_\Lambda$ on $\det(V^+)$ and Φ is a section of V^+; $\hat{A}$ is the induced connection on the SO(3) bundle $\mathfrak{g}_{\mathfrak{t}} \subset \mathfrak{su}(V)$; $F^+_{\hat{A}}$ is the self-dual component of the curvature of $\hat{A}$; the term $(\Phi \otimes \Phi^*)_{00}$ is the component of $\Phi \otimes \Phi^*$ lying in $\rho(\Lambda^+) \otimes \mathfrak{g}_{\mathfrak{t}}$; $D_A : C^\infty(V^+) \to C^\infty(V^-)$ is the Dirac operator. We let $\mathcal{M}_{\mathfrak{t}}$ be the space of SO(3) monopoles for $\mathfrak{t} = (\rho, V)$, modulo gauge transformations.

2.4. **Singularities in SO(3)-monopole space.** We now classify the fixed points of the circle action on $\mathcal{M}_{\mathfrak{t}}$ given by complex multiplication on the spinor components.

An SO(3) monopole (A, Φ) is a *Yang-Mills* or *instanton* solution if

$$\Phi \equiv 0 \quad \text{and} \quad F^+_{\hat{A}} = 0.$$

Hence, there is a moduli subspace of SO(3) instantons,

$$M^w_\kappa \hookrightarrow \mathcal{M}_{\mathfrak{t}},$$

where $\kappa = -\frac{1}{4} p_1(\mathfrak{g}_{\mathfrak{t}})$ and $w \in H^2(X; \mathbb{Z})$ lifts $w_2(\mathfrak{g}_{\mathfrak{t}})$.

An SO(3) monopole (A, Φ) is a *Seiberg-Witten* or *reducible* solution if

$$A = B \oplus B \otimes A_L \quad \text{on} \quad V = W \oplus W \otimes L,$$

for some line bundle L, and $\Phi = \Psi \oplus 0$ with Ψ a section of W^+ obeying

$$\begin{aligned} \mathrm{Tr}(F^+_B) - \rho^{-1}(\Psi \otimes \Psi^*)_0 - F^+_{A_\Lambda} &= 0, \\ D_B \Psi &= 0. \end{aligned}$$

Hence, there are moduli subspaces of Seiberg-Witten monopoles for $\mathfrak{s} = (\rho, W)$, whenever $V = W \oplus W \otimes L$:

$$M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}.$$

3. Invariants of smooth 4-manifolds

We sketch definitions of the Donaldson series [17] and Seiberg-Witten series [25].

3.1. **Donaldson invariants.** Set $\mathbb{A}(X) = \mathrm{Sym}(H_0(X; \mathbb{R}) \oplus H_2(X; \mathbb{R}))$, so $z \in \mathbb{A}(X)$ is a linear sum of monomials

$$x^m \beta_1 \beta_2 \cdots \beta_{\delta - 2m},$$

with $x \in H_0(X; \mathbb{Z})$ positive generator and $\beta_i \in H_2(X; \mathbb{R})$. Cohomology classes on M^w_κ can be defined via a map [1], [3],

$$\mu_p : H_i(X; \mathbb{R}) \to H^{4-i}(M^w_\kappa; \mathbb{R}).$$

The Donaldson invariant is then a linear function

$$D^w_X : \mathbb{A}(X) \to \mathbb{R},$$

where, for a monomial z with $\deg(z) = 2\delta$,

$$D^w_X(z) = \langle \mu_p(z), [M^w_\kappa] \rangle$$

with $\mu_p(z) = \mu_p(x)^m \smile \mu_p(\beta_1) \smile \cdots \smile \mu_p(\beta_{\delta - 2m})$.

