Deformations of large fundamental groups

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1 Introduction

J. Kollár has introduced the notion of a variety with a large fundamental group in [24]. Let us recall the definition:

Definition 1.1 *A smooth projective variety X has a large fundamental group if $im[\pi_1(res(Z)) \to \pi_1(X)]$ is an infinite group for every irreducible and positive dimensional subvariety Z of X, where $res(Z)$ denotes any resolution of Z.*

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For the sake of a clear exposition we name the subvarieties that brake the largeness of the fundamental group.

Definition 1.2 *A subvariety $Z \subset X$ is π_1- small iff $im[\pi_1(res(Z)) \to \pi_1(X)]$ is a finite group. In case Z is one dimensional we will call it π_1-small curve.*

Kollár has introduced the Definition 1.1 in his studies of a conjecture by Shafarevich: The universal covering of every smooth projective variety is holomorphically convex.

In this paper we address the following question: Is the property of a variety having a large fundamental group stable under Kähler deformations?

The next theorems are the main tools we will use in our approach to the question raised above. They are of different nature which give us extra flexibility to tackle these type of problems. These results stand by themselves and have applications outside this article.

Theorem 1.1 *Let $f : X \to D$ be a Kähler deformation of smooth Kähler variety X_t over disc D. Assume that there exists a homomorphisms $\rho : \pi_1(X_t) \to GL(n, \mathbb{C})$ with a kernel H. Then there exists a relative Shafarevich morphism $f_{sh} : sh^H(X) \to D$ for the family $f : X \to D$, such that when restricted to a point $t \in D$ it becomes $sh^H(X_t$*

For description of the Shafarevich morphism see the subsection 2.1.

Theorem 1.2 *Let $f : X \to D$ be a Kähler deformation of $X_0 = f^{-1}(0)$, where X_0 has a Stein universal covering. Fix any $N \in \mathbb{N}$, then by shrinking D we can assume*

$$Vol_{\tilde{w}_t}(Z) > N$$

For all compact subvarieties of the universal covering $\tilde{X}_t$ of X_t with $t \in D$ ($\tilde{w}_q$ is the induced the Kähler metric on $\tilde{X}_t$).

Theorem 1.3 *Let $\rho : X \to T$ be a proper morphism with equidimensional fibers between analytic spaces. If $X_{t_o} = \rho^{-1}(t_0)$ does not contain a π_1- small irreducible component then there exists an open subset U, $t_0 \in U \subset T$ s.t. all $X_t = \rho^{-1}(t)$, $t \in U$, do not contain π_1-small irreducible components.*

The following consequence of Theorem 1.2 gives restriction on the π_1-small subvarieties that occur in a small Kähler deformations of a variety with Stein universal covering.

Theorem 1.4 *Let $f : X \to D$ be a Kähler deformation of $X_0 = f^{-1}(0)$ where X_0 has a Stein universal covering. Fix any $N \in \mathbb{N}$, then by shrinking D we can assume*

$$Vol_{w_t}(Z).(\#im[\pi_1(res(Z) \to \pi_1(X_q)] + 1) > N$$

For all π_1-small subvarieties Z of X_t with $t \in D$.

In this paper we use the results stated above mainly to study the behavior of large fundamental groups under deformations of Kähler surfaces. Here is one of the main theorems we obtain:

Theorem 1.5 *Let X be a Kähler surface with $\rho : \pi_1(X) \to GL(N, \mathbb{C})$ an infinite linear representation. Then having a large fundamental group is stable under small Kähler deformations of X.*

Remark 1.1 *All known examples of Kähler surfaces with infinite fundamental groups have an infinite linear representation of their fundamental groups.*

Theorem 1.5 and the corollary stated below give partial answers to the following conjectures proposed in [4].

Conjecture 1.1 *Let X be a compact Kähler manifold whose universal covering is Stein. Then small Kähler deformations have large fundamental group.*

If the Shafarevich conjecture is correct conjecture 1.1 implies

Conjecture 1.2 *Every small deformation X_t of a smooth projective variety X with a Stein universal covering $\widetilde{X}$ has an universal covering $\widetilde{X_t}$, which is still Stein.*

Corollary 3.4 Let X be a compact Kähler surface with an infinite linear representation $\rho : \pi_1(X) \to GL(N, \mathbb{C})$. Suppose the infinite cover defined by $Ker(\rho)$ is Stein then a small Kählerian deformation of X still has a Stein universal covering.

It was shown in [4] that conjecture 1.2 is not correct without the Kähler assumption, we also note that what was lost in that example was not the property of the fundamental group being large but the holomorphic convexity.

The organization of the paper is the following:

i) In section 2 we discuss and prove the main tools we need for the proofs.

ii) In section 3 we give the proofs of theorem 1.4 , 1.5 and some corollaries.

iii) In section four some more applications are discussed.

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2 Tools

In this section we discuss some techniques used in the proofs. We start with some facts from nonabelian Hodge theory.

2.1 The relative Shafarevich morphism in families

In this subsection we recall some results about the Shafarevich varieties and develop some tools of nonabelian Hodge theory, which are also of independent interest.

One can easily notice that the Shafarevich conjecture is equivalent to the following two conditions:

1) There exists a normal analytic space $sh(\widetilde{X})$ with no compact complex subspaces and a proper morphism with connected fibers $sh : \widetilde{X} \longrightarrow sh(\widetilde{X})$. This morphism is known as the Cartan-Remmert reduction of $\widetilde{X}$.

2) $sh(\widetilde{X})$ is a Stein space.

Kollár observed in [23] and [24] that 1) is equivalent to:

There exists a normal variety $sh(X)$ and a proper morphism with connected fibers $sh : X \longrightarrow sh(X)$, which contracts exactly the subvarieties Z in X having the property that $im[\pi_1(res(Z)) \longrightarrow \pi_1(X)]$ is finite.

Clearly the variety $sh(X)$ has a large fundamental group. Campana [7] and Kollár [24], [23] have shown that a rational map with connected fibers $sh : X \longrightarrow sh(X)$ always exists. This indicates that as long as the information contained in the fundamental group is the essential one we can restrict ourselves to two big classes of varieties:

1) Varieties with a finite fundamental group;

2) Varieties with a large fundamental group.

It appears that varieties with large fundamental group are an interesting class to study. Much work in this direction has been done by F. Bogomolov, T. Napier, R. Narasimhan, M. Nori, T. Pantev, M. Ramachandran (see e.g. [14],[22], [20],[24], [23], [27]). Recently using ideas of Campana and Kollár the following statements were proved (see [19], [20]).

Theorem 2.A Let X be a Kähler manifold which has an almost faithful representation of its fundamental group to $GL(n, \mathbb{C})$. Then the Shafarevich morphism $sh : X \to sh(X)$ exists.

Remark 2.1 *The variety $sh(X)$ is birationally isomorphic to the one defined by Campana and Kollár.*

Theorem 2.B Let X be a Kähler surface which has an almost faithful representation ρ of its fundamental group to any affine complex group. Then the Shafarevich conjecture is true for X.

The above theorems were proved using nonabelian Hodge theory developed recently by Corlette [8], Gromov and Schoen [15] and Simpson [31], [34], [33].

Kollár has also defined the H-Shafarevich morphism. Let $H \triangleleft \pi_1(X)$ be a normal subgroup. We will say that a subgroup $\Gamma \subset \pi_1(X)$ is almost contained in H if the intersection $\Gamma \cap H$ has finite index in Γ and we will write $\Gamma \lesssim H$. Then the H-Shafarevich variety is a normal variety $sh^H(X)$ and a proper map with connected fibers $sh^H : X \longrightarrow sh^H(X)$, which contracts exactly the subvarieties Z in X having the property that $im[\pi_1(res(Z)) \longrightarrow \pi_1(X)] \lesssim H$.

Now if we have an infinite representation of the fundamental group of X in $GL(n, \mathbb{C})$ with a kernel H we get:

Theorem 2.C([19]) There exists a normal variety $sh^H(X)$ and a proper map with connected fibers $sh^H : X \longrightarrow sh^H(X)$, which contracts exactly the subvarieties Z in X having the property that $im[\pi_1(res(Z)) \longrightarrow \pi_1(X)] \lesssim H$. Moreover there exists a finite covering and a desingularization of $sh^H(X)$ which makes it a fibration of abelian varieties over a variety of a general type.

The variety $sh^H(X)$ can be seen as a nonabelian analogue of the Albanese variety. Now we give a version of the above theorem in families.

Theorem 1.1 Let $f : X \to D$ be a Kähler deformation of a smooth Kähler surface X_t over disc D. Assume that there exists a homomorphisms $\rho : \pi_1(X_t) \to GL(n, \mathbb{C})$ with a kernel H. Then there exists (may be after a base change) a relative Shafarevich morphism $f_{sh} : sh^H(X) \to D$ for the family $f : X \to D$, such that when restricted to a point $t \in D$ it becomes $sh^H(X_t)$.

