HEIGHTS FOR p–ADIC MEROMORPHIC FUNCTIONS AND VALUE DISTRIBUTION THEORY

by

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§ 1 Introduction.

1.1. In recent years many papers concern with the relation between number theory and value distribution theory (Nevanlinna theory) (see [L], [V]1, [V]2, [W], [O]1, [O]2). In [V]1 P. Vojta gives a "dictionary" for translating the results of Nevanlinna theory in the one–dimensional case to diophantine approximations. Due to this dictionary we can regard the Roth's theorem as an analog of Nevanlinna Second Main Theorem. P. Vojta has also made quantitative conjectures which generalize Roth's theorem to higher dimensions by relating the Second Main Theorem of Nevanlinna in higher dimensions (Griffiths–Stoll–Carlson–King) to the theory of heights. One can say that P. Vojta proposed an " arithmetic Nevanlinna Theory" in higher dimensions. In the philosophy of Hasse-Minkowski principle one would naturally have interest to determine how Nevanlinna theory would look in the p–adic case.

1.2. In [H]1, [H]2, [H–M] we constructed a p–adic analog of Nevanlinna theory. In this paper we introduce the notion of heights for p–adic meromorphic functions and thereby study p–adic holomorphic functions as well as meromorphic ones. By using the notion of heights, in several problems we only need to consider the behavior of functions when the argument passes "critical points". This makes it easier to prove both the p–adic interpolation theorem and p–adic analogs of two Main Theorems of Nevanlinna theory. The notion of heights and the p–adic analog of Nevanlinna theory in higher dimensions will be described in a future paper.

1.3. We first recall some facts from classical Nevanlinna theory ([N], [Hay]). Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$ and $a \in \mathbb{C}$ be a complex number. One asks the following question: How "large" is the set of points $z \in \mathbb{C}$ at which $f(z)$ takes the value a or values "close to a" ? For every value a Nevanlinna has constructed the following functions.

Let $n(f,a,z)$ denote the number of points $z \in \mathbb{C}$ for which $f(z) = a$ and $|z| \leq r$, counting with multiplicity. We set

$$N(f,a,r) = \int_0^r \frac{n(f,a,t)-n(f,a,0)}{t}\,dt + n(f,a,0)\log r$$

$$m(f,a,r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\varphi})-a|}\,d\varphi \ ,$$

where

$$\log^+ x = \begin{cases} \log x & \text{if } x > 1 \\ 0 & \text{if } x \leq 1 \ , \end{cases}$$

and that

$$T(f,a,r) = N(f,a,r) + m(f,a,r) \ .$$

Nevanlinna's First Main Theorem asserts that for every meromorphic function $f(z)$ there exists a function $T(f,r)$ such that for all $a \in \mathbb{C}$,

$$T(f,a,r) = T(f,r) + h(f,a,r) \quad ,$$

where $h(f,a,r)$ is a bounded function of r. Since the function $T(f,r)$ does not depend on a, we can roughly say that a meromorphic function takes every value a the same number of times.

Nevanlinna's Second Main Theorem asserts that generally $m(f,a,r)$ is small compared with $T(f,r)$ and consequently $N(f,a,r)$ approximates $T(f,r)$. Namely, one defines the defect of a as follows:

$$\delta(a,f) = \varliminf_{r\to\infty} \frac{m(f,a,r)}{T(f,r)} = 1 - \varlimsup_{r\to\infty} \frac{N(f,a,r)}{T(f,r)} \quad .$$

Then the set of defect values, i.e. those a such that $\delta(a) > 0$, is finite or countable, in addition $\Sigma\, \delta(a) \le 2$, where the sum extends over all defect values.

1.4. In § 2 we define the height for p–adic holomorphic functions. The p–adic Poisson-Jensen formula is described in terms of heights. In § 3 we concern with the problems of p–adic interpolation of holomorphic functions. We define the height of discrete sequences of points and give a necessary and sufficient condition for a sequence of points to be an interpolating sequences of a given function. In § 4 we define the height for meromorphic functions and prove the p–adic analog of two Nevanlinna Main Theorems.

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§ 2. Height of p–adic holomorphic functions.

2.1. Let p be a prime number, $\mathbb{Q}_p$ the field of p–adic numbers, and $\mathbb{C}_p$ the p–adic completion of the algebraic closure of $\mathbb{Q}_p$. The absolute value in $\mathbb{Q}_p$ is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the additive valuation on $\mathbb{C}_p$ which extends ord_p. Let D be the open unit disk in $\mathbb{C}_p$:

$$D = \{z \in \mathbb{C}_p;\ |z| < 1\}$$

Let $f(z)$ be a p–adic holomorphic functions on D represented by a convergent series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Since we have

$$\lim_{z\to\infty} \{v(a_n) + nv(z)\} = \infty$$

for all $v(z) = t > 0$, it follows that for every $t > 0$ there exists an n for which $v(a_n) + nt$ is minimal. Let $n^+_{f,t}$, $n^-_{f,t}$ be the smallest and the largest values of n at which $v(a_n) + nt$ attains its minimum. We set:

$$h^+_{f,t} = n^+_{f,t} \cdot t$$
$$h^-_{f,t} = n^-_{f,t} \cdot t$$
$$h_{f,t} = h^-_{f,t} - h^+_{f,t}$$

2.2. Definition. We call $h^+_{f,t}$, $h^-_{f,t}$, $h_{f,t}$ the right local height, left local height, local height of the function $f(z)$ at $t = -\log_p|z|$ respectively.

2.3. Definition. The global height $f(z)$ is defined by

$$H(f,t) = \min_{0 \le n < \infty} \{v(a_n) + nt\}$$

2.4. Remarks. 1) In [H]1 we called $H(f,t)$ the Newton polygon of the function $f(z)$. However the term "Newton polygon" is used in the literature for an another object. We use here " the height" which would be more suitable in this context.