3.2. **Kronheimer-Mrowka structure theorem.** One says that a 4-manifold X has *KM-simple type* if for some w, and all $z \in \mathbb{A}(X)$,

$$D^w_X(x^2 z) = 4D^w_X(z).$$

One defines the *Donaldson series* by setting

$$\mathbf{D}^w_X(h) = D^w_X((1 + \tfrac{1}{2}x)e^h), \quad h \in H_2(X;\mathbb{R}). \tag{2}$$

We recall the celebrated

Theorem 3.1. [17] *If X has KM-simple type, then there exist $a_r \in \mathbb{Q}$ and $K_r \in H^2(X;\mathbb{Z})$, the KM-basic classes, such that*

$$\mathbf{D}^w_X(h) = e^{\frac{1}{2}Q_X(h,h)} \sum_{r=1}^{s} (-1)^{\frac{1}{2}(w^2 + w\cdot K_r)} a_r e^{\langle K_r, h\rangle}, \quad h \in H_2(X;\mathbb{R}). \tag{3}$$

3.3. **Seiberg-Witten invariants.** The *Seiberg-Witten invariants* comprise a function,

$$SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z},$$

where

$$SW_X(\mathfrak{s}) = \langle \mu_{\mathfrak{s}}(x)^{\max}, [M_{\mathfrak{s}}]\rangle,$$

and $\mu_{\mathfrak{s}}(x) \in H^2(M_{\mathfrak{s}};\mathbb{Z})$ is a cohomology class associated to a circle action. One says that a 4-manifold X has *SW-simple type* if

$$SW_X(\mathfrak{s}) = 0$$

when $M_{\mathfrak{s}}$ has positive dimension and calls $c_1(\mathfrak{s}) = c_1(W^+)$ an *SW-basic class* if $SW_X(\mathfrak{s}) \neq 0$. We define the *Seiberg-Witten series* by

$$\mathbf{SW}^w_X(h) = \sum_{\mathfrak{s}\in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) e^{\langle c_1(\mathfrak{s}), h\rangle}, \quad h \in H_2(X;\mathbb{R}). \tag{4}$$

Witten's prediction then takes the form stated in Conjecture 1.1.

4. SO(3) monopole cobordism

4.1. **Bubbling and Uhlenbeck compactness.** In order to apply the moduli space $\mathcal{M}_{\mathfrak{t}}$ of SO(3) monopoles as a cobordism, we must use a compactification. If $\{A_\alpha, \Phi_\alpha\}_{\alpha\in\mathbb{N}} \subset \tilde{\mathcal{M}}_{\mathfrak{t}}$, then the sequence converges to an *ideal SO(3) monopole* $(A_\infty, \Phi_\infty, \mathbf{x})$ in $\tilde{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)$ if

- $(A_\alpha, \Phi_\alpha) \to (A_\infty, \Phi_\infty)$ in C^∞ on $X \setminus \{\mathbf{x}\}$, modulo gauge transformations,
- $|F_{A_\alpha}|^2 \to |F_{A_\infty}|^2 + 8\pi^2 \sum_{x\in\mathbf{x}} \delta(x)$, as measures.

We let $\bar{\mathcal{M}}_{\mathfrak{t}}$ be closure of $\mathcal{M}_{\mathfrak{t}}$ with respect to the Uhlenbeck topology, implicit above, in the space of ideal SO(3) monopoles,

$$\bigsqcup_{\ell=0}^{N} \left(M_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)\right),$$

where $\mathfrak{t}_\ell = (\rho, V_\ell)$ is a spin^u structure with characteristic classes

$$p_1(\mathfrak{g}_{\mathfrak{t}_\ell}) = p_1(\mathfrak{g}_{\mathfrak{t}}) + 4\ell, \quad w_2(\mathfrak{g}_{\mathfrak{t}_\ell}) = w_2(\mathfrak{g}_{\mathfrak{t}}), \text{ and } c_1(\mathfrak{t}_\ell) = c_1(\mathfrak{t}),$$

and $c_1(\mathfrak{t}) := c_1(\det^{\frac{1}{2}}(V^+))$. The space $\bar{\mathcal{M}}_{\mathfrak{t}}$ is smoothly stratified, with a *top* or *zeroth level* $\mathcal{M}_{\mathfrak{t}}$, and *lower levels* $\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)$, $\ell \geq 1$.

