FORMULAS FOR A_n AND B_n–SOLUTIONS OF WDVV EQUATIONS

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Abstract

The simplest non-trivial solutions of WDVV equations are A_n and B_n-potentials, which describe metrics of K.Saito on spaces of versal deformation of A_n and B_n-singularities. These are some polynomials, which were known for $n \leqslant 4$. In this paper we find the potentials for all A_n and B_n-singularities. We give recurrente formula for coefficients of KP and n-KdV hierarchy.

Keywords: WDVV equations hierarchy, singularities.

1991 MSC: 81

1. Introduction

1. WDVV equations appeared in papers of E.Witten [W1], R.Dijkgraaf, E.Verlinde, H.Verlinde [DVV] as equations for primary free energy of two-dimensional topological field theory. The simplest non-trivial solutions of WDVV equations are A_n and B_n-potentials, which describe metrics of K.Saito on spaces of versal deformation of A_n and B_n-singularities. According to E.Witten [W2] the coefficients of A_n–potentials are intersection numbers of Mumford–Morita–Miller classes of a moduli space of spheres with punctures and n–spin structures.

The A_n-potential has the following description[D1]. Let us consider the space of polynomials $M_n = \{z^{n+1} + a_1 z^{n-1} + ... + a_n | a_i \in \mathbb{C}\}$. The tangent space T_p the point $p = p(z) \in M$ is a set of polinomials $Q_n = \{b_1 z^{n-1} + ... + b_n | b_i \in \mathbb{C}\}$. Define on T_p a structure of a Frobenius algebra, considering that $v_1 * v_2 = v_1 \cdot v_2 (\text{mod } dp/dz)$ is multiplication of the algebra and $g(v_1, v_2) = \operatorname*{Res}_{z=\infty} \frac{v_1 \cdot v_2}{dp/dz}$ its scalar product. Then M_n is a Frobenius manifold. It has the plane coordinats $(t^1, ..., t^n)$ such that $g(\frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j}) = \delta_{i+j,n+1}$ and $E(t^i) = \frac{n+2-i}{n+2} t^i$, where the vector field E is given by $E = \sum_{k=1}^n \frac{k+1}{n+1} a_k \frac{\partial}{\partial a^k}$ in coordinats $\{a_1, ..., a_n\}$. Moreover, there exist a polynom F such that $\frac{\partial}{\partial t^i} * \frac{\partial}{\partial t^j} = \sum_{k=1}^n \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^{n+1-k}} \frac{\partial}{\partial t^k}$. This polynom is called A_n-potential.

According to [DVV], A_n-potential is the string solution of n-KdV (Gelfand-Dikii) hierarhy. According to [K], A_n-potentialis is the string solution of dispersionless (quasi-classical) limit of n-KdV hierarhy. It is follow directly from our reprezantation that this is the same polynomial. Moreover, we give a recurrent formulas for calculation of all these polynomials. Up to now A_n potential was know only for $n \leqslant 4$ [W2,D1,D2]. We also give recurrence formulas for coefficients of KP and n-KdV hierarchy. Results of J.-B.Zuber [Z] give a reduction of B_n–potentials to A_n–potentials.

We formulate the main result of the paper in paragraph 2. In paragraph 3 we prove auxiliary combinatorial lemma. It is used in paragraph 4 for representation of KP hierarhy as an equation on a Baker-Akhiezer function. We use a representation of the Baker-Akhiezer function by τ–function and describe KP hierarhy as an infinite system of differential equations on $v = -\log\tau$ in paragraph 5. We give an analog of this representation for n-KdV hierarhy in paragraph 6 . Using this representation we prove the main theorem in paragraph 7.

Acknowledgement. The final part of this paper was written during the author stay at Max-Planck-Institut für Mathematik in Bonn. I would like to thank this institution for support and hospitality. I would like to thank B.Dubrovin and K.Vaninsky for fruitful discussions.

This work was carried out with the financial support by grants: RFBR 98-01-00612 and INTAS 96-0713.

2. Main theorems

For calculation of the A_{n-1}-potential we use combinatorial coefficients $P_n(i_1...i_m)$, where n and s_j are natural numbers.Define

$$P_n(i) = n, \quad P_n(s_1...s_m) = C_n^m - \sum_{q=1}^{m-1} P_n(s_1...s_q) C^{m-q}_{n-q-(s_1+\cdots+s_q)},$$

where

$$0! = 1, C_p^t = \frac{p!}{t!(p-t)!} \text{ for } p \geqslant t \geqslant 0 \text{ and } C_p^t = 0 \text{ in another cases.}$$

Let us assume $x_0 = 0$. For $q > 0$ we recurrently define polynomials x_{-q} $(x_1, ..., x_{n-1})$ putting

$$x_{-q} = -\frac{1}{n} \sum_{m=2}^{\infty} \sum P_n(s_1 \dots s_m) x_{n-s_1} \cdots x_{n-s_m},$$

where the second sum is taken over all natural numbers $(s_1, ..., s_m)$ such that $\sum_{i=1}^m s_i = q + n + 1 - m$.