Proof Let us first repeat the definition of the variety $sh^H(X)$ following [19]. With some abuse of the notations we will denote X_0 by X. We start with a representation $\rho(\pi_1(X)) \to GL(n, \mathbb{C})$. We will use several reductions.

Step 1.

First we consider the case when ρ has reductive image S - the Zariski closure of $\rho(\pi_1(X))$ in $GL(n, \mathbb{C})$. For the moment we will make two simplifying assumptions:

- S is semisimple;
- $\rho(\pi_1(X)) \subset S \cap GL(n, \overline{\mathbb{Q}})$

These assumptions together with the fact that $\pi_1(X)$ is a finitely generated group imply that $\rho : \pi_1(X) \to S$ is defined over a finite extension E of $\mathbb{Q}$. Let O be the ring of integers in E. Put $O_{\mathfrak{p}}$ for the localization of O at some prime ideal $\mathfrak{p}$ in O and $\widehat{O}_{\mathfrak{p}}$ for the completion of $O_{\mathfrak{p}}$ with respect to the natural discrete valuation. Denote by $E_{\mathfrak{p}}$ the field of fractions of $\widehat{O}_{\mathfrak{p}}$ and let $S(E_{\mathfrak{p}})$ be the group of $E_{\mathfrak{p}}$-points of S. Since $E_{\mathfrak{p}}$ is a local field we can use the Bruhat-Tits theory to attach to

$S(E_{\mathfrak{p}})$ an affine building $B_{\mathfrak{p}}$. Similarly we consider the symmetric space $\mathrm{Symm}_{\nu} = S(E_{\nu})/(\text{maximal compact subgroup})$ at an archimedean place ν. Due to our assumptions ρ induces a representation $\rho' : \pi_1(X) \to S(E_{\mathfrak{p}})$ with $\ker(\rho) = \ker(\rho')$. Here we introduce a definition:

Definition 2.1 *We call the representation $\rho' : \pi_1(X) \to S(E_{\mathfrak{p}})$ unbounded if its image is not contained in a maximal compact subgroup of $S(E_{\mathfrak{p}})$.*

We are in a position to apply the formalism of nonabelian Hodge theory. We introduce the technique of spectral coverings to show the existence of nonabelian analogues of the Albanese map. This technique is based on the existence of equivariant pluriharmonic maps to symmetric spaces and buildings proved by Corlette [8] in the case of symmetric spaces and by Gromov and Schoen [15] in the case of buildings.

The following theorem was proved in [20]:

Theorem 2.D Let X be a smooth projective variety which has a Zariski dense unbounded representation $\rho' : \pi_1(X) \to S(E_{\mathfrak{p}})$. Then the Shafarevich morphism $sh^H : X \to sh^H(X) = Y$ exists and $\dim Y \leq rank S(E_{\mathfrak{p}})$.

For the proof of this theorem one needs to utilize the algebraic geometric information contained in the harmonic map data. This is achieved by constructing the so called spectral cover, which we proceed to describe.

Using the existence results in [15] we assign to every unbounded representation $\rho' : \pi_1(X) \to S(E_{\mathfrak{p}})$ a nontrivial ρ'-equivariant harmonic map $U : \widetilde{X} \to B_{\mathfrak{p}}$ to the affine Bruhat-Tits building of $S(E_{\mathfrak{p}})$. In every apartment Ap of $B_{\mathfrak{p}}$ we have a canonical choice of coordinates modulo the action of the affine Weyl group $\widetilde{W}$. This means that locally we can choose one coordinate z_1 and consider also $e_1(z_1), \ldots, e_w(z_1)$, where $e_1, \ldots, e_w$ are all the elements of the usual Weyl group. Let $U^*(T^*Ap)$ be the pullback of the cotangent bundle of Ap and let d denote the exterior differentiation on Ap. Then we can define, modulo the action of W, the differential forms $de_1(z_1), \ldots, de_w(z_1)$ globally on the whole building. (Observe that after taking differentials we can make everything invariant under the translation part of the affine Weyl group.) Consider the complexified differentials $U^*de_1(z_1), \ldots, U^*de_w(z_1)$. Due to the harmonicity of U the (1,0) part of each of the forms $U^*de_1(z_1), \ldots, U^*de_w(z_1)$ is holomorphic. This follows from the fact that by theorem 7.3 [15] U is actually pluriharmonic outside the singular set. We will denote these new holomorphic one forms again by $U^*de_1(z_1), \ldots, U^*de_w(z_1)$. Since U is an equivariant map these holomorphic one forms descend to multivalued forms on X. Choose a regular weight α for $S(E_{\mathfrak{p}})$. It determines a W-invariant $\mathbb{C}[x]$-valued function $P_\alpha : \mathfrak{h}\otimes\mathbb{C} \to \mathbb{C}[x]$, $P_\alpha(x) := \prod_{\mu\in W\alpha}(x-\mu)$ on the Cartan algebra $\mathfrak{h}$ of $S(E_{\mathfrak{p}})$. Let $\kappa_1,\ldots,\kappa_w$ be the coefficients of P_α. The κ_i's are W-invariant polynomials on $\mathfrak{h}\otimes\mathbb{C}$. By using the standard identification of the complexifications of apartment Ap and the Cartan algebra $\mathfrak{h}$ we can evaluate $\kappa_1,\ldots,\kappa_r$ on the multi-valued forms $U^*de_1(z_1),\ldots,U^*de_w(z_1)$ to obtain sections h_i of $Symm^i\Omega^1_X$, $i=1,\ldots,w$ well defined and holomorphic outside of the singularities of the harmonic map U. Furthermore, according to [15] the map U is Lipschitz and therefore we get that the multi-holomorphic differentials $h_1,\ldots,h_w$ are bounded. Using theorem 3.2 [20] we conclude that $h_1,\ldots,h_w$ extend to holomorphic differentials on the whole X.

Let $p:\mathcal{X}\to X$ be the total space of the cotangent bundle T^*X and let $\lambda\in H^0(\mathcal{X},p^*\Omega^1_X)$ be the tautological section. We define a spectral covering corresponding to the elements

$$\left\{h_i \in H^0(X, Symm^i\Omega^1_X)\right\}_{i=1}^{w}$$

to be the zero scheme $\mathfrak{S}$ of the section

$$\lambda^w + p^*(h_1)\lambda^{w-1} + \ldots + p^*(h_w) \in H^0(\mathcal{X}, p^*Symm^w\Omega^1_X).$$

Notice that different choices of a regular weight α result in birationally isomorphic $\mathfrak{S}$. $\mathfrak{S}$ is a projective subscheme of $\mathcal{X}$ and by construction the natural morphism $\mathfrak{S}\to X$ is a Galois cover with Galois group W. For more detailed account on spectral coverings see [11] and [10].

The statement of Theorem 1.1 now follows from the abundance of holomorphic one forms on the spectral covering S. We consider the map $s : S\to ALB(S)/B$, where $ALB(S)$ is the Albanese variety of S and B is the subvariety of $ALB(S)$ generated by the image in $ALB(S)$ of the degeneracy set of the linearly independent one forms on S. Denote by k the dimension of the image R of S in $ALB(S)/B$. If the degeneracy set of the linearly independent holomorphic one forms on R is a divisor in R. Then we are in a position to apply a version of Castelnuovo de Francis theorem see [29]. If the degeneracy set is of codimension at least two then by the theorem of Napier and Ramachandran [28] we can find a curve M in R that avoids this set and such that the image of $\pi_1(M)$ in $\pi_1(R)$ is a subgroup of finite index. In this situation one easily gets a contradiction with the Zariski denseness of the representation $\rho' : \pi_1(X)\to S(E_{\mathfrak{p}})$. Morally the above argument says that the information about the representation $\rho' : \pi_1(X)\to S(E_{\mathfrak{p}})$ is contained in the degeneracy set of the linearly independent holomorphic one forms on S or in other words in the ramification divisor of the spectral covering.

Finally let us mention that one gets a factorization not of X but of a blow up of some finite nonramified covering X' of X. To get the existence of the Shafarevich morphism $sh^H : X \to sh^H(X) = Y$ one argues as follows. By the above Castelnuovo de Franchis argument we get that there exists a normal projective variety Y and a map $f : X \to Y$ with connected fibers so that for some finite etale cover $X' \to X$ the variety $Sh^H(X')$ exists and the Shafarevich map $sh^H : X' \to Sh^H(X')$ is just the Stein factorization of the composition $X' \to X \to Y$. The blowups of X' mentioned above are not essential here. They will be needed later to make the variety $Sh^H(X')$ smooth, which is used in the Hodge theoretic arguments later when we use the theory of MHS in the proof. The finite cover X' though is essential since we need to deal with the orbi-phenomenon. Now it only remains to observe that Y must be $Sh^H(X)$ because any subvariety in X has a finite étale cover that is a subvariety in X' and because the map $Sh^H(X') \to Y$ is finite. Observe that there is no difference between $Sh^H(X')$ and Y from the view point of the Shafarevich map. The difference becomes visible only from the view point of the representation ρ. Namely $\pi_1(X') \to S(E_\mathfrak{p})$ factors completely through $\pi_1(Sh^H(X'))$ but $\rho' : \pi_1(X) \to S(E_\mathfrak{p})$ factors only through an orbi-extension of $\pi_1(Y)$. In other words the subvarieties of X whose fundamental groups are commensurable with the kernel of ρ do not see this orbistructure.

