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A Hurwitz theory avatar of open-closed strings

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ABSTRACT

We review and explain an infinite-dimensional counterpart of the Hurwitz theory realization [1] of algebraic open-closed string model a lá Moore and Lizaroiu, where the closed and open sectors are represented by conjugation classes of permutations and the pairs of permutations, i.e. by the algebra of Young diagrams and bipartite graphes respectively. An intriguing feature of this Hurwitz string model is coexistence of two different multiplications, reflecting the deep interrelation between the theory of symmetric and linear groups S_∞ and $GL(\infty)$.

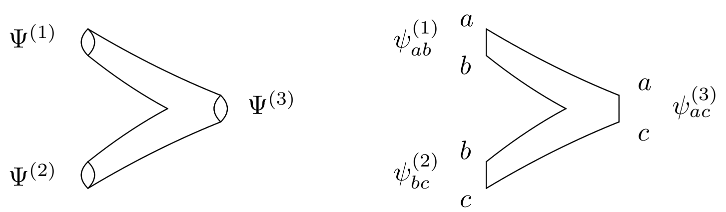
It is an old idea (see [2],[3],[4],[5] for a nice presentation) to formulate the open-closed string theory in purely algebraic terms (see sect.1 for details). This allows one to consider much simpler examples of the same phenomenon and involve basic mathematical constructions into the string theory framework.

In this paper we analyze (in sect.2) from this perspective the theory of closed (ordinary) and open Hurwitz numbers, which is actually the representation theory of symmetric (permutation) groups S_n (for initial steps in this direction see [6, 1]). In the infinite-dimensional case (S_∞) there appear two multiplications $\circ$ and $*$ (one with unity, the other one without) induced respectively by multiplication of permutations and differential operators, which is now well understood in the "closed-string" sector, but awaits similar understanding in the "open-string" one. We discuss this issue in sect.3.

1 Open-closed duality in terms of Cardy-Frobenius algebras [2, 3, 4, 5, 7]

In string theory, the multiplication in the algebra of fields is associated with the sewing operation and with pant diagrams, Fig.1.

Closed-string sector: algebra A Open-string sector: algebra B



$$\Psi^{(3)} = \Psi^{(1)} \cdot \Psi^{(2)} \qquad\qquad \psi^{(3)}_{ac} = \psi^{(1)}_{ab} \cdot \psi^{(2)}_{bc}$$

Here Ψ's are the fields in the closed sector and ψ_{ab} are those in the open one, we denote their algebras A and B correspondingly. The principal difference between the open and closed sectors is that in the former case the fields carry a pair of additional indices from the set of "boundary conditions" (or "D-branes"). In result $B = \oplus\mathcal{O}_{ab}$ splits into a combination of spaces corresponding to different boundary conditions. The sewing in the picture determines the algebra multiplication $\mathcal{O}_{ab} \otimes \mathcal{O}_{bc}$ which belongs to $\mathcal{O}_{ac}$ (no sum over b). Multiplications of all other elements are zero (e.g. $\mathcal{O}_{ab} \otimes \mathcal{O}_{cc} \to 0$). Diagonal subspaces $\mathcal{O}_{aa}$ are subalgebras of B, naturally

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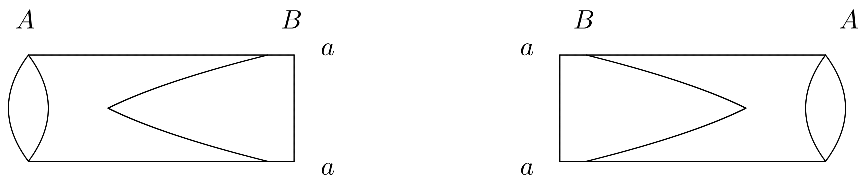
associated with particular D-branes. They can be labeled both by a pair of indices aa ar by single index a (very much like Cartan elements of the Lie algebras SL).

Multiplication operations satisfy a number of obvious relations [5]:

- **Closed-string sector (algebra A) :** associativity, commutativity
- **Open-string sector (algebra B) :** associativity

In the closed string sector there are also an identity element $\mathbf{1}_A$ and a linear form $\Big\langle \dots \Big\rangle_A$. Similarly, in the open sector in each space $\mathcal{O}_{aa}$ there are an identity element $\mathbf{1}_a$ and a linear form $\Big\langle \dots \Big\rangle_a$, this latter providing at the same time the pairings of two elements $\psi_{ab} \in \mathcal{O}_{ab}$ and $\psi'_{ba} \in \mathcal{O}_{ba}$: $\Big\langle \psi_{ab} \cdot \psi'_{ba} \Big\rangle_a = \Big\langle \psi'_{ba} \cdot \psi_{ab} \Big\rangle_b$. Note that the identity element of the whole algebra B is given by the sum $\mathbf{1}_B = \sum_a \mathbf{1}_a$.

