POLYLOGARITHMS, DEDEKIND ZETA FUNCTIONS, AND THE ALGEBRAIC *K*-THEORY OF FIELDS

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§1. Introduction. The Dedekind zeta function $\zeta_F(s)$ of an algebraic number field F is the most important invariant of F. Its Euler product tells how the unramified primes of $\mathbb{Q}$ split in F. Information about the ramified primes and about the behavior at infinity is contained in three integers Δ, n_+ and n_-: the first is the absolute value of the discriminant of F and is a positive integer whose prime divisors are precisely the primes ramifying in F, and the other two ($= r_1 + r_2$ and r_2 in the standard notation) give the dimensions of the $(+1)$- and (-1)-eigenspaces of complex conjugation on $F \otimes_{\mathbb{Q}} \mathbb{R}$ $(= \mathbb{R}^{r_1} \times \mathbb{C}^{r_2})$. These invariants are in turn determined by $\zeta_F(s)$

via its functional equation

$$\zeta_F^*(s) := \Delta^{s/2}\big(\pi^{-s/2}\Gamma(\frac{s}{2})\big)^{n_+}\big(\pi^{-s/2}\Gamma(\frac{s+1}{2})\big)^{n_-}\zeta_F(s) = \zeta_F^*(1-s). \quad (1)$$

In the case $F = \mathbb{Q}$, the function $\zeta_F(s) = \zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ was first studied by Euler, who showed that its values at negative odd integers are rational and that its values at positive even integers are rational multiples of powers of π: $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, etc. More generally, if F is totally real ($n_- = 0$), then the Klingen-Siegel theorem says that the values of $\zeta_F(s)$ at negative odd integers are again rational numbers, or equivalently (by (1)), that the values at positive values of s are rational multiples of $\pi^{n_+ s}/\sqrt{\Delta}$. If F is not totally real, then one does not expect rationality results of this form. However, the Dirichlet class number formula says that the value of $\zeta_F(s)/\zeta_{\mathbb{Q}}(s)$ at $s = 1$ equals $\pi^{n_-}/\sqrt{\Delta}$ times a rational linear combination of $(n_+ - 1)$-fold products of logarithms of numbers in F (we cannot ask directly for the value of $\zeta_F(s)$ at $s = 1$ since there is a simple pole there), while the result proved in [17] by 3-dimensional hyperbolic geometry is that the value of $\zeta_F(s)$ at $s = 2$ equals $\pi^{2n_+}/\sqrt{\Delta}$ times a rational linear combination of n_--fold products of *dilogarithms* of algebraic numbers (in F or an extension of F of degree at most 4). These two cases suggest the conjecture that *the value of $\zeta_F(m)$ for an arbitrary number field F and integer $m > 1$ can be expressed in terms of the polylogarithm function* $Li_k(x)$, defined by

$$Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad (k \in \mathbb{N}, \quad |x| \le 1) \qquad (2)$$

(and by analytic continuation if $|x| > 1$). This conjecture was stated and a few numerical examples supporting it were given in [19], but a precise formulation was not given. It is the purpose of this paper to provide such a formulation and to describe the evidence supporting it. The conjecture will take the form that $\zeta_F(m)$ is equal up to a rational factor to $\pi^{mn_\pm}/\sqrt{\Delta}$ times the determinant of an $n_\mp \times n_\mp$-matrix whose entries are $\mathbb{Z}$-linear combinations of numbers $P_m(x)$ with $x \in F$, where the sign $\pm$ is determined by $(-1)^m = \pm 1$ and where $P_m : \mathbb{C} \to \mathbb{R}$ is a certain combination of the polylogarithm functions $Li_k(x)$, $k \le m$ (the detailed formulation will say precisely which matrices one can take). The statement can be generalized to include values of Artin L-functions at $s = m$ and in this form reduces to a weakened form of Stark's conjectures for $m = 1$.

Finally, we must bring in the third ingredient of the title, algebraic K-theory. For each integer $i \ge 1$ we have the groups $K_i(F)$ and $K_i(\mathcal{O}_F)$, the

ith algebraic K-groups of F and of its ring of integers. For $i = 1$ we have $K_1(F) = F^\times$ and $K_1(\mathcal{O}_F) = \mathcal{O}_F^\times \approx \mathbb{Z}^{n_+-1}$ (here and in the sequel, we will use $A \approx B$ for two abelian groups A and B to mean that there is a map $A \to B$ with finite kernel and cokernel). For $i > 1$, it was shown by Borel [4] that

$$K_i(F) \approx K_i(\mathcal{O}_F) \approx \begin{cases} 0 & \text{if } i \text{ is even,} \\ \mathbb{Z}^{n_\mp} & \text{if } i = 2m-1 \text{ is odd, } (-1)^m = \pm 1 \end{cases} \qquad (3)$$

and that in the latter case there is a natural map (the regulator mapping) of $K_i(F)$ into $\mathbb{R}^{n_\mp}$ whose image is a lattice with covolume a rational multiple of $\zeta_F(m)/\pi^{mn_\pm}\sqrt{\Delta}$. (Note that, by (1), the rank $\rho = n_\mp$ of $K_{2m-1}(F)$ equals the order of vanishing of $\zeta_F(s)$ at $s = 1-m$ and that $\zeta_F(m)/\pi^{mn_\pm}\sqrt{\Delta}$ is a rational multiple of $\lim_{s\to 1-m} \zeta_F(s)/(s-1+m)^\rho$.) From this point of view, our conjecture says that the algebraic K-group $K_{2m-1}(F)$ is a subquotient of the free abelian group on $F^\times$, the regulator mapping being given by the polylogarithm function P_m evaluated on the different embeddings of F into $\mathbb{C}$. (There are $n_\mp$ essentially different such embeddings as far as the function P_m is concerned, since $P_m(\overline{x}) = \mp P_m(x)$ for $x \in \mathbb{C}$.) This will be discussed in more detail in the body of the paper.

The paper contains a rather large number of numerical examples (mostly for $m = 3$) motivating and substantiating the various forms of the main conjecture. The reader who is willing to take the motivation on faith can skip straight to §§7–8, where the general formulation of the conjecture is given, and to the further examples and discussion in the following two sections.

In the final section of the paper we will also describe briefly the progress which has been made on our conjecture since the Texel conference: Deligne and Beilinson have reformulated and refined it using ideas from motivic cohomology, Goncharov has proved (most of) the case $m = 3$ and in particular has shown that $\zeta_F(3)$ for any number field F can be expressed in terms of the modified trilogarithm function P_3, and Beilinson has constructed a map from the subquotient of the free abelian group on $F^\times$ specified in the conjecture to the group $K_{2m-1}(F)$ such that the polylogarithm corresponds to the Borel regulator.

I would like to express my thanks to Christophe Soulé, Spencer Bloch, Herbert Gangl and Alexander Goncharov for useful discussions about the material in this paper, and especially to Pierre Deligne and Sasha Beilinson for trying to explain to me the mysteries of algebraic K-theory and the philosophy of motivic cohomology and for helping me formulate a reasonably precise version of the conjectural relationship between zeta values, polylogarithms, and K-groups.

§2. The dilogarithm, hyperbolic geometry, and the Bloch group. For $m = 2$ the *Bloch-Wigner function* is the modified dilogarithm function defined by

$$D(x) = \Im\big(Li_2(x) + \log|x|\log(1-x)\big) \qquad (|x| \le 1),$$
$$D(x) = -D(1/x) \qquad (|x| \ge 1).$$

It is a continuous function from the Riemann sphere $\mathbf{P}^1(\mathbf{C})$ to $\mathbf{R}$ and is (real-) analytic except at the points 0, 1 and ∞, where it has the value 0. It satisfies a number of functional equations, most notably the six-fold symmetry property

$$D(x) = D\Big(\frac{x-1}{x}\Big) = D\Big(\frac{1}{1-x}\Big) = -D\Big(\frac{1}{x}\Big) = -D(1-x) = -D\Big(\frac{x}{x-1}\Big) \qquad (x \in \mathbf{P}^1(\mathbf{C})) \tag{4}$$

and the "five-term relation" of Spence and Abel

$$D(x) + D(y) + D\Big(\frac{1-x}{1-xy}\Big) + D(1-xy) + D\Big(\frac{1-y}{1-xy}\Big) = 0 \qquad (x, y \in \mathbf{P}^1(\mathbf{C})) \tag{5}$$

(which include equations (4) by specializing to $y = 0$ or ∞). These can be expressed in a more natural way by thinking of the argument of D as the cross-ratio of four points a, b, c, d on the complex line, i.e. by defining $\tilde{D}(a,b,c,d) = D\big(\frac{a-c}{a-d}\frac{b-d}{b-c}\big)$, in which case they take the form

$$\tilde{D}(a_{\pi(1)}, a_{\pi(2)}, a_{\pi(3)}, a_{\pi(4)}) = \operatorname{sign}(\pi)\, \tilde{D}(a_1, a_2, a_3, a_4) \qquad (\pi \in \mathfrak{S}_4),$$
$$\sum_{i \pmod 5} \tilde{D}(a_i, a_{i+1}, a_{i+2}, a_{i+3}) = 0 \qquad (a_i \in \mathbf{P}^1(\mathbf{C}), \quad a_{i+5} = a_i). \tag{6}$$

This in turn has a geometric interpretation: if we think of $\mathbf{P}^1(\mathbf{C})$ as the boundary of hyperbolic 3-space, then $\tilde{D}(a,b,c,d)$ is the hyperbolic volume of the ideal hyperbolic tetrahedron $\Delta(a,b,c,d)$ with vertices a, b, c and d, and (6) just says that this volume is independent of the numbering of the vertices (except that the orientation changes under odd renumberings) and that the five tetrahedra $\Delta(a_i, a_{i+1}, a_{i+2}, a_{i+3})$ add up algebraically to the zero 3-cycle.

The relationship between $\zeta_F(2)$ and $D(x)$ was proved in [17] using this interpretation of $D(x)$ as a hyperbolic volume. For instance, in the case $n_- = 1$ we showed that

$$\zeta_F(2) \sim \frac{\pi^{2n_+}}{\sqrt{\Delta}} \sum_i D(x_i)$$

(here and from now on, $a \sim b$ means that $a/b \in \mathbb{Q}^\times$) for some numbers $x_i \in \overline{\mathbb{Q}}$ of degree ≤ 4 over F. This is because one can associate to F a hyperbolic 3-manifold M_F whose volume is $\sim \Delta^{\frac{1}{2}}\zeta_F(2)/\pi^{2n_+}$, and then triangulate M_F into ideal tetrahedra whose invariants (= cross-ratios of their vertices, taken with the proper orientation) x_i are at most quartic over F. If $n_- > 1$, then a similar argument shows that

$$\zeta_F(2) \sim \frac{\pi^{2n_+}}{\sqrt{\Delta}} \sum_i \prod_\sigma D(\sigma(x_i)), \tag{7}$$

where σ ranges over (extensions of) the n_- non-real embeddings of F into $\mathbb{C}$. (Note that $D(\overline{x}) = -D(x)$, so $D(x) = 0$ for x real.)

Now the invariants x_i of an ideal triangulation of a complete hyperbolic 3-manifold are not arbitrary: they must satisfy the relation

$$\sum_i [x_i] \wedge [1 - x_i] = 0 \qquad \in \Lambda^2(\mathbb{C}^\times), \tag{8}$$

where $\Lambda^2(\mathbb{C}^\times)$ denotes the second exterior power of $\mathbb{C}^\times$, thought of as a module over $\mathbb{Z}$. (This relation was noticed several years ago by Thurston and is mentioned in [9]. It also follows from Corollary 2.4 of [13], which describes the combinatorics of triangulations of hyperbolic 3-manifolds.) This suggests the following definition. For any field F, denote by $\mathcal{F}_F$ the free abelian group on $F^\times$ (i.e., the set of finite linear combinations $\sum_i n_i[x_i]$ with $n_i \in \mathbb{Z}$, $x_i \in F^\times$; we will also identify $\mathcal{F}_F$ with the quotient of the free abelian group on $\mathbb{P}^1(F)$ by the subgroup $\langle [0], [\infty] \rangle$), and define

$$\mathcal{A}(F) = \Big\{ \sum_i n_i[x_i] \in \mathcal{F}_F \mid \sum_{x_i \ne 1} n_i[x_i] \wedge [1 - x_i] = 0 \text{ in } \Lambda^2(F^\times) \otimes_{\mathbb{Z}} \mathbb{Q} \Big\}. \tag{9}$$

In other words, $\mathcal{A}(F)$ is the kernel of the map $\beta : \mathcal{F}_F \to \Lambda^2(F^\times) \otimes_{\mathbb{Z}} \mathbb{Q}$ defined by $\beta([x]) = [x] \wedge [1 - x]$ for $x \ne 1$, $\beta([1]) = 0$. It is easily checked that the expression

$$S_{xy} = [x] + [y] + \Big[\frac{1-x}{1-xy}\Big] + [1 - xy] + \Big[\frac{1-y}{1-xy}\Big], \tag{10}$$

corresponding to the 5-term relation (5), belongs to $\mathcal{A}(F)$ whenever it makes sense (i.e. for all x, $y \in \mathbb{P}^1(F)$ except $(x,y) = (0,\infty)$, $(\infty,0)$, or $(1,1)$; if x or y or xy equals 0, 1 or ∞, we use our convention $[0] = [\infty] = 0$). We set

$$\mathcal{C}(F) = \big\langle S_{xy} \big\rangle_{x,y \in \mathbb{P}^1(F),\ (x,y) \ne (0,\infty), (\infty,0), (1,1)} \tag{11}$$

and define the *Bloch group* $\mathcal{B}(F)$ as the quotient $\mathcal{A}(F)/\mathcal{C}(F)$. Taking $(x,y)=(0,0)$, $(1,\infty)$ $(x,0)$ and (x,∞), we find that the relations $[1]=0$ and $[x]+[1-x]=[x]+[1/x]=0$ $(\forall x\in \mathbb{P}^1(F))$ hold in $\mathcal{B}(F)$. The functional equation (5) implies that the function D can be defined on $\mathcal{B}(F)$ for any subfield $F\subseteq \mathbb{C}$ by $\sum n_i[x_i]\mapsto \sum n_i D(x_i)$. Therefore if F is a number field, $[F:\mathbb{Q}]=n_+ + n_-$ as usual, then there is a map $D_F:\mathcal{B}(F)\to \mathbb{R}^{n_-}$ defined by $\sum n_i[x_i]\mapsto \left(\sum n_i D(\sigma(x_i))\right)_\sigma$, where σ ranges over the non-real embeddings $F\hookrightarrow \mathbb{C}$ (taking one of each complex conjugate pair).

The algebraic theorem that explains—and strengthens—equation (7) is now the following: for each number field F, there is a map $\mathcal{B}(F)\to K_3(F)$ with finite kernel and cokernel whose composition with the Borel regulator mapping $K_3(F)\to \mathbb{R}^{n_-}$ is D_F. This result is due to Bloch and Suslin; cf. Suslin's ICM survey talk [15]. Together with Borel's theorem, it implies that $\mathcal{B}(F)$ has rank n_- and that (7) is true with arguments in F itself (rather than quartic over F) and with the sum of n_--fold products replaced by the $n_-\times n_-$ determinant $\det\left(D(\sigma(\xi_\nu))\right)_{\nu,\sigma}$, where $(\xi_\nu)_{1\le\nu\le n_-}$ are linearly independent elements of $\mathcal{B}(F)$ and σ ranges over the non-conjugate non-real embeddings $F\hookrightarrow \mathbb{C}$.

A more leisurely expository account of the material in this section is given in [18].

§3. The trilogarithm and $\zeta_F(3)$. We would like to get an analogous statement for other special values of $\zeta_F(s)$, involving higher polylogarithms and some generalization of the Bloch groups. Specifically, we would like to define for each number field F groups $\mathcal{A}_m(F)$ and $\mathcal{C}_m(F)$ such that the quotient $\mathcal{B}_m(F)=\mathcal{A}_m(F)/\mathcal{C}_m(F)$ is a model for the K-group $K_{2m-1}(F)$ and the regulator map on $K_{2m-1}(F)$ corresponds to the mth polylogarithm function on $\mathcal{B}_m(F)$. By analogy with the case $m=2$, $\mathcal{A}_m(F)$ should be defined as the subgroup of elements of $\mathcal{F}_F$ satisfying some algebraic relation, while $\mathcal{C}_m(F)$ should come from the functional equations of the polylogarithm function. To guess the right form, we look at numerical examples, starting with the case $m=3$, to which the next several sections are devoted. As mentioned in the introduction, the reader not interested in the numerical motivation of the conjecture can skip to §6.