Lemma 2.A There exists at least one place ν for which the representation $\rho' : \pi_1(X) \to S(E_\nu)$ is unbounded.

If we assume that for all places ν the corresponding representations are bounded then we get that our initial representation $\rho : \pi_1(X) \longrightarrow S$ is finite. Indeed, only finitely many among the maps $U_\mathfrak{p}$ and U_ν are nontrivial since $\rho'(\pi_1(X))$ is finitely generated and hence only finitely many elements of O occur in the denominators of the traces of its generators. Combined with the fact that $E_\mathfrak{p}$ is discrete in $\prod_\mathfrak{p} E_\mathfrak{p} \times \prod_\nu E_\nu$ this gives that the image of ρ' acts discretely on a finite product $\prod_\mathfrak{p} B_\mathfrak{p} \times \prod_\nu \mathrm{Symm}_\nu$ via the diagonal action (for the proof of discreteness see [6]). Thus $\rho'(\pi_1(X))$ is simultaneously discrete and compact, hence finite, which contradicts the Zariski denseness of ρ.

If we have an unbounded representation at the infinite place then we can use the results of Zuo [37] and Mok [29] (in case of discrete representation) to get a factorization. Next we remove the two simplifying assumptions we made at the beginning of Step 1. The above arguments show that we get the existence of the Shafarevich morphism for every semisimple representation defined over $\overline{\mathbb{Q}}$. To treat a general semisimple ρ we would like to deform $\rho : \pi_1(X) \longrightarrow S$ to a representation that is defined over $\overline{\mathbb{Q}}$. Observe that existence of a nontrivial deformation of $\rho : \pi_1(X) \longrightarrow S$ is not an issue since the rigid representations are always defined over $\overline{\mathbb{Q}}$.

Lemma 2.B In a small neighborhood of $\rho : \pi_1(X) \longrightarrow S$ we can find a Zariski dense representation defined over $\overline{\mathbb{Q}}$.

The proof of this lemma can be found in [22]. Observe that when deforming we might enlarge the kernel of the original representation. Now we need the following:

Lemma 2.C If the Shafarevich morphism exists for every representation defined over $\overline{\mathbb{Q}}$ in a neighborhood of $\rho : \pi_1(X) \longrightarrow S$ in the moduli space of representations then it exists for $\rho : \pi_1(X) \longrightarrow S$.

Proof See [20].

The passage from semisimple to a reductive representation is almost straightforward. Indeed up to a finite index every reductive group is a product of a semisimple group and a torus. If this torus is trivial we go to Step 1. If it is not this means that we have a nontrivial Albanese variety $Alb(X)$. So we consider the map $X \to Y \times Alb(X)$, where Y is the factorization variety for the semisimple representation. The image of X under this map will be the Shafarevich variety for our reductive representation.

Step 2.

Now we move to the case of representations to a linear group L. This case requires some new ideas. Every linear group L is an extension of its nilpotent radical N and its reductive quotient G

$$1 \to N \to L \to G \to 1.$$

If G is a trivial group then we just use the Albanese map. If G is a nontrivial group using the existence of a mixed Hodge structure - MHS on the relative completion of $\pi_1(X)$ described in [34] one can prove the following:

Theorem 2.E Let X be a smooth projective variety such that it admits an infinite representation $\rho : \pi_1(X) \to L$ to a complex linear group with kernel H. Assume that the representation $\rho : \pi_1(X) \to G$ factors through a representation of the fundamental group of a variety Y. Then $\rho : \pi_1(X) \to L$ factors through a representation of the fundamental group of a variety Z which admits a finite nonramified covering $Z' \to Z$ birationally isomorphic to a fibration of abelian varieties over Y.

Now we introduce a MHS on the relative completion of $\pi_1(X)$. Let X be a quasi-projective variety, $x \in X$ a base point and let $\rho : \pi_1(X, x) \to G$ be a Zariski dense representation to a complex reductive group G.

Definition 2.2 *Let X be smooth. We say that ρ arises from a* complex variation of Hodge structures *(CVHS) of weight k if the following conditions are satisfied:*

(i) *There exists a compactification $\overline{X}$ of X with $X_\infty := \overline{X} - X$ - a divisor with simple normal crossings, such that ρ is quasi-unipotent around X_∞.*

(ii) *There exists a faithful representation $G \subset GL(V)$ such that for the corresponding $\mathcal{C}^\infty$ local system $(\mathcal{V}, D)$ one has a $\mathcal{C}^\infty$ decomposition $\mathcal{V} = \oplus_{r+s=k} \mathcal{V}^{r,s}$ with respect to which D satisfies Griffiths' transversality condition*

$$D : \mathcal{V}^{r,s} \to A^{0,1}(\mathcal{V}^{r+1,s+1}) \oplus A^{1,0}(\mathcal{V}^{r,s}) \oplus A^{0,1}(\mathcal{V}^{r,s}) \oplus A^{1,0}(\mathcal{V}^{r-1,s+1}).$$

(iii) *There exists a D-parallel hermitian form on $\mathcal{V}$ which makes the above decomposition orthogonal and which on $\mathcal{V}^{r,s}$ is positive definite if r is even and negative definite if r is odd.*

The condition on the monodromy at infinity in the definition is an a-priori condition on the representation ρ and is somewhat independent from the conditions **(ii)** and **(iii)**. In the case of real VHS it is automatically satisfied in view of Borel's theorem. Furthermore, it is straightforward to check that the quasi-unipotency does not depend on the particular choice of a compactification $\overline{X}$.

The above definition can be extended in the usual way to the singular case by means of hypercoverings.

Definition 2.3 *For any variety X we say that ρ arises from a* complex variation of Hodge structures *(CVHS) of weight k if there exists a smooth hypercovering $X_\bullet$ of X such that for any element $p : Z \to X$ in $X_\bullet$ we have that $p^*\rho$ arises from a CVHS of weight k.*

Similarly one defines a variation of complex mixed Hodge structures MHS (see [35]) and Mixed twistor structures MTS (see [34]).

Remark 2.2 For any morphism $f : X \to Y$ with X - an arbitrary variety and Y - smooth and projective and for any representation $\rho : \pi_1(Y, f(x)) \to G$ arising from a CVHS of weight k the natural pullback representation $f^*\rho : \pi_1(X, x) \to G$ arises from a CVHS of weight k by default.

Let now X be any variety and let $\rho : \pi_1(X) \to G$ be a Zariski dense representation in a complex reductive group. The Malcev completion of $\pi_1(X, x)$ relative to ρ is the projective limit

$$\hat{\pi}_\rho(X,x) := \lim_{\longleftarrow} \left\{ \begin{tikzcd} 1 \arrow[r] & R^u M \arrow[r] & M \arrow[r] & G \arrow[r] & 1 \\ & & \pi_1(X,x) \arrow[u] \arrow[ur, "\rho"'] & & \end{tikzcd} \;\middle|\; \begin{array}{l} M \text{ - complex} \\ \text{linear algebraic} \\ \text{group, } R^u M \text{ -} \\ \text{the unipotent} \\ \text{radical of } M. \end{array} \right\}.$$

One has the following fundamental fact.

Proposition 2.A Let $\rho : \pi_1(X, x) \to G$ be a Zariski dense representation arising from a CVHS $\mathbb{V} := (\mathcal{V}, D)$. Then the Lie algebra $lie\hat{\pi}_\rho(X, x)$ admits a canonical functorial complex MHS in the sense that it is equipped with three opposed filtrations $(W_\bullet, F^\bullet, \overline{F}^\bullet)$ such that for every k the graded quotient $Gr_k(W_\bullet)$ corresponds to a CVHS of weight k on X which belongs to the full tensor subgategory $\mathcal{C}_\mathbb{V}$ in the category of CVHS on X that is generated by $\mathbb{V}$ and the Tate twists.

Remark 2.3 The functoriality of the MHS in the Proposition 2.A means that for any morphism of varieties $f : (Z, z) \to (X, f(z))$ the map induced on completions

$$\hat{\pi}_{f^*\rho}(Z, z) \longrightarrow \hat{\pi}_\rho(X, f(z))$$

is a morphism of MHS of type $(0, 0)$. Also one expects that for a smooth projective X the weight filtration on $lie\hat{\pi}_\rho(X, x)$ is non-zero only in weights $\leq$ weight$(\mathcal{V})$.

The main geometric application of the MHS on $lie\hat{\pi}_\rho(X, x)$ for us will be the following lemma.