There is also the third crucial ingredient in the construction: **the open-closed duality** which comes from the possibility to interpret the annulus diagram in two dual ways. To this end, one needs to somehow relate the closed and open sectors. This is achieved by treating D-branes as states in the closed sector A via the diagram:



Algebraically, the requirement is that there are the homomorphisms

$$\phi_a : \quad A \longrightarrow \mathcal{O}_{aa}, \tag{1}$$

one per each D-brane, and the dual maps

$$\phi^a : \quad \mathcal{O}_{aa} \longrightarrow A \tag{2}$$

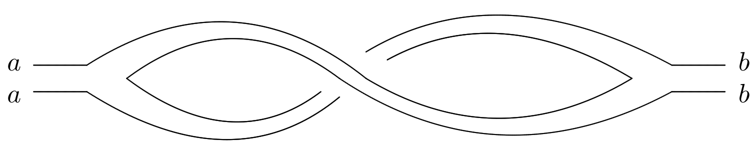
such that $\Big\langle \phi^a(\psi_{aa})\Psi) \Big\rangle_A = \Big\langle \psi_{aa}\phi_a(\Psi) \Big\rangle_a$. The homomorphism ϕ_a preserves the identity: $\phi_a(\mathbf{1}_A) = \mathbf{1}_a$ and is central: $\phi_a(\Psi)\psi_{ab} = \psi_{ab}\phi_b(\Psi)$.

In terms of this homomorphisms one can write the open-closed duality in the form of the Cardy condition:

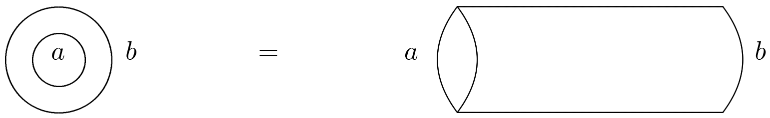
$$\sum_i \psi^i_{ab}\psi_{aa}\bar{\psi}^i_{ba} = \phi_b(\phi^a(\psi_{aa})) \tag{3}$$

where ψ^i_{ab} is a basis in $\mathcal{O}_{ab}$ and $\bar{\psi}^i_{ba}$ is its conjugated under the pairing.

The l.h.s. of this equation produces from the element ψ_{aa} an element of $\mathcal{O}_{bb}$ via the double twist diagram



which can be obtained in the closed string channel (the r.h.s. of (3)) as



The Cardy condition can be also rewritten in the "converted form" (as an identity between combinations of correlation functions). To do this, first of all, we adjust our notation for the needs of Hurwitz theory and denote the elements of A and B through Δ and Γ. We also extend in the evident way the action of homomorphism to the whole diagonal part $B_d = \sum_a \mathcal{O}_{aa}$ of B: $\phi \equiv \sum_a \phi_a$ and similarly extend the linear form $\Big\langle \psi_{ab} \Big\rangle_B = \delta_{ab} \Big\langle \psi_{ab} \Big\rangle_a$ which immediately allows one to define the pairing for any two elements of B.

Then the Cardy relation can be rewritten as follows

$$\sum_{\Gamma \in B} \Big\langle \Gamma_{aa} \cdot \Gamma \cdot \Gamma_{bb} \cdot \bar{\Gamma} \Big\rangle_B \;=\; \sum_{\Delta \in A} \Big\langle \Gamma_{aa} \cdot \phi(\Delta) \Big\rangle_A \Big\langle \phi(\bar{\Delta}) \cdot \Gamma_{bb} \Big\rangle_A \tag{4}$$

The bars denote the duals: $\Big\langle \Gamma \cdot \bar{\Gamma} \Big\rangle_B = 1$ and $\Big\langle \Delta \cdot \bar{\Delta} \Big\rangle_A = 1$. Below we use the Cardy relation exactly in this form, only we omit the indices A and B in the linear forms.

2 Hurwitz theory [6, 1, 8, 9]

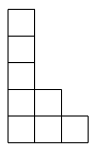
In Hurwitz theory the closed-string algebra is that of the Young diagrams (conjugation classes of permutations). This implies that the open-string fields will be labeled by pairs of Young diagrams with some additional data. Following [4] we identify them with bipartite graphs, conjugation classes of pairs of permutations.

A special feature of Hurwitz theory is additional decompositions of algebras $A = \oplus_n A_n$ and $B = \oplus_n B_n$. Homomorphisms $A_n \longrightarrow B_n$ and Cardy relations are straightforward only for particular values of n, while entire bi???-algebra has a more sophisticated structure, which is only partly exposed in the present paper and deserves further investigation.