2) We have

$$H(f,t) = \min_{\substack{v(z)=t \\ 0 \le n < \infty}} \{-\log_p|a_n| - n \log_p|z|\} .$$

2.5. Lemma. 1) If $h_{f,t} = 0$ then the function $f(z) \neq 0$ when $v(z) = t$ and one has

$$|f(z)| = p^{-H(f,t)}$$

2) If $h_{f,t} \neq 0$, then $f(z)$ has zeros at $v(z) = t$ and $h_{f,t} = t \cdot$ {number of zeros at $v(z) = t$ }

3) In any finite segment $[r,s]$, $0 < r < s < +\infty$ there are only finitely many t satisfying $h_{f,t} \neq 0$. Such points t are called the critical points of $f(z)$.

Proof. 1) Assume that $h_{f,t} = 0$, then $n^+_{f,t} = n^-_{f,t}$ and $v(a_n) + nt$ attains its minimum for a unique $\bar{n} = n^+_{f,t} = n^-_{f,t}$. We have $H(f,t) = v(a_{\bar{n}}) + \bar{n}t = v(\sum_{n=0}^{\infty} a_n z^n)$ at $v(z) = t$.

2) and 3) follow from definition 2.2, 2.3 and the properties of the Netwon polygon of f(z) (see [M]. [H]1).

2.6. Example. Consider the function

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$$

For every $t > 0$ we have

$$v((-1)^{n-1}/n) + nt) \begin{cases} = nt - \log n/\log p & \text{if } n = p^k \\ > nt - \log n/\log p & \text{if } n \neq p^k \end{cases}$$

Hence, for any $t > 0$, $n^+_{\log,t}$ and $n^-_{\log,t}$ have the form p^k for some $k \geq 0$. It is easy to see that $n^+_{\log,t} \neq n^-_{\log,t}$ if and only if $n^+_{\log,t} = p^{k-1}$ and $n^-_{\log,t} = p^k$ for some k. In this case we have

$$v((-1)^{p^{k-1}-1}/p^{k-1}) + p^{k-1}t = v((-1)^{p^k-1}/p^k) + p^k t \ .$$

Thus, the function $\log(1+z)$ has critical points $t_k = \frac{1}{p^k - p^{k-1}}$ $(k = 1,2,\ldots)$ and we have: $h^+_{\log,t_k} = \frac{1}{p-1}$, $h^-_{\log,t_k} = \frac{p}{p-1}$, $h_{\log,t_k} = 1$, $h_{\log,t} = 0$ for all $t \neq t_k$ $(k = 1,2,\ldots)$, $H(\log,t) = \frac{1}{p-1} + [\log_p(p-1)t]$, where $[x]$ denotes the largest integer being equals or less than x.

2.7. Theorem. (the p–adic Poisson–Jensen formula). Let $f(z)$ be a holomorphic function in the unit disk and let $t_0 > t > 0$. Then we have:

$$H(f,t_0) - H(f,t) = h^-_{f,t_0} - h^+_{f,t} + \sum_{t_0>s>t} h_{f,s} \qquad (1)$$

Proof: Let $t_0 > t_1 > t_2 > \dots > t_n \geq t$ be all the critical points of the function $f(z)$. Note that the height $H(f,s)$ is a linear function of s in every segment $[t_{k+1}, t_k]$ and we have $n^-_{f,t_k} = n^+_{f,t_{k+1}}$, $H(f,s) = v(a_{n^+_{f,t_{k+1}}}) + n^+_{f,t_{k+1}} s = v(a_{n^-_{f,t_k}}) + n^-_{f,t_k} s$. From this it follows that $H(f,t_k) - H(f,t_{k+1}) = [v(a_{n^-_{f,t_k}}) + n^-_{f,t_k} t_k] - [v(a_{n^+_{f,t_{k+1}}}) + n^+_{f,t_{k+1}} t_{k+1}] = n^-_{f,t_k}(t_k - t_{k+1})$. $H(f,t_0) - H(f,t) = H(f,t_0) - H(f,t_1) + H(f,t_1) - H(f,t_2) + \dots + H(f,t_n) - H(f,t) = (n^-_{f,t_0} t_0 - n^-_{f,t_0} t_1) + (n^-_{f,t_1} t_1 - n^-_{f,t_2} t_2) + \dots + (n^-_{f,t_n} t_n - n^-_{f,t_n} t) = h^-_{f,t_0} + t_1(n^-_{f,t_1} - n^-_{f,t_0}) + t_2(n^-_{f,t_2} - n^-_{f,t_1}) + \dots + t_n(n^-_{f,t_n} - n^-_{f,t_{n-1}}) - h^+_{f,t} = h^-_{f,t_0} - h^+_{f,t} + \sum_{t_0>s>t} h_{f,s}$.

Theorem 2.7 is proved.

2.8. Remark. Note that the formula (1) is analogous to the classical Poisson-Jensen formula. In fact, suppose that $t_0 = \infty$, $f(0) \neq 0$ and t is not a critical point of the function $f(z)$. Then we have $H(f,t_0) = -\log_p|f(0)|$, $H(f,t) = -\log_p|f(z)|$ on the circle $|z| = p^{-t}$, $h^-_{f,t_0} = 0$, $\sum_{t_0>s>t} h_{f,s} - h^+_{f,t} = \sum -\log_p|z_i|$, where the sum extends over all the zeros z_i of the function $f(z)$ in the disk $|z| \leq p^{-t}$. Then formula (1) takes the following form: $\log_{v(z)=t} |f(z)| - \log_p|f(0)| = \sum -\log_p|z_i|$. Recall that the classical Poisson–Jensen formula is the following:

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\,d\theta - \log|f(0) = \sum_{\substack{a\in D\\ a\neq 0}} -(\mathrm{ord}_a f)\log|a| \quad ,$$

where D is the unit disk in $\mathbb{C}$ and $\mathrm{ord}_a f$ is the order of $f(z)$ at a.