This polynomial has the form

$$x_{-q}=\sum_{m=1}^{\infty}\sum B_{qi_1\cdots i_m}x_{i_1}\cdots x_{i_m},$$

where $B_{qi_1,...,i_m}$ are constants and the second sum is taken over numbers $(i_1,...,i_m)$ such that $1\leqslant i_1\leqslant ...\leqslant i_m\leqslant n-1$.

Define

$$A_{i_1\cdots i_m}=-\frac{1}{i_1p}B_{i_1\cdots i_m}$$

and p is the number of j such that $i_j=i_1$.

Consider

$$F_A=\sum_{m=3}^{\infty}\sum A_{i_1\cdots i_m}x_{i_1}\cdots x_{i_m},$$

$$F_B=\sum_{m=3}^{\infty}\sum A_{2i_1-1\cdots 2i_m-1}x_{i_1}\cdots x_{i_m},$$

where the second sum is taken over all $1\leqslant i_1\leqslant ...\leqslant i_m\leqslant n-1$.

Theorem 1. *The polynomial F_A is a A_{n-1}–potential. The polynomial F_B is a B_{n-1}–potential. They satisfy the equations*

$$\sum_{\gamma=1}^{n-1}\frac{\partial^3F}{\partial x_\alpha\partial x_\beta\partial x_\gamma}\frac{\partial^3F}{\partial x_{n-\gamma}\partial x_\eta\partial x_\xi}=\sum_{\gamma=1}^{n-1}\frac{\partial^3F}{\partial x_\xi\partial x_\beta\partial x_\gamma}\frac{\partial^3F}{\partial x_{n-\gamma}\partial x_\eta\partial x_\alpha},\ (\alpha,\beta,\eta,\xi=1,...,n-1),$$

$$\frac{\partial^3F}{\partial x_1\partial x_\alpha\partial x_\beta}=\delta_{\alpha+\beta,n},\quad \sum_{j=1}^{n-1}d_i\frac{\partial F}{\partial x_i}=dF,$$

where $d_i=1+\frac{1-i}{n}$, $d=2+\frac{2}{n}$ for $F=F_A$ and $d_i=1+\frac{1-i}{n-1}$, $d=2+\frac{1}{n-1}$ for $F=F_B$.

3. Combinatorial lemma

For natural $s,i_1,...,i_n$ and integer not negative $j_1,...,j_n$ we define $P_s\begin{pmatrix} i_1 & \cdots & i_n\\ j_1 & \cdots & j_n\end{pmatrix}$ by recurrence formula:

$$1)P_s\begin{pmatrix} i_1 & \cdots & i_n\\ 0 & \cdots & 0\end{pmatrix}=0;\quad 2)P_s\begin{pmatrix} i\\ j\end{pmatrix}=C_s^j\quad\text{for } j>0;$$

$$3) P_s \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix} = \frac{1}{n!} C_s^{j_1+\dots+j_n} \frac{(j_1+\dots+j_n)!}{j_1!\cdots j_n!} -$$

$$\sum_{q=1}^{n-1} P_s \begin{pmatrix} i_1 & \dots & i_q \\ j_1 & \dots & j_q \end{pmatrix} \frac{1}{(n-q)!} C_{s-(i_1+\dots+i_q+j_1+\dots+j_q)}^{j_{q+1}+\dots+j_n} \frac{(j_{q+1}+\dots+j_n)!}{j_{q+1}!\cdots j_n!}$$

for $(j_1, ..., j_n) \neq (0, ..., 0)$.

Let $\begin{bmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{bmatrix}$ be the set of all matrices, which appear from $\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ by permutation of columns. Let $\begin{Vmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{Vmatrix}$ be the number of such matrices. Define

$$P_s \begin{bmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{bmatrix} = \sum P_s \begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix},$$

where the sum is taken by all

$$\begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix} \in \begin{bmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{bmatrix}$$

Lemma 1 *Let $m > 0, k > 0$ and $j_n \geqslant 1$ for $n \leqslant m$. Then*

$$P_s \begin{bmatrix} i_1 & \dots & i_m & i_{m+1} & \dots & i_{m+k} \\ j_1 & \dots & j_m & 0 & \dots & 0 \end{bmatrix} =$$

$$= \begin{cases} 0, & \text{if } s \geqslant i_1 + ... + i_m + j_1 + ... + j_m, \\ \frac{1}{k!} \begin{Vmatrix} i_{m+1} & \dots & i_{m+k} \\ 0 & \dots & 0 \end{Vmatrix} P_s \begin{bmatrix} i_1 & \dots & i_m \\ j_1 & \dots & j_m \end{bmatrix}, & \text{if } s < i_1 + ... + i_m + j_1 + ... + j_m. \end{cases}$$