2.1 Closed sector (algebra A)

Each permutation from the symmetric group S_n is a composition of cycles: for example, $6(34)(1527) \in S_7$ is the permutation

$$\begin{array}{c} \\ \downarrow \\ \\ \end{array} \quad \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ & & & & & & \\ i & j & k & l & m & n & p \\ m & p & l & k & j & n & i \end{array} \quad \in \quad [521] \quad =$$



The lengths of cycles form an integer partition of n, and the ordered set of lengths is the Young diagram $\Delta = \{\delta_1 \geq \delta_2 \geq \ldots \geq \delta_{l(\Delta)} > 0\}$ of the size (number of boxes) $|\Delta| = \delta_1 + \delta_2 + \ldots + \delta_{l(\Delta)} = n$. The above-mentioned permutation is associated in this way with the Young diagram [521].

Conversely, given a Young diagram Δ, one can associate with it a direct sum of all permutations of the type Δ from the symmetric group $S_{|\Delta|}$, e.g.

$$[521] = \oplus i(jk)(lmnpq)$$

where the sum goes over all $i, \ldots, q = 1, \ldots, 7$, which are all different, $i \neq \ldots \neq q$. In other words, the Young diagrams label the elements of *the center of the group algebra* of the symmetric group S_n. The multiplication (composition) of permutations induce a multiplication of Young diagrams of the same size, which we denote through $\circ$. For example,

$$A_1^\circ : \qquad [1] \circ [1] = [1],$$

A_2°	[11]	[2]
[11]	[11]	[2]
[2]	[2]	[11]

A_3°	[111]	[21]	[3]
[111]	[111]	[21]	[3]
[21]	[21]	$3 \cdot [111] + 3 \cdot [3]$	$2 \cdot [21]$
[3]	[3]	$2 \cdot [21]$	$2 \cdot [111] + [3]$

$$\dots \tag{5}$$

This multiplication is associative and commutative, and all the structure constants are positive integers, reflecting the combinatorial nature of this algebra A_n°. It describes the closed sector of the Hurwitz model of string theory. Actually, at the next stage Δ plays the role of index a in the open sector.

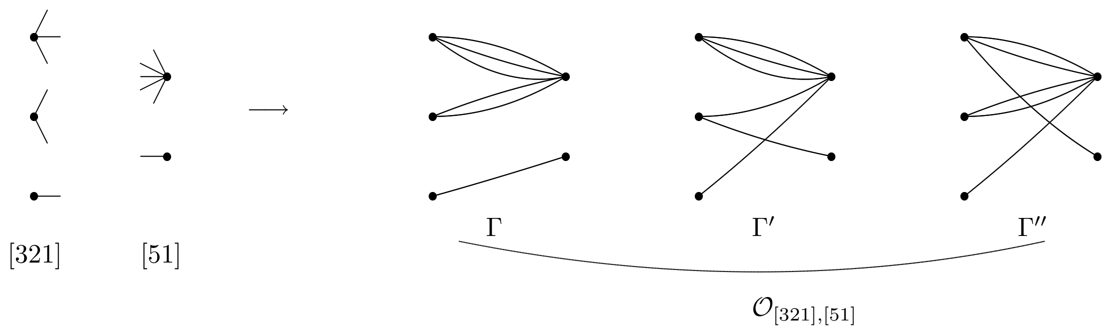
One can also say that the Young diagrams label the conjugation classes of permutations: $\mu \sim g\mu q^{-1}$.

2.2 Open sector (algebra B)

One can similarly consider the common conjugation classes of *pairs* of permutations of the same size:

$$[\mu, \nu] \sim [g\mu g^{-1}, g\nu g^{-1}], \quad \mu, \nu, g \in S_n$$

Note that conjugation g is the same for μ and ν. Such classes are labeled by *the bipartite graphs.* For example, take two permutations from S_6, say, $i(jk)(lmn) \in [321]$ and $i(jklmn) \in [51]$. Represent the two Young diagrams by two columns of vertices, each vertex corresponds to a cycle and has a valence, equal to the length of the cycle:



After that a conjugation class gets associated with a graph obtained by connecting the vertices. Clearly, in our example there are three different bipartite graphs, i.e. three different conjugation classes: $\Gamma, \Gamma', \Gamma'' \in \mathcal{O}_{[321],[51]}$.

Note that the sizes of Young diagrams are equal to the numbers of edges in the graph: $|\Gamma| = \#(\text{edges in } \Gamma)$.