4.2. **Strata in space of SO(3) monopoles.** For the top level, one has a stratification

$$\mathcal{M}_{\mathfrak{t}} = M^w_\kappa \sqcup \mathcal{M}^{*,0}_{\mathfrak{t}} \sqcup \bigcup_{\mathfrak{t}=\mathfrak{s}\oplus\mathfrak{s}'} M_{\mathfrak{s}}. \tag{5}$$

The complement $\mathcal{M}^{*,0}_{\mathfrak{t}}$ in $\mathcal{M}_{\mathfrak{t}}$ of the Yang-Mills and Seiberg-Witten solutions is a smooth manifold, cut out transversely by the SO(3) monopole equations.

A stratification of the form (5) arises in each level, $\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)$, of the compactification $\bar{\mathcal{M}}_{\mathfrak{t}}$. Though dimension-counting arguments rule out contributions from the instanton moduli subspace $M^w_{\kappa-\ell}$ of $\mathcal{M}_{\mathfrak{t}_\ell}$, Seiberg-Witten moduli subspaces of $\mathcal{M}_{\mathfrak{t}_\ell}$ can contribute to Donaldson invariants computed using $M^w_\kappa \subset \mathcal{M}_{\mathfrak{t}}$. To apply the cobordism, we

- Define a link $\bar{\mathbf{L}}^w_{\mathfrak{t},\kappa} \subset \bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ of the instanton moduli subspace,
$$\bar{M}^w_\kappa \subset \bar{\mathcal{M}}_{\mathfrak{t}},$$
by restricting to spinors with L^2 norm equal to a small positive constant, and
- Use gluing theory [9], [10] to construct links $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}} \subset \bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ of ideal Seiberg-Witten moduli subspaces,
$$M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X) \subset \bar{\mathcal{M}}_{\mathfrak{t}}.$$
These links are considerably more difficult to construct than $\bar{\mathbf{L}}^w_{\mathfrak{t},\kappa}$.

4.3. **SO(3)-monopole cobordism formula.** The cobordism $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ now yields the raw identity,

$$\left\langle \mu_p(z) \smile \mu_c^{\delta_c-1}, [\bar{\mathbf{L}}^w_{\mathfrak{t},\kappa}] \right\rangle = - \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} \left\langle \mu_p(z) \smile \mu_c^{\delta_c-1}, [\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}] \right\rangle, \tag{6}$$

where $\mu_c \in H^2(\mathcal{M}^{*,0}_{\mathfrak{t}};\mathbb{Z})$ is a class associated to a circle action on $\mathcal{M}^{*,0}_{\mathfrak{t}}$. One finds that

$$\langle \mu_p(z) \smile \mu_c^{\delta_c-1}, [\bar{\mathbf{L}}^w_{\mathfrak{t},\kappa}] \rangle$$

is a multiple of the Donaldson invariant, $D^w_X(z)$. The sum in (6) is over all $\mathfrak{s} \in \mathrm{Spin}^c(X)$, with $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ empty unless $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X) \subset \bar{\mathcal{M}}_{\mathfrak{t}}$, for some $\ell(\mathfrak{t},\mathfrak{s}) \geq 0$.

The difficult aspect of using (6) to derive Witten's formula (1) is to show that

$$\langle \mu_p(z) \smile \mu_c^{\delta_c-1}, [\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}] \rangle$$

is the correct multiple of the Seiberg-Witten invariant $SW_X(\mathfrak{s})$; the degree of difficulty grows rapidly with $\ell \geq 0$.