§ 3. Heights of sequences of points and p–adic interpolation.

3.1. The construction of the p–adic zeta–function by interpolating from a set of integers ([K–L]) caused many people to become interested in the problem of p–adic interpolation. In [H] 1 we find a necessary and sufficient condition for a discrete sequence of points in the unit disk D to be an interpolating sequence of a given function $f(z)$. This is the first theorem of p–adic interpolation of unbounded functions. In this section we formulate and prove the interpolation theorem in terms of heights of p–adic holomorphic functions.

3.2 Definition. Let $g(z)$ be a holomorphic function in the unit disk D. We denote by $o(g)$ the class of holomorphic functions in D satisfying the following condition

$$\sup_{|z|=r} |f(z)| = o\left(\sup_{|z|=r} |g(z)| \right)$$

when $r \longrightarrow 1-0$.

3.3. Corollary. $f \in 0(g)$ if and only if

$$\lim_{t\to 0} \{H(f,t) - H(g,t)\} = \infty$$

3.4. Now let $u = \{u_0, u_1, \dots\}$ be a sequence of points in D. In what follows we shall only consider sequences u for which the number of points u_i satisfying $v(u_i) \geq t$ is finite for every $t > 0$. We shall always assume that $v(u_i) \geq v(u_{i+1})$ $(i = 0,1,\dots,)$.

3.5. Definition. For every $t > 0$ the heights $h^+_{u,t}$, $h^-_{u,t}$, $h_{u,t}$, $H(u,t)$ are defined by: $h^{\pm}_{u,t} = n^{\pm}_{u,t} \cdot t$, where $n^+_{u,t}$ $(n^-_{u,t})$ is the number of points u_i such that $v(u_i) > t$ (resp. $v(u_i) \geq t$), $h_{u,t} = h^-_{u,t} - h^+_{u,t}$ and $H(u,t) = h^+_{u,t} - h^-_{u,t_0} - \sum\limits_{s>t} h_{u,s}$, where $t_0 = v(u_0)$. We shall always assume that $\lim\limits_{t \to 0} H(u,t) = -\infty$.

3.6. Example. For the sequence of primitive p^m–roots of unity, $m = 1,2,\dots$ we have:

$$h^{\pm}_{u,t} = h^{\pm}_{\log,t},\ h_{u,t} = h_{\log,t},\ H(u,t) = H(\log,t) \ .$$

3.7. Remark. If $u = \{u_i\}$ is the sequence of zeros of the function $f(z)$, then we have $H(f,t) - H(u,t) = 0(1)$ when $t \longrightarrow 0$.

3.8 Definition. The sequence $u = \{u_i\}$ is called an interpolating sequence of $f(z)$ if the sequence of interpolation polynomials for f on u converges to $f(z)$.

3.9. Theorem. The sequence $u = \{u_i\}$ is an interpolating sequences of the function $f(z)$ if and only if

$$\lim_{t \to 0} [H(f,t) - H(u,t)] = \infty$$

Proof. For simplicity we assume that u is a sequence of distinct points. In the case of dealing with sequences of non–distinct points we need a minor modification of the proof.

Recall that the interpolation polynomials $\{P_k(z)\}$ for the function $f(z)$ on the sequence u are determined by the following relations:

$$\deg P_k \le k;\ \ P_k(u_i) = f(u_i),\ \ i = 0,\dots,k\ .$$

We set $S_k(z) = P_{k+1}(z) - P_k(z)$.

First of all we prove the following

3.10. Lemma. For all $t_0 > 0$ and for all k such that $t_k = v(u_k) < t$ we have

$$|\,[H(S_k,t_k) - H(u,t_k)] - [H(S_k,t_0) - H(u,t_0)]\,| \le t_0$$

Proof of Lemma 3.10. By the Poisson–Jensen formula we have

$$H(S_k,t_0) - H(S_k t_k) = h^-_{S_k,t_0} - h^+_{S_k,t_k} + \sum_{t_0 > s > t_k} h_{S_k,s}$$

From this it follows that

$$[H(S_k,t_0) - H(u,t_0)] - [H(S_k,t_k) - H(u,t_k)] =$$

$$= (h^-_{S_k,t_0} - h^-_{u,t_0}) - (h^+_{S_k,t_k} - h^+_{u,t_k}) + (\sum_{t_0 > s > t_k} (h_{S_{k,s}} - h_{u,s}))$$

From the definitions of $h_{S_{k,t}}$, $h_{u,t}$ for k such that $v(u_k) < t$ we have

$$\sum_{t_0 > s > t_k} (h_{S_k,s} - h_{u,s}) = 0$$

$$0 \leq n^{\pm}_{S_k,t_0} - n^{\pm}_{u,t_0},\ n^{\pm}_{S_k,t_k} - n^{\pm}_{u,t_k} \leq 1$$

From this Lemma 3.10 follows.

We now return to prove Theorem 3.9.