We first need an analogue of the Bloch-Wigner function $D(x)$. A modified polylogarithm function which is one-valued for all $x\neq 0,\infty$ was defined implicitly by Ramakrishnan [14] and explicitly in [18] (and, in a modified form which we will discuss in §7, in [16]), and discussed in some detail in [19]. It is given for $|x|\le 1$ by

$$D_m(x)=\Re_m\left(\sum_{r=0}^{m-1}\frac{(-1)^r}{r!}\log^r|x|\,Li_{m-r}(x)-\frac{(-1)^m}{2m!}\log^m|x|\right),\qquad (12)$$

where $\Re_m$ denotes $\Re$ or $\Im$ depending whether m is odd or even, and for $|x| \geq 1$ by the functional equation

$$D_m(\frac{1}{x}) = (-1)^{m-1} D_m(x) \qquad (x \in \mathbb{C}). \tag{13}$$

The function $D_m : \mathbb{P}^1(\mathbb{C}) \smallsetminus \{0, \infty\} \to \mathbb{R}$ is real-analytic except at $x = 1$ and satisfies the functional equation

$$D_m(\overline{x}) = (-1)^{m-1} D_m(x) \qquad (x \in \mathbb{C}). \tag{14}$$

Thus, unlike the case $m = 2$ where D_m vanished on $\mathbb{R}$ and hence only the non-real embeddings of F played a role, D_m for m odd is non-trivial on $\mathbb{R}$ and we can start by looking at totally real fields and $m = 3$, the function $D_3(x)$ being given for x real by

$$D_3(x) = Li_3(x) - \log|x|\, Li_2(x) - \frac{1}{2}\log^2|x| \log(1-x) + \frac{1}{12}\log^3|x| \tag{15}$$

if $-1 \leq x \leq 1$ and by (13) otherwise. Since 3 is odd, the Klingen-Siegel theorem does not apply and we do not a priori know anything about $\zeta_F(3)$ even in the totally real case. For $F = \mathbb{Q}$, of course, the relation $\zeta_F(3) = \zeta(3) = Li_3(1) = D_3(1)$ is trivial, so we begin by looking at real quadratic fields $F = \mathbb{Q}(\sqrt{D})$. Here $n_+ = 2$, $n_- = 0$, so the group $\mathcal{B}_3(F)$ should be 2-dimensional. This means that we expect $\sqrt{D}\,\zeta_F(3)$ to be expressible as the determinant of a 2×2 matrix whose entries are combinations of trilogarithms. This is unsuitable for numerical work, since it is hard to guess the entries of a matrix of size bigger than 1×1 knowing only its determinant. However, we know that F is abelian over $\mathbb{Q}$ and that $\zeta_F(3)$ splits as $\zeta(3)$ times $L(3, \chi_D)$ for a certain Dirichlet series $L(s, \chi_D) = \sum \chi_D(n) n^{-s}$, and it is reasonable to expect that the Galois group of F over $\mathbb{Q}$ acts on $\mathcal{B}_3(F)$ and splits it into two one-dimensional spaces $\mathcal{B}_3(F)^+$ and $\mathcal{B}_3(F)^-$ on which the determinant of the regulator mapping is equal (or rationally proportional) to $\zeta(3)$ and $\sqrt{D}\,L(3, \chi_D)$, respectively. Thus we expect that $\sqrt{D}\,L(3, \chi_D)$ can be written as a linear combination of numbers $D_3(x) - D_3(x')$, where x' denotes the conjugate of $x \in F$ over $\mathbb{Q}$ (here we are thinking of F as embedded once and for all in $\mathbb{R}$, say by $\sqrt{D} > 0$).

To find such a formula empirically, we take a collection of "simple" numbers $x \in F$, where "simple" means that x and $1 - x$ factor into prime ideals of small norm, and search for numerical relations among $\sqrt{D}\,L(3, \chi_D)$ and the numbers $D_3(x) - D_3(x')$ using the L^3 (Lenstra-Lenstra-Lovasz) algorithm. We recall that this is an algorithm which finds short vectors in a finite-dimensional lattice; to use it for the problem of finding linear relations over $\mathbb{Z}$ among a collection of given real numbers $\alpha_1, \ldots, \alpha_r$, one applies it to the lattice $\mathbb{Z}^r$ with the length form

$$\|(n_1, \ldots, n_r)\|^2 = n_1^2 + \cdots + n_r^2 + M(n_1\alpha_1 + \cdots n_r\alpha_r)^2$$

with a very large constant M, in which case any short vector will correspond to an r-tuple of reasonably sized integers n_j with $\sum n_j\alpha_j$ equal to 0 or extremely small. Applying the algorithm to the fields $F = \mathbb{Q}(\sqrt{5})$ $(D = 5)$ and $F = \mathbb{Q}(\sqrt{2})$ $(D = 8)$ and to reasonably chosen collections of numbers $x \in F$, we found the relations

$$\frac{75\sqrt{5}}{32} L(3, \chi_5) \stackrel{?}{=} D_3(2-\sqrt{5}) - D_3(2+\sqrt{5}) + 3\big[D_3(\tfrac{1+\sqrt{5}}{2}) - D_3(\tfrac{1-\sqrt{5}}{2})\big],$$

$$\begin{aligned}\frac{20\sqrt{2}}{3} L(3, \chi_8) \stackrel{?}{=}& D_3(4+2\sqrt{2}) - D_3(4-2\sqrt{2}) + 9\big[D_3(\sqrt{2}) - D_3(-\sqrt{2})\big] \\ &- 9\big[D_3(2+\sqrt{2}) - D_3(2-\sqrt{2})\big] - 6\big[D_3(1+\sqrt{2}) - D_3(1-\sqrt{2})\big].\end{aligned}$$

In both cases, we performed the search using a larger collection of numbers $x \in F$ than the ones which actually appear, and the relation given was the only one found by the computer. In both cases, too, the search was carried out "honestly" in the sense that the L^3 algorithm was first applied with a reasonably large value of M (about 10^{40}) and using a reasonably high precision for the numbers $L(3, \chi_D)$ and $D_3(x)$ (about 20 digits), and the relation found by the computer then checked independently by recalculating all of the numbers involved to a much higher precision (about 50 digits); thus despite the question marks over the equal sign, the above equalities carry a much higher level of certainty than many theorems.

As the next step, we looked for an algebraic relation analogous to (8) which is satisfied by the two linear combinations $[2-\sqrt{5}] - \cdots - 3[\frac{1+\sqrt{5}}{2}]$ and $[4+\sqrt{2}] - \cdots + 6[1-\sqrt{2}]$ occurring in the above relations and which is *not* satisfied by any other linear combination of the $[x] - [x']$ over which the search was performed. As a reasonable guess, we assumed that this relation would take place in $\bigotimes^3 F^\times$ and would involve only the numbers x and $1 - x$. After some effort, we found that there was a unique relation of this form which worked, namely

$$\sum_i n_i\, [x_i] \otimes [x_i] \otimes [\frac{x_i}{(1-x_i)^2}] \equiv 0, \tag{16}$$

where "$\equiv$" means "equal modulo torsion." This suggests the definition

$$\mathcal{A}_3^{(1)}(F) = \ker\big(\beta_3^{(1)} : \mathcal{F}_F \to (F^\times \otimes F^\times \otimes F^\times) \otimes \mathbb{Q}\big),$$

$$\beta_3^{(1)} : [x] \mapsto [x] \otimes [x] \otimes [\frac{x}{(1-x)^2}], \qquad [1] \mapsto 0.$$

(We have written the superscript "(1)" because this will not be our final definition of $\mathcal{A}_3(F)$.) As before, we can extend D_3 from $\mathbb{C}^\times$ to $\mathcal{A}_3^{(1)}(\mathbb{C})$ by $D_3(\sum n_i[x_i]) = \sum n_i D_3[x_i]$; then for each embedding σ of a number field F into $\mathbb{C}$ we get a map D_3^σ from $\mathcal{A}_3^{(1)}(F)$ to $\mathbb{R}$ by composing $\mathcal{A}_3^{(1)}(F) \to \mathcal{A}_3^{(1)}(\mathbb{C})$ with D_3. We can now formulate:

CONJECTURE (FIRST VERSION). *Let F be a totally real field of degree n and discriminant D. Then the image of the map $\prod_{\sigma: F \hookrightarrow \mathbb{R}} D_3^\sigma : \mathcal{A}_3^{(1)}(F) \to \mathbb{R}^n$ is a lattice (= cocompact $\mathbb{Z}$-submodule) whose covolume is a rational multiple of $\sqrt{D}\,\zeta_F(3)$. In particular, $\sqrt{D}\,\zeta_F(3)$ is a rational linear combination of n-fold products $D_3(x^{(1)}) \cdots D_3(x^{(n)})$, $x \in F$.*

Of course, what we really want is that there should be a canonically defined subgroup $\mathcal{C}_3^{(1)}(F)$ of $\mathcal{A}_3^{(1)}(F)$ and an isomorphism (up to torsion) from the K-group $K_5(F)$ to the quotient group $\mathcal{A}_3^{(1)}(F)/\mathcal{C}_3^{(1)}(F)$ whose composite with $\prod D_3^\sigma$ is the Borel regulator map; however, we will wait with the formal statement of this until we give our final definition of $\mathcal{A}_3(F)$.

One nice aspect of the conjecture is that it makes non-trivial predictions even in the case $F = \mathbb{Q}$, which because of the identity $\zeta(3) = D_3(1)$ had at first sight appeared to be of no interest. For instance, let

$$X = X_{\{2,3\}} = \left\{-1,\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ -\frac{1}{3},\ \frac{1}{4},\ \frac{3}{4},\ -\frac{1}{8},\ \frac{1}{9},\ \frac{8}{9}\right\}$$

be the set of all rational numbers x with $|x| \leq 1$ (since x and $1/x$ are equivalent from the point of view of D_3) such that x and $1-x$ contain only the prime numbers 2 and 3 in their numerator and denominator. The point of the restriction on the prime factors of x and $1-x$ is that it makes it easier to satisfy the conditions defining $\mathcal{A}_3^{(1)}(\mathbb{Q})$: specifically, each element $\beta_3^{(1)}([x]) = [x] \otimes [x] \otimes [x/(1-x)^2]$ $(x \in X)$ is (up to 2-torsion) a linear combination of the six elements

$$\begin{gathered}[2]\otimes[2]\otimes[2],\ \ ([2]\otimes[3]+[3]\otimes[2])\otimes[2],\ \ [3]\otimes[3]\otimes[2],\\ [2]\otimes[2]\otimes[3],\ \ ([2]\otimes[3]+[3]\otimes[2])\otimes[3],\ \ [3]\otimes[3]\otimes[3],\end{gathered}$$

so any seven of the $[x]$ are linearly dependent modulo $\mathcal{A}_3^{(1)}(\mathbb{Q})$. The space $\mathcal{F}_X = \{\sum_{x \in X} n_x[x]\}$ must therefore have an intersection with $\mathcal{A}_3^{(1)}(\mathbb{Q})$ of rank at least 5. A brief computation shows that the rank is in fact exactly

5, a basis being given by

$$\xi_1 = [-1], \quad \xi_2 = 4[\frac{1}{2}] + 4[-\frac{1}{2}] - [\frac{1}{4}],$$
$$\xi_3 = 4[\frac{1}{3}] + 4[-\frac{1}{3}] - [\frac{1}{9}], \quad \xi_4 = 9[\frac{1}{2}] + 18[-\frac{1}{2}] - [-\frac{1}{8}], \tag{17}$$
$$\xi_5 = 3[\frac{1}{2}] + 2[\frac{1}{3}] - 6[\frac{2}{3}] - [-\frac{1}{3}] - 3[\frac{3}{4}] + [\frac{8}{9}].$$

Calculating independently with the L^3 algorithm to search for numerical linear relations between the values $D_3(x)$ ($x \in X$) and $\zeta(3)$, we found exactly the same 5-dimensional space of relations, the numerical values of the $D_3(\xi_j)$ being

j	1	2	3	4	5
$D_3(\xi_j)$	$-\frac{3}{4}\zeta(3)$	0	0	$-\frac{35}{8}\zeta(3)$	$-\frac{67}{24}\zeta(3)$

(18)

Finally, we can get other examples of the conjecture by returning to the two quadratic fields we started with. In $\mathbb{Q}(\sqrt{5})$ the images of the elements $2-\sqrt{5}$, $2+\sqrt{5}$, $\frac{1}{2}(1+\sqrt{5})$ and $\frac{1}{2}(1-\sqrt{5})$ under $\beta_3^{(1)}$ are (up to torsion)

$$\begin{aligned} [\phi^{-3}] \otimes [\phi^{-3}] \otimes [\phi^{-1}/4] &= -18[\phi] \otimes [\phi] \otimes [2] - 9[\phi] \otimes [\phi] \otimes [\phi], \\ [\phi^{3}] \otimes [\phi^{3}] \otimes [\phi/4] &= -18[\phi] \otimes [\phi] \otimes [2] + 9[\phi] \otimes [\phi] \otimes [\phi], \\ [\phi] \otimes [\phi] \otimes [\phi^{3}] &= \quad 3[\phi] \otimes [\phi] \otimes [\phi], \quad \text{and} \\ [\phi^{-1}] \otimes [\phi^{-1}] \otimes [\phi^{-3}] &= - \ 3[\phi] \otimes [\phi] \otimes [\phi], \end{aligned}$$

respectively, where $\phi = \frac{1}{2}(1+\sqrt{5})$, so the two elements

$$[2-\sqrt{5}] - [2+\sqrt{5}] + 3([\frac{1+\sqrt{5}}{2}] - [\frac{1-\sqrt{5}}{2}]), \quad [\frac{1+\sqrt{5}}{2}] + [\frac{1-\sqrt{5}}{2}]$$

belong to $\mathcal{A}_3^{(1)}(F)$. The first is Galois anti-invariant and gives the relation $D_3(\xi) \sim \sqrt{5}\,L(3,\chi_5)$ we already saw, while the second is Galois invariant and leads to the relation

$$D_3(\frac{1+\sqrt{5}}{2}) + D_3(\frac{1-\sqrt{5}}{2}) = \frac{1}{5}\zeta(3).$$

Similarly, for $\mathbb{Q}(\sqrt{2})$ we find that the 8 numbers $x = 4+2\sqrt{2}, \ldots$ occurring in our original numerical relation involve (up to sign) only two numbers $1+\sqrt{2}$ and $\sqrt{2}$ in the factorizations of x and $1-x$, so just as with $X_{\{2,3\}}$ there are 6 conditions which a linear combination of them must satisfy to belong to $\ker \beta_3^{(1)}$, and consequently at least two linearly independent combinations belonging to the kernel. Doing the calculation, we find that there are exactly two, the Galois anti-invariant combination $[4+2\sqrt{2}]-[4-2\sqrt{2}]+\cdots$ we already exhibited and the Galois invariant combination

$$\begin{aligned}\xi &= [4+2\sqrt{2}]+[4-2\sqrt{2}]-9([\sqrt{2}]+[-\sqrt{2}]) \\ &\quad -9([2+\sqrt{2}]+[2-\sqrt{2}])-3([1+\sqrt{2}]+[1-\sqrt{2}]),\end{aligned}$$

for which $D_3(\xi)$ should be rationally proportional to $\zeta(3)$ and in fact equals $-\frac{125}{8}\zeta(3)$. Thus for both our quadratic fields F we have given elements in $\mathcal{A}_3^{(1)}(F)$ whose images under $\prod D_3^\sigma : \mathcal{A}_3^{(1)}(F) \to \mathbb{R}^2$ are rational multiples of $(\zeta(3), \zeta(3))$ and of $(\sqrt{D}\, L(3,\chi_D), -\sqrt{D}\, L(3,\chi_D))$ and hence which span a lattice with covolume $\sim \sqrt{D}\,\zeta_F(3)$, in accordance with the conjecture.