Proof: Prove at first the lemma for $m = 1$ using induction by k. For $m = k = 1$

$$P_s \begin{bmatrix} i_1 \; i_2 \\ j_1 \; 0 \end{bmatrix} = P_s \begin{pmatrix} i_1 \; i_2 \\ j_1 \; 0 \end{pmatrix} + P_s \begin{pmatrix} i_2 \; i_1 \\ 0 \; j_1 \end{pmatrix} =$$

$$\frac{1}{2} C_s^{j_1} - P_s \begin{pmatrix} i_1 \\ j_1 \end{pmatrix} \cdot C_{s-(i_1+j_1)}^0 + \frac{1}{2} C_s^{j_1} - P_s \begin{pmatrix} i_2 \\ 0 \end{pmatrix} \cdot C_{s-i_2}^0 = C_s^{j_1} - C_s^{j_1} C_{s-(i_1+j_1)}^0 =$$

$$= \begin{cases} 0, & \text{if } s \geqslant i_1 + j_1, \\ C_s^{j_1} = P_s \begin{bmatrix} i_1 \\ j_1 \end{bmatrix}, & \text{if } s < i_1 + j_1. \end{cases}$$

To prove the lemma for $m=1, k=N$, we assume that it is proved for $m=1, k<N$. If $s \geqslant i_1+j_1$, then

$$P_1\begin{bmatrix} i_1 & i_2 & \dots & i_{k+1} \\ j_1 & 0 & \dots & 0 \end{bmatrix} = \frac{1}{k!}C_s^{j_1}\left\|\begin{matrix} i_2 & \dots & i_{k+1} \\ 0 & \dots & 0 \end{matrix}\right\| - P_s\begin{pmatrix} i_1 \\ j_1 \end{pmatrix}\cdot C^0_{s-(i_1+j_1)}\cdot\frac{1}{k!}\left\|\begin{matrix} i_2 & \dots & i_{k+1} \\ 0 & \dots & 0 \end{matrix}\right\| = 0.$$

If $s < i_1+j_1$ then

$$P_1\begin{bmatrix} i_1 & i_2 & \dots & i_{k+1} \\ j_1 & 0 & \dots & 0 \end{bmatrix} = \frac{1}{k!}C_s^{j_1}\left\|\begin{matrix} i_2 & \dots & i_{k+1} \\ 0 & \dots & 0 \end{matrix}\right\| - A\cdot C^0_{s-(i_1+j_1)} =$$

$$\frac{1}{k!}\left\|\begin{matrix} i_2 & \dots & , i_{k+1} \\ 0 & \dots & 0 \end{matrix}\right\| P_s\begin{bmatrix} i_1 \\ j_1 \end{bmatrix}.$$

Thus the lemma is proved for $m=1$.

To prove the lemma for $m=N$, we assume that it is proved for $m<N$. Then

$$P_s\begin{bmatrix} i_1 & \dots & i_m & i_{m+1} & \dots & i_{m+k} \\ j_1 & \dots & j_m & 0 & \dots & 0 \end{bmatrix} =$$

$$\sum_{\begin{pmatrix} \alpha_1 & \dots & \alpha_m \\ \beta_1 & \dots & \beta_m \end{pmatrix} \in \begin{bmatrix} i_1 & \dots & i_m \\ j_1 & \dots & j_m \end{bmatrix}} \Big(\frac{1}{(m+k)!}C_s^{\beta_1+\dots+\beta_m}\frac{(\beta_1+\dots+\beta_m)!}{\beta_1!\cdots\beta_m!}C^k_{m+k}\cdot$$

$$\left\|\begin{matrix} i_{m+1} & \dots & i_{m+k} \\ 0 & \dots & 0 \end{matrix}\right\| -$$

$$-\sum_{q=1}^{m} P_s\begin{pmatrix} \alpha_1 & \dots & \alpha_q \\ \beta_1 & \dots & \beta_q \end{pmatrix}\frac{1}{(m+k-q)!}C^{\beta_{q+1}+\dots+\beta_m}_{s-(\alpha_1+\dots+\alpha_q+\beta_1+\dots+\beta_q)}\frac{(\beta_{q+1}+\dots+\beta_m)!}{\beta_{q+1}!\cdots\beta_m!}C^k_{m+k-q}$$