Lemma 2.D Let X be a smooth projective variety and let $f : (Z, z) \to (X, x)$ be a (based) morphism from a reduced curve Z. Let $\rho : \pi_1(X, x) \to G$ be a Zariski dense representation arising from a CVHS $\mathbb{V}$ of weight k. Then the map

$$f_* : \hat{\pi}_{f^*\rho}(Z, z) \longrightarrow \hat{\pi}_\rho(X, x)$$

is trivial if and only if the maps

$$f^* : H^1(X, \mathbb{U}) \longrightarrow H^1(Z, f^*\mathbb{U})$$

are trivial for every irreducible $\mathbb{U} \in \mathrm{Ob}(\mathcal{C}_{\mathbb{V}})$.

Next we will need the following general statement:

Theorem 2.F Let r be a local system underlying a complex variation of MHS. There is a MHS on the completion of the local ring $\mathcal{O}_r$ of the moduli space of all representation of fixed rank equal to those of r as well as on every quotient ring $\widehat{\mathcal{O}}_r/m^k$. Moreover $\widehat{\mathcal{O}}_r/m^k$ is a mixed Hodge representation of the relative completion of $\pi_1(X)$ w.r.t. r.

The second statement is a consequence of the fact that the weight filtration goes by powers of m^k. This was first conjectured by Deligne. The proof follows from the work of Hain [13]. In [13] the above statement is proved for a real VHS. Adding to r its complex conjugate we put ourselves in a position to apply Hain's result. A different proof was given by Simpson in [34].

Proposition 2.1 *Let $\pi_1(X)$ be a linear group. Then there exists a CVHS $\mathcal{V} \to X$ such that $\pi_1(X)$ embeds in the relative completion with respect to $\mathcal{V}$.*

Proof Let $\rho : \pi_1(X) \to GL(n, \mathbb{C})$ be a faithful representation. Denote by R an irreducible component of the the space of all rank n representations of $\pi_1(X)$ containing ρ. Every element $a \in \pi_1(X)$ can be interpreted as a $GL(n, \mathbb{C})$-valued function on R, $a : R \to GL(n, \mathbb{C})$, $\mu \mapsto \mu(a)$. Since ρ is faithful such an a is uniquely determined by its value at ρ. On the other hand according to Corollary 6.12 of [32] the Higgs bundle corresponding to the semi-simplification of ρ can be deformed (in its irreducible component) to a Higgs bundle which is fixed under the action of $\mathbb{C}^\times$. We need to worry a bit about the change of the complex structure when we move from the moduli space of representations to the moduli space of Higgs bundles. It could be that we have a fixed under the action of $\mathbb{C}^\times$ CVHS on every component of the moduli space of Higgs bundles but not on every component of the moduli space of representations (or moduli space of of local systems).

One can show that this does not happen, namely in every component the moduli space of semisimple local systems of rank n on X there exists a local system underlying a complex variation of Hodge structures. Indeed the moduli space Z of λ-connections realizes the moduli of Higgs bundles as a deformation of the moduli space of semisimple local systems on X. The irreducible components of Z are the closures of the products of the irreducible components of the moduli of local systems with $\mathbb{C}^\times$ and every such component intersects the fiber of Z over 0 (the moduli of Higgs bundles) in a union of components. But the natural $\mathbb{C}^\times$-action on Z restricts to the dilation action on the Higgs bundles and this yields the statement. We can thus conclude that R contains a point $\mathcal{V}$ which underlies a complex variation of Hodge structures.

Assume now $a \in \pi_1(X)$ be an element that goes to the identity in the relative completion of $\pi_1(X)$ w.r.t. $\mathcal{V}$. Then due to Theorem 2.E a goes to the identity in $GL(n, \widehat{\mathcal{O}}_{\mathcal{V}}/m^k)$ for every k and hence the function a equals the identity in $GL(n, \widehat{\mathcal{O}}_{\mathcal{V}})$. In particular $a = e$ on R and thus $\rho(a) = \rho(e)$ which is a contradiction.

Q.E.D.

Now we finish the definition of the variety $sh^H(X) = sh^H(X_0)$.

Let X be any smooth projective variety and let $\rho : \pi_1(X) \to GL(n, \mathbb{C})$ be an infinite linear representation. Due to Proposition 2.1 there exists a CVHS $\mathbb{V}$ with the property $\ker[\pi_1(X) \to \hat{\pi}_{\mathbb{V}}(X)] = \ker(\rho)$. Consider the Shafarevich variety Y for the reductive local system $\mathbb{V}$. In the same way as in [19] we can construct a $\mathbb{Z}$-VHS on Y and a morphism f from X to the corresponding family of abelian varieties. Again Lemma 2.D implies that f contracts precisely the subvarieties in X whose fundamental group is commensurable with $\ker(\rho)$.

In the same way one defines $sh^H(X_t)$ for every $t \in D$.

Now after defining $sh^H(X_t)$ we move to the proof of the theorem. Again we do a couple of reductions.

1) We study the the semisimple part of the representation $\rho(\pi_1(X))$ in $GL(n, \mathbb{C})$ and define a global local system on X (see [21].)

Lemma 2.1 *Theorem 1.1 is correct in the case of semisimple representation $\rho(\pi_1(X))$ in $GL(n, \mathbb{C})$.*

This global local system defined above defines a global Higgs field (see [20]) and therefore a global spectral covering on X after some base change.

Indeed the degree of nonreduceness of the spectral covering, defined in theorem 2.D, might vary with t but by doing a base-change $X' \to D'$ we can arrange that the degree becomes the same. (Abusing the notations we will denote the family after the base change also by $X \to D$.)

This gives the Shafarevich variety for the semisimple part of $\rho(\pi_1(X))$ in $GL(n, \mathbb{C})$.

2) Using the Albanese morphisms we prove.

Lemma 2.2 *Theorem 1.1 is correct in case of nilpotent representation representation $\rho(\pi_1(X))$ in $GL(n, \mathbb{C})$.*

Now we combine the results from the previous two lemmas. To finish the proof we need a global deformation of the representation $\rho(\pi_1(X))$ to a CVHS on X' which is a $CVHS$ for every t. In other words we need a relative version of proposition 2.1. To accomplish that we argue as follows. The Kähler deformation $f : X \to D$ produces a fibration $f : M(X_t, GL(n, \mathbb{C})) \to D$, where $M(X_t, GL(n, \mathbb{C}))$ is the moduli space of rank n local systems on X_t. Therefore we can find a flat local system ρ for the fibration $f : M(X_t, GL(n, \mathbb{C})) \to D$ so that the corresponding relative completions constitute a variation of Mixed twistor structures - MTS (introdiced by C. Simpson in [34]) of $\hat{\pi}_\rho(X_t, x)$. As it follows from lemma 2.4 applied to the variation of MTS $\hat{\pi}_\rho(X_t, x)$ the later produces a fibration of $sh^H(X_t)$ over D. As an intermediate step we need a strictness property for the MTS, which is an easy consequesnce from [34]. For more details see [20].

This finishes the proof of theorem 1.1.

Q.E.D.

As we saw we have a pluriharmonic map from the covering X' of X corresponding to $\rho : \pi_1(X) \to GL(n, \mathbb{C})$ to $\prod_{\mathfrak{p}} B_{\mathfrak{p}} \times \prod_{\nu} \mathrm{Symm}_\nu$. Using this harmonic map we construct the Shafarevich variety Y for the reductive local system $\mathbb{V}$. Now using the representation $\pi_1(X) \to \hat{\pi}_{\mathbb{V}}(X)$ we construct a family of abelian varieties Y' over Y. Therefore we obtain a pluriharmonic map PU from X' to the universal covering of Y'.

Definition 2.4 *We say that $\rho : \pi_1(X) \to GL(n, \mathbb{C})$ is a large linear representation if the corresponding pluriharmonic map $PU : X' \to Y'$ has a generically maximal rank.*

On the level of groups this means that generic non-zero dimesional subvariety subvariety Z has nontrivial $\rho : \pi_1(Z) \to GL(n, \mathbb{C})$.

The above considerations also give the following universality property of $sh^H(X)$. Let X be a smooth projective variety such that it admits an infinite representation $\rho : \pi_1(X) \to L$ to a complex linear group with kernel H. Assume that the representation $\rho : \pi_1(X) \to G$ factors through a representation of the fundamental group of a variety Y. Let

$$1 \to N \to L \to G \to 1$$

the representation of G as an extension of nilpotent group N and a reductive group G. Let L' be any infinite subgroup of L so that

$$1 \to N' \to L' \to G' \to 1$$

and N' and G' are subgroups of N and G respectively.

Corollary 2.1 *The Shafarevich morphism for the representation $\rho : \pi_1(X) \to L$ factors through the Shafarevich morphism for the representation $\rho' : \pi_1(X) \to L'$.*

2.2 Volume tools

We study the Kähler deformations of compact Kähler manifolds X_0 whose universal coverings are Stein. A Kähler deformation of X_0 means a Kähler manifold X and a smooth proper morphism $f : X \to D$ with $X_0 = f^{-1}(0)$, where D is the disc, s.t. the induced Kähler metric on the fibers X_t varies smoothly.