§4. **The trilogarithm (continued).** The examples we have just given show that the conjecture formulated in §3 is reasonable for totally real fields. However, when we pass to general number fields, then the group $\mathcal{A}_3^{(1)}(F)$ is too big and we must impose a further condition on elements of $\mathcal{F}_F$ to get elements in K-theory. This phenomenon, which is immediately seen in the numerical examples, was predicted by P. Deligne. To see what happens, consider an element $\xi = \sum n_i[x_i] \in \mathcal{F}_F$ and suppose that ξ belongs to $\ker \beta_3^{(1)}$, i.e., that equation (16) holds. Let $v : F^\times \to \mathbb{Z}$ be any homomorphism (for instance, the valuation at a prime p in case $F = \mathbb{Q}$). Then applying $v \otimes \mathrm{Id} \otimes \mathrm{Id}$ to (16), we find

$$\sum_i n_i\, v(x_i)\, [x_i] \otimes [\frac{x_i}{(1-x_i)^2}] \equiv 0.$$

Applying to this the projection $F^\times \otimes F^\times \to \Lambda^2(F^\times)$, we deduce that

$$\sum_i n_i\, v(x_i)\, [x_i] \wedge [1-x_i] \equiv 0,$$

i.e., that $\sum n_i v(x_i)[x_i]$ belongs to the group $\mathcal{A}(F)$ defined in §2. If π denotes the canonical projection from $\mathcal{A}(F)$ to $\mathcal{B}(F) = \mathcal{A}(F)/\mathcal{C}(F) \approx K_3(F)$, then the extra condition we must impose is

$$\pi\Bigl(\sum_i n_i\, v(x_i)\, [x_i]\Bigr) \equiv 0. \tag{19}$$

(In the case of totally real fields, this extra condition was not necessary because $\mathcal{B}(F) \otimes \mathbb{Q} = \{0\}$.) Then we have:

CONJECTURE (SECOND VERSION). *Let F be an arbitrary number field of degree $n = n_+ + n_-$, Δ the absolute value of the discriminant of F. Define $\mathcal{A}_3^{(2)}(F)$ as the set of $\xi = \sum_i n_i\,[x_i] \in \mathcal{A}_3^{(1)}(F)$ such that (19) holds for all $v \in Hom(F^\times, \mathbb{Z})$. Then the image of the map $\prod_{\sigma: F \hookrightarrow \mathbb{R}} D_3^\sigma : \mathcal{A}_3^{(2)}(F) \to \mathbb{R}^{n_+}$, where the product is taken over all real embeddings and one of each pair of complex conjugate embeddings of F into $\mathbb{C}$, is a lattice whose covolume is a rational multiple of $\sqrt{\Delta}\,\zeta_F(3)/\pi^{3n_-}$. In particular, $\sqrt{\Delta}\,\zeta_F(3)/\pi^{3n_-}$ is a rational linear combination of n_+-fold products $\prod_\sigma D_3(\sigma(x))$, $x \in F$.*

We now give an example of the necessity of the condition (19), taking for F the non-real cubic field of largest discriminant, namely the field $F = \mathbb{Q}(\theta)$ where $\theta^3 - \theta - 1 = 0$. (We cannot take F to be an imaginary quadratic field because in that case the complex conjugation belongs to the Galois group of F over $\mathbb{Q}$, so any ξ can be decomposed as $\frac{1}{2}(\xi - \bar{\xi})$ plus $\frac{1}{2}(\xi + \bar{\xi})$; the first has a trivial image under D_3, so cannot be used for the regulator, and the second is always in the kernel of D_2, so the condition is automatically satisfied.) We have the real embedding given by $\theta^{(1)} = 1.32471\cdots$ and the two complex embeddings $\theta^{(2)} = \dfrac{\theta^{(1)}}{2}\left(\dfrac{\sqrt{-23}}{2\theta^{(1)}+3} - 1\right)$ and $\theta^{(3)} = \overline{\theta^{(2)}}$. The zeta-function of F splits as $\zeta(s)L(s)$ where the L-series $L(s) = 1 - 2^{-s} - 3^{-s} + 6^{-s} + \cdots$ can be given either as a difference of Epstein zeta functions

$$L(s) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \left(\frac{1}{(m^2 + mn + 6n^2)^s} - \frac{1}{(2m^2 + mn + 3n^2)^s} \right)$$

or as the Mellin transform of the modular form $\eta(\tau)\eta(23\tau) = q - q^2 - q^3 + q^6 + \cdots$ of level 23 and weight 1. Using either representation and standard formulas, we can calculate $L(3)$ to a high accuracy:

$$L(3) = 0.84395\ 21532\ 37338\ 53825\ 22215\ 69424\ 77200\ 00563\ \cdots$$

(we have resisted the impulse to give this many digits of any of the numbers occurring up to now!). If $\xi = \sum n_i[x_i]$ is any element of $\mathcal{F}_F$, we denote by $\xi^{(1)}$, $\xi^{(2)}$, $\xi^{(3)}$ the images of ξ in $\mathcal{F}_\mathbb{C}$ under the various embeddings $F \hookrightarrow \mathbb{C}$; then for a good ξ we expect the numbers

$$\delta^+(\xi) = \frac{D_3(\xi^{(1)}) + 2D_3(\xi^{(2)})}{\zeta(3)}, \quad \delta^-(\xi) = \frac{D_3(\xi^{(1)}) - D_3(\xi^{(2)})}{\Lambda_3}, \qquad (20)$$

where $\Lambda_3 = \dfrac{23^{5/2}}{96\pi^3}\,L(3)$, to be rational. Here the factor $23^2/96$ has been inserted for convenience, and the justification for the linear combinations of $D_3(\xi^{(1)})$ and $D_3(\xi^{(2)})$ comes from the Artin L-function version of our conjecture which will be mentioned later (§10C); note that we do not need $D_3(\xi^{(3)})$ because it equals $D_3(\xi^{(2)})$. Consider the six numbers $x_1, \ldots, x_6 = \theta, -\theta, \theta^2, \theta^3, -\theta^4$ and θ^5. Each of them has the property that x_i and $1-x_i$ are, up to sign, powers of θ, so each $\beta_3^{(1)}(x_i)$ is a multiple of $[\theta]\otimes[\theta]\otimes[\theta]$, the multiples being 9, −5, 16, 9, −96 and −75, respectively. There are therefore 5 linearly independent elements $5[\theta]+9[-\theta]$, ... belonging to $\ker(\beta_3^{(1)})$. However, when we compute the corresponding values of $\delta^{\pm}(\xi)$ numerically, we do not find rational numbers. (And if we do not believe the splitting (20) of the expected 2-dimensional space $\mathcal{B}_3(F)$ into two 1-dimensional $\mathbb{Q}$-vector spaces, we can take two different ξ from our list and compute the 2×2 determinant giving the covolume of the lattice they generate; the answer is not a rational multiple of $23^{1/2}\zeta_F(3)/\pi^3$.) On the other hand, each $[x_i]$ belongs to $\mathcal{A}_2(F)$ (since x_i and $1-x_i$ are powers of θ and $[\theta]\wedge[\theta]=0$), so the value of $D_2(x_i^{(2)})$ for each i is a rational multiple of $23^{3/2}\zeta_F(2)/\pi^4$, the multiples being 1, −2, −2, 2, 1 and −1, respectively. Hence, taking v to be any homomorphism which is non-zero on θ, we see that the values of $v(x_i)\,\pi_2([x_i])$ are proportional to 1, −2, −4, 6, 4 and −5, respectively. Thus (19) imposes an extra condition on ξ and cuts down the space of solutions to dimension 4, a basis being given by $4[\theta]+4[-\theta]-[\theta^2]$, $3[\theta^5]+27[\theta]-2[\theta^3]$, $13[\theta^5]+30[-\theta]+125[\theta]$ and $5[-\theta^4]-130[\theta]-22[\theta^5]$. For these four elements we do indeed find that the values of $(\delta^+(\xi),\delta^-(\xi))$ belong to $\mathbb{Q}^2$ (or even, with our normalization of Λ, to $\mathbb{Z}^2$), the values being $(0,0)$, $(-3,69)$, $(13,259)$ and $(-12,-351)$, respectively.

§5. **The trilogarithm (concluded).** We now have a conjecture in the case $m=3$ which works for all number fields. However, it turns out that there is still a further improvement which can be made. To see this, we assume for convenience that we are back in the totally real case, so that a "good" element $\xi=\sum n_i[x_i]\in\mathcal{F}_F$ is one satisfying (16). Since $z\mapsto\log|z|$ is a homomorphism from $\mathbb{C}^\times$ to $\mathbb{R}$, it follows that

$$\sum_i n_i \log^2|x_i|\,\log\Big|\frac{x_i}{(1-x_i)^2}\Big| = 0.$$

Consequently we can modify the function $D_3(x)$ by adding to it any multiple of the function $\log^2|x|\,\log|x/(1-x)^2|$ without affecting the validity of our conjecture. The obvious choice is the function

$$\widetilde{D}_3(x) = D_3(x) - \frac{1}{12}\log^2|x|\log\Big|\frac{x}{(1-x)^2}\Big|, \tag{21}$$

because according to (15) it is given for $-1\le x\le 1$ by

$$\widetilde{D}_3(x) = Li_3(x) - \log|x|\, Li_2(x) - \frac{1}{3}\log^2|x|\,\log(1-x) \tag{22}$$

in which the term $\log^3|x|$ which blows up at $x=0$ has disappeared.

Now a miracle occurs. Replacing D_3 by $\widetilde{D}_3$ does not affect the validity of our previous conjecture, as already mentioned, because the two functions agree on $\mathcal{A}_3^{(1)}(F)$ anyway. But as a result of this change we find that there are suddenly many more combinations $\sum n_i[x_i]$ which work than before. For instance, return to the 11-element set $X = X_{\{2,3\}}$ which was already used as an example in §3. Applying the L^3 algorithm as before to find linear relations over $\mathbb{Z}$, now between the values $\widetilde{D}_3(x)$ $(x \in X)$ and $\zeta(3)$, we find a 9-dimensional space of solutions rather than a 5-dimensional space as before, a basis being given by the 5 old elements (17) together with the 4 new elements

$$\begin{aligned}&\xi_6 = [\tfrac{1}{2}], \quad \xi_7 = 2[\tfrac{1}{3}] - 4[\tfrac{2}{3}] - 2[-\tfrac{1}{3}] + [\tfrac{3}{4}],\\ &\xi_8 = 4[-\tfrac{1}{2}] + [-\tfrac{1}{3}] + [\tfrac{3}{4}], \quad \xi_9 = [-\tfrac{1}{2}] + [\tfrac{1}{3}] + [\tfrac{2}{3}],\end{aligned} \tag{23}$$

the corresponding values of $\widetilde{D}_3(\xi_j)$ being equal to $\frac{7}{8}\zeta(3)$, 0, $-\frac{5}{2}\zeta(3)$, and $\zeta(3)$, respectively.

Now, just as before, we look for an algebraic condition which picks out of the 11-dimensional space $\mathcal{F}_X$ precisely the 9-dimensional subspace spanned by $\xi_1, \dots, \xi_9$, and again we find exactly one which works: the relation (16) must be replaced by the relation

$$\sum_i n_i\,[x_i] \otimes \Big([x_i] \wedge [1-x_i]\Big) \equiv 0, \tag{24}$$

which now takes place in $F^\times \otimes \Lambda^2(F^\times)$ and, as before, is required to hold only up to torsion. In other words, our new candidate for the numerator of the Bloch group in the case $m=3$ (for F totally real) is the group

$$\begin{aligned}\mathcal{A}_3^{(3)}(F) &= \ker\big(\beta_3 : \mathcal{F}_F \to (F^\times \otimes \Lambda^2(F^\times)) \otimes \mathbb{Q}\big),\\ \beta_3 &: [x] \mapsto [x] \otimes ([x] \wedge [1-x]), \qquad [1] \mapsto 0,\end{aligned}$$

and we make the

CONJECTURE (THIRD VERSION). *Let F be a totally real field of degree n and discriminant D. Then the image of the map $\prod_{\sigma: F \hookrightarrow \mathbb{R}} \widetilde{D}_3^\sigma : \mathcal{A}_3^{(3)}(F) \to \mathbb{R}^n$ is a lattice with covolume $\sim \sqrt{D}\,\zeta_F(3)$.*

If F is not totally real, of course, then we must add to the definition of $\mathcal{A}_3^{(3)}(F)$ the same supplementary condition as in §4, namely, that (19) should hold for every linear map $F^\times \to \mathbb{Z}$. In other words, if $\xi = \sum n_i[x_i]$ satisfies (24), then the element $\iota_v(\xi)$, where $\iota_v : \mathcal{F}_F \to \mathcal{F}_F$ is the map defined on generators by $[x] \mapsto v(x)\,[x]$, belongs to $\mathcal{A}(F) = \ker([x] \mapsto [x] \wedge [1-x])$, and we want the image of ξ under the composite mapping

$$\mathcal{A}_3^{(3)}(F) \xrightarrow{\iota_v} \mathcal{A}(F) \xrightarrow{\pi} \mathcal{A}(F)/\mathcal{C}(F) = \mathcal{B}(F) \approx K_3(F)$$

to vanish for all v. This can be expressed more algebraically by saying that the image of ξ under the composite mapping

$$\mathcal{A}_3^{(3)}(F) \xrightarrow{\iota} F^\times \otimes \mathcal{A}(F) \xrightarrow{\mathrm{Id}\otimes\pi} F^\times \otimes \mathcal{B}(F)$$

vanishes, where $\iota : \mathcal{F}_F \to F^\times \otimes \mathcal{F}_F$ is the map defined on generators $[x]$ ($x \in F$) by $[x] \mapsto [x] \otimes [x]$. Thus we have finally:

Conjecture. *Let F be a number field of degree $n = n_+ + n_-$, Δ the absolute value of the discriminant of F. Define $\mathcal{A}_3(F) = \ker\big((Id \otimes \pi) \circ \iota : \mathcal{A}_3^{(3)}(F) \to F^\times \otimes \mathcal{B}(F)\big)$ and $\widetilde{D}_3^F = \prod_\sigma \widetilde{D}_3^\sigma : \mathcal{A}_3(F) \to \mathbb{R}^{n_+}$ (product over the embeddings of F into $\mathbb{C}$ up to complex conjugation). Then $\widetilde{D}_3^F(\mathcal{A}_3(F))$ is a lattice in $\mathbb{R}^{n_+}$ with covolume $\sim \sqrt{\Delta}\,\zeta_F(3)/\pi^{3n_-}$.*

We can summarize the various definitions of the last three sections and their interrelationships by the commutative diagram

$$\begin{array}{ccccccc} & D_3^F & \mathcal{F}_F & \supseteq & \mathcal{A}_3^{(1)}(F) & \supseteq & \mathcal{A}_3^{(2)}(F) \\ & \swarrow & & & & & \\ \mathbb{R}^{n_+} & & & & \cap & & \cap \\ & \nwarrow & & & & & \\ & \widetilde{D}_3^F & \mathcal{F}_F & \supseteq & \mathcal{A}_3^{(3)}(F) & \supseteq & \mathcal{A}_3(F) \end{array} \tag{25}$$

where $\mathcal{A}_3^{(1)}(F)$ and $\mathcal{A}_3^{(3)}(F)$ denote the kernels of $[x] \mapsto [x]\otimes[x]\otimes[x/(1-x)^2]$ and $[x] \mapsto [x] \otimes ([x] \wedge [1-x])$, $\mathcal{A}_3(F)$ is the kernel of $[x] \mapsto [x] \otimes \pi_2([x])$, $\mathcal{A}_3^{(2)}(F) = \mathcal{A}_3^{(1)}(F) \cap \mathcal{A}_3(F)$, and the horizontal inclusions are equalities when F is a totally real field.