$$\left\|\begin{matrix} i_{m+1} & \dots & i_{m+k} \\ 0 & \dots & 0 \end{matrix}\right\|\Big) = P_s\begin{bmatrix} i_1 & \dots & i_m \\ j_1 & \dots & j_m \end{bmatrix}\frac{1}{k!}\left\|\begin{matrix} i_{m+1} & \dots & i_{m+k} \\ 0 & \dots & 0 \end{matrix}\right\| -$$

$$P_s\begin{bmatrix} i_1 & \dots & i_m \\ j_1 & \dots & j_m \end{bmatrix} C^0_{s-(i_1+\dots+i_m+j_1+\dots+j_m)}\frac{1}{k!}\left\|\begin{matrix} i_{m+1} & \dots & i_{m+k} \\ 0 & \dots & 0 \end{matrix}\right\| =$$

$$= \begin{cases} 0, & \text{if } s \geqslant i_1+\dots+i_m+j_1+\dots+j_m, \\ P_s\begin{bmatrix} i_1 & \dots & i_m \\ j_1 & \dots & j_m \end{bmatrix}\frac{1}{k!}\left\|\begin{matrix} i_{m+1} & \dots & i_{m+k} \\ 0 & \dots & 0 \end{matrix}\right\|, & \text{if } s < i_1+\dots+i_m+j_1+\dots+j_m. \square \end{cases}$$

4. Equations for Baker-Akhiezer function

Consider the KP hierarchy. This is a condition of compatibility of an infinite system of differential equations

$$\frac{\partial\psi}{\partial x_n} = L_n\psi, \tag{1}$$

where

$$L_n = \frac{\partial^n}{\partial x_1^n} + \sum_{i=2}^{n} B_n^i(x)\frac{\partial^{n-i}}{\partial^{n-i}x_1},$$

and ψ is a function of type

$$\psi(x,k) = \exp(\sum_{j=1}^{\infty} x_j k^j)(1 + \sum_{i=1}^{\infty}\xi_i k^{-i}),$$

(here $k \in \mathbb{C}$ belong to some neighbourhood of ∞ and $x = (x_1, x_2, ...)$ — is a finite sequence).

We use notation

$$\partial_i = \frac{\partial}{\partial x_i},\ \partial = \partial_1.$$

A direct calculation gives

Lemma 2. *Conditions of compatibility of (1) are*

$$B_s^t = -\sum_{i=1}^{t-1} C_s^i \partial^i \xi_{t-i} - \sum_{j=2}^{t-1} B_s^j \sum_{i=0}^{t-j-1} C_{s-j}^i \partial^i \xi_{t-i-j}, \tag{2}$$

$$\partial_n \xi_i = \sum_{j=1}^{n+i-1} C_n^j \partial^j \xi_{i+n-j} + \sum_{k=2}^{n} B_n^k \sum_{j=0}^{n-k} C_{n-k}^j \partial^j \xi_{i+n-j-k}. \quad \square \tag{3}$$

In this case ψ is called a *Baker-Akhiezer function.*

Consider now the function

$$\ln\psi(x,k) = \sum_{j=1}^{\infty} x_j k^j + \sum_{j=1}^{\infty} \eta_j k^{-j},$$

where

$$\xi_j = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1+\cdots+i_n=j} \eta_{i_1}\cdots\eta_{i_n}.$$

Lemma 3. *Let $2 \leqslant t \leqslant s$. Then*

$$B_s^t = -\sum_{n=1}^{\infty} \sum P_s \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix} \partial^{j_1} \eta_{i_1} \cdots \partial^{i_n} \eta_{i_n},$$

where the second sum is taken over all matrices $\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ *such that* $i_m \geqslant 1, j_m \geqslant 1$ *and* $i_1 + \cdots + i_n + j_1 + \cdots + j_n = t$.

Proof: By induction on t. For $t = 2$ according to (2), $B_s^2 = -s\partial\xi_1 = -P_s\binom{1}{1}\partial\eta_1$. To prove the lemma for $t = N$ consider that it is proved for $t < N$. According to (2)

$$B_s^t = -\sum_{i=1}^{t-1} C_s^i \partial^i \Big(\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1 + \cdots + i_n = t-i} \eta_{i_1} \cdots \eta_{i_n} \Big) +$$

$$+ \sum_{j=2}^{t-1} \Big(\sum_{n=1}^{\infty} \sum_{i_1 + \cdots + j_n = j} P_s \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix} \partial^{j_1} \eta_{i_1} \cdots \partial^{j_n} \eta_{i_n} \Big) \cdot$$

$$\Big(\sum_{i=0}^{t-j-1} C_{s-j}^i \partial^j \Big(\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1 + \cdots + i_n = t-i-j} \eta_{i_1} \cdots \eta_{i_n} \Big) \Big) =$$

$$= -\sum_{n=1}^{\infty} \sum \Big(\frac{1}{n!} C_s^{j_1 + \cdots + j_n} \frac{(j_1 + \cdots + j_n)!}{j_1! \cdots j_n!} -$$