5. Application of the cobordism

We consider following distinct cases, arranged in increasing order of complexity:

- There are no Seiberg-Witten moduli spaces with non-zero invariants in $\bar{\mathcal{M}}_{\mathfrak{t}}$, so the intersection $(M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)) \cap \bar{\mathcal{M}}_{\mathfrak{t}}$ is empty for all $\ell \geq 0$ and $SW_X(\mathfrak{s}) \neq 0$. The Donaldson invariants defined by $\bar{M}^w_\kappa \subset \bar{\mathcal{M}}_{\mathfrak{t}}$ are then zero and, eventually, this observation leads to a vanishing result:
$$\mathbf{D}^w_X(h) \equiv 0 \equiv SW^w_X(h) \pmod{h^{c(X)-2}}, \quad h \in H_2(X;\mathbb{R}).$$
- Contributions from $M_{\mathfrak{s}} \subset \mathcal{M}_{\mathfrak{t}}$ lead to Witten's formula, mod $h^{c(X)}$.
- Contributions from $M_{\mathfrak{s}} \times X \subset \bar{\mathcal{M}}_{\mathfrak{t}}$ lead to Witten's formula, mod $h^{c(X)+2}$.
- Contributions from $M_{\mathfrak{s}} \times \mathrm{Sym}^2(X) \subset \bar{\mathcal{M}}_{\mathfrak{t}}$ lead to Witten's formula, mod $h^{c(X)+4}$.
- Calculation of contributions $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X) \subset \bar{\mathcal{M}}_{\mathfrak{t}}$ for $\ell \geq 3$ should lead to a full verification of Witten's formula.

We have considered the cases $\ell = 0$ and $\ell = 1$ in detail [7], [8], [11] and we would expect the case $\ell = 2$ to follow in a similar manner, by exploiting work of Leness [18] on the wall-crossing formula for Donaldson invariants.

At present, we can compute the general shape of the contributions for $\ell \geq 3$, but complete, direct computations of those contributions appears to be a difficult problem.

5.1. **Level-zero Seiberg-Witten contributions.** Let $B \subset H^2(X;\mathbb{Z})$ be the set of Seiberg-Witten basic classes and define $B^\perp \subset H^2(X;\mathbb{Z})$ as the Q_X-orthogonal complement of B. We call a 4-manifold X *abundant* if $Q_X|_{B^\perp}$ has a hyperbolic sublattice. Every compact, complex algebraic, simply connected surface with $b_2^+ \geq 3$ is abundant [7].

Theorem 5.1. [8] *Assume X is abundant, has $b_1 = 0$, odd $b_2^+ \geq 3$, and SW-simple type. Suppose $\Lambda \in B^\perp$ exists with $\Lambda^2 = 2 - (\chi + \sigma)$. For such Λ and $w \in H^2(X;\mathbb{Z})$ with $w - \Lambda \equiv w_2(X) \pmod 2$ one has, for all $h \in H_2(X;\mathbb{R})$,*

$$\mathbf{D}_X^w(h) \equiv 0 \equiv \mathbf{SW}_X^w(h) \pmod{h^{c(X)-2}}, \tag{7}$$

$$\mathbf{D}_X^w(h) \equiv 2^{2-c(X)} e^{\frac{1}{2}Q_X(h,h)} \mathbf{SW}_X^w(h) \pmod{h^{c(X)}}. \tag{8}$$

The vanishing assertion (7) for the Seiberg-Witten series is a statement that the Moore-Mariño-Peradze conjecture holds for (abundant) 4-manifolds of Seiberg-Witten simple type [6], [19], [20].