¿From now on $\tilde{X}$ denotes the universal covering of X and $\sigma : \tilde{X} \to X$ denotes the respective covering map. If $\tilde{Z}$ is a compact subvariety of $\tilde{X}$ we call its reduced image by σ Z and $\hat{Z}$ the normalization of Z. We denote the degree of the mapping $\sigma|_{\tilde{Z}} : \tilde{Z} \to Z$ by $d_{\tilde{Z}}$. Clearly the existence of a compact subvariety $\tilde{Z}$ is equivalent to $im[\pi_1(res(Z)) \to \pi_1(X)]$ being finite. We denote the order of this subgroup of $\pi_1(X)$ by $\#im[\pi_1(res(Z)) \to \pi_1(X)]$. We introduce a notation:

Definition 2.5 (Proposition). *Let L be a positive line bundle on a complex manifold X, and let Z be a compact subvariety of X of dimension d. Then:*

$$Vol_L(Z) = \int_{Z-SingZ} dV_L = \frac{1}{d!}\int_{Z-SingZ} (i^*\nu)^d = deg_Z(L)$$

where dV_L is the volume element coming from a metric associated to a curvature form ν for L. In the case X is Kähler we have:

$$Vol_w(Z) = \frac{1}{d!}\int_{Z-SingZ} (i^*w)^d = [w]^d([Z])$$

where w is the Kähler form, $i : Z \to X$ is the inclusion and $[w] \in H^2(X,\mathbb{C})$ is the Kähler class. We also denote by $[Z] \in H_{2d}(X,\mathbb{Z})$ is the homology class of Z and $[w]^d([Z])$ is the d-th-cup product of $[w]$ evaluated at $[Z]$.

Remark 2.4 *Let $f : X \to D$ be a deformation of the compact manifold $X = f^{-1}(0)$ over the disc D. Then we also have a deformation of the universal covering of X_0, $\tilde{X}_0$, $\tilde{f} = f \circ \sigma : \tilde{X} \to D$ where $\sigma : \tilde{X} \to X$ is the universal covering of X. This deformation is naturally C^∞ trivial. Pick $x \in \tilde{X}$ and D_x a fundamental domain containing x. The C^∞ trivialization of $\tilde{f} : \tilde{X} \to$ induced from the C^∞ trivialization of $f : X \to D$ gives a natural way to identify x and D_x to sets x_q and D_{x_t} in $\tilde{X}_q$, notice that D_{x_t} is a fundamental domain of $\tilde{X}_t$.*

Theorem 2.1 *Let $f : X \to D$ be a Kähler deformation of $X_0 = f^{-1}(0)$, where X_0 has a Stein universal covering. Fix any $N \in \mathbb{N}$, then by shrinking D we can assume:*

$$Vol_{\tilde{w}_t}(\tilde{Z}) > N$$

for all compact subvarieties $\tilde{Z}$ of $\tilde{X}_t$ where $\tilde{X}_t$ is the universal covering of X_t and $\tilde{w}_t$ is the induced Kähler metric on $\tilde{X}_t$.

Proof The deformation of the Stein covering space $\tilde{X}_0$ is locally pseudo-trivial, i.e. for any relatively compact open subset $U_0 \subset \tilde{X}_0$ we can find an open set $U \subset X$ such that $U \cap \tilde{X}_0 = U_0$, which defines a holomorphic trivial deformation of U_0 [1]. The open sets $U_t = U \cap \tilde{X}_t$ are obtained by moving U_0 along a holomorphic vector field. Choose U_0 such that it contains a very big ball $B_x(r)$, i.e. $r >> 0$, around a point $x \in \tilde{X}_0$. Since a fundamental domain D_x containing x is relatively compact we have $D_x \subset B_x(r')$, for some r'. We also have the smooth variation of the U_t, the metric $\tilde{w}_t$, x_q and D_{x_t} with t (as noted in the previous remark). Hence for sufficiently small D and $t \in D$ the following is true:

(1)

$$B_{x_t}(r) \subset U_t$$

(2)

$$D_{x_t} \subset B_{x_t}(r').$$

Condition (1) implies that all compact subvarieties $\tilde{Z}$ of $\tilde{X}_q$ passing through the fundamental domain D_{x_q} can not be contained in $B_{x_q}(r)$. This follows from U_q being biholomorphic to an open subset of the Stein $\tilde{X}_0$. Condition (2) states that any subvariety $\tilde{Z} \subset X_q$ passing through D_{x_q} must have $[r - r']$ (the integer part of $r - r'$) disjoint pieces of the type $\tilde{Z} \cap B_{x_{it}}(1)$.

Recall the standard fact saying that for all k-dimensional subvarieties S and points x of a Kähler manifold X with bounded geometry, $Vol_{2k}(S \cap B_x(1)) > C > 0$, where C depends only on the bound for the geometry of X. Since all subvarieties $\tilde{Z}$ can be translated to a subvariety passing through D_{x_q}, we can restrict ourselves to the case of subvarieties passing through D_{x_q}. The theorem follows from:

$$Vol_{\tilde{w}_t}(\tilde{Z}) > \sum_{i=1}^{[r-r']} Vol_{\tilde{w}_t}(\tilde{Z} \cap B_{x_{it}}(1)) > [r - r'].C$$

and the fact that r can be made as big as wanted by making D sufficiently small. Notice that the minimum of the geometry bounds C_t described above, $C = inf_{t\in\Delta}\{C_q\} \neq 0$ since the metrics vary smoothly.

Q.E.D.

2.3 Topological tools

In this subsection we prove a theorem that will be very useful to control the the appearance π_1-small subvarieties. In the proof we use the topological stratification plus a "lower semicontinuous" property of the homotopy groups of the fibers of a proper morphism.

Theorem 2.2 *Let $\rho : X \to T$ be a proper morphism with equidimensional fibers between analytic spaces. If $X_{t_o} = \rho^{-1}(t_0)$ does not contain a π_1- small* *irreducible component then there exists an open subset U, $t_0 \in U \subset T$ s.t. all $X_t = \rho^{-1}(t)$, $t \in U$, do not contain π_1-small irreducible components.*

Proof Let $\bigcup_{l=1}^{k_t} X_t^l = X_t = \rho^{-1}(t)$ be the irreducible decomposition of the fiber X_t and $\hat{X_t^l}$ denote the normalization of X_t^l. By the hypothesis of the theorem:

$$\# im[\pi_1(\hat{X_{t_0}^l}) \to \pi_1(X)] = \infty$$

for $1 \leq l \leq k_{t_0}$. We want to show that there is an open subset U s.t. $t_0 \in U \subset T$ and all irreducible components X_t^l of the fibers over U satisfy:

$$\# im[\pi_1(\hat{X_t^k}) \to \pi_1(X)] = \infty.$$

There is an open neighborhood U of $t_0 \in T$ and a finite stratification of U, that is $U = \bigcup_i S_i$ with S_i locally closed irreducible subsets of U, such that $\rho_i : \rho^{-1}(S_i) \to S_i$ are topological trivial fibrations. This statement follows from [36](Theorems 2.2; 3.3; 4.14) plus the fact that the morphism is proper. Now we pick a neighborhood $V \subset U$ of t_0 s.t. $S_i \cap V \neq$ iff $t_0 \in \overline{S}_i$. Let us analyze each morphism $\overline{\rho}_i : {\rho_i}^{-1}(\overline{S}_i) \to \overline{S}_i$. The theorem will follow if we show that the general fiber of the morphisms $\overline{\rho}_i$ satisfies the conditions of the theorem, i.e.:

$$im[\pi_1(\hat{X_{gen}}) \to \pi_1(X)] = \infty.$$

Pick an irreducible component of ${\rho_i}^{-1}(\overline{S}_i)$ that dominates $\overline{S}_i$. Normalize both the irreducible component picked before and $\overline{S}_i$ and name them $\hat{{\rho_i}^{-1}(\overline{S}_i)}$ and $\hat{\overline{S}}_i$ respectively. We get an induced morphism of normal irreducible varieties $\hat{\overline{\rho}}_i : \hat{{\rho_i}^{-1}(\overline{S}_i)} \to \hat{\overline{S}}_i$. Notice that the irreducible components of the fiber $\hat{{\overline{\rho}_i}^{-1}}(t_0)$ are partial normalizations of some irreducible components of the fiber $X_{t_0} = \rho^{-1}(t_0)$. Hence the hypothesis of the theorem is still valid to the morphism $\overline{\rho}_i'$.