$$\sum_{q=1}^{n-1} P_s \begin{pmatrix} i_1 & \dots & i_q \\ j_1 & \dots & j_q \end{pmatrix} \frac{1}{(n-q)!} C_{s-(i_1 + \cdots i_q + j_1 + \cdots + j_q)}^{j_{q+1} \cdots j_n} \frac{(j_{q+1} + \cdots j_n)!}{j_{q+1}! \cdots j_n!} \Big) \partial^{j_1} \eta_{i_1} \cdots \partial^{j_n} \eta_{i_n} =$$

$$= -\sum_{n=1}^{\infty} \sum P_s \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix} \partial^{j_1} \eta_{i_1} \cdots \partial^{j_n} \eta_{i_n},$$

where the second sums are taken over all matrices $\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ such that $i_1 + \cdots + i_n + j_1 + \cdots + j_n = t$, $i_m \geqslant 1, j_m \geqslant 0$. According to lemma 1 it is possible to consider the last sum with $j_m > 0$ for all m. $\square$

Lemma 4.

$$\partial_s \eta_r = \sum_{n=1}^{\infty} \sum P_s \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix} \partial^{j_1} \eta_{i_1} \cdots \partial^{j_n} \eta_{i_n},$$

where the second sum is taken by all matrices $\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ *such that* $i_m \geqslant 1, j_m \geqslant 1$ *and* $i_1 + \dots + i_n + j_1 + \dots + j_n = r + s$.

Proof: By induction on r. According to (3) and lemma 3 for $r = 1$

$$\partial_s \eta_1 = \partial_s \xi_1 = \sum_{j=1}^{\infty} C_s^j \partial^j \xi_{s+1-j} + \sum_{k=2}^{s} B_s^k \sum_{j=0}^{s-k} C_{s-k}^j \partial^j \xi_{1+s-j-k} =$$

$$\sum_{j=1}^{s} C_s^j \partial^j \xi_{s+1-j} - \sum_{k=2}^{\infty} \Big(\sum_{n=1}^{\infty} \sum_{i_1+\dots+j_n=k} P_s \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix} \partial^{j_1} \eta_{i_1} \cdots \partial^{j_n} \eta_{i_n} \Big) \cdot$$

$$\Big(\sum_{j=0}^{s-k} C_{s-k}^j \partial^j \xi_{1+s-j-k} \Big) = \sum_{n=1}^{\infty} \sum P_s \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix} \partial^{j_1} \eta_{i_1} \cdots \partial^{j_n} \eta_{i_n},$$

where the second sum in the last formula is taken over all matrices $\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ such that $i_1 + \dots + i_n + j_1 + \dots + j_n = s + 1,\ i_m \geqslant 1,\ j_m \geqslant 0$. According to lemma 1 in this sum it is sufficient to consider only matrices, where $j_m > 0$ for all m.

To prove now the lemma for $r = N$,we assume that it is proved for $r < N$. According to (3)

$$\partial_s \Big(\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1+\dots+i_n=r} \eta_{i_1} \cdots \eta_{i_n} \Big) = \sum_{j=1}^{s+r-1} C_s^j \partial^j \xi_{s+r-j} + \sum_{k=2}^{\infty} B_s^k \sum_{j=0}^{s-k} C_{s-k}^j \partial^j \xi_{r+s-j-k}.$$

Thus according to lemma 3, lemma 1 and inductive hypothesis,

$$\partial_s \eta_r = \sum_{j=1}^{\infty} C_s^j \partial^j \Big(\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1+\dots+i_n=s+r-j} \eta_{i_1} \cdots \eta_{i_n} \Big) +$$

$$+ \sum_{k=2}^{\infty} \Big(\sum_{i_1+\dots+j_n=k} P_s \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix} \partial^{j_1} \eta_{i_1} \cdots \partial^{j_n} \eta_{i_n} \Big) \cdot$$

$$\sum_{j=0}^{s-k} C_{s-k}^j \partial^j \Big(\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1+\dots+i_n=r+s-j-k} \eta_{i_1} \cdots \eta_{i_n} \Big) - \partial_s \Big(\sum_{n=2}^{\infty} \frac{1}{n!} \sum_{i_1+\dots+i_n=r} \eta_{i_1} \cdots \eta_{i_n} \Big) =$$

$$= \sum_{n=1}^{\infty} \sum P_s \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix} \partial^{j_1} \eta_{i_1} \cdots \partial^{j_n} \eta_{i_n},$$

where the second sum in the last formula is taken over all matrices $\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ such that $i_1 + \dots + i_n + j_1 + \dots + j_n = s + r$, $i_m \geqslant 1, j_m \geqslant 1$. $\square$