1) Necessity. Suppose that $H(f,t) - H(u,t)$ does not tend to infinity. Then we can find a sequence $\{s_i\}$ such that $H(f,s_i) - H(u,s_i)$ is bounded. Hence there is an integer k_0 such that for $k \geq k_0$ we have

$$H(S_k,s_0) - H(u,s_0) > \sup\{H(f,s_i) - H(u,s_i)\} + 1 + s_0 \ .$$

In view of Lemma 3.10 for $k \geq k_0$ and all $i \geq 0$ we have:

$$H(S_k,s_i) - H(u,s_i) \geq \sup\{H(f,s_i) - H(u,s_i)\} + 1$$

and hence,

$$H(S_k,s_i) - H(f,s_i) \geq 1 \tag{2}$$

We set $M_0 = \inf_{0 \leq k \leq k_0} H(S_k,0)$. Since $\lim_{t \to 0} H(u,t) = -\infty$ it suffices to consider the case when $f(z)$ is unbounded, i.e. $\lim_{t \to 0} H(f,t) = -\infty$. Then there exists a number N_0 such that for all $N \geq N_0$ we have

$$H(f,s_N) \leq M_0 - 1$$

Since $H(S_k,s_N) \geq H(S_k,0)$, we have

$$H(S_k,s_N) - H(f,s_N) \geq M_0 - H(f,s_N) \geq 1$$

Thus, the inequality (2) holds for all $k \geq 0$ and all $n \geq N_0$. By assumption we have

$$f(z) = \sum_{k=0}^{\infty} S_k(z) \quad ,$$

and this implies the obvious inequality

$$H(f,s_n) \geq \min_{k \geq 0} \{H(S_k,s_N)\} \quad .$$

This contradicts (1) and proves the necessity.

2) Sufficiency. We first prove the following

3.11. <u>Lemma</u>. For any k we have $H(S_k,t_k) \geq H(f,t_k)$ or $H(S_k,t_{k+1}) \geq H(f,t_{k+1})$

<u>Proof of Lemma</u> 3.11. By Lazard's lemma ([Laz]) we have:

$$f(z) = \varphi(z) \prod_{i=0}^{k} (z-u_i) + Q_k(z)$$

where $\deg Q_k(z) \leq k$, $H(Q_k,t_k) \geq H(f,t_k)$. On the other hand, $Q_k(u_i) = f(u_i)$, $i = 0,\ldots,k$, and then $Q_k(z) \equiv P_k(z)$. Thus, $H(P_k,t_k) \geq H(f,t_k)$. Similarly, $H(P_{k+1},t_{k+1}) \geq H(f,t_{k+1})$. If $v(u_{k+1}) = v(u_k)$, i.e. $t_k = t_{k+1}$, then we have $H(S_k,t_k) \geq H(f,t_k)$. Assume that $t_k \neq t_{k+1}$. If $H(P_{k+1},t_k) \geq H(f,t_k)$ then we have

$H(S_k,t_k) \geq H(f,t_k)$. Otherwise, $H(P_{k+1},t_k) < H(P_k,t_k)$. Since $t_k \neq t_{k+1}$ we have $n_{\overline{P}_{k+1},t_{k+1}} = k+1$ and $n_{\overline{P}_{k+1},t_k} = k \geq n_{\overline{P}_k,t_k}$. Thus we have

$$H(P_k,t_{k+1}) = H(P_k,t_k) - n_{\overline{P}_k,t_k}(t_k - t_{k+1}) \geq$$
$$\geq H(P_{k+1},t_k) - n_{\overline{P}_{k+1},t_k}(t_k - t_{k+1}) = H(P_{k+1},t_{k+1})$$

and then $H(S_k,t_{k+1}) \geq H(f,t_{k+1})$.

We now return to the proof of sufficiency. In view of Lemma 3.10, for an arbitrary N we have $H(S_n,t_N) \geq H(u,t_N) + t_N + H(S_n,t_n) - H(u,t_n)$ for $t_n = v(u_n) < t_N$.
By Lemma 3.11 we have either $H(S_n,t_n) \geq H(f,t_n)$ or $H(S_n,t_{n+1}) \geq H(f,t_{n+1})$, and then we obtain

$$H(S_n,t_N) \geq H(u,t_N)+t_N+\min\{ [H(f,t_n)-H(u,t_n)] , H(f,t_{n+1})-H(u,t_{n+1})] \ .$$

From this and the assumption we have

$$\lim_{n\to\infty} H(S_n,t_N) = \infty \ ,$$

i.e. $\lim\limits_{n\to\infty} S_n(z) = 0$, and hence there exists $P(z) = \lim\limits_{n\to\infty} P_n(z)$. It remains to prove that $P(z) \equiv f(z)$. Since u is an interpolating sequence of $P(z)$, we must have

$$\lim_{t\to 0} [H(P,t) - H(u,t)] = \infty$$

By setting $g(z) = P(z) - f(z)$ we obtain

$$\lim_{t\to 0} [H(g,t) - H(u,t) = \infty \qquad (3)$$

On the other hand, as $g(u_i) = 0$ for $i = 0,1,2,\dots$, we find (3) contradicts Remark 3.7. Then $g(z) \equiv 0$ and Theorem 3.9 is proved.

We can formulate Theorem 3.9 in terms of local heights.

3.12. Corollary. The sequence $u = \{u_i\}$ is an interpolating sequence of the function $f(z)$ if

$$\lim_{t\to 0} \{ \sum_{s>t} h_{u,s} - \sum_{s>t} h_{f,s}\} = \infty$$

and $h^+_{u,t} - h^+_{f,t}$ is bounded when $t \longrightarrow 0$.

In fact, under these conditions it follows from the Poisson–Jensen formula and the definition of $H(u,t)$ that $\lim_{t\to 0} \{H(f,t) - H(u,t)\} = \infty$.

3.13 Remark. One can find the function $f(z)$, the sequence of points u such that $\lim_{t\to 0} \{ \sum_{s>t} h_{u,s} - \sum_{s>t} h_{f,s}\} = \infty$ while $h^+_{u,t} - h^+_{f,t}$ is unbounded and $H(f,t) - H(u,t)$ does not converge to infinity.

3.14 Corollary. The sequence u is an interpolating sequence for all functions in $o(f)$ if the functions

$$n^{\pm}_{f,t} - n^{\pm}_{u,t}$$

are bounded.