5.2. **Level-one Seiberg-Witten contributions.** With more sophisticated analytical tools, specifically gluing theory, we can compute contributions from $M_\mathfrak{s} \times X \subset \bar{\mathcal{M}}_\mathfrak{t}$ and these computations lead to the

Theorem 5.2. [11] *Same hypotheses as Theorem 5.1, but now suppose $\Lambda \in B^\perp$ exists with $\Lambda^2 = 4 - (\chi + \sigma)$. For such Λ and $w \in H^2(X;\mathbb{Z})$ with $w - \Lambda \equiv w_2(X) \pmod 2$ one has, for all $h \in H_2(X;\mathbb{R})$,*

$$\mathbf{D}_X^w(h) \equiv 0 \equiv \mathbf{SW}_X^w(h) \pmod{h^{c(X)-2}}, \tag{9}$$

$$\mathbf{D}_X^w(h) \equiv 2^{2-c(X)} e^{\frac{1}{2}Q_X(h,h)} \mathbf{SW}_X^w(h) \pmod{h^{c(X)+2}}. \tag{10}$$

We expect an identity similar to (10), but mod $h^{c(X)+4}$, by computing contributions for $\ell = 2$, when $\Lambda^2 = 6 - (\chi + \sigma)$. The restrictive hypotheses on existence of classes Λ with prescribed even squares can be dropped if one can consider contributions for arbitrary $\ell \geq 0$.

5.3. **Seiberg-Witten contributions from arbitrary levels.** More generally, we expect the following to hold:

Conjecture 5.3. [11] For Λ, w in $H^2(X;\mathbb{Z})$ with $w - \Lambda \equiv w_2(X) \pmod 2$, and all $d \geq 0$ and $h \in H^2(X;\mathbb{R})$,

$$\begin{aligned} D_X^w(h^d) = 2^{2-c(X)} & \sum_{\mathfrak{s} \in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) \\ & \times \sum_{i=0}^{m} \boldsymbol{\delta}_i\left(\langle c_1(\mathfrak{s}), h\rangle, \langle \Lambda, h\rangle\right) Q_X^{\ell - i}(h,h), \end{aligned} \tag{11}$$

where $\ell = \ell(c_1(\mathfrak{s}), \Lambda)$ and $m(\ell, d) = \min(\ell, [d/2])$ above. The functions $\boldsymbol{\delta}_i$ are homogeneous polynomials in two variables, whose coefficients are polynomials in $2\chi \pm 3\sigma$, $(c_1(\mathfrak{s}) - \Lambda)^2$, Λ^2, and $(c_1(\mathfrak{s} - \Lambda) \cdot c_1(\mathfrak{s})$. A similar formula holds for the invariants $D_X^w(x^m h^{d-2m})$, $0 \leq m \leq [d/2]$.

Conjecture 5.3 is easier to prove than Witten's, but is still powerful enough to prove that the Seiberg-Witten invariants determine the Donaldson invariants. We have carried out most of the work required to prove this conjecture, though some details remain to be checked.

Witten's formula (1) should follow from Conjecture 5.3 by indirect calculations of the remaining unknown coefficients, exploiting

- Known Donaldson and Seiberg-Witten invariants for complex surfaces, or
- A possible generalization of the wall-crossing calculations of Göttsche [15] for Donaldson invariants when $b_2^+ = 1$ (and 1-parameter families) to the case of $b_2^+ > 1$ (and b_2^+-parameter families).

6. Outline of proof of Witten's formula

We shall first sketch how to compute the rough form of the pairings,

$$\langle \mu_p(z) \smile \mu_c^{\delta_c - 1}, [\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}]\rangle, \tag{12}$$

or, at least why these pairings have the form

$$SW_X(\mathfrak{s}) \times (\text{Factors depending only on topology}).$$

We use our gluing theory [9], [10] to construct a topological model for a neighborhood in $\bar{\mathcal{M}}_{\mathfrak{t}}$ and hence a link, $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}} \subset \bar{\mathcal{M}}_{\mathfrak{t}}/S^1$, of the 'stratum'

$$M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X) \subset \bar{\mathcal{M}}_{\mathfrak{t}}.$$

Given this topological model for $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$, we can then apply intersection theory methods to partly compute the pairings (12).

This suffices to prove the 'rough version' of Witten's formula (11), which is enough to show that the Seiberg-Witten invariants determine the Donaldson invariants. We have completed much of this step, though many details remain to be checked. We shall illustrate the method below, often assuming $\ell = 1$ for the sake of simplicity [11].

