Abundance Conjecture for 3-folds: Case $\nu = 1$

by

Yoichi Miyaoka

Dedicated to Professor F. Hirzebruch on his 60th birthday

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Introduction.

A normal projective variety is said to be <u>minimal</u> if it has only terminal singularities and its canonical divisor $K_X \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ is nef. A recent result of S. Mori [Mr] asserts the existence of a minimal model for a given complex algebraic 3-fold except for uniruled one.

In [My] the author proved a minimal 3-fold has non-negative Kodaira dimension; when combined with Mori's theorem mentioned above this amounts to the following characterization of 3-folds with $\kappa = -\infty$:

Theorem. A complex algebraic 3-fold has Kodaira dimension $-\infty$ if and only if it is uniruled.

A natural question now arises: <u>What is the characterization of 3-folds with</u> $\kappa = 0$? More specifically:

(*) Does a 3-fold with $\kappa = 0$ have a minimal model with numerically trivial canonical divisor?

To make things more explicite, let us introduce an invariant $\nu(X)$, the <u>numerical Kodaira dimension</u>, of a minimal variety X . By definition,

$$\nu(X) = \min \{d \in \mathbb{Z};\ c_1(K_X)^{d+1} = 0 \in H^{2d+2}(X,\mathbb{Q})\} .$$

Clearly ν takes value in $\{0,1,\ldots,\dim X\}$. For example, $\nu(X) = 0$ is equivalent to the numerical triviality of K_X ; $\nu(X) = \dim X$ if and only if K_X is <u>big</u>, i.e. $K_X^{\dim X} > 0$.

As is easily seen, the question (*) would be affirmatively answered if we could verify

(**) (Abundance conjecture) $\kappa(X) = \nu(X)$.

The inequality $\kappa(X) \leq \nu(X)$ follows from a formal argument, yet the inequality of the converse direction is not so trivial. Furthermore (**) involves an important implication; via his powerful "base point freeness theorem", Y. Kawamata [Kw] pointed out that the linear system $|mK_X|$ is free from base points for sufficiently divisible m , provided the abundance conjecture (**) is true.

In an extremal case $\nu = 0$ or 3, the equality $\kappa = \nu$ for a minimal 3-fold can be checked rather easily. The objective of the present paper is to show the equality in one of the intermediate cases: $\nu = 1$.

Main Theorem. Let X be a minimal 3-fold with $\nu(X) = 1$. Then $\kappa(X) = 1$ and there is a positive integer m such that

$\mathcal{O}_X(mK_X)$ is generated by global sections.

Our proof is based on the analysis of an effective Cartier divisor $D \in |mK_X| (m > 0)$, the existence of which is guaranteed by $\kappa(X) \geq 0$ [My]. We are interested in the analytic and infinitesimal neighbourhoods of D as well as D itself. A direct analysis of them seems a little bit too tough; to simplify the situation, we need three reduction steps described below.

Let $U \subset X$ be a sufficiently small analytic neighbourhood of D. Then we have:

(0.1) (Gorenstein reduction, see § 1) There is a finite covering $\gamma : V \longrightarrow U$ étale off $\mathrm{Sing}(U)$ such that $K_V = \gamma^* K_U$ is Cartier.

(0.2) (Semi-stable reduction, see § 2) There is a proper, generically finite covering $\sigma : W \longrightarrow V$, étale off $\mathrm{supp}(\gamma^* D)$, such that W is smooth and that $\sigma^*\gamma^* D$ is a multiple of a reduced divisor $\tilde{D}$ with only simple normal crossings.

(0.3) (Minimal model à la Kulikov-Persson-Pinkham, § 3) After finitely many contractions of components of $\tilde{D}$ and elementary transformations, a smooth "minimal model" $(W_0, \tilde{D}_0)$ of $(W, \tilde{D})$ is reached. The natural image $\tilde{D}_0$ of $\tilde{D}$ in W_0 is still a divisor with only simple normal crossings and $\tilde{D}_0 | \tilde{D}_0 \approx K_{W_0} | \tilde{D}_0 \approx 0$.

Once we come across this situation, it is combinatorics to determine the structure of $\tilde{D}_0$ as an analytic space. A theorem of R. Friedman shows that $\tilde{D}_0$ is actually a degeneration of smooth surfaces with $\kappa = 0$. This implies that $K_{W_0}|\tilde{D}_0$ and $\tilde{D}_0|\tilde{D}_0$ are both torsions in $\mathrm{Pic}(\tilde{D}_0)$ so that there exists an étale covering $\tau : M \longrightarrow W_0$ such that $K_M|S \sim S|S \sim 0$, where $S = \tau^*\tilde{D}_0$. Finally, we study the infinitesimal neighbourhoods of S in M :

(0.4) The infinitesimal displacements of S in M is not obstructed. In particular,

$$\dim H^0(nS, \mathcal{O}_{nS}(kS)) = n \quad \text{for} \quad n \in \mathbb{N},\ k \in \mathbb{Z} ,$$

whence it follows that

$$\dim H^0(nD, \mathcal{O}_{nD}(nD)) \sim O(n) .$$

Main Theorem is a direct consequence of (0.4), see § 4.