In fact, from the proof of the Poisson–Jensen formula it follows that if for all $t > 0$ we have $n^-_{f,t} - n^-_{u,t} < M$ then $H(u,t) - H(f,t) < H(u,t_0) - H(f,t_0 + M t_0$ for $t < t_0$. Let g be a function of calss $o(f)$. We have

$H(g,f) - H(u,t) = [H(g,t) - H(f,t)] - [H(u,t) - H(f,t)] > [H(g,t) - H(f,t)] -$

$- [H(u,t_0) - H(f,t_0)] - M t_0 \longrightarrow \infty$ when $t \longrightarrow 0$.

3.15. Corollary. The sequence $\{\gamma-1\}$ where $\gamma^{p^n} = 1$, $n = 1,2,\ldots$ is an interpolating sequence for all functions of class $o(\log)$.

In fact, take for $f(z)$ the function $\log(1+z)$ and let u be the sequence in Corollary 3.15. Then $n^{\pm}_{f,t} - n^{\pm}_{u,t} = 0$ (see example 3.6).

A similar result holds for functions of class $o(\log^k)$. Note that the p–adic L–functions associated to cusps forms are p–adic holomorphic functions of class $o(\log^k)$ for some k (see [Vish]).

3.16. Corollary. Let $\{u_i\} \subset D$ and $\{\alpha_i\} \subset \mathbb{C}_p$ be two sequences of values in D and $\mathbb{C}_p$. Let $\{P_n(z)\}$ be the sequence of polynomials satisfying the conditions: $\deg P_n(z) \leq n$, $P_n(u_i) = \alpha_i$, $i = 0,\ldots,n$. Then we have the following

1) If $H(P_n,0) - H(u,t_n) \longrightarrow \infty$ when $n \longrightarrow \infty$, there exists a holomorphic function $f(z)$ such that $f(u_i) = \alpha_i$, $i = 0,1,2,\ldots$, $f(z) = \lim_{n\to\infty} P_n(z)$

2) Conversely, if there exists a holomorphic function $g(z) = \lim_{n\to\infty} P_n(z)$, then

$$H(P_n,0) - H(u,t_n) + n\, t_n \longrightarrow \infty$$

when $n \longrightarrow \infty$.

Proof. We have $H(P_n,t_n) \geq H(P_n,0)$ and $H(P_n,t_n) - H(u,t_n) \longrightarrow \infty$ when $n \longrightarrow \infty$. Arguments similar to those used to prove Theorem 3.9 give us for every fixed N :

$$H(S_n,t_N) - H(u,t_N) \geq \min\{ [H(P_n,t_n) - H(u,t_n),\ H(P_n,t_{n+1}) - M(u,t_{n+1})]$$

Consequently, $\lim_{n\to\infty} H(S_n,t_N) = \infty$ and there exists $f(z) = \lim P_n(z)$. Obviously that $f(u_i) = \alpha_i$, $i = 0,1,2,...$

Conversely, if there exists a holomorphic function $g(z) = \lim P_n(z)$, then we have $H(P_n,t_n) \geq H(g,t_n)$ and then $H(P_n,0) \geq H(P_n,t_n) - n\,t_n \geq H(g,t_n) - n\,t_n$; $H(P_n,0) - H(u,t_n) + n\,t_n \geq H(g,t_n) - H(u,t_n) \longrightarrow \infty$, since u is an interpolating sequence of the function $g(z)$.

3.17. Remark. In many cases we have $n\,t_n < \infty$. For example when u is the sequence $\{\gamma-1\}$ with $\gamma p^n = 1$, Corollary 3.16 gives a necessary and sufficient condition.

§4. Height for p–adic meromorphic function

4.1 Let $\varphi(z)$ be a meromorphic function on D. By definition, $\varphi(z) = f(z)/g(z)$, where $f(z)$ and $g(z)$ are holomorphic functions on D not having common zeros. We set

$$H(\varphi,t) = H(f,t) - H(g,t)$$

we call $H(\varphi,t)$ the global height of the function $\varphi(z)$. As in the case of holomorphic functions, the (right, left) local height $\varphi(z)$ at t is defined by $h^+_{\varphi,t} = h^+_{f,t} - h^+_{g,t}$; $h^-_{\varphi,t} = h^-_{f,t} - h^-_{g,t}$, $h_{\varphi,t} = h^-_{\varphi,t} - h^+_{\varphi,t}$.

4.2 Remark. $h_{\varphi,t} > 0$ ($h_{\varphi,t} < 0$) if and only if $\varphi(z)$ has zeros (poles) at $v(z) = t$.

4.3 The characteristic function. For $a \in \mathbb{C}_p$ we set

$$m(\varphi,a,t) = H^+(\varphi - a,t) = \max\{H(\varphi - a,t),0\}$$
$$N(\varphi,a,t) = \sum_{s>t} n(\varphi,a,s)(s-t)$$

where $n(\varphi,a,s)$ denotes the number of points $z \in D$ such that $v(z) = t$ and $\varphi(z) = a$, here every such point is counted according to its multiplicity as a root of $\varphi(z) = a$. We set

$$T(\varphi,a,t) = N(\varphi,a,t) + m(\varphi,a,t),$$

and moreover that

$$N(\varphi,t) = \sum_{s>t} h_{g,s} - h^+_{g,t}$$
$$m(\varphi,t) = H^+(1/\varphi,t)$$
$$T(\varphi,t) = N(\varphi,t) + m(\varphi,t)$$

we call $T(\varphi,t)$ the characteristic function of the meromorphic function $\varphi(z)$.