5. KP hierarchy

According to [DKJM] the Baker-Akhiezer function ψ is

$$\psi(x,k) = \exp(\sum x_j k^j) \frac{\tau(x_1 - k^{-1}, x_2 - \frac{1}{2}k^{-2}, x_3 - \frac{1}{3}k^{-3}, ...)}{\tau(x_1, x_2, x_3, ...)}$$

for some function $\tau(x_1, x_2, ...)$. This gives a possibility to describe the KP hierarchy as an infinite system of differential equations on $v(x,k) = -\ln \tau(x,k)$, see[N]. Indeed,

$$\sum_{j=1}^{\infty} \eta_j k^{-j} = \ln \psi(x,k) - \sum_{j=1}^{\infty} x_j k^j = -v(x_1 - k^{-1}, x_2 - \frac{1}{2}k^{-2}, ...) + v(x) =$$

$$\sum_{n=1}^{\infty} \sum_{i_1 + \dots + i_n = j} \frac{(-1)^{n+1}}{n! i_1 \dots i_n} \partial_{i_1} \cdots \partial_{i_n} v(x) k^{-j}.$$

Therefore,

$$\eta_r = \sum_{n=1}^{\infty} \sum_{i_1 + \dots + i_n = r} \frac{(-1)^{n+1}}{n! i_1 \dots i_n} \partial_{i_1} \cdots \partial_{i_n} v. \tag{4}$$

Theorem 2. *There exist universal rational coefficients*

$$R_r \begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}, R_{ij} \begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}$$

such that

$$\eta_r = \frac{1}{r} \partial_r v + \sum_{n=1}^{\infty} \sum R_r \begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix} \partial_{s_1} \partial^{t_1} v \cdots \partial_{s_n} \partial^{t_n} v, \tag{5}$$

$$\partial_i \partial_j v = \sum_{n=1}^{\infty} \sum R_{ij} \begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix} \partial_{s_1} \partial^{t_1} v \cdots \partial_{s_n} \partial^{t_n} v, \tag{6}$$

where the second sums are taken over all matrices $\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}$ *such that* $s_m, t_m \geqslant 1$, *and the sum* $s_1 + \dots + s_n + t_1 + \dots + t_n$ *is equal* r *for (5) and* $i + j$ *for (6). Moreover,*

$$R_{ij} \begin{pmatrix} s_1 & \dots & s_n \\ 1 & \dots & 1 \end{pmatrix} = \frac{j}{s_1 \cdots s_n} P_i(s_1 ... s_n).$$

Proof: By induction on k and $i+j$. For $i+j=2$ the theorem is obvious. For $r=1$ it follows from (4). To prove theorem for $i+j=N$ and $r=N-1$, consider that it is proved for $i+j<N$ and $r<N-1$. Later we consider that $s_m, t_m \geqslant 1$ and $\sigma_n = s_1+\cdots+s_n+t_1+\cdots+t_n$. Then according to (4) and (6)

$$\eta_r = \frac{1}{r}\partial_r v + \sum_{n=2}^{\infty} \sum_{s_1+\cdots+s_n=r} \frac{(-1)^{n+1}}{n! s_1\cdots s_n} \partial_{s_1}\cdots\partial_{s_n} v(x) =$$

$$\frac{1}{r}\partial_r v + \sum_{n=1}^{\infty} \sum_{\sigma_n=r} R_r \begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix} \partial_{s_1}\partial^{t_1} v \cdots \partial_{s_n}\partial^{t_n} v.$$

Thus according to (5), (6) and lemma 4

$$\frac{1}{j}\partial_i\partial_j v = \partial_i \eta_j - \partial_i \Big(\sum_{n=1}^{\infty} \sum_{\sigma_n=j} R_j \begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix} \partial_{s_1}\partial^{t_1} v \cdots \partial_{s_n}\partial^{t_n} v \Big) =$$

$$\sum_{n=1}^{\infty} \sum_{\sigma_n=i+j} P_i \begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix} \partial^{t_1}\eta_{s_1} \cdots \partial^{t_n}\eta_{s_n} -$$

$$\partial_i \Big(\sum_{n=1}^{\infty} \sum_{\sigma_n=j} R_j \begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix} \partial^{t_1}\partial_{s_1} v \cdots \partial^{t_n}\partial_{s_n} v \Big) =$$

$$\sum_{n=1}^{\infty} \sum_{s_1+\cdots+s_n+n=i+j} P_i \begin{pmatrix} s_1 & \dots & s_n \\ 1 & \dots & 1 \end{pmatrix} \partial(\frac{1}{s_1}\partial_{s_1} v) \cdots \partial(\frac{1}{s_n}\partial_{s_n} v) +$$

$$\sum_{n=1}^{\infty} \sum_{\sigma_n=i+j, t_1+\cdots+t_n>n} R_{ij} \begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix} \partial_{s_1}\partial^{t_1} v \cdots \partial_{s_n}\partial^{t_n} v.$$