The passage from this stage to Witten's formula requires us to compute the many universal, but unknown coefficients in the rough version (11) of Witten's formula. Thus we shall also indicate how one should be able to use existing results [12], [13], [17] for the Donaldson and Seiberg-Witten invariants to determine the remaining unknown coefficients in our expressions for the pairings (12).

6.1. Neighborhood of Seiberg-Witten stratum.

Our gluing theory [9], [10] allows us to construct a model for a neighborhood of the level $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ in a local, 'virtual' moduli space,

$$\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} := \tilde{N}_{\mathfrak{t}_\ell,\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_{\mathfrak{s}}} \bar{\mathrm{Gl}}_{\mathfrak{t}_\ell}(\delta). \tag{13}$$

A neighborhood of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ in the true moduli space $\bar{\mathcal{M}}_{\mathfrak{t}}$ then takes the shape

$$\boldsymbol{\gamma}\left(\boldsymbol{\chi}^{-1}(0) \cap \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}\right) = \bar{\mathcal{M}}_{\mathfrak{t}} \cap \boldsymbol{\gamma}(\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}),$$

The stratum $M_{\mathfrak{s}}$ in $\mathcal{M}_{\mathfrak{t}_\ell}$ has a 'virtual' normal bundle,

$$N_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}},$$

and an obstruction bundle with section $\boldsymbol{\chi}$,

$$\Xi_{\mathfrak{t},\mathfrak{s}} \to \mathcal{M}_{\mathfrak{t}_\ell}.$$

The gluing map,

$$\boldsymbol{\gamma} : \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} \to \text{Configuration space of ideal pairs containing } \bar{\mathcal{M}}_{\mathfrak{t}},$$

gives a homeomorphism from $\boldsymbol{\chi}^{-1}(0)\cap\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ onto a neighborhood of $M_{\mathfrak{s}}\times\mathrm{Sym}^{\ell}(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}$.

6.2. **Instanton component of the topological model.** When $\ell=1$, the instanton gluing-data component is given by

$$\bar{\mathrm{Gl}}_{\mathfrak{t}_\ell}(\delta)=\left(\mathrm{Fr}(\mathfrak{g}_{\mathfrak{t}_1})\times_X\mathrm{Fr}(T^*X)\times\bar{M}^{s,\natural}_1(S^4,\delta)\right)/(\mathrm{SO}(3)\times\mathrm{SO}(4)). \tag{14}$$

Here, $M^{s,\natural}_k(S^4,\delta)$ is the moduli space of k-instantons on S^4, framed at south pole s, with mass center at the north pole, and scale $\le\delta$. For $k=1$ there are homeomorphisms,

$$M^{s,\natural}_1(S^4,\delta)\cong(0,\delta]\times\mathrm{SO}(3),$$
$$\bar{M}^{s,\natural}_1(S^4,\delta)\cong c(\mathrm{SO}(3)),$$

where $c(\mathrm{SO}(3))$ is the cone on $\mathrm{SO}(3)$.

Our gluing-model (13) admits an Uhlenbeck stratification, given below when $\ell=1$:

$$\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}=\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}\sqcup(N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)-M_{\mathfrak{s}})\times X\sqcup M_{\mathfrak{s}}\times X.$$