The previous paragraphs show that we can reduce the proof of the theorem to the following:

Let $\rho : X \to T$ be a proper morphism of irreducible normal complex spaces which is a topological fiber bundle over a Zariski dense open subset of T. If $X_{t_0} = \rho^{-1}(t_0)$ satisfies the hypothesis of the theorem then general fiber satisfies the thesis of the theorem. The conditions on the morphism stated above permit us to use theorem 2.12 from [25], which states that:

$$Cok[\pi_1(X_{gen}) \to \pi_1(X_{t_0})] < \infty.$$

The hypothesis of the theorem says on the other hand that:

$$im[\pi_1(X_{t_0}) \to \pi_1(X)] = \infty.$$

Hence one gets $im[\pi_1(X_{gen}) \to \pi_1(X)]$ is infinite, which concludes the proof.

Q.E.D.

We also give proof of the following topological result which is a straightforward generalization of the results of [26], [7] , and [24].

Proposition 2.2 *Let X be a normal projective variety. Then $\pi_1(X)$ is finite if and only if X is chain connected via π_1-small curves.*

Proof One implication is obvious. Suppose that X is chain connected via π_1-small curves. By [24] (Theorem 1.8) there exists an open subset X_0 and a proper morphism $f : X_0 \to Z$, the Shafarevich map, such that if $z \in Z$ then $X_z = f^{-1}(z)$ is π_1-small. This map has also the property that if $z \in Z$ is very general all normal cycles which are π_1-small and intersect X_z must be contained in X_z. Pick a point $x_0 \in X_z$, with z very general, and x_1 any other point in X. There is a chain of π_1-small curves $\{C_1, C_2, ..., C_k\}$ with $C_i \cap C_{i+1} \neq \{\emptyset\}$, $x_0 \in C_1$ and $x_1 \in C_k$. The property of the Shafarevich map mentioned above implies that $C_1 \subset X_z$, which in itself implies $C_2 \subset X_z$ and so on and hence $x_1 \in X_z$. This implies $f(X) = point$ and we are done.

Q.E.D.

3 Theorems on large fundamental groups

3.1 Proof of theorem 1.4

Theorem 1.4 gives restrictions on π_1-small subvarieties that can appear in a small deformation of a variety whose universal covering is Stein.

Theorem 1.4 Let $f : X \to D$ be a Kähler deformation of $X_0 = f^{-1}(0)$ where X_0 has a Stein universal covering. Fix any $N \in \mathbb{N}$, then by shrinking D we can assume

$$Vol_{\tilde{w}_t}(Z).(\#im[\pi_1(\hat{Z}) \to \pi_1(X_q)] + 1) > N$$

For all π_1-small subvarieties of X_t with $t \in D$.

Proof By theorem 2.1 for any $N \in \mathbb{N}$ there is a sufficiently small disc $\Delta \subset Q$ around q_0 such that all compact subvarieties $\tilde{Z} \subset X_q$ have: $Vol_{\tilde{w}_q}(\tilde{Z}) > N$. Using the equalities

$$Vol_{\tilde{w}_q}(\tilde{Z}) = d_{\tilde{Z}}.Vol_{w_q}(Z)$$

and

$$d_{\tilde{Z}} = \#im[\pi_1(\hat{Z}) \to \pi_1(X_q)] + 1$$

we finish the proof of the theorem.

Q.E.D.

Corollary 3.1 *Let X_0 be a projective manifold whose universal covering is Stein. Suppose $f : X \to D$ is a deformation of $X_0 = \pi^{-1}(q_0)$ with a relative ample line bundle L on X. Fix any $N \in \mathbb{N}$, then by shrinking D we can assume that all compact subvarieties $\tilde{Z} \subset X_q$ have:*

$$deg_{L_t}(Z).(\#im[\pi_1(res(Z)) \to \pi_1(X_t)] + 1) > N$$

where $L_t = L|_{X_t}$.

Proof It is straightforward consequence of theorem 1.4 for the metric induced from the relative ample line bundle L and the Definition-Proposition 2.5.

Q.E.D.

Corollary 3.2 *Let X_0 be a compact Kähler manifold whose universal covering is Stein and $f : X \to D$ be a Kähler deformation of $X_0 = f^{-1}(0)$. Suppose that the fundamental group, $\pi_1(X_0)$, has an upper bound for the order of its finite subgroups. Then for sufficiently a small disc $D' \subset D$ we have:*

$$Vol_{w_t}(Z) > N$$

for all π_1-small subvarieties $Z \subset X_t$, $t \in D'$. In the case when X is projective with ample line bundle L one has:

$$deg_{L_t}(Z) > N$$

Proof Let b be the upper bound for the order of the finite subgroups of $\pi_1(X_0)$. The previous corollary states that there exists a $D' \subset D$ such that:

$$Vol_{w_t}(Z).(\#im[\pi_1(res(Z)) \to \pi_1(X)] + 1) > N.b$$

and

$$deg_{L_t}(Z).(\#im[\pi_1(res(Z)) \to \pi_1(X_t)] + 1) > N.b$$

But by hypothesis $im[\pi_1(res(Z)) \to \pi_1(X)] < b$.
Q.E.D.

The above corollary states that for a small Kähler (projective) deformations of a Kähler (projective) manifold as in the corollary the property of having a large fundamental group can be only broken by subvarieties with very big volume (degree). Hence if we have an upper bound for the degree of the subvarieties that can violate the the property of a large fundamental group, we can show that large fundamental groups are preserved under small Kähler deformations. We use this idea to give a proof of Theorem 1.6.

3.2 Proof of theorem 1.5

In this subsection we prove theorem 1.5. We use the tools developed earlier to study π_1-small curves in deformations of Kähler surfaces and give two different proofs of Theorem 1.5 (one of the proofs requires additional hypothesis on the surface). All surfaces are compact unless stated otherwise. The existence of a relative Shafarevich morphism plus the "lower semicontinuity" of the the homotopy groups in a family gives the first proof.

Theorem 1.5 Let X_0 be a Kähler surface with $\rho : \pi_1(X_0) \to GL(N, \mathbb{C})$ an infinite linear representation. Then having a large fundamental group is stable under small Kähler deformations of X_0.

Proof Let $f : X \to D$ be a deformation of $X_0 = f^{-1}(0)$ and $\rho : \pi_1(X_0) \to Gl(n, \mathbf{C})$ be an infinite linear representation. Theorem 1.1 states that there exists a relative ρ-Shafarevich morphism for the family $f : X \to D$, $s^\rho_D : X \to s^\rho_D(X)$. Let g denote the product morphism $g = s^\rho_D \times f : X \to s^\rho_D(X) \times D$.

Since the π_1-small curves must lie in the fibers of g, we need to show that there exists an open neighborhood U, $0 \in U \subset D$ s.t. for all $t \in U$ the fibers $X_{y,t} = g^{-1}(y, t)$ satisfy the homotopy condition:

$$im[\pi_1(res(X_{y,t})) \to \pi_1(X)] = \infty$$

There are 3 cases:

Case 1. All $X_{y,0}$ are 0-dimensional. The dimension of the fibers of f is an uppersemicontinuous function in the Zariski analytic topology of $g(X)$. This implies that there exists an open neighborhood U of $0 \in \Delta$ such that all fibers for $t \in U$ $X_{y,t}$ are 0-dimensional. Hence there are no π_1-small curves.

Case2. We have finite number of 1-dimensional $X_{y,0}$. From the uppersemicontinuouty mentioned above it follows that $S = \{s \in g(X)$, such that $dim g^{-1}(s) = 1\}$ is a closed analytic subspace of $g(X)$. By shrinking D we can assume that all irreducible components of S pass through $g(X_0)$. This implies that all possible π_1-small curves lie in a family of curves of X containing a curve of X_0. Using the projection $p_2 : s^\rho_D(X) \times D \to D$ we can actually see these families as families of curves over D whose central fibers are not π_1-small. Theorem 1.3 then implies that for each family there is an open neighborhood U, $0 \in U \subset D$ s.t. all curves over U are not π_1-small. The result then follows since there are only a finite number of families to consider.

Case 3. All $X_{y,0}$ are 1-dimensional. In this case we can apply immediately Theorem 1.3 to the morphism g. Since the fibers over $g(X_0)$ are not π_1-small curves, Theorem 1.3 implies that every point of $g(X_0)$ has a neighborhood U s.t all curves over U are not π_1-small. Now since $g(X_0)$ is compact the result follows. **Q.E.D.**

We now prove Theorem 1.5 under some additional hypothesis using the volume tools (Theorem 1.4)

Theorem 1.6 Let X_0 be a Kähler surface with $\rho : \pi_1(X_0) \to GL(N, \mathbb{C})$ an infinite linear representation, finite torsion $\pi_1(X_0)$ and Stein universal covering. Then a small Kähler deformation of X_0 still has a large fundamental group.

Proof Let $\rho : \pi_1(X_0) \to GL(N, \mathbb{C})$ be an infinite linear representation of $\pi_1(X)$. We use the relative ρ-Shafarevich morphism to control the volume of the π_1-small curves and then use Corollary 3.2 to get a contradiction on the existence of π_1-small curves.