4.4. Theorem Let $\varphi(z)$ be a meromorphic function in D. Then for every $a \in \mathbb{C}_p$ we have

$$T(\varphi,a,t) = T(\varphi,t) + 0(1) .$$

We first prove the following

4.5. Lemma. Let φ, φ_i $(i = 1,2,\ldots,k)$ be meromorphic functions on D. Then we have:

1) $m(\sum_{i=1}^{k} \varphi_i, t) \leq \max_i \{m(\varphi_i, t)\}$

2) $m(\prod_{i=1}^{k} \varphi_i, t) \leq \sum_{i=1}^{k} m(\varphi_i, t)$

3) $N(\sum_{i=1}^{k} \varphi_i, t) \leq \sum_{i=1}^{k} N(\varphi_i, t)$

4) $N(\prod_{i=1}^{k} \varphi_i, t) \leq \sum_{i=1}^{k} N(\varphi_i, t)$

5) $T(\sum_{i=1}^{k} \varphi_i, t) \leq \sum_{i=1}^{k} T(\varphi_i, t)$

6) $T(\prod_{i=1}^{k} \varphi_i, t) \leq \sum_{i=1}^{k} T(\varphi_i, t)$

7) $T(\varphi, t)$ is a decreasing function of t

8) $T(\varphi, t)$ is a bounded function if and only if $\varphi(z)$ is a ration of two bounded holomorphic functions.

Proof. 1) and 2) follow from the properties of the height and the definition of the function $m(\varphi, t)$. 3) and 4) are proved by the remark that $N(\varphi, t)$ is the sum of valuations of poles of $\varphi(z)$ in the disk $|z| \leq p^{-t}$. 5) and 6) are consequences of 1), 2), 3), 4).

We now prove 7). First of all we show that $N(\varphi, t)$ is a decreasing function. Assume $t' \geq t'' > 0$ and in the segment (t'', t') there is no critical point of $g(z)$. Then we have

$$N(\varphi,t') = \sum_{s>t'} h_{g,s} - h^{+}_{g,t'} = \sum_{s>t''} h_{g,s} - h_{g,t'} - h^{+}_{g,t'} =$$

$$\sum_{s>t''} h_{g,s} - h^{-}_{g,t'} = \sum_{s>t''} h_{g,s} - n^{-}_{g,t'}t' =$$

$$\sum_{s>t''} h_{g,s} - n^{+}_{g,t''}t' \leq \sum_{s>t''} h_{g,s} - h^{+}_{g,t''} = N(\varphi,t'') \quad (4)$$

Since every segment $[t'',t']$ can be divided into a finite number of segments on which $g(z)$ does not have critical points, (4) shows that $N(\varphi,t)$ is a decreasing function.

Now assume that $m(\varphi,t') = 0$, then $T(\varphi,t') = N(\varphi,t') \leq N(\varphi,t'') \leq T(\varphi,t'')$. When $m(\varphi,t') > 0$ we have $H(1/\varphi,t') > 0$ and $H(\varphi,t') < 0$, i.e. $m(1/\varphi,t') = 0$. Then we have

$$T(1/\varphi,t') = N(1/\varphi,t') \leq N(1/\varphi,t'') \leq T(1/\varphi,t'') \quad (5)$$

Note that the Poisson–Jensen formula is valid for meromorphic functions when the heights $h^{\pm},h,H$ are defined as above. We take t_0 so that for $t > t_0$ the function $\varphi(z)$ does not have critical points and hence $h_{\varphi,s} = 0$ for $s > t_0$. We have

$$H(\varphi,t_0) - H(\varphi,t) = h^{-}_{\varphi,t_0} - (h^{+}_{\varphi,t} - h^{+}_{g,t}) + \sum_{s>t} h_{g,s} =$$

$$= h^{-}_{\varphi,t_0} + [\sum_{s>t} h_{f,s} - h^{+}_{f,t}] - [\sum_{s>t} h_{g,s} - h^{+}_{g,t}]$$

From this it follows that

$$T(\varphi,t) - T(1/\varphi,t) = H(\varphi,t_0) - h^-_{\varphi,t_0} \tag{6}$$

By combining (5) and (6) we obtain $T(\varphi,t') \leq T(\varphi,t'')$. To prove (8) we assume that $\varphi(z) = f(z)/g(z)$, where $f(z)$ and $g(z)$ are two bounded holomorphic functions. From (6) it follows that

$$N(g,t) + m(g,t) = N(1/g,t) + m(1/g,t) + H(g,t_0) - h^-_{g,t_0} .$$

Then we have

$$\begin{aligned} N(1/g,t) &= m(g,t) - m(1/g,t) + N(g,t) + H(g,t_0) - h^-_{g,t_0} \\ &= -H(g,t) - h^-_{g,t_0} + N(g,t) \end{aligned}$$

Since g is bounded, so are $N(g,t)$, $H(g,t)$ and $N(\varphi,t) = N(1/g,t)$. Then $T(\varphi,t) = N(\varphi,t) + m(\varphi,t)$ is bounded also.

Now suppose $T(\varphi,t)$ is bounded. Then $N(\varphi,t)$ is bounded, and since $T(1/\varphi,t)$ is bounded, so is $N(1/\varphi,t)$. Suppose that $\varphi(z) = f(z)/g(z)$. It follows from (6) that

$$\begin{aligned} & m(f,t) - m(1/f,t) = N(1/f,t) - N(f,t) + 0(1) \\ & H(f,t) = N(f,t) + m(f,t) - N(1/f,t) + 0(1) \end{aligned}$$

Since $N(1/f,t) = N(1/\varphi,t)$ is bounded, we have $H(f,t) > -\infty$, and consequently $f(z)$ is bounded. Similarly $g(z)$ is bounded.

