AN ELLIPTIC ANALOGUE OF THE GAUSS HYPERGEOMETRIC FUNCTION

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The Gauss hypergeometric function is one of the key special functions in mathematics. It can be defined in many ways, e.g., by either an infinite series or the Euler integral representation [1]:

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} x^n = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a} dt, \tag{1}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol and $\Gamma(x)$ is the Euler gamma function. Admissible domains of parameters for these representations are not indicated for brevity. This function satisfies the second-order differential equation with three singular points

$$x(1-x)y''(x) + (c-(a+b+1)x)y'(x) - aby(x) = 0. \tag{2}$$

When, say, a is a negative integer, the series terminate and define the well-known system of Jacobi orthogonal polynomials.

Various extensions of this function have been proposed in the literature on the basis of generalized plain or q-hypergeometric series [1]. The theory of quantum and classical completely integrable systems led to a new class of functions of hypergeometric type related to elliptic curves [3, 5, 6, 9] (for a brief review of the related results, see [8]).

For $|p|, |q| < 1$, the infinite products

$$(z;q)_\infty = \prod_{j=0}^{\infty}(1-zq^j), \quad (z;p,q)_\infty = \prod_{j,k=0}^{\infty}(1-zp^jq^k)$$

are well defined. The odd Jacobi theta function has the form

$$\theta_1(u|\tau) = \sum_{k\in\mathbb{Z}+1/2} e^{\pi i\tau k^2 + 2\pi i k(u-1/2)} = ip^{1/8}e^{-\pi i u}\,(p;p)_\infty\,\theta(e^{2\pi i u};p), \tag{3}$$

where $p = e^{2\pi i\tau}$ and $\theta(z;p) := (z;p)_\infty(pz^{-1};p)_\infty$. We have

$$\theta(pz;p) = \theta(z^{-1};p) = -z^{-1}\theta(z;p) \tag{4}$$

and $\theta(z;p) = 0$ for $z = p^k,\ k \in \mathbb{Z}$. With the help of compact notation

$$\theta(a_1,\ldots,a_k;p) := \theta(a_1;p)\cdots\theta(a_k;p), \quad \theta(at^{\pm};p) := \theta(at;p)\theta(at^{-1};p)$$

the addition formula (a Riemann relation) for theta functions takes the form

$$\theta(xw^{\pm}, yz^{\pm};p) - \theta(xz^{\pm}, yw^{\pm};p) = yw^{-1}\theta(xy^{\pm}, wz^{\pm};p). \tag{5}$$

Euler's gamma function can be defined as a special meromorphic solution of the functional equation $f(u+\omega_1) = uf(u)$ for a nonzero constant ω_1. q-Gamma functions are connected to solutions of the equation $f(u+\omega_1) = (1-e^{2\pi i u/\omega_2})f(u)$ with $q = e^{2\pi i\omega_1/\omega_2}$. For $|q|<1$, one of its solutions has the form $\Gamma_q(u) = 1/(e^{2\pi i u/\omega_2};q)_\infty$ defining the standard q-gamma function (we skip consideration of the case $|q|=1$). Elliptic gamma functions are analogously connected to the equation

$$f(u+\omega_1) = \theta(e^{2\pi i u/\omega_2};p)f(u). \tag{6}$$

The ratio

$$\Gamma(z;p,q) = \frac{(pqz^{-1};p,q)_\infty}{(z;p,q)_\infty} \tag{7}$$

for $z = e^{2\pi i u/\omega_2}$ yields a particular solution of (6) [5]. Due to the symmetry between p and q, this elliptic gamma function satisfies two linear difference equations of the first order

$$\Gamma(qz;p,q) = \theta(z;p)\Gamma(z;p,q), \quad \Gamma(pz;p,q) = \theta(z;q)\Gamma(z;p,q).$$

The reflection formula for it has the form $\Gamma(z;p,q)\ \Gamma(pq/z;p,q) = 1$, and $\Gamma(z;0,q) = 1/(z;q)_\infty$. There is a partner of this function well defined for $|q|=1$ [8]. According to [2], the elliptic gamma functions are the $SL(3;\mathbb{Z})$-group modular objects. Also, they can be represented as particular combinations of four Barnes multiple gamma functions of the third order.