Bipartite graphs of the same size can be multiplied: the product $\Gamma_1 \circ \Gamma_2$ is non-vanishing, when the right Young diagram of Γ_1 coincides with the left Young diagram of Γ_2:

$$\Delta^r(\Gamma_1) = \Delta^l(\Gamma_2)$$

The product is then a sum of graphs with

$$\Delta^l(\Gamma_1 \circ \Gamma_2) = \Delta^l(\Gamma_1), \qquad \Delta^r(\Gamma_1 \circ \Gamma_2) = \Delta^r(\Gamma_2),$$

obtained by connecting the edges entering the same vertex in all possible ways. Formally,

$$[\mu, \nu] \circ [\mu' \nu'] = \sum_g [\mu, g\nu' g^{-1}] \cdot \delta(v, g\mu' g^{-1}) \tag{6}$$

This multiplication is still associative, but no longer commutative.

Technically one can label a bipartite graph by two cyclic representations with appropriately identified indices. For example, the three graphs from $\mathcal{O}_{[321],[51]}$ in the above example are:

$$\Gamma = [i(jk)(lmn),\ i(jklmn)], \qquad \Gamma' = [i(jk)(lmn),\ j(iklmn)], \qquad \Gamma'' = [i(jk)(lmn),\ l(ijkmn)]$$

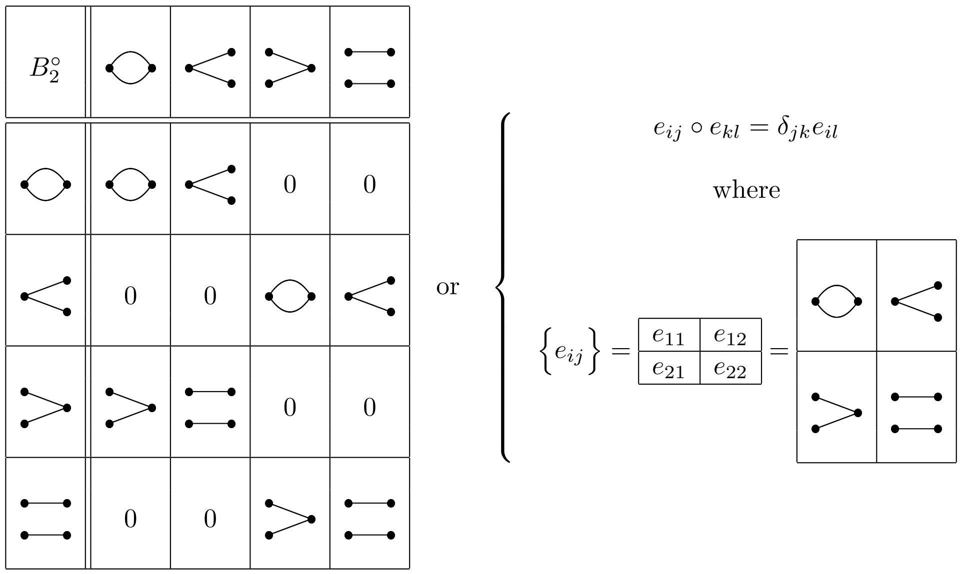
To multiply the so represented graphs one simply needs to appropriately rename the indices. For example, multiplying $\Gamma'' \in \mathcal{O}_{[321],[51]}$ with a graph from $\mathcal{O}_{[51],[2211]}$, one does the following:

$$[i(jk)(lmn),\ l(ijkmn)] \circ [i(jklmn),\ ij(kl)(mn)] = [i(jk)(lmn),\ l(ijkmn)] \circ [l(ijkmn),\ lj(ik)(mn)] = [i(jk)(lmn),\ lj(ik)(mn)]$$

This algebra of bipartite graphs is the open-sector algebra B_n° of the Hurwitz theory.