We are now in a position to prove theorem 4.4. We have

$$m(\frac{1}{\varphi-a},t) + N(\frac{1}{\varphi-a},t) = T(\frac{1}{\varphi-a},t) = T(\varphi-a,t) + 0(1)$$

Using Lemma 4.5 we obtain

$$T(\varphi - a,t) \le T(\varphi,t) + \log_p^+ a$$
$$T(\varphi,t) \le T(\varphi - a,t) + \log_p^+ a .$$

Since $T(\varphi,a,t) = T(\varphi - a,t)$, Theorem 4.4 is proved.

4.6. Theorem Let $\varphi(z)$ be a non–constant meromorphic function on D, $a_1,\dots,a_q$ be distinct numbers of $\mathbb{C}_p$. Then we have

$$m(\varphi,t) + \sum_{i=1}^{q} m(\frac{1}{\varphi - a_i},t) \le 2T(\varphi,t) - N_1(t) + 0(1)$$

where $N_1(t) = N(1/\varphi',t) + 2N(\varphi,t) - N(\varphi',t)$

Proof. Define

$$F(z) = \sum_{i=1}^{q} \frac{1}{\varphi(z) - a_i} .$$

By setting $\delta = \min_{i \neq k} |a_i - a_k|$ we first prove that

$$H^+(F,t) \ge \sum_{i=1}^{q} H^+(\varphi - a_i,t) - q \log^+ 1/\delta$$

If for all i, $|\varphi(z) - a_i| \geq \delta$ then this inequality is trivial since the value on the right is negative. Assume for some k, $|\varphi(z) - a_k| < \delta$ then for $i \neq k$ one has

$$|\varphi(z) - a_i| = \max\{|\varphi - a_k|, |a_i - a_k|\} \geq \delta$$

and hence

$$\frac{1}{|\varphi - a_k|} > 1/\delta, \quad \frac{1}{|\varphi - a_i|} \leq 1/\delta \quad \text{for every } i \neq k.$$

Consequently,

$$|F(z)| = \frac{1}{|\varphi(z) - a_k|}.$$

From this it follows that

$$H^+(F,t) = H^+(\frac{1}{|\varphi - a_k|},t) \geq \sum_{i=1}^{q} H^+(\varphi - a_i,t) - q \log^+ 1/\delta$$

On the other hand we have

$$m(F,t) = m(\frac{1}{\varphi} \cdot \frac{\varphi}{\varphi'} \cdot \varphi' F,t) \leq m(\frac{1}{\varphi'},t) + m(\frac{\varphi}{\varphi'},t) + m(\varphi' F,t)$$

In view of formula (6) we have

$$m(\frac{\varphi}{\varphi'},t) = m(\frac{\varphi'}{\varphi},t) + N(\frac{\varphi'}{\varphi'},t) - N(\frac{\varphi}{\varphi''},t) + 0(1)$$

$$m(\frac{1}{\varphi},t) = T(\varphi,t) - N(\frac{1}{\varphi},t) + 0(1)$$

Thus,

$$m(F,t) \leq T(\varphi,t) - N(1/\varphi,t) + m(\frac{\varphi'}{\varphi},t) + N(\frac{\varphi}{\varphi'},t) + m(\varphi' F,t) + 0(1)$$

$$\sum_{i=1}^{q} m(\frac{1}{\varphi - a_i},t) + m(\varphi,t) \leq T(\varphi,t) - N(\frac{1}{\varphi'},t) + N(\frac{\varphi'}{\varphi},t)$$

$$- N(\frac{\varphi}{\varphi'},t) + m(\frac{\varphi'}{\varphi},t) + T(\varphi,t) - N(\varphi,t) + m(\varphi' F,t) + 0(1)$$

It follows from formula (6) that

$$N(\frac{\varphi'}{\varphi},t) - N(\frac{\varphi}{\varphi'},t) = m(\frac{\varphi}{\varphi'},t) - m(\frac{\varphi'}{\varphi},t) + 0(1) = H(\varphi,t) - H(\varphi',t) + 0(1) =$$

$$= H(\varphi,t) - H(\varphi',t) + 0(1) = N(\frac{1}{\varphi'},t) - N(\varphi,t) - N(\frac{1}{\varphi'},t) + N(\varphi',t)$$

Thus we have

$$m(\varphi,t) + \sum_{i=1}^{q} m(\frac{1}{\varphi - a_i},t) \leq 2T(\varphi,t) - N_1(t) + m(\frac{\varphi'}{\varphi},t) + m(\sum_{i=1}^{q} \frac{\varphi'}{\varphi - a_i},t) + 0(1),$$

$$N_1(t) = N(1/\varphi',t) + 2N(\varphi,t) - N(\varphi',t)$$

It remains to prove that $m(\frac{\varphi'}{\varphi},t) + m(\sum_{i=1}^{q} \frac{\varphi'}{\varphi - a_i},t)$ is bounded. We prove the following.

4.7 Lemma. For every meromorphic function $\varphi(z)$ we have

$$m(\frac{\varphi'}{\varphi},t) \leq 2t$$

Proof. Assume $\varphi(z)$ is holomorphic in D:

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$$

Then we have

$$H(\varphi',t) = \min_{1 \leq n < \infty} \{v(na_n) + (n-1)t\} \geq \min_{0 \leq n < \infty} \{v(na_n) + nt\} - t = H(\varphi,t) - t.$$

Now let $\varphi = f/g$ where f and g are holomorphic functions. Then for every $t > 0$ we have

$$m(\varphi'/\varphi,t) = m(\frac{gf'-fg'}{g^2} \cdot \frac{g}{f},t) = m(\frac{f'}{f} - \frac{g'}{g},t) \leq m(\frac{f'}{f},t) + m(\frac{g'}{g},t) =$$

$$H^+(\frac{f'}{f},t) + H^+(\frac{g}{g'},t) = \max\{H(f,t) - H(f',t),0\} + \max\{H(g,t) - H(g',t),0\} \leq 2t$$

This completes the proof of Lemma 4.7 and Theorem 4.6.