We now give numerical examples of the above versions of the conjecture for the same number fields $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\theta)$ ($\theta^3 - \theta - 1 = 0$) that were used to illustrate the first two versions. For $F = \mathbb{Q}(\sqrt{5})$ we had four elements $2 \pm \sqrt{5}$ and $(1 \pm \sqrt{5})/2$ and two linear combinations of them which satisfied (16) and hence belonged to $\mathcal{A}_3^{(1)}(F)$. For the last two of these elements x, both x and $1-x$ belong (up to torsion) to the group generated by $\phi = (1+\sqrt{5})/2$, so $[x]\otimes\big([x]\wedge[1-x]\big)$ vanishes trivially. This gives a new element $\big[(1+\sqrt{5})/2\big]-\big[(1-\sqrt{5})/2\big]$. It is anti-invariant under the action of $\mathrm{Gal}(F/\mathbb{Q})$, so should map under $\widetilde{D}_3$ to a rational multiple of $\sqrt{5}\,L(3,\chi_5)$. The multiple turns out to be 25/24, giving the identities

$$\widetilde{D}_3\left(\frac{1\pm\sqrt{5}}{2}\right) = \frac{1}{10}\,\zeta(3) \pm \frac{25\sqrt{5}}{48}\,L(3,\chi_5).$$

For $F=\mathbb{Q}(\sqrt{2})$ the situation is similar. Here we had used a collection of 8 elements for which x and $1-x$ belonged to a sub $\mathbb{Z}$-module of $F^\times$ of rank 2. This led to 6 linear conditions needed to satisfy (16) and hence to 2 linear combinations ξ which mapped to zeta values under D_3. Equation (24), on the other hand, imposes only 2 conditions, since each $[x]\otimes\big([x]\wedge[1-x]\big)$ is up to torsion a linear combination of the two elements $[\varepsilon]\otimes\big([\varepsilon]\wedge[\pi]\big)$ and $[\pi]\otimes\big([\varepsilon]\wedge[\pi]\big)$ ($\varepsilon = 1+\sqrt{2}$, $\pi=\sqrt{2}$), so we now get a 6-dimensional space of linear combinations which work for $\widetilde{D}_3$. As a basis of this space we can take the two previous elements and the four new ones

$$\begin{gathered}[\sqrt{2}]+[-\sqrt{2}],\ [2+\sqrt{2}]+[2-\sqrt{2}]+[1+\sqrt{2}]+[1-\sqrt{2}],\\ [1+\sqrt{2}]-[1-\sqrt{2}],\ \text{and } [2+\sqrt{2}]-[2-\sqrt{2}]+[\sqrt{2}]-[-\sqrt{2}].\end{gathered}$$

The first two are Galois invariant and map under $\widetilde{D}_3$ to $\frac{7}{32}\,\zeta(3)$ and $\frac{57}{32}\,\zeta(3)$, respectively, while the latter two are Galois anti-invariant and both map to $\frac{4\sqrt{2}}{3}\,L(3,\chi_8)$. Finally, look again at the field $F=\mathbb{Q}(\theta)$ and the six numbers $x_1=\theta,\ \dots,\ x_6=\theta^5$ of §4. All six lie in $\mathcal{A}_3^{(3)}(F)$, since all have $\{x,1-x\}\subset\langle\pm1,\theta\rangle$, and for the same reason the requirement that $\pi_2(\iota_v(\xi))=0$ (or equivalently, that $D\big(\iota_v(\xi^{(2)})\big)=0$, where $(\cdot)^{(2)}$ is a non-real embedding of F into $\mathbb{C}$) imposes only a single linear condition. The space of solutions is therefore now 5-dimensional, rather than 4-dimensional as before; as a basis we can take the four elements we had before together with the new solution $[-\theta]+2[\theta]$, or, better, a new simpler basis $[-\theta]+2[\theta]$, $[\theta^2]+4[\theta]$, $[\theta^3]-6[\theta]$, $[-\theta^4]-4[\theta]$, $[\theta^5]+5[\theta]$ with images (0,3), (0,−12), (3,−15), (2,−13) and (1,13) under the (conjectural) mapping (δ^+,δ^-) : $\mathcal{A}_3(F)\to\mathbb{Z}^2$ defined by equation (20). Examples of this kind can of course be multiplied indefinitely, but become numerically quite intricate even for relatively simple number fields. For instance, for the field $F=\mathbb{Q}(\sqrt{-11})$ the conjecture predicts that $\widetilde{D}_3(\xi)$ for any element ξ of $\mathcal{A}_3(F)$ should be a rational multiple of $\zeta(3)$ (because $n_+=1$, the same value as for $\mathbb{Q}$; note that $\zeta(3)\sim\sqrt{D}\,\zeta_F(3)/\pi^3$). By looking at the 26 numbers $x\in F\smallsetminus\mathbb{Q}$ for which x and $1-x$ involve only the three primes 2 and $\frac{1}{2}(1 \pm \sqrt{-11})$ (this seems to be the smallest set which works), we obtain enough numbers and few enough conditions to produce non-trivial relations $\widetilde{D}_3(\xi) \sim \zeta(3)$, but all involve several terms and fairly complicated coefficients, a typical example being

$$176\widetilde{D}_3(\frac{4+\sqrt{-11}}{8}) - 1320\widetilde{D}_3(\frac{1+\sqrt{-11}}{2}) - 3456\widetilde{D}_3(\frac{-1+\sqrt{-11}}{3})$$
$$- 800\widetilde{D}_3(\frac{4+\sqrt{-11}}{3}) + \widetilde{D}_3(\frac{1+\sqrt{-11}}{32}) - 53\widetilde{D}_3(\frac{31+\sqrt{-11}}{32}) = \frac{1264}{3}\zeta(3).$$

If we also use the $x \in \mathbb{Q}$ for which x and $1-x$ involve only the primes 2 and 3, we get simpler relations, e.g.

$$\widetilde{D}_3(\frac{5+\sqrt{-11}}{4}) - 2\,\widetilde{D}_3(\frac{-1+\sqrt{-11}}{3}) + \widetilde{D}_3(\frac{1}{3}) + \frac{3}{2}\,\widetilde{D}_3(\frac{-1}{2}) = \zeta(3).$$

§6. Functional equations of the trilogarithm and the group $\mathcal{C}_3(F)$. The conjectures which we have stated are thus supported by considerable numerical evidence. However, if we compare them with the results described in §2 for the dilogarithm, then we see that they are still incomplete: we should still define a subgroup $\mathcal{C}_3(F)$ analogous to $\mathcal{C}(F)$ such that $\mathcal{B}_3(F) = \mathcal{A}_3(F)/\mathcal{C}_3(F)$ is conjecturally isomorphic (up to torsion) to the K-group $K_5(F)$, with $\widetilde{D}_3^F$ corresponding to the Borel regulator mapping. In analogy with $\mathcal{C}(F)$, which was defined (eq. (11)) as the space spanned by the 5-term relation satisfied by $D(x)$, we would expect $\mathcal{C}_3(F)$ to be spanned by the functional equations of the trilogarithm. We now come to the second remarkable property of $\widetilde{D}_3$: all of the functional equations of the trilogarithm become "clean" (i.e., contain no lower-order terms) if we use the trilogarithm function $\widetilde{D}_3$. Specifically, the classical functional equations of the trilogarithm are the easy 1-variable relations

$$I_x = [x]-[\frac{1}{x}], \quad D_x = [x^2]-4[x]-4[-x], \quad T_x = [x]+[1-x]+[1-\frac{1}{x}]-[1]$$

and the 2-variable relation

$$S^{(3)}_{xy} = [\frac{x(1-y)^2}{y(1-x)^2}] + [xy] + [\frac{x}{y}] - 2[\frac{x(1-y)}{y(1-x)}] - 2[\frac{x(1-y)}{x-1}]$$
$$- 2[\frac{y(1-x)}{y-1}] - 2[\frac{1-y}{1-x}] - 2[x] - 2[y] + 2[1], \qquad (26)$$

found by Spence (1809) and Kummer(1840); here x and y are free variables and by "relations" we mean that for each of the given expressions $\xi =$

$\sum n_i[x_i]$ the sum $\sum n_i Li_3(x_i)$ is a linear combinâtion of lower order terms (here, products of logarithms of rational functions of x and y). The number of such lower-order terms is fairly large, but goes down considerably when the functional equation is expressed in terms of D_3 rather than Li_3; for instance, we have

$$D_3(I_x) = 0, \qquad D_3(T_x) = \frac{1}{12}\log|x(1-x)|\,\log\left|\frac{x}{(1-x)^2}\right|\,\log\left|\frac{x^2}{1-x}\right|$$

and

$$D_3(S^{(3)}_{xy}) = -\frac{1}{4}\log|xy|\,\log\left|\frac{x}{y}\right|\,\log\left|\frac{x(1-y)^2}{y(1-x)^2}\right|$$

(as observed in [18]), with 0, 1, and 1 products of logarithms on the right, whereas the corresponding formulas for Li_3 itself ([10], eqs. (6.7), (6.11) and (6.96)) have 2, 3, and 7 lower-order terms, respectively. When we pass to $\widetilde{D}_3$, however, there are no lower terms at all:

$$\widetilde{D}_3(I_x) = \widetilde{D}_3(D_x) = \widetilde{D}_3(T_x) = \widetilde{D}_3(S^{(3)}_{xy}) = 0 \qquad \forall x,\, y.$$

This property of $\widetilde{D}_3$ was noted by Lewin ([11], 3.2) and will be generalized in §7, where we will show that $\sum n_i \widetilde{D}_3(x_i)$ is constant for any collection of integers n_i and rational functions x_i satisfying the identity (24).

We thus have two ways to try to define the group $\mathcal{C}_3(F)$. One, by analogy with (11), is to find a sufficient set of functional equations for $\widetilde{D}_3$ analogous to the 5-term relation for the Bloch-Wigner function. All of the relations given above, as well as various other functional equations given in [10], can be obtained from the Spence-Kummer relation (26) by specialization, and we will present further evidence in a moment that this relation generates the full kernel of $\widetilde{D}_3^F : \mathcal{A}_3(F) \to \mathbb{R}^{n_+}$ for number fields. We can thus define, in full analogy with (11),

$$\mathcal{C}_3(F) = \left\langle S^{(3)}_{xy} \right\rangle_{x,y\,\in\,\mathbb{P}^1(F);\ \text{no indeterminate terms}} \tag{27}$$

(the phrase "no indeterminate terms" means that we exclude pairs like $(x,y) = (0,0)$ where one or more of the terms in $S^{(3)}_{xy}$ becomes $0/0$). A more modest definition, since we do not know that all functional equations follow from the Spence-Kummer ones, would be to define $\mathcal{C}_3(F)$ as the group spanned by all functional equations of $\widetilde{D}_3$ with the variables specialized to F. For higher polylogarithms, where we definitely do not possess generating collections of functional equations, we are of course forced to use the latter definition; we will therefore have to give a precise formulation of it for the general case in §7 and do not give a formulation here. Whichever way we define $\mathcal{C}_3$, we define the third Bloch group $\mathcal{B}_3(F)$ as the quotient $\mathcal{A}_3(F)/\mathcal{C}_3(F)$ and make the conjecture

CONJECTURE. *The map* $\widetilde{D}_3^F : \mathcal{B}_3(F) \to \mathbb{R}^{n_+}$ *for any number field* F *is an isomorphism onto a sublattice of* $\mathbb{R}^{n_+}$ *with covolume* $\sim \sqrt{\Delta}\,\zeta_F(3)/\pi^{3n_-}$.

Of course, what we really want is that there is a canonical map $K_5(F) \to \mathcal{B}_3(F)$ which is an isomorphism up to torsion and whose composite with $\widetilde{D}_3^F$ gives the Borel regulator mapping; we have not stated the conjecture in this form because our numerical evidence only concerns the image of the trilogarithm map $\widetilde{D}_3^F$ and gives no information about algebraic K-theory.

To support this conjecture, as well as to provide yet further evidence for the conjectures of §§3–5, let us look once more at the field $F = \mathbb{Q}$, this time allowing the prime factors 2, 3 and 5 in the numerators and denominators of x and $1-x$. The set $X = X_{\{2,3,5\}}$ of x having this property and with $|x| \le 1$ contains exactly 50 rational numbers, the one with the largest denominator being $\frac{3}{128}$. For $x \in X$, we write $x = \pm 2^{\alpha_2}3^{\alpha_3}5^{\alpha_5}$, $1 - x = \pm 2^{\beta_2}3^{\beta_3}5^{\beta_5}$ and set $\gamma_{qr} = \begin{vmatrix} \alpha_q & \alpha_r \\ \beta_q & \beta_r \end{vmatrix}$ for $q,\ r \in \{2,3,5\}$. The product $\alpha_p\gamma_{qr}$ gives the coefficient of $[p] \otimes ([q] \otimes [r])$ in $\beta_3([x])$. There are 9 such products (since $\gamma_{qr} = -\gamma_{rq}$), but only 8 of them are linearly independent since we have the relation

$$\alpha_2\,\gamma_{35} - \alpha_3\,\gamma_{25} + \alpha_5\,\gamma_{23} = \begin{vmatrix} \alpha_2 & \alpha_3 & \alpha_5 \\ \alpha_2 & \alpha_3 & \alpha_5 \\ \beta_2 & \beta_3 & \beta_5 \end{vmatrix} = 0.$$

Also, $\gamma_{25} \equiv \gamma_{35} \pmod 2$, as one sees by an easy argument using the Legendre symbol at 5. We therefore have 8 linearly independent integral invariants $\nu_1(x), \dots, \nu_8(x)$ defined by

$$\begin{pmatrix} \nu_1(x) \\ \nu_2(x) \\ \nu_3(x) \\ \nu_4(x) \\ \nu_5(x) \\ \nu_6(x) \\ \nu_7(x) \\ \nu_8(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -2 & -3 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_2\gamma_{23} \\ -\alpha_3\gamma_{23} \\ \frac{1}{2}\alpha_5(\gamma_{35} - \gamma_{25}) \\ -\alpha_5\gamma_{35} \\ -\alpha_2\gamma_{25} \\ \alpha_3\gamma_{35} \\ -\frac{1}{2}\alpha_2(\gamma_{25} + \gamma_{35}) \\ \frac{1}{2}\alpha_3(\gamma_{25} + \gamma_{35}) \end{pmatrix}.$$

(Example, left to the reader: the invariants $\nu_1(x), \dots, \nu_8(x)$ for $x = \frac{3}{128}$ are 251, 5, −216, 288, −144, 12, −72, −9.) The triangular matrix and the order of the invariants on the right of this equation have been chosen so that $\nu_i(x_j) = \delta_{i,j}$ (Kronecker delta) for the 8 rational numbers $x_1, \dots, x_8 = -\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$, $-\frac{1}{5}$, $\frac{2}{5}$, $-\frac{2}{3}$, $\frac{1}{10}$, $\frac{1}{6}$. Hence the element $[x] - \sum_{i=1}^{8} \nu_i(x)\,[x_i]$ of $\mathcal{F}_{\mathbb{Q}}$ belongs to $\ker(\beta_3)$ for any $x \in X$, and if our conjecture is right we must have

$$\tilde{D}_3(x) = \sum_{i=1}^{8} \nu_i(x)\, \tilde{D}_3(x_i) + \frac{m(x)}{12}\, \zeta(3) \tag{28}$$

for some rational number $m(x)$ whose value can be calculated numerically. For instance, using the values of $\nu_i(\frac{3}{128})$ just given, one finds $m(\frac{3}{128}) = 7518$. In fact, $m(x)$ turns out to be integral for all $x \in X$ except $\frac{1}{2}$. What's more, the integrality of $m(x)$ is a theorem, not merely a numerical fact, since H. Gangl in Bonn has checked that the relation (28) *is a consequence of the Spence-Kummer relation for all 50 elements* $x \in X$. This provides rather strong evidence for the conjecture presented above that the kernel of $\tilde{D}_3^F$ coincides with the group generated by the Spence-Kummer relation for all $x, y \in F$. Gangl has performed similar calculations for the much larger set $X = X_{\{2,3,5,7\}}$ and again all relations $\tilde{D}_3(\xi) = \mu\, \zeta(3)$ ($\xi \in \mathcal{A}_3(\mathbb{Q}) \cap \mathcal{F}_X$, $\mu \in \mathbb{Q}$) which he checked turned out to be deducible from the Spence-Kummer relation and to have the property that μ is a multiple of $1/24$, though it need now no longer be an even multiple even if ξ does not contain $[\frac{1}{2}]$; for instance, the image of

$$4\left[\frac{1}{3}\right] - 12\left[\frac{1}{6}\right] - 3\left[\frac{1}{8}\right] + 3\left[\frac{1}{36}\right] + 2\left[-\frac{1}{2}\right] - 18\left[-\frac{1}{6}\right] + \left[-\frac{1}{48}\right] + \left[\frac{2}{9}\right]$$

under $\tilde{D}_3$ is $\dfrac{89}{24}\, \zeta(3)$. In any event, it seems to be the case that the image of the map $\tilde{D}_3 : \mathcal{A}_3(\mathbb{Q}) \to \mathbb{R}$ is the lattice $\dfrac{1}{24}\, \zeta(3)\, \mathbb{Z}$ and that the kernel is the group generated by the Spence-Kummer relation.

§7. Higher order polylogarithm functions and functional equations. Like $D_3(x)$, the function $D_m(x)$ defined by (12) has a $\log^m$ singularity at $x = 0$ (and hence, by the functional equation (13), also at $x = \infty$) if m is odd, but this can be removed in the same way as in the case $m = 3$ by adding a suitable multiple of $\log^{m-1}|x| \log|\frac{x}{(1-x)^2}|$, i.e. by looking at

$$\tilde{D}_m(x) = \Re_m\left(\sum_{r=0}^{m-1} \frac{(-\log|x|)^r}{r!}\, Li_{m-r}(x) + \frac{(-\log|x|)^{m-1}}{m!} \log|1 - x| \right). \tag{29}$$

This is the form of the function $D_m(x)$ given by Wojtkowiak [16], who denotes it $-\mathrm{L}(x)$. Observe that the function $\tilde{D}_m$ vanishes at $x = 0$ and—since the addition of the $(-1)^{m-1}$-symmetric term $\frac{1}{2m!} \log^{m-1}|x| \log|\frac{x}{(1-x)^2}|$ does not affect the validity of the functional equation (13)—also at $x = \infty$. We now show that $\tilde{D}_m$ for every m satisfies "clean" functional equations, i.e., functional equations involving no lower-order polylogarithms or ordinary logarithms.