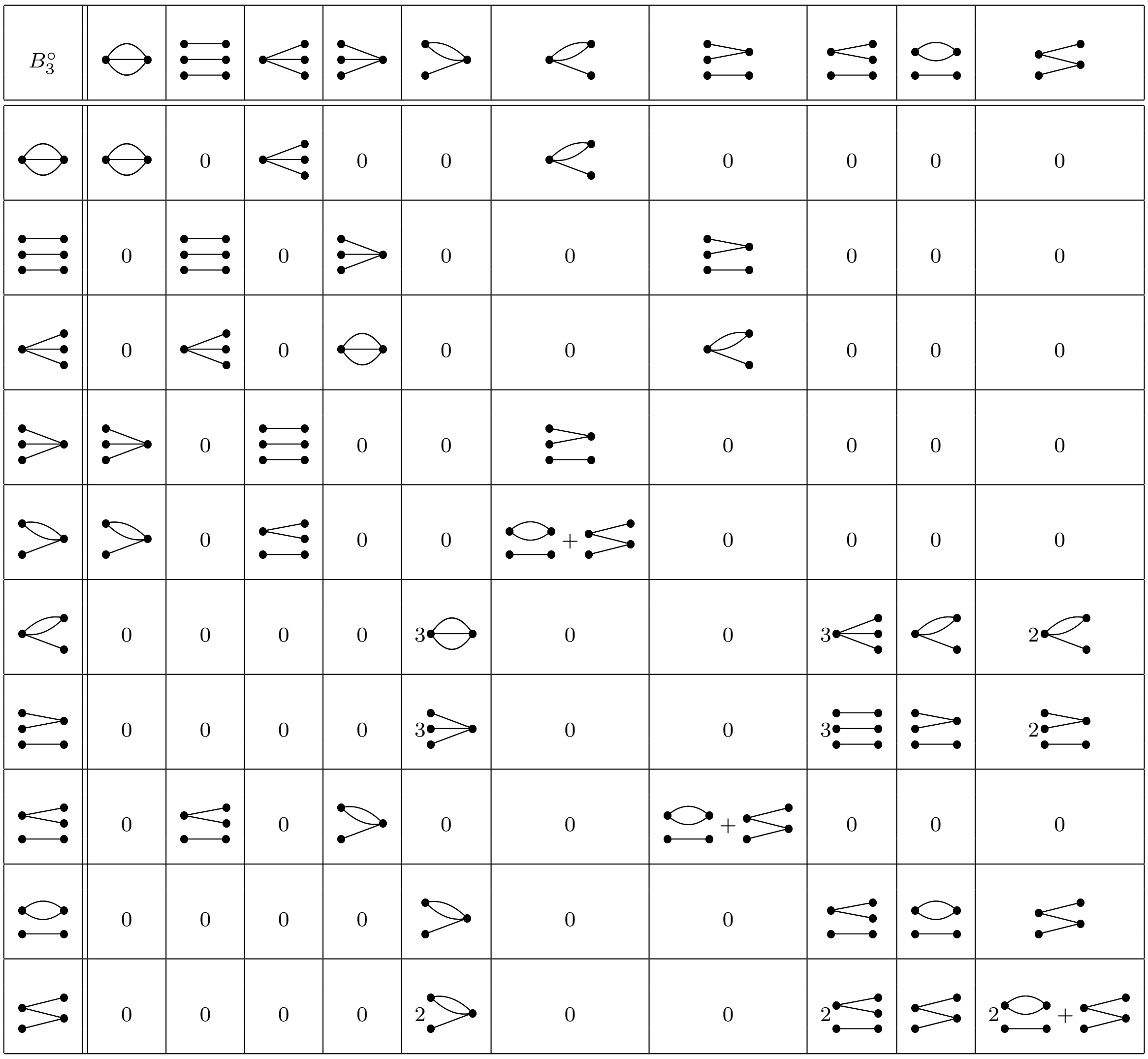
The simplest pieces of multiplication table are:

(7)



. . . (8)

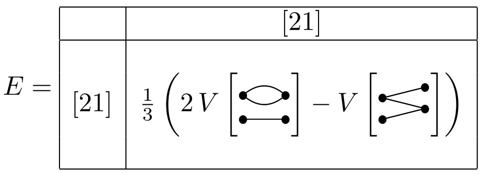
and, a little more complicated:

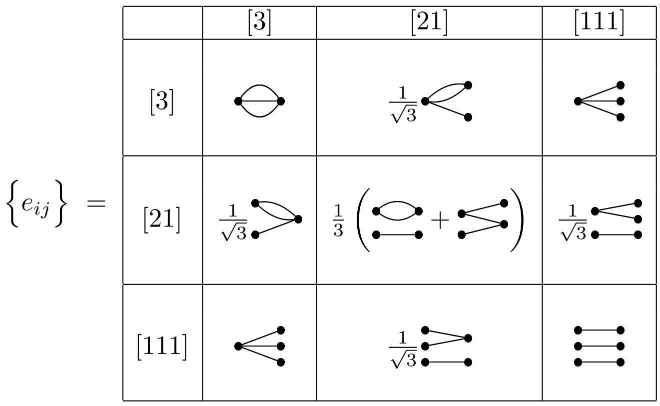


This table coincides with the combinatorial multiplication table 1 from [4] (with misprint corrected in the right lowest corner). It can be also represented as the sum of the matrix algebras $M_3 \oplus M_1$:

$$e_{ij} \circ e_{kl} = \delta_{jk} e_{il},$$
$$E \circ e_{ij} = e_{ij} \circ E = 0,$$
$$E \circ E = E$$

where

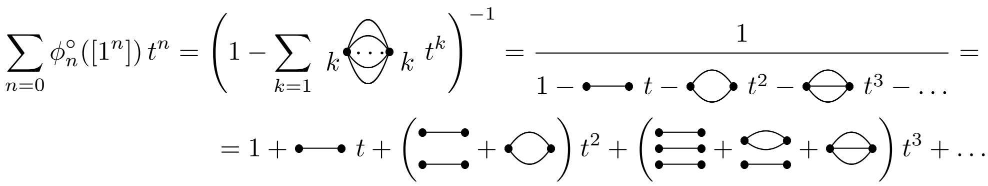




(9)