The obvious relation $P_i \begin{pmatrix} s_1 & \dots & s_n \\ 1 & \dots & 1 \end{pmatrix} = P_i(s_1 ... s_n)$ concludes the proof. □

Consequence 1. *The system (6) is equivalent of KP hierarhy.*

Remark. The first equation from (6) is the "integrated over x_1" KP equation

$$\partial_2^2 v = \frac{4}{3}\partial_3\partial_1 v - \frac{1}{3}\partial_1^4 v + 2(\partial_1^2 v)^2.$$

The relation (6) was first deduced in [DN] from the Hirota hierarchy [DKJM]. The method produces another formulas for coefficients R_{ij}. The equality of coefficients provides nontrivial combinatorial identities. □

Remark. From the consequence it follows that any formal solution of KP hierarhy is defined defined up to a constant by an arbitrary infinite set of series of one variable $f_i(x_1) = \partial_i v|_{x_2=x_3=\cdots=0} \quad (i = 1, 2, ...)$.

6. Gelfand–Dikii hierarchy

According to [S], the set of solution of n–Gelfand–Dikii$(n - -KdV)$ hierarchy bijectively corresponds to the set of solutions of KP hierarchy independent from x_n. In this case according to theorem 2

$$0 = \partial_m \partial_n v = \frac{mn}{m+n-1} \partial_{n+m-1} \partial v + \sum_{m=1}^{\infty} \sum R_{mn} \begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m \end{pmatrix} \partial_{s_1} \partial^{t_1} v \cdots \partial_{s_m} \partial^{t_m} v, \quad (7)$$

where $1 \leqslant s_j \leqslant n+m-2,\ t_j \geqslant 1$. This gives recurrence formulas expressing $\partial_k \partial v$ for $k > n$ via $\partial_r \partial v$ for $r < n$. Thus, we have relations

$$\partial \partial_{n+r} v = \sum_{m=1}^{\infty} \sum N^m_{1(n+1)} \begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m \end{pmatrix} \partial_{s_1} \partial^{t_1} v \cdots \partial_{s_m} \partial^{t_m} v, \quad (8)$$

where $t_j \geqslant 1, s_j < n, \quad \sum_{j=1}^m (s_j + t_j) = n + r + 1$.

Example. For $n = 2$ the system (8) is reducied to the $K\partial V$ hierarchy.

Comparing the systems (8) and (6), we find

Theorem 3. *The n–Gelfand–Dikii hierarchy is equivalent to a system of differential equations in the form*

$$\partial_i \partial_j v = \sum_{m=1}^{\infty} \sum N^m_{ij} \begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m \end{pmatrix} \partial_{s_1} \partial^{t_1} v \cdots \partial_{s_m} \partial^{t_m} v, \quad (9)$$

where $i, j \geqslant 1,\ 1 \leqslant s_\alpha \leqslant n-1,\ t_\alpha \geqslant 1, \sum_{i=1}^m (s_\alpha + t_\alpha) = i + j$ and $N^m_{ij} \begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m \end{pmatrix}$ are some universal rational coefficients.

Examples. 1. For $n = 3$ the first equation from the system (9) is the Boussinesq equation

$$\partial_2^2 v = -\frac{1}{3} \partial^4 v + 2(\partial^2 v)^2.$$

2. For $n = 4$ the first equations of the system (9) are

$$\partial_2^2 v = \frac{4}{3}\partial_3\partial v - \frac{1}{3}\partial^4 v + 2(\partial^2 v)^2,$$

$$\partial_3\partial_2 v = -\frac{3}{2}\partial_2\partial^3 v + 3\partial_2\partial v\partial^2 v,$$

$$\partial_3^2 v = -\frac{1}{4}\partial_3\partial^3 v + \frac{1}{8}\partial^6 v + \frac{9}{8}(\partial_2\partial v)^2 - \frac{9}{8}(\partial^3 v)^2 - \frac{9}{4}\partial^4 v\partial^2 v + 3(\partial^2 v)^3.\Box$$

Remark The structure of the system (9) demonstrats that any formal solutions of n–Gelfand–Dikii equation is defined up to a constant by an arbitrary set of $n-1$ series of one variable $f_i(x_1) = \partial_i v|_{x_2=x_3=\cdots=0} \quad (i = 1, ..., n-1)$.