PROPOSITION 1. *Let $\{n_i,\ x_i(t)\}$ be a collection of integers n_i and rational functions of one variable $x_i(t)$ satisfying the identity*

$$\sum_i n_i\,[x_i(t)]^{m-2}\otimes\big([x_i(t)]\wedge[1-x_i(t)]\big)=0 \tag{30}$$

in $\left(Sym^{m-2}(\mathbb{C}(t)^\times)\otimes\Lambda^2(\mathbb{C}(t)^\times)\right)\underset{Z}{\otimes}\mathbb{Q}$. Then $\sum_i n_i\,\widetilde{D}_m(x_i(t))=$ constant.

PROOF: Using the identities

$$\frac{\partial\log|x|}{\partial x}=\frac{1}{2x},\quad \frac{\partial Li_j(\overline{x})}{\partial x}=0,\quad \frac{\partial Li_j(x)}{\partial x}=\begin{cases}\dfrac{1}{x}Li_{j-1}(x) & (j\geq 2)\\[2mm] \dfrac{1}{1-x} & (j=1)\end{cases},$$

we find after a short calculation that

$$\frac{\partial\widetilde{D}_m(x)}{\partial x}=\frac{\alpha_m}{x}\,\widetilde{D}_{m-1}(x)+\beta_m\,\log^{m-2}|x|\left(\frac{\log|1-x|}{x}+\frac{\log|x|}{1-x}\right),$$

where α_m denotes $(-1)^m/2\sqrt{-1}$ and β_m is $\sqrt{-1}/2(m-1)!$ for m even, $(m-1)/2m!$ for m odd. Hence

$$\begin{aligned}\frac{\partial}{\partial t}\sum_i n_i\,\widetilde{D}_m(x_i(t)) &= \alpha_m\sum_i\frac{x_i'(t)}{x_i(t)}\,\widetilde{D}_{m-1}(x_i(t))\\ +\beta_m\sum_i\log^{m-2}|x_i(t)|&\left(\frac{x_i'(t)}{x_i(t)}\log|1-x_i(t)|-\frac{(1-x_i(t))'}{1-x_i(t)}\log|x_i(t)|\right).\end{aligned}$$

The second sum vanishes because of (30) and because the maps $\lambda: x(t)\mapsto \log|x(t)|$ and $\lambda': x(t)\mapsto x'(t)/x(t)$ are linear on $\mathbb{C}(t)^\times$. For the first sum, we use the fact that, by the fundamental theorem of algebra, any rational function $x(t)$ factors as $c\prod(t-\alpha)^{v_\alpha(x)}$ and hence $x'(t)/x(t)=\sum v_\alpha(x)/(t-\alpha)$, where the product and the sum extend over all $\alpha\in\mathbb{C}$ (only finitely many terms being non-trivial) and $v_\alpha(x)$ denotes the order of x at α. Hence the first sum can be written as

$$\sum_{\alpha\in\mathbb{C}}\frac{1}{t-\alpha}\left(v_\alpha(x_i(t))\,\widetilde{D}_{m-1}(x_i(t))\right).$$

But the formal sum $\sum n_i\,v_\alpha(x_i(t))\,[x_i(t)]$ satisfies the analogue of (30) with m replaced by $m-1$ (if $m=2$, then this makes no sense, but then there is nothing to prove since $\widetilde{D}_1\equiv 0$), so the expression in the inner sum is a constant by the proposition applied inductively to $\widetilde{D}_{m-1}$. Substituting $t = \alpha$, we see that this constant is zero, since whenever $v_\alpha(x_i(t))$ is not zero the function $\widetilde{D}_{m-1}$ is being evaluated at either 0 or ∞, where it vanishes. Hence $(\partial/\partial t)(\sum n_i \widetilde{D}_m(x_i(t))) = 0$ and similarly, of course, for $\partial/\partial\bar{t}$. □

REMARK: There is a similar result, slightly simpler to prove, for polylogarithms of a real variable. For $-1 \le x \le 1$ define

$$L_m(x) = \sum_{r=0}^{m-1} \frac{(-\log|x|)^r}{r!} Li_{m-r}(x) + \frac{(-\log|x|)^{m-1}}{m!} \log|1-x|, \qquad (31)$$

and extend to $\mathbb{R}$ by $L_m(\frac{1}{x}) = (-1)^{m-1} L_m(x)$. This function agrees for m odd with the restriction of $\widetilde{D}_m$ to $\mathbb{R}$ (for m even, of course, $\widetilde{D}_m|\mathbb{R}$ vanishes identically). It was defined (for $x < 1$ and by a recurrence rather than a closed formula) by Lewin ([11], eq. (16)), who conjectured that it always satisfies "clean" functional equations and verified this for the functional equations found by Kummer for the polylogarithms of order ≤ 5. In our language this takes the form that $\sum n_i L_m(x_i(t))$ is piecewise constant (the pieces being the intervals between the real roots of the functions $x_j(t) - 1$ if m is even) for any collection of $n_i \in \mathbb{Z}$ and $x_i(t) \in \mathbb{R}(t)$ satisfying (30). The proof, as already remarked, is similar to that of Proposition 1, but even simpler: we now have

$$L_m'(x) = \frac{(-1)^{m-1}}{2m(m-2)!} \log^{m-2}|x| \left(\frac{\log|1-x|}{x} + \frac{\log|x|}{1-x} \right),$$

so

$$\frac{d}{dt} \sum_i n_i L_m(x_i(t)) = \frac{(-1)^{m-1}}{m(m-2)!}$$
$$\times (\underbrace{\lambda \otimes \ldots \otimes \lambda}_{m-1} \otimes \lambda')(\sum_i n_i \underbrace{x_i \otimes \ldots \otimes x_i}_{m-2} \otimes (x_i \otimes (1-x_i) - (1-x_i) \otimes x_i))$$
$$= 0$$

where $\lambda : x \mapsto |x|$ and $\lambda' : x \mapsto x/x'$ as before. Note that this time the fundamental theorem of algebra, which is false over $\mathbb{R}$, was not needed since dL_m/dx, unlike $\partial\widetilde{D}_m/\partial x$, does not involve the $(m-1)$st polylogarithm. □

The analogue of Proposition 1 for the function D_m is:

PROPOSITION 2. *Let $\{n_i, x_i(t)\}$ be a collection of integers n_i and rational functions of one variable $x_i(t)$ satisfying the identity*

$$\sum_i n_i \, [x_i(t)]^{m-1} \otimes [\frac{x_i(t)}{(1-x_i(t))^2}] = 0 \qquad (32)$$

in $\left(Sym^{m-1}(\mathbb{C}(t)^\times)\bigotimes \mathbb{C}(t)^\times\right)\bigotimes_{Z}\mathbb{Q}$. *Then* $\sum\limits_i n_i\, D_m(x_i(t)) = constant.$

PROOF: Just as before, except that in the formula for $\partial\widetilde{D}_m(x)/\partial x$ the expression $\left(\frac{\log|1-x|}{x}+\frac{\log|x|}{1-x}\right)$ becomes $\log|x|\left(\frac{1}{x}+\frac{2}{1-x}\right)$, and $x'\left(\frac{1}{x}+\frac{2}{1-x}\right) = \lambda'\left(\frac{x}{(1-x)^2}\right)$. Alternatively, one can deduce this proposition directly from Proposition 1 since clearly $\sum n_i D_m(x_i)$ and $\sum n_i \widetilde{D}_m(x_i)$ agree for any $\sum n_i[x_i]$ satisfying (32). □

We end this section by discussing a different generalization of $\widetilde{D}_3$ which also satisfies "clean" functional equations, namely, the function

$$P_m(x) = \Re_m\left(\sum_{j=0}^{m}\frac{2^j\,B_j}{j!}\left(\log|x|\right)^j Li_{m-j}(x)\right), \tag{33}$$

where B_j is the jth Bernoulli number ($B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, ...). (The letter "P" here stands for **P**olylogarithm; the "D" of $D_m(x)$ goes back to the Bloch-Wigner **D**ilogarithm and is illogical for $m > 2$!) The motivation for the choice of coefficients, which at first sight looks a little arbitrary, is as follows. We want a function of the form $P_m(x) = D_m(x)$+(lower order terms) which, like D_m, is one-valued, real-analytic on $\mathbb{P}^1(\mathbb{C})\smallsetminus\{0,1,\infty\}$, and $(-1)^{m-1}$-symmetric with respect to $x\mapsto 1/x$ or $x\mapsto\overline{x}$, but which is continuous on all of $\mathbb{P}^1(\mathbb{C})$. To ensure the one-valuedness we make the Ansatz

$$P_m(x) = \sum_{j=0}^{m-1}\gamma_j\left(\log|x|\right)^j D_{m-j}(x) \tag{34}$$

with some universal coefficients $\gamma_0 = 1$, γ_1, ... yet to be determined. The requirement of $(-1)^{m-1}$-symmetry with respect to $x\mapsto 1/x$ or $x\mapsto\overline{x}$ is then equivalent to the requirement that $\gamma_j = 0$ for j odd. On the other hand, the definition of D_m can be written

$$D_m(x) = \Re_m\left(\sum_{j=0}^{m}\frac{(-1)^j}{j!}\left(\log|x|\right)^j Li_{m-j}(x)\right),$$

with the convention $Li_0(x)\equiv -\frac{1}{2}$, so we can rewrite (34) as

$$P_m(x) = \Re_m\left(\sum_{j=0}^{m}\beta_j\left(\log|x|\right)^j Li_{m-j}(x)\right)$$

with $\{\beta_j\}$ defined by the generating function identity

$$\sum_{j=0}^{\infty}\gamma_j\,x^j\cdot e^{-x} = \sum_{j=0}^{\infty}\beta_j\,x^j. \tag{35}$$

Now $\log^j|x|\,Li_{m-j}(x)$ is $\mathrm{o}(1)$ as $x \to 0$ for $j < m$ (since $Li_{m-j}(x) = \mathrm{O}(x)$), and the term $j = m$ gives 0 if m is even because then $\Re_m(Li_{m-j}) = \Re_m(-\frac{1}{2}) = 0$. Consequently $P_m(x)$ is automatically continuous (with value 0) at $x = 0$ when m is even, but is finite at $x = 0$ for m odd if and only if $\beta_m = 0$. We therefore demand that $\beta_m = 0$ for all odd $m > 1$. Replacing x by $-x$ in (35) and subtracting, we deduce (since $\gamma_j = 0$ for j odd)

$$\left(\sum_{j=0}^{\infty} \gamma_j\, x^j\right)\left(e^{-x} - e^{x}\right) = 2\,\beta_1\, x,$$

whence

$$\beta_1 = -1\,, \quad \sum_{j=0}^{\infty} \gamma_j\, x^j = \frac{x}{\sinh x}\,, \quad \sum_{j=0}^{\infty} \beta_j\, x^j = \frac{2x}{e^{2x}-1}\,, \quad \beta_j = \frac{2^j B_j}{j!}\,.$$

This explains the choice of coefficients in equation (33) and proves that the function it defines is continuous (with value 0) at $x = 0$ and hence, by the functional equation, at $x = \infty$; at $x = 1$, of course, it is continuous with value 0 for m even, $\zeta(m)$ for m odd. Furthermore, from the formula $\left(\sum \gamma_j x^j\right)^{-1} = \sum x^{2j}/(2j+1)!$ we deduce that the functions P_m and D_m are related by

$$D_m(x) = \sum_{0 \le 2j < m} \frac{\log^{2j}|x|}{(2j+1)!}\, P_{m-2j}(x); \tag{36}$$

thus $D_0(x) = P_0(x) = 0$, $D_1(x) = P_1(x) = \frac{1}{2}\log\left|\frac{x}{(1-x)^2}\right|$, $D_2(x) = P_2(x) = D(x)$, $D_3(x) = P_3(x) + \frac{1}{6}P_1(x)\log^2|x|$, etc.

PROPOSITION 3. *The statement of Proposition 1 remains valid if $\widetilde{D}_m$ is replaced by P_m.*

PROOF: Let $\mathcal{P}_m(x) = \sum \beta_r \log^r|x|\big(Li_{m-r}(x) - (-1)^m Li_{m-r}(\overline{x})\big)$, which differs from $P_m(x)$ only by a factor of ± 2 or $\pm 2i$. Using the identities

$$-\sum_{k=1}^{r} \beta_k\, \beta_{r-k} = r\beta_r + 2\,\beta_{r-1}, \quad -\sum_{k=1}^{r} (-1)^k \beta_k \beta_{r-k} = r\beta_r \qquad (r \ge 1),$$

which are easily proved from the generating function $\sum \beta_r x^r = 2x/(e^{2x}-1)$, we find after a short calculation that

$$\frac{\partial}{\partial x}\mathcal{P}_m(x) = \frac{\alpha_m}{x}\,\mathcal{P}_{m-1}(x) + \beta_m\,\log^{m-2}|x|\left(\frac{\log|1-x|}{x} + \frac{\log|x|}{1-x}\right),$$

and similarly for $\dfrac{\partial}{\partial \overline{x}}$. The rest of the proof is like that of Proposition 1. $\square$

Finally, we mention a different way to see the function $P_m(z)$ which was pointed out to me by Deligne and Beilinson, and which appears in the interpretation of the polylogarithm in terms of variations of mixed Hodge structures. Let $g(z)$ ($z \neq 0, 1$) denote the $n \times n$ triangular matrix ($n > m$) with entries $g_{rs}(z) = 0$ if $r < s$, $\dfrac{1}{(r-s)!}\left(\dfrac{\log z}{2\pi i}\right)^{r-s}$ if $0 < s \leq r$, and $\dfrac{1}{(2\pi i)^r} Li_r(z)$ if $s = 0$. This matrix is well-defined if we have defined the polylogarithms using a common path from 0 to z in $\mathbb{C} \smallsetminus \{0,1\}$, and the many-valuedness of the logarithms and polylogarithms is expressed by the fact that $g(z)$ is multiplied on the right by a triangular matrix with rational entries when z moves around a loop in $\mathbb{C}^\times \smallsetminus \{1\}$. Since such a matrix is in particular real, the matrix $h(z) = \overline{g(z)}\, g(z)^{-1}$ has entries which are single-valued functions of z. We write $h(z) = e^{iP(z)}$ where $P(z)$ is a real triangular matrix. It is easy to see that the entries of $P(z)$ all vanish except for the subdiagonal entries, which equal $\dfrac{1}{\pi} \log|z|$, and the entries in the first column, and a short calculation shows that the mth entry in the first column is $\dfrac{\pm 1}{(2\pi)^m} P_m(z)$.

§8. Formulation of the main conjecture. In this section we state conjectures for the higher order polylogarithms analogous to the conjectures for the trilogarithm presented in §§3–6. We shall give two versions, analogous to the different versions for the trilogarithm; the second one seems preferable, but neither one obviously implies the other.

For any field F let $\mathcal{F}_F$ denote the free abelian group on $F^\times$, or alternatively, the quotient of the free abelian group on $\mathbb{P}^1(F) = F \cup \{\infty\}$ by the relations $[0] = [\infty] = 0$, and let $\iota : \mathcal{F}_F \to F^\times \otimes \mathcal{F}_F$ be the map defined on generators by $[x] \mapsto [x] \otimes [x]$. We define subgroups $\mathcal{A}_m^{(1)}(F) \subset \mathcal{F}_F$ ($m \geq 1$) recursively by

$$\mathcal{A}_1^{(1)}(F) = \ker(\beta_1^{(1)}), \qquad \mathcal{A}_m^{(1)}(F) = \iota^{-1}\big(F^\times \otimes \mathcal{A}_{m-1}^{(1)}(F)\big) \quad (m \geq 2)$$

or in one step as $\ker(\beta_m^{(1)})$, where $F_{\mathbb{Q}}^\times = F^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\beta_m^{(1)}$ is the map $\mathcal{F}_F \to \mathrm{Sym}^{m-1}(F_{\mathbb{Q}}^\times) \otimes F_{\mathbb{Q}}^\times$ defined on generators by

$$\beta_m^{(1)}([x]) = [x]^{m-1} \otimes \Big[\frac{x}{(1-x)^2}\Big] \qquad (x \in \mathbb{P}^1(F) \smallsetminus \{0,1,\infty\})$$

and $\beta_m^{(1)}([x]) = 0$ for $x \in \{0,1,\infty\}$. Define $\mathcal{C}_m^{(1)}(F)$ to be the subgroup of $\mathcal{A}_m^{(1)}(F)$ generated by all images $(\phi_\alpha - \phi_\beta)\big(\mathcal{A}_m^{(1)}(\mathbb{Q}(t))\big)$ as α and β range over $\mathbb{P}^1(F)$, where $\phi_\alpha : \mathcal{F}_{\mathbb{Q}(t)} \to \mathcal{F}_F$ is the evaluation map defined on generators by $[x(t)] \mapsto [x(\alpha)]$. By virtue of Proposition 2 of the last section, $\mathcal{C}_m^{(1)}(F)$ is the group spanned by the functional equations (with arguments in $\mathbb{Q}(t)$) of the mth polylogarithm function D_m; this is like eq. (11) for $m = 2$, except that we no longer know the functional equations of D_m explicitly. In any case, it follows that if F is a subfield of $\mathbb{C}$, then the map $D_m : \mathcal{F}_F \to \mathbb{R}$ vanishes on $\mathcal{C}_m(F)$. Finally, and with apologies for the many superscripts, we define

$$\begin{aligned}\mathcal{A}_m^{(2)}(F) &= \iota^{-1}\big(F^\times \otimes \mathcal{C}_{m-1}^{(1)}(F)\big)\\ &= \ker\big(\mathcal{A}_m^{(1)}(F) \xrightarrow{\iota} F^\times \otimes \mathcal{A}_{m-1}^{(1)}(F) \xrightarrow{\mathrm{Id}\otimes\pi_{m-1}^{(1)}} F^\times \otimes \mathcal{F}_F/\mathcal{C}_{m-1}^{(1)}(F)\big),\end{aligned}$$

where $\pi_{m-1}^{(1)}$ is the obvious projection, and $\mathcal{B}_m^{(1)}(F)$, the (first version of the) mth *Bloch group*, as the quotient $\mathcal{A}_m^{(2)}(F)/\mathcal{C}_m^{(1)}(F)$.