We now return to the Second Main Theorem. We set

$$\overline{N}(\varphi,t) = N(\varphi',t) - N(\varphi,t)$$

Then $\overline{N}(\frac{1}{\varphi-a},t)$ is the number of distinct zeros of $\varphi(z)-a$ in $|z| \le p^{-t}$. We set

$$\delta(a) = \lim_{t\to 0} \frac{m(1/\varphi-a,t)}{T(\varphi,t)} = 1 - \overline{\lim_{t\to 0}} \frac{N(1/\varphi-a,t)}{T(\varphi,t)}$$

$$\theta(a) = \lim_{t\to 0} \frac{N(1/\varphi-a,t) - \overline{N}(1/\varphi-a,t)}{T(\varphi,t)}$$

$$\Theta(a) = 1 - \overline{\lim_{t\to 0}} \frac{\overline{N}(1/\varphi-a,t)}{T(\varphi,t)}$$

4.8 Theorem Let $\varphi(z)$ be a meromorphic function on D. Then the set of values $a \in \mathbb{C}_p$ such that $\Theta(a) > 0$ is finite or countable and furthermore we have

$$\sum_a (\delta(a) + \theta(a) \le \sum \Theta(a) \le 2$$

Proof. given $\epsilon > 0$, for t sufficiently close to zero it holds

$$N(\frac{1}{\varphi-a},t) - \overline{N}(\frac{1}{\varphi-a},t) > (\theta(a) - \epsilon)T(\varphi,t)$$

$$N(\frac{1}{\varphi-a},t) < (1-\delta(a) + \epsilon)T(\varphi,t)$$

Hence

$$\overline{N}(\frac{1}{\varphi-a},t) < (1 - \delta(a) - \theta(a) + 2\epsilon)T(\varphi,t)$$

Thus

$$\Theta(a) \ge \delta(a) + \theta(a)$$

Let $a_1,\dots,a_q$ be arbitrary distinct numbers of $\mathbb{C}_p$. Adding $N(\varphi,t) + \sum_{i=1}^{q} N(\frac{1}{\varphi-a_i},t)$ to both sides of the inequality in Theorem 4.6 we obtain:

$$(q+1)T(\varphi,t) \leq \sum_{i=1}^{q} N(\frac{1}{\varphi-a_i},t) + N(\varphi,t) - N_1(t) + 2T(\varphi,t) + 0(1)$$

From this it follows that

$$(q-1)T(\varphi,t) \leq \sum_{i=1}^{q} N(\frac{1}{\varphi-a_i},t) + \overline{N}(\varphi,t) - N(\frac{1}{\varphi'},t) + \text{[illegible]} + 0(1)\,.$$

We note that $N(\frac{1}{\varphi-a_i},t)$ is the sum of valuations $v(z_k)$ of zeros z_k of the function $\varphi-a_i$, but every zero of order ℓ of $\varphi - a_i$ is a zero of order $\ell - 1$ of φ', we obtain:

$$\sum_{i=1}^{q} N((\frac{1}{\varphi-a_i},t) - N((\frac{1}{\varphi'},t) \leq \sum_{i=1}^{q} \overline{N}((\frac{1}{\varphi-a_i},t)\,.$$

Consequently,

$$(q-1)T(\varphi,t) \leq \sum_{i=1}^{q} \overline{N}((\frac{1}{\varphi-a_i},t) + \overline{N}(\varphi,t) + 0(1).$$

Therefore,

$$\sum_{i=1}^{q} \overline{\lim_{t\to 0}} \frac{\overline{N}(1/\varphi - a_i, t)}{T(\varphi, t)} + \overline{\lim_{t\to 0}} \frac{\overline{N}(\varphi,t)}{T(\varphi,t)} \geq q-1,$$

$$\sum_{i=1}^{q} (1-\Theta(a_i)) + 1 - \Theta(\infty) \geq q-1$$

$$\sum_{i=1}^{q} \Theta(a_i) + \Theta(\infty) \leq 2$$

This proves Theorem 4.8.

4.9. From the First and Second Main Theorems we have a number of corollaries about properties of p–adic meromorphic functions. Since the proofs in many cases are similar to those in the complex case, we formulate them without proofs.

For each $a \in \mathbb{C}_p$ we let $E_a(\varphi)$ denote the set of points $z \in D$ for which $\varphi(z) = a$, where every points is taken as many times as its multiplicity of being a root of the equation $\varphi(z) - a = 0$.

Corollary. Suppose that $\varphi_1(z)$ and $\varphi_2(z)$ are two meromorphic functions on D for which there exist three distinct values $a_1, a_2, a_3 \in \mathbb{C}_p$ such that $E_{a_i}(\varphi_1) \equiv E_{a_i}(\varphi_2)$, $i = 1,2,3$. Assume moreover at least one of them is not a ratio of two bounded holomorphic functions. Then $\varphi_1 \equiv \varphi_2$.

4.10 Corollary. Let $R(u)$ be a rational function of degree d and $f(z)$ be a meromorphic function on $\{z \in \mathbb{C}_p, |z| < R\}$, $R \leq \infty$. Then we have

$$T(R(f),t) = dT(f,t) + 0(1)$$

when $t \longrightarrow -\log_p R$

4.11. Corollary. A meromorphic function $f(z)$ is transcendental if and only if

$$\lim_{t\to-\infty} \frac{T(f,t)}{-t} = \infty$$

4.12 Corollary. For a meromorphic function on D we have

$$\sum_{a\in\mathbb{C}_p} \Theta(a,f^{(k)}) \leq 1 + \frac{1}{k+1}$$

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