The very well poised elliptic hypergeometric integrals with even number of parameters are defined by the expression

$$I^{(m)}(t_1,\ldots,t_{2m+6}) = \kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^{2m+6}\Gamma(t_j z^{\pm};p,q)}{\Gamma(z^{\pm 2};p,q)}\frac{dz}{z}, \tag{8}$$

where $\mathbb{T}$ denotes the positively oriented unit circle, $\kappa = (p;p)_\infty(q;q)_\infty/4\pi i$, and the parameters are constrained by the balancing condition $\prod_{j=1}^{2m+6} t_j = (pq)^{m+1}$. Here we use the compact notation

$$\Gamma(a_1,\ldots,a_k;p,q) := \Gamma(a_1;p,q)\cdots\Gamma(a_k;p,q),$$
$$\Gamma(tz^{\pm};p,q) := \Gamma(tz;p,q)\Gamma(tz^{-1};p,q), \quad \Gamma(z^{\pm 2};p,q) := \Gamma(z^2;p,q)\Gamma(z^{-2};p,q).$$

We assume that $|t_j|<1$, then $\mathbb{T}$ separates sequences of poles of the integrand of $I^{(m)}$ converging to zero from their reciprocals going to infinity. Analytical continuation of these integrals define meromorphic functions which can be considered as elliptic analogues of the plain hypergeometric functions ${}_{m+1}F_m$. Indeed, the first member of this hierarchy is known explicitly

Theorem 1. [6]

$$I^{(0)}(t_1,\ldots,t_6) = \prod_{1\le j<k\le 6}\Gamma(t_j t_k;p,q). \tag{9}$$

This is the most general known univariate exact integration formula generalizing Euler's beta integral [1]. In appropriate $p \to 0$ limit, one obtains the Rahman q-beta integral, which is a one-parameter extension of the Askey-Wilson integral [1].

The elliptic beta integral (9) leads to the following recurrence relation

$$I^{(m+1)}(t_1, \ldots, t_{2m+8}) = \frac{\prod_{2m+5 \le k < l \le 2m+8} \Gamma(t_k t_l; p, q)}{\Gamma(\rho_m^2; p, q)} \tag{10}$$
$$\times \kappa \int_{\mathbb{T}} \frac{\prod_{k=2m+5}^{2m+8} \Gamma(\rho_m^{-1} t_k w^{\pm}; p, q)}{\Gamma(w^{\pm 2}; p, q)} I^{(m)}(t_1, \ldots, t_{2m+4}, \rho_m w, \rho_m w^{-1}) \frac{dw}{w},$$

where $\rho_m^2 = \prod_{k=2m+5}^{2m+8} t_k / pq$. This recurrence is a special realization of an integral analogue of the Bailey chains discovered in [7] (for an application of the Bailey chains technique to the proof of Rogers-Ramanujan type identities, see, e.g. [1]). It allows us to find a multiple integral representation for $I^{(m)}$ similar to the one for ${}_{m+1}F_m$.

For $m = 0$, substitution of the explicit expression for $I^{(0)}$ (9) in the right-hand side of (10) yields the identity

$$V(\underline{t}) = \prod_{1 \le j < k \le 4} \Gamma(t_j t_k, t_{j+4} t_{k+4}; p, q)\, V(\underline{s}), \tag{11}$$

where $V(\underline{t}) := I^{(1)}(t_1, \ldots, t_8)$ is our elliptic analogue of the Gauss hypergeometric function and

$$\left\{ \begin{array}{ll} s_j = \rho t_j, & j = 1,2,3,4 \\ s_j = \rho^{-1} t_j, & j = 5,6,7,8 \end{array} \right. ; \quad \rho = \sqrt{\frac{pq}{t_1 t_2 t_3 t_4}} = \sqrt{\frac{t_5 t_6 t_7 t_8}{pq}},$$

and $|t_j|, |s_j| < 1$. This reflection transformation in the parameter space appears to belong to the Weyl group for the root system E_7 [4]. It represents an elliptic analogue of the Bailey transformation for four non terminating ${}_{10}\varphi_9$-series. Sums of residues of particular finite sequences of the V-function integrand define elliptic hypergeometric series. However, infinite sums of such residues in general do not converge and, therefore, the infinite series representation for the V-function is not well defined.