In this paper, we work in the category of analytic spaces.

Acknowledgements. This paper was motivated by M. Reid's suggestion that the analysis of $D \in |mK_X|$ eventually leads to the proof of (**); in this sense the present approach owes much to him. The idea was conceived at University of Pisa and worked out at Columbia University and Max-Planck-Institut at

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1. Gorenstein reduction

In order to show the Gorenstein reduction (0.1), let us start with some elementary observations.

(1.1) Lemma. Let $(Z,0)$ be a germ of terminal 3-fold singularity of index r. Then

$$H_1(Z,\mathbb{Z}) = 0 \ ,$$

$$H_1(Z-0,\mathbb{Z}) \cong \mathbb{Z}/r\mathbb{Z} \ ,$$

$$\mathrm{Pic}(Z) = (1) \ ,$$

$$\mathrm{Pic}(Z-0)_{tor} \cong \mathrm{Hom}(H_1(Z-0,\mathbb{Z}),\mathbb{C}^*)_{tor} \cong \mu_r \ .$$

Proof. $(Z,0)$ is a μ_r-quotient of a compound Du Val singularity $(\tilde{Z},\tilde{0})$ and $\pi_1(\tilde{Z}-\tilde{0}) = (1)$ by Milnor's theorem [Ml, Theorem 6.6]. □

(1.2) Lemma. Let $(Z,0)$ be as above and S an effective Cartier divisor passing through 0. Then the restriction mapping

$$\mathrm{Pic}(Z-0)_{tor} \longrightarrow \mathrm{Pic}(S-0)_{tor}$$

is injective.

Proof. Let $f : \tilde{Z} \longrightarrow Z$ be the "canonical" μ_r-covering as in the proof of (1.1). $\tilde{S} = f^*S$ is a <u>connected</u> Cartier divisor on $\tilde{Z}$, while $\tilde{0} = f^{-1}(0)$ is a single point and hence of codimension 2 in $\tilde{S}$. Therefore $\tilde{S}-\tilde{0}$ is connected, which implies the surjectivity of $\pi_1(S-0) \longrightarrow \pi_1(Z-0)$ and of $H_1(S-0,\mathbb{Z}) \longrightarrow H_1(Z-0,\mathbb{Z})$. Thus we infer the injectivity of

$$\mathrm{Pic}(Z-0)_{tor} \cong \mathrm{Hom}(H_1(Z-0,\mathbb{Z}),\mathbb{C}^*) \longrightarrow F = \mathrm{Hom}(H_1(S-0,\mathbb{Z}).\mathbb{C}^*)) .$$

The group F is naturally identified with that of flat line bundles $\subset \mathrm{Pic}(S-0)$.

(1.3) Corollary. In the same notation as in (1.2), $\alpha K_Z|_S$ is Cartier on S if and only if $r|\alpha$ ($\alpha \in \mathbb{Z}$, γ = index of $(Z,0)$).

Proof. $\alpha K_Z|_S$ is Cartier if and only if $\mathcal{O}_{S-0}(\alpha K_Z) \cong \mathcal{O}_{S-0}$, which means that αK_Z is trivial on $Z-0$ by (1.2), i.e. αK_Z is Cartier on Z.

□

Let U be an analytic 3-fold with only finitely many terminal singularities and $D \subset \bar{U}$ an effective Cartier divisor which contains the singular locus $\mathrm{Sing}(U)$.

(1.4) Lemma. Let r denote the index of $\bar{U}$, viz. the L.C.M. of the indices at the singular points. Assume that $c_1(rK_{\bar{U}})|D \in H^2(D,\mathbb{Z})$ is a torsion. Then there are a small neighbourhood $\bar{U}' \subset \bar{U}$ of D and a finite étale covering $g : \bar{U}'' \longrightarrow \bar{U}'$ such that $c_1(rK_{\bar{U}''})|g^*D = 0 \in H^2(g^*D,\mathbb{Z})$.

Proof. Immediate consequence of the natural isomorphism

$$H^2(D,\mathbb{Z})_{tor} \cong H_1(D,\mathbb{Z})_{tor} \cong H_1(\bar{U}',\mathbb{Z})_{tor}$$

for a tubular neighbourhood $\bar{U}'$ of D .

□

(1.5) Lemma. Let the notation and the assumption be as in (1.4). Then there exists a finite cyclic μ_r-covering $h : D^* \longrightarrow g^*D$ which has the following two properties:

(1.5.1) h is étale off $\mathrm{Sing}(\bar{U}'') \subset g^*D$;

(1.5.2) The branch index of h at $P \in g^*D$ is exactly the local index of $\bar{U}''$ at P ; in other words, D^* is locally a disjoint union of canonical covers over P .