When $\ell\ge 2$, the symmetric product has its usual stratification and one can also construct bundles $\mathrm{Gl}_{\mathfrak{t}_\ell}(\delta,\Sigma)\to\Sigma$ with instanton moduli space fibers, for each stratum $\Sigma\subset\mathrm{Sym}^{\ell}(X)$, following the prescription of Friedman and Morgan [14] and developing the idea of Kotschick and Morgan [16] and Mrowka [21]. The difficult part is to assemble these local gluing data bundles into a space of global gluing data, $\mathrm{Gl}_{\mathfrak{t}_\ell}(\delta)$, with a projection map onto $\mathrm{Sym}^{\ell}(X)$. One essentially has

$$\bar{\mathrm{Gl}}_{\mathfrak{t}_\ell}(\delta)=\bigcup_{\Sigma\subset\mathrm{Sym}^{\ell}(X)}\mathrm{Gl}_{\mathfrak{t}_\ell}(\delta,\Sigma),$$

where we patch together the local bundles $\mathrm{Gl}_{\mathfrak{t}_\ell}(\delta,\Sigma)$ using (non-canonical) transition maps. Of course, the difficulty is to ensure that the transition maps obey the cocycle condition when $\ell\ge 3$.

6.3. **Link of the Seiberg-Witten stratum.** We define the link in two steps, first considering the Seiberg-Witten component of the link of $M_{\mathfrak{s}}\times\mathrm{Sym}^{\ell}(X)\subset\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1$,

$$\bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t},\mathfrak{s}}:=\left(\partial\tilde{N}_{\mathfrak{t}_\ell,\mathfrak{s}}(\varepsilon)\times_{\mathcal{G}_{\mathfrak{s}}}\bar{\mathrm{Gl}}_{\mathfrak{t}_\ell}(\delta)\right)/S^1,$$

and, second, defining the instanton component of the link of $M_{\mathfrak{s}}\times\mathrm{Sym}^{\ell}(X)\subset\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1$,

$$\bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}:=\left(\tilde{N}_{\mathfrak{t}_\ell,\mathfrak{s}}(\varepsilon)\times_{\mathcal{G}_{\mathfrak{s}}}\partial\bar{\mathrm{Gl}}_{\mathfrak{t}_\ell}(\delta)\right)/S^1,$$

a complex disk bundle over $M_{\mathfrak{s}}\times\partial\bar{\mathrm{Gl}}_{\mathfrak{t}_\ell}(\delta)/S^1$. When $\ell=1$, the boundary $\partial\bar{\mathrm{Gl}}_{\mathfrak{t}_\ell}(\delta)$ is defined by restricting to instantons on S^4 with scale δ. We then define the link of $M_{\mathfrak{s}}\times\mathrm{Sym}^{\ell}(X)$ in the virtual moduli space, $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1$,

$$\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}:=\bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t},\mathfrak{s}}\cup\bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}.$$

Finally, we obtain

$$\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}:=\boldsymbol{\gamma}\left(\boldsymbol{\chi}^{-1}(0)\cap\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}\right)=(\bar{\mathcal{M}}_{\mathfrak{t}}/S^1)\cap\boldsymbol{\gamma}(\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}),$$

the link of $M_{\mathfrak{s}}\times\mathrm{Sym}^{\ell}(X)$ in the true moduli space $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$.

6.4. **Factorization in Seiberg-Witten link pairings.** The virtual moduli space method reduces the computation of the link pairing (12) to that of the pairing on the right-hand side below,

$$\langle \mu_p(z) \smile \mu_c^{\delta_c}, [\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}]\rangle = \langle \mu_p(z) \smile \mu_c^{\delta_c} \smile e, [\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}]\rangle, \tag{15}$$

where e is Euler class of the total obstruction bundle over $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$, with section $\boldsymbol{\chi}$, and $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}} \cong \boldsymbol{\chi}^{-1}(0) \cap \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$. We next show [11] that the pairing (15) with $[\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}]$ is expressible in terms of pairings with

$$[M_{\mathfrak{s}} \times \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_\ell}(\delta)/S^1].$$

In particular, we can show that the latter pairings are in turn products of

- Pairings with $[M_{\mathfrak{s}}]$, giving multiples of the Seiberg-Witten invariant, $SW_X(\mathfrak{s})$, and
- Pairings with $[\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_\ell}(\delta)/S^1]$.

The pairings with $[\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_\ell}(\delta)/S^1]$ depend only the topology of the 4-manifold, X, and universal data.