According to [K] the dispersionless (quasiclassical) limit of n–Gelfand–Dikii hierarchy is given by n–Gelfand–Dikii hierarchy with out all monomials

$$R_{ij}\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}\partial_{s_1}\partial^{t_1} v\cdots\partial_{s_n}\partial^{t_n} v,$$

where $\sum_{i=1}^n t_j > n$.

Thus, from theorem 2 it follows

Consequence 1. *Dispersionless (quasiclassical) limit of n–Gelfand–Dikii hierarchy is given by the infinite system*

$$\frac{1}{j}\partial_i\partial_j v = \sum_{m=1}^{\infty}\sum_{s_1+\cdots+s_m=i+j-m} P_i(s_1...s_m)\frac{1}{s_1}\partial\partial_{s_1} v\cdots\frac{1}{s_m}\partial\partial_{s_m} v. \tag{10}$$

where i,j=1,2,... ,

and the equation $\partial_n v = 0$

7. Proof of theorem 1

According to [K] one of A_{n-1}-potentials is

$$\widetilde{F}(x_1, ..., x_{n-1}) = v(x_1, \frac{x_2}{2}, ..., \frac{x_{n-1}}{n-1}, 0, ...),$$

where v satisfies the dispersionless n–Gelfand–Dikii hierarchy .

From consequence 1 it follows

$$0=\partial_\ell\partial_n\widetilde{F}=\sum_{m=1}^{\infty}\sum_{s_1+\cdots+s_m=\ell+n-m}P_n(s_1\dots s_m)\partial\partial_{s_1}\widetilde{F}\cdots\partial\partial_{s_m}\widetilde{F}=$$

$$n\partial\partial_{n+\ell-1}\widetilde{F}+\sum_{m=2}^{\infty}\sum_{s_1+\cdots+s_m=\ell+n-m}P_n(s_1\dots s_m)\partial\partial_{s_1}\widetilde{F}\cdots\partial\partial_{s_m}\widetilde{F}.$$

Thus

$$\partial\partial_{n+\ell-1}\widetilde{F}=-\frac{1}{n}\sum_{m=2}^{\infty}\sum_{s_1+\cdots+s_m=\ell+n-m}P_n(s_1\dots s_m)\partial\partial_{s_1}\widetilde{F}\cdots\partial\partial_{s_m}\widetilde{F}. \tag{8}$$

According to [W2]

$$\partial\partial_s\widetilde{F}=x_{n-s}\quad\text{and}\quad\partial\partial_{n+s}\widetilde{F}=-s\partial_s\widetilde{F}$$

for $s<n$. Thus because $\partial\partial_n\widetilde{F}=0$ that according to (11) $\partial\partial_{n+\ell-1}\widetilde{F}=x_{1-\ell}$ for all ℓ. Therefore

$$\partial_s\widetilde{F}=-\frac{1}{s}x_{-s}=\partial_sF_A$$

and $F_A-\widetilde{F}$ — is a linear form. According to [D2, p.16] this form is equal 0. According [Z] $F_B(x_1,...,x_n)=F_A(x_1,0,x_2,0,...,x_{n-1},0,x_n)$.□

These are the potentials find from our formulas

$$(A_2)\quad F=\frac{1}{2}x_1^2x_2+\frac{1}{24}x_2^4,$$

$$(A_3)\quad F=\frac{1}{2}x_1^2x_3+\frac{1}{2}x_1x_2^2+\frac{1}{4}x_2^2x_3^2+\frac{1}{60}x_3^5,$$

$$(A_4)\quad F=\frac{1}{2}x_1^2x_4+x_1x_2x_3+\frac{1}{6}x_2^3+\frac{1}{12}x_3^4+\frac{1}{2}x_2x_3^2x_4+\frac{1}{4}x_2^2x_4^2+\frac{1}{6}x_3^2x_4^3+\frac{1}{120}x_4^6,$$

$$(A_5)\quad F=\frac{1}{2}x_1^2x_5+\frac{1}{2}x_2^2x_3+\frac{1}{4}x_2^2x_5^2+x_2x_3x_4x_5+x_1x_2x_4+\frac{1}{6}x_2x_4^3+\frac{1}{2}x_1x_3^2+\frac{1}{2}x_3^2x_4^2+$$
$$+\frac{1}{6}x_3^3x_5+\frac{1}{6}x_3^2x_5^3+\frac{1}{2}x_3x_4^2x_5^2+\frac{1}{6}x_4^4x_5+\frac{1}{8}x_4^2x_5^4+\frac{1}{210}x_5^7.$$

$$(B_2)\quad F=\frac{1}{2}x_1^2x_2+\frac{1}{60}x_2^5,$$

$$(B_3)\quad F=\frac{1}{2}x_1^2x_3+\frac{1}{2}x_1x_2^2+\frac{1}{6}x_2^3x_3+\frac{1}{6}x_2^2x_3^3+\frac{1}{210}x_3^7.$$

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