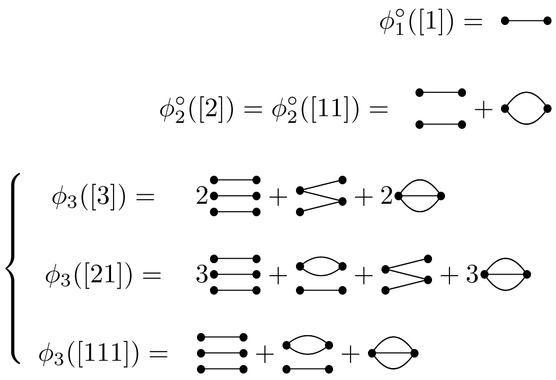
2.3 Relation between A_n and B_n

As we discussed in the first section, the $\circ$-homomorphism $\phi_n^\circ : A_n^\circ \longrightarrow B_n^\circ$ converts the Young diagrams from A_n° into a certain linear combination of graphs from $\oplus_\Delta \mathcal{O}_{\Delta,\Delta}$ (but not Δ into $\mathcal{O}_{\Delta,\Delta}$ with the same Δ). The identity element of A_n°, i.e. $[1^n] = [\underbrace{1,\ldots,1}_n]$ is mapped into the identity element of B_n° which is given by the formal series:



(10)

More generally:



$$\dots \qquad (11)$$

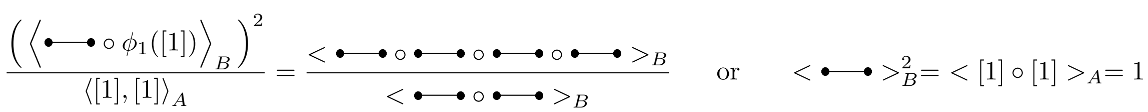
Remarkably, the homomorphism ϕ_n° has a non-trivial kernel (coinciding with the non-trivial ideal in A_n°). In particular,

$$\begin{aligned} \ker\ \phi_1 &= \emptyset \\ \ker\ \phi_2 &= [2]-[11] \\ \ker\ \phi_3 &= [3]-[21]+[111] \\ &\dots \end{aligned} \qquad (12)$$

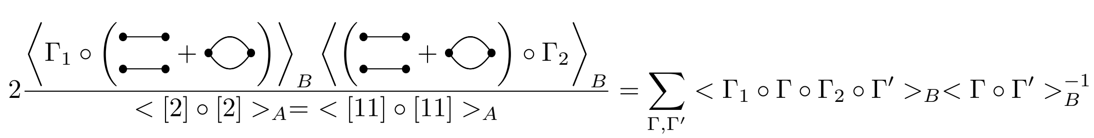
For each n the Cardy relation (4) is satisfied, provided all the sums are over elements from A_n° and B_n° with the same n:

$$\sum_{\Delta,\Delta'} <\Gamma_1\circ\phi(\Delta)>_B \Big(<\Delta\circ\Delta'>_A\Big)^{-1} <\phi(\Delta')\circ\Gamma_2>_B = \sum_{\Gamma,\Gamma'} <\Gamma_1\circ\Gamma\circ\Gamma_2\circ\Gamma'>_B \Big(<\Gamma\circ\Gamma'>_B\Big)^{-1} \qquad (13)$$

For example:



(14)



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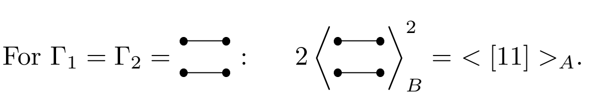
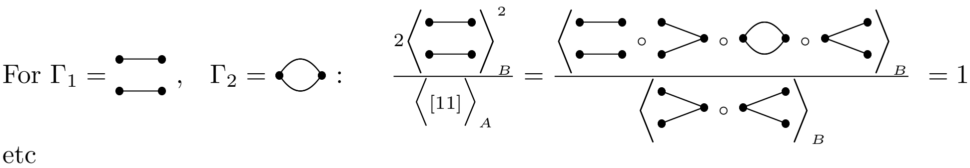




etc

3 Unification of all A_n's and B_n's

For unification purpose one can consider the linear spaces $\mathcal{A} = \otimes_n A_n$ and $\mathcal{B} = \otimes_n B_n$ which can be considered as semi-infinite sequences of Young diagrams and bipartite graphs respectively, containing exactly one element (perhaps, vanishing) of each size. The $\circ$-multiplication is then done termwise:

$$\begin{pmatrix} \Delta_1 \in A_1^\circ \\ \Delta_2 \in A_2^\circ \\ \Delta_3 \in A_3^\circ \\ \Delta_4 \in A_4^\circ \\ \dots \end{pmatrix} \circ \begin{pmatrix} \Delta_1' \\ \Delta_2' \\ \Delta_3' \\ \Delta_4' \\ \dots \end{pmatrix} = \begin{pmatrix} \Delta_1 \circ \Delta_1' \\ \Delta_2 \circ \Delta_2' \\ \Delta_3 \circ \Delta_3' \\ \Delta_4 \circ \Delta_4' \\ \dots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Gamma_1 \in B_1^\circ \\ \Gamma_2 \in B_2^\circ \\ \Gamma_3 \in B_3^\circ \\ \Gamma_4 \in B_4^\circ \\ \dots \end{pmatrix} \circ \begin{pmatrix} \Gamma_1' \\ \Gamma_2' \\ \Gamma_3' \\ \Gamma_4' \\ \dots \end{pmatrix} = \begin{pmatrix} \Gamma_1 \circ \Gamma_1' \\ \Gamma_2 \circ \Gamma_2' \\ \Gamma_3 \circ \Gamma_3' \\ \Gamma_4 \circ \Gamma_4' \\ \dots \end{pmatrix} \tag{16}$$

thus providing the new infinite algebras $\mathcal{A}^\circ$ and $\mathcal{B}^\circ$. The $\circ$-homomorphism $\phi^\circ : \mathcal{A}^\circ \longrightarrow \mathcal{B}^\circ$ is also defined termwise, and the Cardy relation also holds termwise, i.e. in the operator form (3) rather than in the converted one (4).

The original spaces of Young diagrams and graphs, $A = \oplus_n A_n$ and $B = \oplus_n B_n$ can be embedded into $\mathcal{A}^\circ$ and $\mathcal{B}^\circ$ with the maps

$$\begin{array}{lll} \rho : & A = \oplus_n A_n \longrightarrow & \mathcal{A} = \otimes_n A_n \\ \sigma : & B = \oplus_n B_n \longrightarrow & \mathcal{B} = \otimes_n B_n \end{array} \tag{17}$$

The first embedding, ρ maps the element $\Delta \in A_n$ to the column with zero first $n-1$ entries and the $(n+k)$-th entry of the form

$$\rho_{n+k}[\Delta] = \frac{(r_\Delta + k)!}{k!\ r_\Delta!}\ [\Delta, \underbrace{1, \dots, 1}_{k}] \tag{18}$$

where r_Δ is the number of lines of the unit length in Δ.

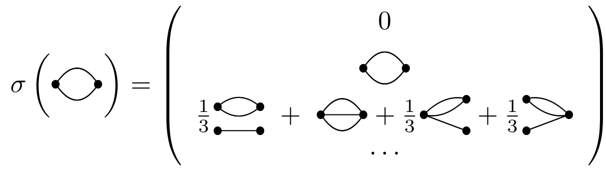
Similarly, the σ-embedding maps the element $\Gamma \in B_n$ to the column whose entries $\sigma_n(\Gamma)$ are

$$\sigma_n(\Gamma) = \begin{cases} \sum_{\Gamma_n \in \mathcal{E}_n(\Gamma)} \dfrac{|\mathbf{Aut}(\Gamma_n)|}{|\mathbf{Aut}(\Gamma_n \setminus \Gamma)|} \cdot \Gamma_n & n \geq |\Gamma| \\ 0 & n < |\Gamma| \end{cases} \tag{19}$$

We call the graph with all connected components having two vertices as *simple graph*, and call *the standard extension of the graph* the graph obtained by adding simple connected components. Then, $\mathcal{E}_n(\Gamma)$ in (19) denotes the set of all degree n standard extensions of Γ.

In the simplest example, these maps are

$$\rho([2]) = \begin{pmatrix} 0 \\ [2] \\ [21] \\ [211] \\ \dots \end{pmatrix} \quad \text{and}$$



However, these embeddings are not $\circ$-homomorphisms. Still, because of triangular form of the mappings, the images $\rho(A) \subset \mathcal{A}$ and $\sigma(B) \subset \mathcal{B}$ are $\circ$-subalgebras, i.e. $\rho(A) \circ \rho(A) \subset \rho(A)$ and $\sigma(A) \circ \sigma(A) \subset \sigma(A)$, so that one can define a new operation on A and B, which we call $*$-multiplication:

$$\rho(\Delta * \Delta') = \rho(\Delta) \circ \rho(\Delta') \quad \text{and} \quad \sigma(\Gamma * \Gamma') = \sigma(\Gamma) \circ \sigma(\Gamma') \tag{20}$$