Let $g = s_D^\rho \times f : X \to s_D^\rho(X) \times D$ be the same morphism as in the previous theorem. Since g is proper we can assume that by shrinking Δ the set of points $y \in g(X)$ with 1-dimensional $g^{-1}(y)$ is a finite union of irreducible closed sets, $\bigcup_i^n F_i$. Let $X_{F_i} = red(g^{-1}(F_i))$, where $red(g^{-1}(F_i))$ is the reduced analytic space associated with $g^{-1}(F_i)$. The X_{F_i} are finite unions of irreducible components, hence to simplify the proof we can assume X_{F_i} is irreducible. Let $\hat{F}_i$ be the normalization of F_i and $g_{\hat{F}_i} : X_{\hat{F}_i} \to \hat{F}_i$ be the pullback to $\hat{F}_i$ of $g : X_{F_i} \to F_i$.

Theorem 3 of Barlet [2] states that the volume of the fibers of $g_{\hat{F}_i} : X_{\hat{F}_i} \to \hat{F}_i$ relative to any hermitean metric over $X_{\overline{F}_i}$ is locally bounded over $\overline{F}_i$. We consider the metric induced from the Kähler metric on X. This result plus the fact that there are only a finite number of F_i implies that there is neighborhood $O \in U \subset D$ s.t all surfaces X_t, $t \in U$ have π_1-small curves with bounded volume.Recall that corollary 3.2 says that the volume of theπ_1-small curves is larger than any given number by making the deformation sufficiently small. Hence the result follows by contradiction. **Q.E.D.**

The following corollary describes the case where the abelian part of π_1 is enough to settle the question about the Kähler deformations of surfaces with large fundamental groups. In this case the relative Albanese morphism plays the role of the Shafarevich morphism in family.

Corollary 3.3 *Let X be a Kähler surface with a large fundamental group and $b_1(X) \neq 0$. Then a small deformation of X still has a large fundamental group.*

Using the previous results we can now prove the partial answer to the question mentioned in the introduction about the stability of the Steiness of universal covers.

Definition 3.1 *Let $\rho : \pi_1(X) \to GL(n, \mathbb{C})$ be an infinite linear representation, $\pi_1(X)$ is a ρ-large if $\rho(im[\pi_1(res(Z)) \to \pi_1(X)])$ is an infinite group for every irreducible and positive dimensional subvariety Z of X.*

Corollary 3.4 *Let X be a compact Kähler surface with an infinite linear representation $\rho : \pi_1(X) \to GL(N, \mathbb{C})$. Suppose the infinite cover defined by $Ker(\rho)$ is Stein then a small Kählerian deformation of X still has a Stein universal covering.*

Proof This results follows from looking at an intermediate cover of X, more precisely the infinite cover X^ρ defined by $Ker(\rho)$. The starting points are the result of [19] stating that X^ρ is holomorphic convex and the fact that this follows just from ρ being an infinite linear representation. Hence X_t^ρ being holomorphically convex is stable under deformations of X.

To get the result one needs only to show that X_t^ρ remains Stein for small Kählerian deformation of X, since unramified coverings (even infinite, which can be the case for $\tilde{X}_t \to X_t^\rho$) of Stein varieties are still Stein.

By the first paragraph, it only remains to show that having a ρ-large fundamental groups is stable for small Kählerian deformations of X. This is a consequence of the proof of Theorem 1.5. **Q.E.D**

3.3 Deformations of surfaces and π_1-small curves

This subsection gives some results on π_1 -small curves on surface deformations. We are going to use geometric bounds for the degree of curves on surfaces plus Theorem 1.4 to get information on the small curves that can appear along a deformation. The first result states: in a small deformation, as above, the small curves C that can appear in X_t must have at least one of the following very high: geometric genus, selfintersection, or $\#im[\pi_1(\hat{C}) \to \pi_1(X_t)]$, where $\hat{C}$ is the normalization of C .

We also prove that a small deformation of a surface of general type whose universal covering is Stein has no rational curves C with $C^2 \leq 0$. We then note that this result plus a consequence of the Shafarevich Conjecture would rule out the appearance rational curves.

Proposition 3.1 *Let X_0 be a surface of general type whose universal covering is Stein. Suppose $f : X \to D$ is a deformation of $X_0 = f^{-1}(0)$ over* *the disc. Then for any $N \in \mathbb{N}$ there is a sufficiently small disc $D' \subset D$ around 0 s.t. all π_1-small curves of the X_t, $t \in D'$, satisfy:*

(3)

$$max\{g(C), C^2, \#im[\pi_1(\hat{C}) \to \pi_1(X_q)]\} > N$$

Proof First note that K_{X_0} is ample since $\pi_1(X_0)$ being large rules out the existence of (-2)-rational curves. By shrinking the base D, if necessary, we can choose the relative canonical bundle $\omega_{X/Q}$ as the relative ample line bundle required by corollary 2.1. This corollary states that there is a sufficiently small disc $D' \subset D$ around 0 s.t. all small curves of the X_t, $t \in D'$, satisfy:

(4)

$$max\{K_{X_q}.C, \#im[\pi_1(\hat{C}) \to \pi_1(X_q)]\} > N$$

Using the results of [17] in the spirit of the results of [3] we know that the degree of a curve C having geometric genus $g(C)$ and selfintersection C^2 in a surface of general type X is bounded by:

(5)

$$K_X.C < 6c_2(X) - 2c_1^2(X) + 6g(C) - 6 + C^2$$

Since $6c_2(X) - 2c_1^2(X)$ is a topological invariant of the surface it remains constant under deformations. The combination of (4) and (5) yields the desired conclusion.

Q.E.D

The previous proposition implies that if any two of the three entries in (3) can be bounded the other entry must be very high.

The study of rational curves C along small deformations is more attainable because $\#im[\pi_1(\hat{C}) \to \pi_1(X_q)] = 0$ (see the next remark) avoids the necessity to give a bound for the order of the torsion subgroups of $\pi_1(X_0)$.

Remark 3.1 *Let C be a rational curve on a complex manifold X and let C' be an irreducible component of the preimage of C in the universal covering $\sigma : \tilde{X} \to X$. Then C' is compact and the covering map $\sigma|_{C'} : C' \to C$ is of degree one. To see this we just notice that there is a map between the normalizations of C' and C, $\hat{\sigma} : \hat{C}' \to \hat{C}$, that is by construction an unramified covering map. By Hurwitz formula, since $\hat{C}$ is $\mathbb{P}^1$, we have $\hat{C}' = \mathbb{P}^1$ and $deg(\overline{\sigma}) = 1$.*

Proposition 3.2 *Let X_0 be a surface of general type whose universal covering is Stein. Then any small deformation of X_0 has no rational curves C with $C^2 \leq 0$.*

Proof It follows from the previous remark that $\#im[\pi_1(\hat{C}) \to \pi_1(X_q)] = 0$. The hypothesis of the proposition bounds C^2, hence the conclusion is obtained using proposition 3.1.

Q.E.D

Remark 3.2 *The Shafarevich Conjecture plus proposition 3.2 imply that if X is a surface with large fundamental group then any small Kähler deformation of X has no rational curves. If the universal covering $\tilde{Y}$ of a surface Y is holomorphically convex then there is Remmert reduction, $\sigma : \tilde{Y} \to S$. The compact curves $\tilde{C}$ of $\tilde{Y}$ are contained in the fibers of the reduction map σ, hence a $\pi_1 -$ small curve C of Y must have $C^2 \leq 0$.*

4 Some applications

4.1 Some nonvanishing results

We give some applications of the results proved in the previous sections. Let us recall the following theorem:

Theorem 4.A Let X be a smooth projective variety with $\pi_1(X)$ generically large fundamental group. Let D be a big Cartier divisor on X. Then $h^0(X, K_X + D) > 0$.

We also recall:

Definition 4.1 *A line bundle L on X is called big if there is a dense open set $U \subset X$ such that if $x_1, x_2 \in U$ and $m \gg 1$ then there is a section $f \in H^0(X, L^m)$ such that $f(x_1) = 0$ and $f(x_2) \neq 0$. A line bundle L i s called nef if $deg_C L \geq 0$ for every compact irreducible curve $C \subset X$. A Cartier divisor is called big (respectively nef) if the line bundle $L = O_X(D)$ is big (respectively nef).*

The Theorem 4.A was conjectured by Kollár in [24] and proved in [18] and [12]. The most general result in this direction belongs to Takayama. Combining the above theorem with Theorem 1.5 we get:

Theorem 4.1 *Let $X_t \to \Delta$ be a small deformation of a smooth projective surface X_0 and D_t is a big divisor on X_t. Assume that $\pi_1(X_0)$ is a large fundamental group with infinite linear representation. Then for every t in may be a smaller disk Δ', $h^0(X_t, K_{X_t} + D_t) > 0$.*

4.2 Vanishing results

Let X be any smooth projective variety, to recall the notion of $\rho : \pi_1(X) \to GL(n, \mathbb{C})$ being a large linear representation, see Definition 2.4. Note that if $\pi_1(X)$ is a ρ-large fundamental group (Definition 3.1) then rho is a large linear representation.