Now suppose that F is an arbitrary field with $[F : \mathbb{Q}] = n_+ + n_-$ and $\Delta =$ absolute value of discriminant of F as usual. For each embedding σ of F into $\mathbb{C}$ we define D_m^σ as the composite of σ with the map $D_m : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ defined by (12), and define $D_m^F : \mathcal{F}_F \to \mathbb{R}^{n_\mp}$ (where $(-1)^m = \pm 1$) as the product of the D_m^σ over all the real and half the complex embeddings of F (one of each complex conjugate pair) if m is odd and over half the complex embeddings if m is even. By what we just said, the map D_m^F vanishes on $\mathcal{C}_m^{(1)}(F)$, so it defines a map from $\mathcal{B}_m^{(1)}(F)$ to $\mathbb{R}^{n_\mp}$.

MAIN CONJECTURE (FIRST VERSION). *D_m^F is an isomorphism from $\mathcal{B}_m^{(1)}(F)$ onto a lattice in $\mathbb{R}^{n_\mp}$ whose volume is a rational multiple of $\sqrt{\Delta}\,\zeta_F(m)/\pi^{mn_\pm}$.*

For the second version of the conjecture we modify the definitions as done in §5 for the trilogarithm. We define groups $\mathcal{A}_m^{(3)}(F) \subset \mathcal{F}_F$ recursively by

$$\mathcal{A}_2^{(3)}(F) = \ker(\beta_2), \qquad \mathcal{A}_m^{(3)}(F) = \iota^{-1}\big(F^\times \otimes \mathcal{A}_{m-1}^{(3)}(F)\big) \quad (m \ge 3)$$

or in one step as $\ker(\beta_m)$, where $\beta_m : \mathcal{F}_F \to \mathrm{Sym}^{m-2}(F_{\mathbb{Q}}^\times) \otimes \Lambda^2(F_{\mathbb{Q}}^\times)$ is the map defined on generators by

$$\beta_m([x]) = [x]^{m-2} \otimes \big([x] \wedge [1-x]\big) \qquad (x \in \mathbb{P}^1(F) \smallsetminus \{0,1,\infty\})$$

and $\beta_m([x]) = 0$ for $x \in \{0,1,\infty\}$. Define $\mathcal{C}_m(F)$ to be the subgroup of $\mathcal{A}_m^{(3)}(F)$ generated by all images $(\phi_\alpha - \phi_\beta)\big(\mathcal{A}_m^{(3)}(\mathbb{Q}(t))\big)$ as α and β range over $\mathbb{P}^1(F)$, and set

$$\begin{aligned}\mathcal{A}_m(F) &= \iota^{-1}\big(F^\times \otimes \mathcal{C}_{m-1}(F)\big)\\ &= \ker\big(\mathcal{A}_m^{(3)}(F) \xrightarrow{\iota} F^\times \otimes \mathcal{A}_{m-1}^{(3)}(F) \xrightarrow{\mathrm{Id}\otimes\pi_{m-1}} F^\times \otimes \mathcal{F}_F/\mathcal{C}_{m-1}(F)\big),\end{aligned}$$

and $\mathcal{B}_m(F)$, the (better) mth *Bloch group*, as $\mathcal{A}_m(F)/\mathcal{C}_m(F)$.

MAIN CONJECTURE (SECOND VERSION). *$\widetilde{D}_m^F$ is an isomorphism from $\mathcal{B}_m(F)$ onto a lattice in $\mathbb{R}^{n_\mp}$ whose volume is a rational multiple of $\sqrt{\Delta}\,\zeta_F(m)/\pi^{mn_\pm}$.*

The various groups and maps we have defined are related by the commutative diagram (25) with "3" replaced everywhere by "m." In particular, the lattices given in the two conjectures must coincide if both conjectures are true, but the one in the second conjecture *a priori* could be larger. Note also that we could replace the function $\widetilde{D}_m$ by the function P_m in the second conjecture, since D_m and $\widetilde{D}_m$ clearly agree on $\mathcal{A}_m^{(1)}$, while P_m and $\widetilde{D}_m$ agree on $\mathcal{A}_m$ by virtue of (36) and the definition of $\mathcal{A}_m$.

Of course, as explained in §§1–2, what we really want is that the lattice in $\mathbb{R}^{n_\mp}$ arising in the two above versions of the conjecture coincides (up to torsion) with the lattice given by the Borel regulator mapping $K_{2m-1}(F) \to \mathbb{R}^{n_\mp}$, in which case the statement about the covolume is a consequence of Borel's theorem. Thus our final formulation is

MAIN CONJECTURE. *There is a canonical map $\mathcal{B}_m(F) \to K_{2m-1}(F)$, with finite kernel and cokernel, whose composite with the Borel regulator mapping coincides with P_m^F.*

Before turning to examples, we make two remarks about aspects of the above formulations which might have bothered the reader. First of all, it might be objected that the definition of the kernel $\mathcal{C}_m(F)$ is not really constructive, since we have no criterion to determine which elements of $\mathcal{F}_F$ can be obtained as linear combinations of specializations of functional equations of the polylogarithm: for $m \geq 6$ no functional equations at all beyond the trivial inversion and duplication relations are known, and even, say, for $m = 2$, where it is known that the 5-term relation (10) gives everything, it is not clear how to check whether a given element $\sum n_i[x_i] \in \mathcal{F}_F$ is a linear combination of elements of this form. This objection is especially serious since, even if we decided not to worry about the kernel of our hypothetical surjection $\mathcal{A}_m(F) \to K_{2m-1}(F)$, as was done in §§3–5 for the case $m = 3$, we would still need to know $\mathcal{C}_m(F)$ inductively in order to define the next group $\mathcal{A}_{m+1}(F)$. However, in practise we can decide whether a given element is in $\mathcal{C}_m(F)$ by first checking whether it is in $\mathcal{A}_m(F)$ (assuming the group $\mathcal{C}_{m-1}(F)$ known by induction) and then computing its image under the polylogarithm map. According to the conjecture, the answer must lie in a latttice in Euclidean space, and by computing many examples we can both see numerically that this holds—thus confirming the conjecture—and also recognize which elements are zero, since the vanishing of an element in a discrete subgroup of Euclidean space can be ascertained by a finite precision computation. In other words, the numerical verification proceeds by a bootstrap procedure from one level to the next, with the consistency of the entire numerical procedure providing the evidence for the correctness of the conjecture.

The other remark concerns the definition of the kernel using 1-variable functional equations only, i.e., by specializing elements of $\mathcal{A}_m(\mathbb{Q}(t))$ to values $t \in F$ rather than specializing elements of $\mathcal{A}_m(\mathbb{Q}(t_1, \dots, t_N))$ to values $t_1, \dots, t_N \in F$. This may seem strange in view of the fact that the basic functional equations used for the di- and trilogarithms, namely the 5-term relation (10) and the Kummer-Spence relation (26), involve two variables, and it is clear that no single one-variable equation can ever suffice to produce the whole kernel. However, there is no loss of generality involved, since the specialization of a functional equation $\sum n_i P_m(x_i(t_1, \dots, t_N)) = c$ to values $t_1 = \alpha_1 \in F$, ..., $t_N = \alpha_N \in F$ is the same as the specialization of the one-variable functional equations $\sum n_i P_m(x_i(f_1(t), \dots, f_N(t))) = c$ to $t = \alpha_0$, where $F = \mathbb{Q}(\alpha_0)$ and $f_1, \dots, f_N \in \mathbb{Q}(t)$ are chosen such that $\alpha_i = f_i(\alpha_0)$; indeed, this shows that in the definition of $\mathcal{C}_m(F)$ we could have fixed the choice of α and β as α_0 and ∞. We have chosen the one-variable formulation because (i) it is simpler, (ii) it has an algebraic-topological flavor (compare the definitions of homology and homotopy groups in terms of bounding cycles of one dimension higher or of deformations of maps) and thus should lend itself to the comparison with K-theory, and (iii) it sidesteps the question, which we cannot handle anyway, of the existence of a universal functional equation like (10) for the higher polylogarithms, since as soon as we restrict to one-variable equations we are forced to look at infinitely many equations anyway. Of course, one can still speculate that for each m there may be a a single many-variable equation, or even a single two-variable equation like S_{xy} or $S_{xy}^{(3)}$, whose specializations to F generate the group $\mathcal{C}_m(F)$.

We end this section by giving a slightly different formulation of the algebraic structures we have described. For each m we have a filtration

$$\mathcal{F}_F = \mathcal{A}_{m,0}(F) \supset \mathcal{A}_{m,1}(F) \supset \dots \supset \mathcal{A}_{m,m-1}(F) \supset \mathcal{A}_{m,m}(F)$$

with $\mathcal{A}_{m,1}(F) = \mathcal{A}_m^{(3)}(F)$, $\mathcal{A}_{m,m-1}(F) = \mathcal{A}_m(F)$, and $\mathcal{A}_{m,m}(F) = \mathcal{C}_m(F)$ in our previous notation. The groups $\mathcal{A}_{m,i}$ for $i < m$ are defined inductively as $\iota^{-1}(F^\times \otimes \mathcal{A}_{m-1,i}(F))$, while $\mathcal{A}_{m,m}(F)$ is defined as before as the image of $\mathcal{A}_{m,1}(\mathbb{Q}(t))$ under specialization. By induction, one sees that $\mathcal{A}_{m,i}$ for $i > 1$ is the kernel of the (conjecturally surjective) map $\mathcal{A}_{m,i-1}(F) \to \mathrm{Sym}^{m-i}(F^\times) \otimes \mathcal{B}_i(F)$ induced from the map $\iota^{(m-i)} : \mathcal{F} \to \mathrm{Sym}^{m-i}(F) \otimes \mathcal{F}_F$ sending $[x]$ to $[x]^{m-i} \otimes [x]$. The definition of $\mathcal{A}_{m,1}(F) = \ker(\beta_m)$ can be written in a similar form by noting that there is an exact sequence

$$0 \to \mathrm{Sym}^m(F^\times) \xrightarrow{\kappa_0} \mathrm{Sym}^{m-1}(F^\times) \otimes F^\times \xrightarrow{\kappa_1} \mathrm{Sym}^{m-2}(F^\times) \otimes \Lambda^2(F^\times)$$
$$\xrightarrow{\kappa_2} \mathrm{Sym}^{m-3}(F^\times) \otimes \Lambda^3(F^\times) \to \dots, \qquad (37)$$

where κ_i sends $[x_1]\dots[x_{m-i}]\otimes([y_1]\wedge\dots\wedge[y_i])$ to $\sum_{j=1}^{m-i}[x_1]\dots\widehat{[x_j]}\dots[x_{m-i}]\otimes([x_j]\wedge[y_1]\wedge\dots\wedge[y_i])$, and that the image of β_m is in the kernel of κ_2. Thus we can think of β_m instead as a map to $\mathrm{Sym}^{m-1}(F^\times)\otimes F^\times/\mathrm{Sym}^m(F^\times)$. In other words, $\beta_m(\sum n_i[x_i])=0$ iff the element $\sum n_i[x_i]^{m-1}\otimes[1-x_i]$, which is already invariant under permutations of the first $m-1$ variables, is also invariant under permutation of the last two. Then we can summarize by saying that we have injective, and conjecturally surjective, maps

$$\mathcal{A}_{m,i-1}(F)/\mathcal{A}_{m,i}(F)\to\begin{cases}(\mathrm{Sym}^{m-1}(F^\times)\otimes F^\times)/\mathrm{Sym}^m(F^\times) & (i=1)\\ \mathrm{Sym}^{m-i}(F)\otimes\mathcal{B}_i(F^\times) & (1<i\le m)\end{cases}$$

induced by the map $\iota^{(m-i)}:\mathcal{F}_F\to\mathrm{Sym}^{m-i}(F^\times)\otimes\mathcal{F}_F$.

§**9. Examples.** We have given many examples supporting the conjecture for $m=3$ in previous sections. We now give a few examples for larger m.

A. Penta- and heptalogarithms of rational numbers. For our first examples, we look at the field $X=\mathbb{Q}$ and the 11-element set $X=X_{\{2,3\}}$ of §3. We look for non-trivial combinations ξ of elements of x belonging to the group $\mathcal{B}_5(\mathbb{Q})$ and consequently giving a rational multiple of $\zeta(5)$ under P_5; here for the first time we will see the necessity of the analogue of (19) for higher Bloch groups even when F is not totally real. For $x\in X$ we have $x,\ 1-x\in\langle\pm1,2,3\rangle$ and consequently $\beta_5(x)=Q_x([2],[3])\otimes([2]\wedge[3])$ where $Q_x(a,b)$ is a homogeneous cubic polynomial in two variables. For instance, for $x=1/4$ we have

$$[\tfrac{1}{4}]^3\otimes([\tfrac{1}{4}]\wedge[\tfrac{3}{4}])=(-2\,[2])^3\otimes((-2[2])\wedge([3]-2[2]))=16\,[2]^3\otimes([2]\wedge[3])$$

and consequently $Q_{1/4}(a,b)=16a^3$. Computing the values of $Q_x(a,b)$ for each $x\in X$ and forming linear combinations to eliminate these polynomials, we find that the kernel of β_5 on $\mathcal{F}_X$ is 7-dimensional, with generators $[-1]$, $[\frac12]$, $[-\frac13]-2[\frac13]$, $[\frac14]-16[-\frac12]$, $[-\frac18]-162[-\frac12]$, $[\frac19]-48[\frac13]$, and $[\frac89]-9[\frac34]-36[\frac23]-18[-\frac12]-6[\frac13]$. If $\xi=\sum n_i[x_i]$ is one of these elements, then each of the sums $\sum n_i v_2(x_i)^\mu v_3(x_i)^\nu[x_i]$ $(\mu+\nu=2)$ belongs to $\mathcal{A}_3(\mathbb{Q})$ and therefore gives a rational multiple of $\zeta(3)$ under P_3. We must then form linear combinations for which this triple of rational numbers vanishes. This gives 3 linear conditions on our 7 elements and finally a 4-dimensional solution space, with basis $[-1]$, $[\frac14]-16[\frac12]-16[-\frac12]$, $[\frac19]-16[\frac13]-16[-\frac13]$, and $[-\frac18]-126[\frac12]-162[-\frac12]$. The first three elements are uninteresting, since they correspond to the inversion and duplication relations of the trilogarithm, but the last gives the non-trivial relation

$$P_5(-\frac{1}{8})-126\,P_5(\frac{1}{2})-162\,P_5(-\frac{1}{2})=\frac{403}{16}\,\zeta(5).$$

By looking at the larger set $X_{\{2,3,5\}}$ considered in §5, we get many more non-trivial relations of this sort, a typical one being

$$P_5(\tfrac{-9}{16}) - P_5(\tfrac{-1}{24}) + 30P_5(\tfrac{-1}{4}) - 30P_5(\tfrac{-2}{3}) - 3P_5(\tfrac{-1}{3}) - P_5(\tfrac{-1}{8}) + 22P_5(\tfrac{3}{4})$$
$$- P_5(\tfrac{8}{9}) - 24P_5(\tfrac{2}{3}) - 10P_5(\tfrac{3}{8}) + 10P_5(\tfrac{1}{6}) - 18P_5(\tfrac{1}{3}) + 76P_5(\tfrac{1}{2}) = \tfrac{935}{16}\,\zeta(5).$$

We did not have to put question marks above the equality signs in the last two identities because their correctness can be verified analytically using Kummer's functional equation. Christophe Soulé suggested that it might be worth finding at least one example for higher-order logarithms, where no functional equations beyond the inversion and duplication relations are known, just to be sure that the theory does not break down there. To find such an example for the heptalogarithm (since $\mathcal{B}_m(\mathbb{Q})$ is supposed to be 0 for m even), take X to be the set of $x \in \mathbb{Q}$ such that x has only the prime factors 2 and 3 in its numerator and denominator and $1-x$ only the factors 2, 3, 5 and 7, and (to avoid the inversion and duplication relations) also with $|x| < 1$, $x \neq$ square. There are 29 such elements, and 28 relations to be satisfied: 20 to make sure we are in $\ker(\beta_7)$, another 5 to make sure that the images under $[x] \mapsto v_2(x)^\mu v_3(x)^\nu [x]$ $(\mu+\nu=4)$ map to 0 in the 1-dimensional group $\mathcal{B}_3(F)$, and 3 more to make the images under $[x] \mapsto v_2(x)^\mu v_3(x)^\nu [x]$ $(\mu+\nu=2)$ vanish in $\mathcal{B}_5(\mathbb{Q})$. There must therefore be at least one non-trivial solution, and indeed we find exactly one, the value under P_7 being (numerically) a rational multiple of $\zeta(7)$ as expected:

$$25111753072P_7\left(\tfrac{1}{3}\right) + 27461584367P_7\left(\tfrac{-1}{3}\right) - 43524P_7\left(\tfrac{-1}{4374}\right)$$
$$-\frac{470985412}{3}P_7\left(\tfrac{-1}{8}\right) - \frac{17015061}{2}P_7\left(\tfrac{-1}{48}\right) + \cdots \stackrel{?}{=} \frac{1020149599795}{96}\,\zeta(7),$$

the omitted 24 coefficients being integers of between 6 and 12 digits. (This example was found in collaboration with H. Gangl.)