For example,

$$\rho([1]) \circ \rho([1]) = \begin{pmatrix} [1] \\ 2\,[11] \\ 3\,[111] \\ 4\,[1111] \\ \dots \end{pmatrix} \circ \begin{pmatrix} [1] \\ 2\,[11] \\ 3\,[111] \\ 4\,[1111] \\ \dots \end{pmatrix} = \begin{pmatrix} [1] \circ [1] \\ 4\,[11] \circ [11] \\ 9\,[111] \circ [111] \\ 16\,[1111] \circ [1111] \\ \dots \end{pmatrix} = \begin{pmatrix} [1] \\ 4\,[11] \\ 9\,[111] \\ 16\,[1111] \\ \dots \end{pmatrix} = \begin{pmatrix} [1] \\ 2\,[11] \\ 3\,[111] \\ 4\,[1111] \\ \dots \end{pmatrix} + 2 \begin{pmatrix} 0 \\ [11] \\ 3\,[111] \\ 6\,[1111] \\ \dots \end{pmatrix} = \rho([1]) + 2\rho([11])$$

$$\rho([1]) \circ \rho([2]) = \begin{pmatrix} [1] \\ 2\,[11] \\ 3\,[111] \\ 4\,[1111] \\ \dots \end{pmatrix} \circ \begin{pmatrix} 0 \\ [2] \\ [21] \\ [211] \\ \dots \end{pmatrix} = \begin{pmatrix} 0 \\ 2\,[11] \circ [2] \\ 3\,[111] \circ [21] \\ 4\,[1111] \circ [211] \\ \dots \end{pmatrix} = \begin{pmatrix} 0 \\ 2\,[2] \\ 3\,[21] \\ 4\,[211] \\ \dots \end{pmatrix} = 2 \begin{pmatrix} 0 \\ [2] \\ [21] \\ [211] \\ \dots \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ [21] \\ 2\,[211] \\ \dots \end{pmatrix} = 2\rho([2]) + \rho([21])$$

Moreover, the $*$-multiplication is defined for any pair of Young diagrams or of bipartite graphs, without requiring them to have equal sizes:

A^*	$[1]$	$[11]$	$[2]$	$\ldots$
$[1]$	$\underline{[1]}+2\cdot[11]$	$2[11]+3[111]$	$2[2]+[21]$	
$[11]$	$2[11]+3[111]$	$\underline{[11]}+6\cdot[111]+6\cdot[1111]$	$\underline{[2]}+2\cdot[21]+[211]$	
$[2]$	$2[2]+[21]$	$\underline{[2]}+2\cdot[21]+[211]$	$\underline{[11]}+3\cdot[3]+2\cdot[22]$	
$\ldots$				

(21)

One more representation of the $*$-multiplication is in terms of the generating functions

$$J_\Delta(u) = \sum_{k\geq 0} \frac{(r_\Delta+k)!}{k!\ r_\Delta!}\ u^{|\Delta|+k}\ [\Delta,\underbrace{1,\ldots,1}_{k}] \tag{22}$$

In these terms

$$J_{\Delta_1 * \Delta_2}(v) = \oint J_{\Delta_1}(u)\circ J_{\Delta_2}\Big(\frac{v}{u}\Big)\frac{du}{u} \ = \ \sum_\Delta C^\Delta_{\Delta_1\Delta_2} J_\Delta(v) \tag{23}$$

In [8, 9] the algebra A^* was identified with the associative and commutative algebra of the cut-and-join operators,

$$\hat W(\Delta_1)\hat W(\Delta_2) = \sum_\Delta C^\Delta_{\Delta_1\Delta_2}\hat W_\Delta \tag{24}$$

and for $\Delta = [\delta_1\geq\delta_2\geq\ldots\geq\delta_{l(\Delta)}>0] = [\ldots,k+1,\underbrace{k,\ldots,k}_{m_k},k-1,\ldots]$

$$\hat W_\Delta = \frac{1}{\prod_k m_k!\ k^{m_k}}\ \ :\prod_i \mathrm{Tr}\,\hat D^{\delta_i}: \tag{25}$$

familiar also in the theory of matrix models. $\hat D_{\mu\nu}$ is the generator of the regular representation of $GL(\infty)$, see details in [8, 9]. This algebra is isomorphic also to the Ivanov-Kerov algebra [10].

An operator representation of the associative but non-commutative B^* is an open question, to be discussed in the forthcoming paper [11], as well as a $*$-homomorphism $\phi^*:\ A^*\longrightarrow B^*$ and the corresponding Cardy relation.

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