We prove the following lemma, following ideas of Gromov from [14].

Lemma 4.1 *Let X be any smooth projective variety and let $\rho : \pi_1(X) \to GL(n, \mathbb{C})$ be a large linear representation. Then there exists a strictly plurisubharmonic exhaustion function f defined on an open set of $\widetilde{X}$.*

Proof Indeed let d be the distance function on T, the $\mathbb{C}^n$ bundle over $\prod_{\mathfrak{p}} B_{\mathfrak{p}} \times \prod_{\nu} \mathrm{Symm}_{\nu}$, where $\mathbb{C}^n$ is the universal covering of the abelian varieties in the family Y' (see section two). Since T is nonpositively curved d is a convex function. The fact that PU has a maximal rank on an open set U of X' makes the pullback of d by PU a strictly plurisubharmonic function f' on U. Pulling back f' to the universal covering $\widetilde{X}$ of X we get a strictly plurisubharmonic function f on an open set in $\widetilde{X}$.

Q.E.D.

As a corollary of the above lemma we get.

Corollary 4.1 *Let X be a smooth projective variety of dimension n , which admits a large linear representation of its fundamental group in some linear group L. Then $H^0_{L_2}(X, \Omega^k) = 0$ for $0 < k < n$.*

Proof We give a brief proof. Denote by $H^0_{L_2}(X,\Omega^k)$ the holomorphic L_2 k-forms. Let v be a holomorphic L_2 form of degree $p, p < dim_{\mathbb{C}} X = n$. Denote by h the Hessian of f. For the Kähler form ω we can find a constant k so that $k.h^{n-p} \wedge v \wedge \bar{v} \leq \omega^{n-p} \wedge v \wedge \bar{v}$. It is easy to check that $k.h^{n-p} \wedge v \wedge \bar{v}$ is L_1. Integrating by parts and following Gromov [14] we get that $k.h^{n-p} \wedge v \wedge \bar{v} = 0$. But by definition h is a positively definite on an open set of $\widetilde{X}$ (recall that f was a strictly plurisubharmonic exhaustion function an open set of $\widetilde{X}$). By the analytic continuation theorem $v = 0$ on $\widetilde{X}$. **Q.E.D.**

J. Jost and K. Zuo have obtained results of this type in case $\pi_1(X)$ is a Zariski dense subgroup in a complex simple group (see [16]).

Combining the above corollary with theorem 1.3 we get:

Theorem 4.2 *Let $X_t \to \Delta$ be a small deformation of a smooth projective surface X_0. Assume that $\pi_1(X_0)$ has an infinite linear representation $\rho : \pi_1(X) \to GL(n,\mathbb{C})$ and $\pi_1(X_0)$ is ρ-large. Then for every t in may be a smaller disk Δ', we have $H^0_{L_2}(X_t,\Omega^k) = 0$ for $0 < k < n$.*

References

[1] **A. Andreotti, E. Vesentini,** *On the pseudo-rigidity of Stein manifolds.* , Ann. Scuola Norm. Sup. Pisa (3) 16,1962,213–223.

[2] **D. Barlet,***Majoration du volume des fibres génériques et forme gém ètrique du théor ème d'aplatissement* , (Sem. P. Lelong, 1978–1979), pp. 1-18, Lecture Notes in Math., 807, Springer, Berlin, 1980.

[3] **F.Bogomolov,** *Families of curves on a surface of general type* , Dokl. Akad. Nauk SSSR 236 (1977), no. 5, 1041–1044.

[4] **F.Bogomolov, L.Katzarkov,** *Symplectic four manifolds and projective surfaces* , Journal of topology and its applications, v. 88, n. 1-2, 1998, 79- 110.

[5] **F.Bogomolov, B. de Oliveira,** *Stein small deformations of strictly pseudoconvex surfaces.* , Birational algebraic geometry (Baltimore, MD, 1996), 25–41, Contemp. Math., 207, Amer. Math. Soc., Providence, RI, 1997.

[6] **A.Borel and J.P.Serre,** *Cohomologie d'iemmeubles et de S-groupe arithmétiques,* Topology **15** 1976, 211-232.

[7] **F.Campana,** *Coreductions algebrique d'un espace analytic faiblements Kählerienne compact,* Inv. Math. **63** 1981, 187–223.

[8] **K.Corlette,** *Nonabelian Hodge theory,* Proc. Symp. Pure. Math. **54**, part 2, 1993, 125-140.

[9] **P.Deligne,** *Theorie de Hodge II*, Publ. Math. I.H.E.S., 40, (1971), 5-58.

[10] **R.Donagi,** *Spectral covers* , in Current topics in complex algebraic geometry (Berkeley, CA, 1992/93), MSRI Publ. **28** 1996, Cambridge Univ. Press, 65–86.

[11] **R.Donagi,** *Decomposition of Spectral Covers* , Journ. de Algebr. Geometrie, Orsay, Asterisque **218** 1993, 145–175.

[12] **P.Eyssidieux,** *Theorie de l'adjonction L^2 sur le revement universel,* preprint 1997, Universite Paul Sabatier, Toulouse.

[13] **R.Hain,** *The Hodge de Rham theory of the relative completion* , preprint, 1995.

[14] **M.Gromov,** *Kähler hyperbolicity and L_2-Hodge theory* , J. Differential Geom. 33 (1991), no. 1, 263–292.

[15] **M.Gromov, R.Schoen,** *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one* , Publ. Math. I.H.E.S. **76** 1992, 165–246.

[16] **J. Jost, K. Zuo,** preprint 1997.

[17] **S.Lu, Y. Miyaoka** *Bounding curves in algebraic surfaces by genus and Chern numbers* , Amer. J. Math. 119 (1997), no. 3, 487–502.

[18] **L.Katzarkov,** *Nonvanishing theorems for Kähler hyperbolic manifolds.* , preprint 1993.

[19] **L.Katzarkov,** *On the Shafarevich maps* , Proc. of AMS, Vol. 62, Part 2, Alg. geometry, S. Cruz 1995, 173–217.

[20] **L.Katzarkov, T.Pantev, M.Ramachandran,** *Geometric factorization of linear fundamental groups* , in preparation.

[21] **L.Katzarkov,T.Pantev,** *Non - abelian p,p cycles theorem.* preprint 1996.

[22] **L.Katzarkov, M.Ramachadran,** *On the universal coverings of algebraic surfaces,* Ann. Sci. E.N.S., t. 31, f. 4, 1998, 525- 536.

[23] **J.Kollár,** *Shafarevich maps and plurigenera of algebraic varieties* , Invent. Math. **113** 1993, 165–215.

[24] **J.Kollár,** *Shafarevich maps and automorphic forms* ., Princeton University Press, (1995).

[25] **J.Kollár,***Rational curves on algebraic varieties* ., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 32. Springer-Verlag, Berlin, 1996.

[26] **J.Kollár, Y. Miyaoka, S. Mori,** *Rationally connected varieties* ., J. Algebraic Geom. 1 (1992), no. 3, 429–448.

[27] **T.Napier, M.Ramachandran,** *Structure theorems for complete Kähler manifolds and applications to Lefschetz type theorems.* Geom. Funct. Anal. 5 (1995), no. 5,809–851.

[28] **T.Napier, M.Ramachandran,** *The $L^2\overline{\partial}$-method, weak Lefschetz theorems, and the topology of Kähler manifolds*, Amer. Math. Soc. 11 (1998), no. 2, 375–396.

[29] **N.Mok,** *The generalized theorem of Castelnuovo-de Franchis for unitary representations.* , Geometry from the Pacific Rim (Singapore, 1994), 261–284, de Gruyter, Berlin, 1997.

[30] **A.Selberg,** *On discontinuous groups in higher-dimensional symmetric spaces* , Contributions to function theory, Bombay, Tata Institute, 1960, 147–164.

[31] **C.T.Simpson,** *Moduli spaces of representations 1* , IHES - Publications Mathematiques, 79, (1994), 47–129.

[32] **C.T.Simpson,** *Moduli spaces of representations 2* , IHES - Publications Mathematiques, 80, (1995), 5–79.

[33] **C.T.Simpson,** *Nonabelian Hodge theory* , International congress of Mathematics Kyoto, 1990, Proceedings, Springer-Tokyo, (1991), 747–756.

[34] **C.T.Simpson,** *Mixed twistor structures* , alg-geom/9705006.

[35] **J. H. M. Steenbrink,** *A summary of mixed Hodge theory.* Proc. Sympos. Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.

[36] **J.L. Verdier,***Stratifications de Whitney et theoreme de Bertini-Sard.*, Invent. Math. 36 (1976), 295–312.

[37] **K.Zuo,** *Factorization of nonrigid Zariski dense representations of* π_1 *of projective manifolds* , Invent. Math. **118** 1994, 37–46.

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