B. Cyclotomic fields. To get elements in the higher Bloch groups, we have to satisfy algebraic relations like $\sum n_i [x_i]^{m-2} \otimes ([x_i] \wedge [1-x_i]) = 0$ together with supplementary conditions involving the images $\iota_v([x_i]) = v(x_i)[x_i]$ for all homomorphisms $v : F^\times \to \mathbb{Z}$. If all of the x_i are roots of unity, then these relations are automatically satisfied, since both $[x_i] \in F^\times_{\mathbb{Q}}$ and $v(x_i)$ vanish. Thus any combination of roots of unity defines an element of the Bloch group of any order of the field they generate.

Let $F = \mathbb{Q}(\zeta)$ be the cyclotomic field generated by a primitive Nth root of unity ζ $(N > 2)$. For any $M \neq 0$ and $m \geq 1$ we have the "distribution relations"

$$P_m(y) = M^{m-1} \sum_{x^M = y} P_m(x), \qquad (y \in \mathbb{C}) \tag{38}$$

(the special cases $M = -1$ and $M = 2$ are the inversion and duplication relations). Using them for the divisors M of N, we see that the values of $P_m(\zeta^j)$ for $(j, N) > 1$ can be expressed in terms of those with $(j, N) = 1$. so that we need only look at primitive Nth roots of unity. Also, $P_m(\zeta^j) = (-1)^{m-1} P_m(\zeta^{N-j})$, so we can restrict to j in the set $A = \{0 < j < N/2 \mid (j, N) = 1\}$. We have $|A| = \frac{1}{2}\phi(N) = n_+ = n_-$, the expected rank of the Bloch group $\mathcal{B}_m(F)$, so it is reasonable to expect that the elements $[\zeta^j]_{j\in A}$ form a basis. To see that they are linearly independent, we compute their images under the polylogarithm mapping P_m^F. The components of this map are the compositions of P_m with half the embeddings $\sigma : F \to \mathbb{C}$. We take these embeddings to be those given by $\zeta \mapsto \mathbf{e}_N(k) = e^{2\pi i k/N}$ as k runs over A. For each Dirichlet character $\chi \pmod N$ with $\chi(-1) = (-1)^{m-1}$ we let ξ_χ denote the element $\sum_{j\in A} \chi(j)[\zeta^j]$ of $\mathcal{F}_F \otimes \mathbb{C}$. Its image under the mapping P_m^F is the vector

$$P_m^F(\xi_\chi) = \Big(\sum_{j\in A} \chi(j)\, P_m(\mathbf{e}_N(jk))\Big)_{k\in A} = \lambda_\chi \left(\overline{\chi}(k)\right)_{k\in A}$$

where $\lambda_\chi = \sum_{j\in A} P_m(\mathbf{e}_N(j))$. The vectors $\left(\overline{\chi}(k)\right)_{k\in A}$ as χ varies are linearly independent (characters of a finite abelian group are linearly indendent). Therefore the vectors $P_m^F(\zeta^j)$ $(j \in A)$ are linearly independent if and only if all λ_χ are non-zero, and then span a lattice of covolume $\prod_\chi \lambda_\chi$. which we now calculate. Suppose that χ is induced from a primitive character χ_0 modulo a divisor $f = f_\chi$ of N. Then we have the following more or less standard calculation:

$$\begin{aligned}
\lambda_\chi &= \frac{1}{2} \sum_{\substack{1\le j\le N\\ (j,N)=1}} \chi_0(j)\, P_m\big(\mathbf{e}_N(j)\big) \\
&= \frac{1}{2} \sum_{1\le j\le N} \Big(\sum_{d|(j,N)} \mu(d)\Big)\, \chi_0(j)\, P_m\big(\mathbf{e}_N(j)\big) \qquad (\mu(d) = \text{Möbius function}) \\
&= \frac{1}{2} \sum_{d|N} \mu(d) \sum_{1\le r\le N/d} \chi_0(dr)\, P_m\big(\mathbf{e}_{N/d}(r)\big) \\
&= \frac{1}{2} \sum_{\substack{d|N\\ (d,f)=1}} \mu(d)\, \chi_0(d) \sum_{1\le t\le f} \chi_0(t) \sum_{\substack{1\le r\le N/d\\ r\equiv t \pmod f}} P_m\big(\mathbf{e}_{N/d}(r)\big) \\
&= \frac{1}{2} \sum_{d|N/f} \mu(d)\, \chi_0(d) \sum_{1\le t\le f} \chi_0(t) \left(\frac{df}{N}\right)^{m-1} P_m\big(\mathbf{e}_f(t)\big) \qquad \text{(by eq. (38))} \\
&= \frac{1}{2\alpha} \left(\frac{f}{N}\right)^{m-1} \Big(\sum_{d|N/f} \mu(d)\, \chi_0(d)\, d^{m-1}\Big) \sum_{1\le t\le f} \chi_0(t) \sum_{n=1}^{\infty} \frac{\mathbf{e}_f(nt)}{n^m}
\end{aligned}$$

$$= \frac{1}{2\alpha}\left(\frac{f}{N}\right)^{m-1}\left(\sum_{d|N/f}\mu(d)\,\chi_0(d)\,d^{m-1}\right)\cdot\tau(\chi_0)\cdot L(m,\overline{\chi}_0),$$

where $\alpha = 1$ or i depending whether m is odd or even and $\tau(\chi_0) = \sum \chi_0(t)\mathbf{e}_f(t)$ is the Gauss sum associated to χ_0. Each of the factors in the last line is non-zero (the number $\tau(\chi_0)/\alpha$ equals $W_\chi\sqrt{f}$, with W_χ, the root number in the functional equation of $L(s,\chi)$, of absolute value 1), so λ_χ is non-zero for each χ and the vectors we have constructed in $\mathrm{Im}(\beta_m)$ form a basis of $\mathbb{R}^{n_\mp}$. The covolume of the lattice they generate, $\prod \lambda_\chi$, is rationally proportional to $\sqrt{\Delta}\,\zeta_F(m)/\pi^{n_\pm}$ by the formula we have just proven together with the facts that $\zeta_F(s)$ is the product of the L-series $L(s,\chi)$ over *all* Dirichlet characters χ (mod N), that $L(m,\chi)$ for χ with $\chi(-1) = (-1)^m$ is a rational multiple of $W\chi\sqrt{f_\chi}\,\pi^m L(1-m,\chi)$, that $\prod_\chi W_\chi = 1$ and $\prod_\chi f_\chi = \Delta$, and that the numbers $L(1-m,\chi)$ for the characters with $\chi(-1) = (-1)^m$ are algebraic numbers which are permuted among one another by the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We therefore have:

PROPOSITION. *Let F be a cyclotomic field. Then P_m^F maps the subgroup of $\mathcal{B}_m(F)$ generated by the roots of unity in F onto a lattice in $\mathbb{R}^{n_\mp}$ whose volume is a rational multiple of $\sqrt{\Delta}\,\zeta_F(m)/\pi^{mn_\pm}$.* □

If $m = 2$, then it follows from the results quoted in §2 that the the roots of unity generate all of $\mathcal{B}_m(F)$. Beilinson has proved in general that the images of the roots of unity under the polylogarithm map lie in (and hence span) the image of the Borel regulator mapping (cf. §11).

C. "Ladders". It turns out that there are already in the existing literature many numerical examples supporting the conjectures of this paper. These are the so-called "ladder relations" of Lewin and his coworkers (see [1] and the references given there). Suppose given a number u, necessarily algebraic, which satisfies a relation of the form

$$\Lambda: \qquad \prod_{r\geq 1}(1-u^r)^{a_r} = \pm u^N \tag{39}$$

for some integers a_r (Lewin always supposes that the r occurring are divisors of a fixed r_0 with $a_{r_0} = 1$, but this is not needed and spoils the property that the set of relations of the form (39) form a group). Then for each $m \geq 2$ the linear combination $\xi_m(\Lambda) = \sum_r \frac{a_r}{r^{m-1}}[u^r]$ is in the kernel of β_m (even of $\beta_m^{(1)}$). We thus get in a systematic way elements of $\mathcal{A}_m^{(3)}(F)$ (or even $\mathcal{A}_m^{(1)}(F)$) for every m for the field $F = \mathbb{Q}(u)$. The extra conditions defining the subgroup $\mathcal{A}_m(F)$ are also particularly easy to fulfill for the special elements $\xi_m(\Lambda)$, since the subgroup of $F^\times$ generated by their support is 1-dimensional: for each value $m' = 2, \ldots, m-1$ in succession we require that $\xi_{m'}(\Lambda)$ define the 0 element in $\mathcal{B}_{m'}(F)$, in which case $\xi_m(\Lambda)$ will define an element of the next Bloch group $\mathcal{B}_m(F)$. The number of linearly independent conditions this imposes at each stage is alternately n_- and n_+, so if these numbers are small and the number u happens to satisfy a large number of relations (39), we will be able to climb up several steps of the "ladder." Of the many examples given in the papers referred to, we describe only one family which is particularly spectacular. This is for the field $\mathbb{Q}(\theta)$ $(\theta^3 - \theta - 1 = 0)$ already used by us as an example in §4 and §5. We had implicitly used the ladder concept by observing that the numbers $x = \theta$, $-\theta$, θ^3, $-\theta^4$ and θ^5 all have the property that $1 - x$ belongs to the group generated by -1 and θ:

$$1-\theta = -\theta^{-4}, \quad 1+\theta = \theta^3, \quad 1-\theta^3 = -\theta, \quad 1+\theta^4 = \theta^5 \quad 1-\theta^5 = -\theta^4.$$

(one can use $1+x = (1-x^2)/(1-x)$ to put these in the form (39).) In [2], Lewin et al. study the same field and give the seven further ladder relations

$$\frac{1+\theta^6}{1+\theta^2} = \theta^3, \quad \frac{1+\theta^7}{1-\theta^7} = -\theta, \quad \frac{1+\theta^9}{1+\theta^3} = \theta^5, \quad \frac{1+\theta^{10}}{(1+\theta^2)^2} = \theta^3,$$
$$\frac{1+\theta^{14}}{(1+\theta^2)(1-\theta^7)} = -\theta^4, \quad \frac{1+\theta^{15}}{(1+\theta^3)(1+\theta^5)} = \theta^5, \quad \frac{1+\theta^{21}}{(1+\theta^3)^2(1+\theta^7)} = \theta^5.$$

This gives altogether 12 elements $[\theta]$, $[-\theta], \ldots, [-\theta^{21}] - 3[-\theta^7] - 14[-\theta^3]$ of $\mathcal{A}(F)$. Each gives a rational multiple of $23^{3/2}L(2)/\pi^4$ under $P_2(\cdot) = D(\cdot)$ (cf. §4), so we get 11 linear combinations $\sum a_r[u^r]/r$ which give 0 in $\mathcal{B}_2(F)$ and hence lift to elements $\xi = \sum a_r[u^r]/r^2$ of $\mathcal{A}_3(F)$. For each of these the elements $\delta^+(\xi)$ and $\delta_-(\xi)$ defined by (20) are rational, and these must vanish in order to be able to lift again to $\sum a_r[u^r]/r^3 \in \mathcal{A}_4(F)$. Continuing in this way, we find that the number of "viable ladders" drops alternately by $n_- = 1$ or $n_+ = 2$ at each stage, the relevant linear combinations being obtained by computing the images of ξ under $P_m(\cdot)/\zeta(m)$ for m even and the map (20) (with 3 replaced by m) for m odd. This is exactly the structure which was found empirically by the authors of [2], except that they used only the real embedding of F and thus had to find one of the linear relations at each odd stage by searching numerically for a linear combination which gave a rational multiple of $\zeta(m)$ under P_m. (At the even stages they used Lewin's function L_m as defined in §7 rather than the function P_m, which vanishes on $\mathbb{R}$, and always found a rational multiple of $\zeta(m)$ here, too; this is not covered by our conjecture, which only says that $P_m(\xi^{(2)})$ should be a rational multiple of $\sqrt{23}\,L(m)/\pi^{2m}$ in this case.) At the last stage of the computation, for instance, one has two elements

$$\xi_1 = 6230[\theta^{42}] + 15778[\theta^{39}] + 164700[\theta^{28}] + \cdots - 382768243200[\theta]$$
$$\xi_2 = -20440[\theta^{42}] + 122794[\theta^{39}] + 29700[\theta^{28}] + \cdots + 816983193600[\theta]$$

in $\mathcal{A}_9(F)$ (their images under $P_9 = L_9$ equal $\frac{21}{1700}$ times the elements denoted A_9 and B_9, respectively, in [2], where the interested reader can find the remaining coefficients). Computing the values of $\delta^{\pm}(\xi_j)$ as defined by the analogue of (20) with 9 instead of 3, we find that the polylogarithm mapping $P_9^F : \xi \mapsto (P_9(\xi^{(1)}), P_9(\xi^{(2)})) \in \mathbb{R}^2$ maps ξ_1 and ξ_2 to $9850530391\mathbf{e}_1 - 3307\mathbf{e}_2$ and $-21953831855\mathbf{e}_1 + 7055\mathbf{e}_2$, respectively, where

$$\mathbf{e}_1 = 4\,\zeta(9)\,(1,1) \qquad \mathbf{e}_2 = \frac{21}{4}\,23^{19/2}\,\frac{L(9)}{\pi^9}\,(2,-1),$$

in agreement with the result found in [2] that $7055P_9(\xi_1) + 3307P_9(\xi_2)$ is a rational multiple of $\zeta(9)$. Observe that the covolume of the lattice in $\mathbb{R}^2$ spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ is a rational multiple of $\sqrt{23}\,\zeta_F(9)/\pi^9$, in accordance with the general theory.

§**10. Complements.** In this section we discuss some further aspects of the main conjecture.

A. Satisfying the conditions defining $\mathcal{A}_m(F)$. The group $\mathcal{A}_m(F)$ is defined by a number of multilinear conditions on x and $1-x \in F^\times$. To satisfy them, we need a large number of x with x and $1-x$ belonging to a relatively small subspace of $F^\times$. It is not clear a priori if such elements can be found when m is large, and indeed Deligne asked me in a letter of 1988 whether this could always be done. In this subsection we show that the answer is affirmative and discuss some of the numerical aspects involved.

We first count the number of conditions needed. Let S be a subset of $F^\times$ of cardinality s (we may assume without loss of generality that the elements of S are linearly independent) and consider the set

$$X = X(S) = \{x \in \mathbb{Q} \mid x,\, 1-x \in V\},$$

where $V = \langle S\rangle$ is the subspace of $F^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by S (earlier we used the notation X_S for a similarly defined set.) It generates a subgroup $\mathcal{F}_X$ of $\mathcal{F}_F$ of rank equal to $|X|$, the cardinality of X. We want to estimate the number of linear conditions on an element $\xi \in \mathcal{F}_X$ imposed by the requirement that it belong to $\mathcal{A}_m(F)$; if this number is (much) less than $|X|$, then we get (many) elements of $\mathcal{A}_m(F) \cap \mathcal{F}_X$.