We denote by $V(qt_j, q^{-1}t_k)$ elliptic hypergeometric functions contiguous to $V(\underline{t})$ in the sense that t_j and t_k are respectively replaced by qt_j and $q^{-1}t_k$. The following contiguous relation for the V-functions is true

$$t_7 \theta\left(t_8 t_7^{\pm}/q; p\right) V(qt_6, q^{-1}t_8) - (t_6 \leftrightarrow t_7) = t_7 \theta\left(t_6 t_7^{\pm}; p\right) V(\underline{t}), \tag{12}$$

where $(t_6 \leftrightarrow t_7)$ denotes the permutation of parameters in the preceding expression. It is equivalent to the addition formula for theta functions (5) (V's kernel satisfies the same equation).

Considering all possible reflections for the root system E_7, one can obtain many symmetry tansformations for the V-function. Substituting them into (12), one obtains a large set of contiguous relations connecting V-functions with different choices of parameters. An appropriate combination of them yields the equation

$$\mathcal{A}(\underline{t})\Big(U(qt_6, q^{-1}t_7) - U(\underline{t})\Big) + (t_6 \leftrightarrow t_7) + U(\underline{t}) = 0, \tag{13}$$

where we have denoted $U(\underline{t}) = V(\underline{t})/\Gamma(t_6t_8^{\pm}, t_7t_8^{\pm}; p, q)$ and

$$\mathcal{A}(\underline{t}) = \frac{\theta(t_6/qt_8, t_6t_8, t_8/t_6; p)}{\theta(t_6/t_7, t_7/qt_6, t_6t_7/q; p)} \prod_{k=1}^{5} \frac{\theta(t_7t_k/q; p)}{\theta(t_8t_k; p)}. \tag{14}$$

Substituting $t_j = e^{2\pi i g_j}$, one can check that the potential $\mathcal{A}(\underline{t})$ is a modular invariant elliptic function of the variables $g_1, \ldots, g_7$ (g_8 is considered as a dependent variable determined from the balancing condition).

If we set $t_6 = cx, t_7 = c/x$, then the balancing condition $c^2t_1 \cdots t_5t_8 = p^2q^2$ does not depend on x. Replacing $U(\underline{t})$ in (13) by some function $f(x)$, we come to the second order q-difference equation with elliptic function coefficients which is called the elliptic hypergeometric equation. In an appropriate limit it reduces to the classical hypergeometric equation (2). We have already one functional solution of this equation given by the U-function, the second linearly independent solution can be obtained by applying transformations $t_k \to pt_k$ for $k = 1, \ldots, 7$ or modular transformation which do not change the function $\mathcal{A}(\underline{t})$. Particular elliptic hypergeometric series solutions of this equation lead to a nice set of biorthgonal functions similar to the Jacobi polynomials [8].

Interesting properties of the V-function follow from the consideration of recurrence (10) for $m = 1$ with a special choice of parameters (say, $t_5t_7 = t_6t_8 = pq$, when the left hand side is computable by the elliptic beta integral). In particular, one obtains in this way an infinite dimensional module of the Sklyanin algebra. A detailed account of the related questions will be given elsewhere.

Analytical properties of the Gauss hypergeometric function ${}_2F_1$ have their elliptic analogues (modulo changes like a replacement of the differential equation by a finite difference equation). However, many features of this function, like number theoretic or geometric aspects, remain obscure from the elliptic point of view.

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