6.5. **From the rough version to Witten's formula.** A key remaining point is to compute the pairings with $[\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_\ell}(\delta)/S^1]$, for all $\ell \geq 2$. We note that it is already a hard analytical and topological problem just to construct the spaces $\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_\ell}(\delta)$, for $\ell \geq 1$.

We strongly suspect that one should be able to determine the pairings with $[\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_\ell}(\delta)/S^1]$ by exploiting the known Donaldson and Seiberg-Witten invariants for compact, complex algebraic surfaces. For example, when $\ell = 1$, only 3 pairings with $[\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_\ell}(\delta)/S^1]$ are required in the proof of Theorem 5.2.

The pairings with $[\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_\ell}(\delta)/S^1]$ also arose in the Kotschick-Morgan wall-crossing formula for Donaldson invariants when $b_2^+(X) = 1$, where the Riemannian metric varies in 1-parameter families [15], [16]. Is there an analogue of that wall-crossing formula for b_2^+-parameter families when $b_2^+(X) > 1$? If so, we might expect that Göttsche's determination of the Donaldson invariant wall-crossing formula may extend in such a way as to allow us to compute the pairings with $[\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_\ell}(\delta)/S^1]$ without relying heavily on examples.

7. Witten's conjecture and symplectic 4-manifolds

7.1. **Gauge theory and Lefschetz fibrations.** Away from finitely many critical points, a *Lefschetz fibration* $\pi : X \to S$ is smooth fiber bundle over a connected base, whose fibers are closed Riemann surfaces, Σ, of given genus. Possibly after blowing up, all symplectic 4-manifolds admit Lefschetz fibrations [2].

One can ask what is the relationship between the gauge theories and the Lefschetz fibration structure? As a first step, one could use the product, $X = \Sigma \times S$, as a toy model. In this situation, the relationship between the gauge theory moduli spaces and holomorphic maps can be explored via the following techniques:

- Adiabatic limit analysis, by the scaling metric $g = g_\Sigma \oplus g_S$ as $g_\varepsilon = \varepsilon^2 g_\Sigma \oplus g_S$, with $\varepsilon \to 0$, or
- Restriction of stable, holomorphic bundles over $\Sigma \times S$ to $\Sigma \times \{z\}$, as $z \in S$ varies.

This leads to the following identifications:

1. SO(3)-instantons over $\Sigma \times S$ are identified with holomorphic maps, $S \to M_\Sigma$, where M_Σ is the space of flat SO(3) connections over Σ. This identification goes back to Dostoglou and Salamon [5].

2. Seiberg-Witten U(1)-monopoles over $\Sigma \times S$ are identified with holomorphic maps, $S \to M_{\Sigma,d}$, where $M_{\Sigma,d}$ is the space of vortices on the line bundle $L \to \Sigma$ with $d = \langle c_1(L), [\Sigma] \rangle$. This identification has been outlined by Salamon [24].
3. SO(3)-monopoles over $\Sigma \times S$ are identified with holomorphic maps from S to a space of non-abelian vortices over Σ. Non-abelian vortices of this kind have been studied by Bradlow, Garcia-Prada, and others.

Of course, there are many unanswered questions:

- What is the relationship between the gauge theory compactifications and the compactifications of spaces of holomorphic maps?
- What is the relationship between the gauge theory invariants and Gromov-Witten invariants of spaces of flat connections or vortices over Riemann surfaces?
- Can one extend the analysis for the toy model, $\Sigma \times S$, to the case of non-trivial surface bundles or Lefschetz fibrations?
- Can symplectic 4-manifolds provide additional data we can use to determine of Witten's formula?

We note that the relationship between Seiberg-Witten invariants and Lefschetz fibration data bears on a closely related program due to Simon Donaldson and Ivan Smith to give Lefschetz-fibration of proofs of results for symplectic 4-manifolds derived via Seiberg-Witten theory [4].

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