The basic condition that an element $\xi \in \mathcal{F}_X$ must satisfy is that it belongs to the kernel of $\beta_m : \mathcal{F}_X \to \mathrm{Sym}^{m-2}(V) \otimes \Lambda^2(V)$. The number of conditions this imposes is $\dim(\mathrm{Sym}^{m-2}(V)) \dim(\Lambda^2(V)) = \binom{s+m-3}{m-2}\binom{s}{2}$. This can be improved slightly by noting that the image of β_m in fact belongs to the subspace $(\mathrm{Sym}^{m-1}(V) \otimes V)/\mathrm{Sym}^m(V)$ of $\mathrm{Sym}^{m-2}(V) \otimes \Lambda^2(V)$ (cf. the exact sequence (37)), so the actual number of conditions is $\dim(\mathrm{Sym}^{m-1}(V) \otimes V) - \dim(\mathrm{Sym}^m(V)) = (m-1)\binom{m+s-2}{m}$. Once our element ξ belongs to $\ker \beta_m = \mathcal{A}_m^{(3)}$, we must ensure that it lies in the subgroup $\mathcal{A}_m(F)$. This will be the case if it lies successively in the kernel of a sequence of maps to $\mathrm{Sym}^i(V) \otimes \mathcal{B}_{m-i}(F)$ for $i = 2, 3, \ldots$ (cf. the filtration of $\mathcal{A}_m(F)$ given at the end of §8). Since $\dim(\mathrm{Sym}^i(V)) = \binom{i+s-1}{i}$ and $\dim(\mathcal{B}_{m-i}(F)) = n_{(-1)^{m-i-1}}$, this gives finally

$$\dim(\mathcal{F}_X \cap \mathcal{A}_m(F)) \geq |X| - (m-1)\binom{s+m-2}{m} - \sum_{1 \leq i \leq m-2} \binom{i+s-1}{i} n_{(-1)^{m-i-1}}. \tag{40}$$

In any case, the number of conditions which an element of $\mathcal{F}_{X(S)}$ must satisfy in order to belong to $\mathcal{A}_m(F)$ grows polynomially with s. To be sure of obtaining elements of $\mathcal{A}_m(F)$ for each m, therefore, we need to know that we can find sets S of arbitrarily large cardinality s such that $|X(S)|$ grows more than polynomially in s. That this can be done is the content of a beautiful theorem of Erdös-Stewart-Tijdeman [7]. Since the result is not very well known and the proof a little hard to extract from [7], where it is presented as a series of lemmas which are specializations of much more general and complicated propositions stated elsewhere in the paper, we digress from our main theme to state the complete statement and proof.

THEOREM (ERDÖS-STEWART-TIJDEMAN). *For every sufficiently large integer s there is a set S of cardinality s for which the set $X(S)$ has $\geq e^{(4-o(1))\sqrt{s/\log s}}$ elements with an effective $o(1)$ constant (i.e., for each number $c < 4$ the expression $4 - o(1)$ can be replaced by c for s greater than an effectively computable number $s_0(c)$). The set S can be taken to consist only of primes if so desired.*

PROOF: Let $y \gg 0$ and $u \geq 3$ be real numbers and W the set of integers $\leq y^u$ all of whose prime factors are $\leq y$. For the cardinality of W, usually denoted $\psi(y^u, y)$, we have the well-known Canfield-Erdös-Pomerance estimate [5]

$$|W| \geq \left(y \, \frac{e + o(1)}{u \log u}\right)^u,$$

where the dependence of o(1) on u as $u \to \infty$ is effective. It follows that by choosing $u = (2 + o(1))\sqrt{y}/\log y$ we can force $\binom{|W|}{2}/y^u \geq M = e^{(4+o(1))\sqrt{y}/\log y}$. But for each of the $\binom{|W|}{2}$ pairs $(a,c) \in W^2$ with $0 < a < c$, the difference $c-a$ is an integer $< y^u$, so by the pigeonhole principle there must be an integer $b < y^u$ which is representable as $c - a$ in $\geq M$ ways. Set $S = \{p \text{ prime}, p \leq y\} \cup \{b\}$; then the numbers b/c belong to $X(S)$ and there are at least M of them. The cardinality of S is given by $s = \pi(y) + 1 \leq (1 + o(1))\, y/\log y$, so $M \geq e^{(4-o(1))\sqrt{s/\log s}}$. To get the last statement, we replace the choice $S = \{\text{ primes } \leq y\} \cup \{b\}$ by $S = \{\text{ primes } \leq y\} \cup \{\text{prime factors } > y \text{ of } b\}$, noting that the number of prime factors of b greater than y is bounded by $\dfrac{\log b}{\log y} \leq u = o(\dfrac{y}{\log y}) = o(s)$, so that the estimate given is not affected. □

Notice that in the statement and proof we do not necessarily take S to be the set consisting of the first s prime numbers. If we do take this set, however, then the cardinality of $X(S)$ for $s \leq 8$ is given by the following table, due to P. Vojta and R. Gross:

s	1	2	3	4	5	6	7	8
$\lvert X(S)\rvert$	3	21	99	375	1137	3267	8595	21891

(actually, these are lower bounds; there may be more solutions). We also mention that the maximum size M_s of $X(S)$ for any S of cardinality s is $< 1000 \cdot 50^s$ (J.-H. Evertse), so that the limit $\lim_{s\to\infty} \dfrac{\log\log M_s}{\log s}$, if it exists, lies between 1/2 and 1; in [7] it is conjectured that the correct value is in fact 2/3. However, for our purposes only the lower bound on M_s is important.

Finally, we should remark that in practise it would be unwise to look for $\xi \in \mathcal{F}_F$ supported on the whole of $X(S)$, since $|X(S)|$, despite the theorem, grows far too slowly with $|S|$ for this to work well when m is at all large. It is far better to restrict the prime factors of x to a much smaller set than those of $1-x$, since $[x]$ occurs multilinearly and $[1-x]$ only linearly in the defining equations of $\mathcal{A}_m(F)$. Thus in §9A, by looking at the heptalogarithms for non-trivial x with $|x| \in \langle 2,3\rangle$ and $|1-x| \in \langle 2,3,5,7\rangle$ we obtained a system of 28 equations in 29 unknowns, well within the range of the computer. Had we tried instead to use the whole set $X(S)$ for some S, then we would have needed $|X_S| = \frac{1}{2}(|X(S)|+1)$ to be larger than $6\binom{s+5}{7} + \binom{s+1}{2} + \binom{s+3}{4}$ (compare eq. (40)); according to the above table, this first happens for $|S| = 8$ and we would have had to solve a system of $|X_{\{2,3,\dots,19\}}| = 10946$ equations in $6\binom{13}{7} + \binom{9}{2} + \binom{11}{4} = 10662$ unknowns!

B. Rational independence of polylogarithm values. We would like to make, or at least suggest, the conjecture that the only rational dependences over $\mathbb{Q}$ of values of the modified polylogarithm function P_m at arguments in $\overline{\mathbb{Q}} \subset \mathbb{C}$ are those which follow from the functional equations of P_m (including the reality condition $P_m(\overline{x}) = (-1)^{m-1}P_m(x)$), i.e., that the kernel of the map $P_m : \mathcal{F}_{\overline{\mathbb{Q}}} \to \mathbb{R}$ is the subgroup generated by $\mathcal{C}_m(\overline{\mathbb{Q}})$ and the group of elements $[x] + (-1)^m[\overline{x}]$. The corresponding statement when we look only at $\mathcal{A}_m(F) \subset \mathcal{F}_{\overline{\mathbb{Q}}}$ and use all embeddings $F \to \mathbb{C}$ is part of our main conjecture. In that case the polylogarithm mapping is supposed to map isomorphically to a discrete submodule of Euclidean space, so that the injectivity can be "seen" by a finite precision calculation. When we use only one embedding, the image will in general be dense, but the mapping should still be injective. There is no evidence for the conjecture beyond the trivial case $m = 1$. Indeed, so far as I know, there is not a single pair of rational linear combinations of values of the Bloch-Wigner function at algebraic arguments whose ratio is known to be irrational, so the map $P_2 : \mathcal{F}_{\overline{\mathbb{Q}}}/\langle\text{5-term relation}, [x] + [\overline{x}]\rangle \to \mathbb{R}$, far from being injective, might conceivably have a 1-dimensional image over $\mathbb{Q}$! Nevertheless, the underlying algebraic structure of the polylogarithm and the Bloch groups makes the conjecture seem plausible. By virtue of the theorem explained in §9B, a special case of the conjecture would be that the numbers $P_m(\zeta)$ as ζ runs over all roots of unity in $\mathbb{C}$ are linearly independent over $\mathbb{Q}$ except for the relations coming from the distribution relation (38). For $m = 2$ this is a well-known conjecture of Milnor [12].

C. Generalization to Artin L-functions. The whole situation described in this paper should generalize in the more or less obvious way to the pieces of the Bloch group corresponding to various Artin L-series. Specifically, if N is a Galois extension of $\mathbb{Q}$ containing F and $G = \mathrm{Gal}(N/\mathbb{Q})$, $H = \mathrm{Gal}(N/F)$, then one knows that the Dedekind zeta-function of F factors as a product of Artin L-functions $L(s, \rho)$ corresponding to the irreducible components ρ of the representation of G on $\mathbb{C}^{G/H}$, and we expect the Bloch group $\mathcal{B}_m(F)$ to split up in the same way, with the matrix of the polylogarithm map P_m^F splitting up into a direct sum of matrices whose determinants are the appropriate multiples of the numbers $L(m, \rho)$. We do not give the details of the correct formulation, leaving these to the reader or referring him/her to the unpublished paper [9] of B. Gross, where they are described completely (in terms of algebraic K-theory, not of Bloch groups). Observe that the refined conjecture, unlike the statement given in §8, is completely unknown even for $m = 1$, where it reduces to (an imprecise formulation) of Stark's conjecture that $L(1, \rho)$ can always be expressed in closed form in terms of logarithms of algebraic units.

Several cases of the refined conjecture have occurred in this paper: in §3, where the group $\mathcal{B}_3(F)$ for a real quadratic field $F = \mathbb{Q}(\sqrt{D})$ split up into two one-dimensional pieces on which the trilogarithm was proportional to $\zeta(3)$ and $\sqrt{D}\,L(3, \chi_D)$, in §4, where the group $\mathcal{B}_3(F)$ of the non-abelian cubic field $\mathbb{Q}(\theta)$ split up in a similar way (cf. equation (19), noting that the coefficients (1,2) and (1,−1) correspond to the real parts of the trivial and non-trivial characters, respectively, on a group of order 3), and in §9B, where the entire mth Bloch group of a cyclotomic field split up into one-dimensional pieces corresponding to the various characters χ and with values of the polylogarithm proportional to the numbers $L(m, \chi)$. As a further illustration, note that a special case of the refined conjecture is that the value of the zeta-function of an *ideal class* of a number field F at $s = m$ should be expressible in terms of the mth polylogarithm function with algebraic arguments (in the Hilbert class field of F). As an example, take $F = \mathbb{Q}(\sqrt{-23})$. Here there are three ideal classes $\mathcal{A}_0$ (= {trivial ideals}), $\mathcal{A}_1$, and $\mathcal{A}_2 = \mathcal{A}_1^{-1}$. The zeta function of the former is the Epstein zeta-function $\sum(m^2 + mn + 6n^2)^{-s}$ (sum over non-zero pairs of integers (m, n), taken up to sign), while the zeta functions of the other two coincide and equal the Epstein zeta-function $\sum(2m^2 + mn + 3n^2)^{-s}$. Thus $\zeta_F(\mathcal{A}_0, 3) + 2\zeta_F(\mathcal{A}_1, 3) = \zeta_F(3) \sim \pi^3\zeta(3)/\sqrt{23}$, while $\zeta_F(\mathcal{A}_0, 3) - \zeta_F(\mathcal{A}_1, 3) = L(3)$, the value of the L-series of $\mathbb{Q}(\theta)$ discussed in §4. Thus the rationality of the numbers $\delta^{\pm}(\theta)$ defined by (19) says that both $\zeta_F(\mathcal{A}_0, 3)$ and $\zeta_F(\mathcal{A}_1, 3)$ can be expressed in terms of the trilogarithm at arguments in $F(\theta)$, the Hilbert class field of F.

A special case of the refined conjecture, far from obvious, is that if E is a Galois extension of F, then the subspace of $\mathcal{B}_m(E)$ invariant under $\mathrm{Gal}(E/F)$ is precisely $\mathcal{B}_m(F)$. The corresponding property for $K_{2m-1}(F)\otimes \mathbb{Q}$ is known.

§11. Recent developments. In this section, we describe very briefly some recent results concerning the conjecture presented in this paper.

In [8], A. Goncharov essentially proves the conjectures about the trilogarithm presented in §§3–5. Using the connection between K-theory and certain geometric configurations, he is led to a new functional equation for the trilogarithm having 22 terms and depending on 3 parameters, namely $\widetilde{D}_3(G_{\alpha_1\alpha_2\alpha_3}) = 0$ where

$$G_{\alpha_1\alpha_2\alpha_3} = \sum_{i=1}^{3}\Big([\alpha_i] + [\beta_i] - [\beta_i/\alpha_{i-1}] + [\beta_i/\alpha_{i-1}\alpha_i] + [\alpha_i\beta_{i-1}/\beta_{i+1}] + [-\beta_i/\alpha_i\beta_{i-1}] - [\alpha_i\alpha_{i-1}\beta_{i+1}/\beta_i]\Big) - 3\,[1] + [-\alpha_1\alpha_2\alpha_3],$$

where the α_i are arbitrary complex numbers and $\beta_i = 1 - \alpha_i(1 - \alpha_{i-1})$ (indices i taken modulo 3). The Spence-Kummer relation is deduced from this by specializing, say, to $(\alpha_1, \alpha_2, \alpha_3) = (1, x, (1-y)/(1-x))$. It is not known whether, conversely, Goncharov's relation can be deduced from Kummer's. Goncharov expresses the opinion that it cannot, but the evidence presented in §6 seems to suggest the contrary. In any case, Goncharov defines $\mathcal{B}_3(F)$ as the quotient of $\mathcal{A}_3(F)/\langle G_{\alpha_1\alpha_2\alpha_3}\rangle_{\alpha_1,\alpha_2,\alpha_3 \in F}$ and proves that there is a canonical map $K_5(F) \otimes \mathbb{Q} \to \mathcal{B}_3(F) \otimes \mathbb{Q}$, conjecturally an isomorphism, whose composite with $\widetilde{D}_3^F$ is the Borel regulator map.

In [6] and [3], the conjecture formulated in this paper is reinterpreted in terms of motivic cohomology and variations of Hodge structures. In [6], P. Deligne furthermore refines it by replacing the real-valued map P_m by a map, well-defined on $\mathcal{A}_m$, with values in $\mathbb{C}/\pi^m i\mathbb{Q}$. This map behaves correctly under complex conjugation, so gives for a number field F of degree n a map on $\mathcal{B}_m(F)$ with altogether n components, $n_\mp$ of which (the ones we have used) are well-defined real numbers, while the other $n_\pm$ are well-defined only modulo rational multiples, presumably with bounded denominator, of π^m. (An example of this phenomenon occurred in §9C, where the one-dimensional groups $\mathcal{A}_m(\mathbb{Q}(\theta))$ for m even could be detected using the value of Lewin's function L_m, which is well-defined modulo $\mathbb{Q}\pi^m$ on $\mathcal{B}_m$, even though $P_m^F|\mathbb{R}$ vanishes in this case.) In [3], A. Beilinson also constructs an element of a certain K-group that is related under the regulator mapping to polylogarithms. By specializing it to roots of unity he shows that their images under P_m^F lie in the Borel regulator lattice (cf. §9B).

Finally, Beilinson has just informed me that he can construct for all m and F a canonical map from $\mathcal{B}_m(F) \otimes \mathbb{Q}$ to $K_{2m-1}(F) \otimes \mathbb{Q}$ whose composite with the regulator mapping coincides with P_m^F. Thus the image of the polylogarithm map is always contained in the regulator lattice in $\mathbb{R}^{n_\mp}$, and the determinant $\det\big(P_m^F(\xi_j)\big)_j$ for any $n_\mp$ elements $\xi_j \in \mathcal{B}_m(F)$ is a rational multiple of $\sqrt{\Delta}\zeta_F(m)/\pi^m$. It is still not known in general whether the ξ_j can be chosen to make this multiple non-zero, i.e., whether P_m^F is a *surjection* from the Bloch group to the regulator lattice. This can, of course, be checked for any given number field F and value of m by a finite computation, and is true in the case that F is cyclotomic by the discussion in §9B and in the case $m = 3$ by Goncharov's theorem. A consequence of Beilinson's result and the discussion of §10A is that there are infinitely many relations over $\mathbb{Q}$ among the values of the polylogarithm function of arbitrary order at algebraic, or even rational, arguments.

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