Proof. Since $\mathrm{Pic}^0(g^*D) \cong H^1(g^*D,\mathcal{O})/H^1(g^*D,\mathbb{Z})$ is a divisible group, we can find $\tau \in \mathrm{Pic}^0(g^*D)$ such that $rK_{\bar{U}''} - r\tau = 0 \in \mathrm{Pic}^0(g^*D)$. Fix a non-zero section $s \in H^0(g^*D,\mathcal{O}_{g^*D}(rK_{\bar{U}''} - r\tau))$ and construct a μ_r-cover

$$D^* = \mathrm{Specan}\ \{\mathcal{O}_{g^*D} \oplus \mathcal{O}_{g^*D}(\tau - K_{\bar{U}''}) \oplus \ldots \oplus \mathcal{O}_{g^*D}((r-1)(\tau - K_{\bar{U}''}))\}$$

in a standard manner. Then D^* satisfies our requirements by (1.4) since $\mathcal{O}(\tau)$ is locally isomorphic to $\mathcal{O}$.

□

Now we have the following theorem which is slightly more general than (0.1):

(1.6) Theorem. Let $\bar{U}$ be an analytic 3-fold with only finitely many terminal singularities and D an effective Cartier divisor. Let r be the index of $\bar{U}$ and assume that $c_1(rK_{\bar{U}})|_D \in H^2(D,\mathbb{Z})$

is a torsion. Then, for a sufficiently small neighbourhood $\bar{U}' \subset \bar{U}$ of D, there is a finite covering $\gamma : V \longrightarrow \bar{U}'$ which satisfies the following conditions:

(1.6.1) γ is étale off $\mathrm{Sing}(\bar{U}')$;

(1.6.2) The branch index of γ at $P \in D$ is exactly the local index of $\bar{U}$ at P ;

(1.6.3) V is a normal Gorenstein analytic space with only terminal singularities.

Proof. Fix a small neighbourhood $\Delta \subset \bar{U}$ of $\mathrm{Sing}(\bar{U})$. Then choose a sufficiently small neighbourhood $\bar{U}' \subset \bar{U}$ of D in such a way that $D_0 = D - (D \cap \Delta)$ is a deformation retract of $\bar{U}'_0 = U' - (\bar{U}' \cap \Delta)$. By (1.5), we have a finite étale covering

$$\bar{\gamma} : D^*_0 = D^* - h^{-1}(g^{-1}(D \cap \Delta)) \longrightarrow D_0 .$$

Since $\pi_1(D_0) \cong \pi_1(\bar{U}'_0)$, there is an étale covering

$$\gamma_0 : V_0 \longrightarrow \bar{U}'_0$$

which induces $\bar{\gamma}$. On the other hand, we have the canonical covering $\tilde{\Delta} \longrightarrow \Delta$. Recalling that $g \circ h : D^* \longrightarrow D$ is locally the canonical covering, we can patch up V_0 with finitely many copies of components of $\tilde{\Delta}$ to get a finite covering

$$\gamma : V \longrightarrow \bar{U}' \cup \Delta .$$

This construction implies (1.6.1-3).

□

2. Semi-stable reduction

Let Y be a complex 3-manifold, $E \neq 0$ an effective, projective Cartier divisor on Y and $V \subset Y$ a small open neighbourhood of E. Throughout this section, we fix this notation and assume the following extra conditions:

a) The reduced part E_{red} of E is a divisor with only simple normal crossings;

b) $E|E$ is numerically trivial on E ;

c) There exists a divisor H on Y such that $H|E$ is ample.

Let $E = \sum_{i=1}^{s} a_i S_i$ be the decomposition into distinct irreducible components.

(2.1) Lemma. The restriction maps and the degree maps give natural isomorphisms

$$H^4(E,\mathbb{Z}) \xrightarrow[\sim]{\text{rest.}} \bigoplus_{i=1}^{s} H^4(S_i,\mathbb{Z}) \xrightarrow[\sim]{\deg} \mathbb{Z}^s .$$

Proof. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z}_E \longrightarrow \bigoplus_{i=1}^{s} \mathbb{Z}_{S_i} \longrightarrow \bigoplus_{i<j} \mathbb{Z}_{S_i \cap S_j} \longrightarrow \bigoplus_{i<j<k} \mathbb{Z}_{S_i \cap S_j \cap S_k} \longrightarrow 0 .$$

From the fact that the real dimension of $S_i \cap S_j = 2$, the assertion easily follows.

□

We denote by δ the natural isomorphism $H^4(E,\mathbb{Z}) \xrightarrow{\approx} \mathbb{Z}^s$. Let $\rho : H^4_c(V,\mathbb{Z}) \longrightarrow H^4_c(E,\mathbb{Z}) = H^4(E,\mathbb{Z})$ be the restriction map, where the subscript c stands for the cohomology with compact support.

(2.2) Lemma. $\mathrm{Im}(\delta \circ \rho) \subset \{(x_1,\ldots,x_s) \in \mathbb{Z}^s;\ \Sigma a_i x_i = 0\}$.

Proof. Let $\eta \in H^4_c(V,\mathbb{Z})$. Then $\deg(\eta|S_i) = \deg(\eta \cup S_i)$, so that

$$\Sigma a_i \deg(\eta|S_i) = \Sigma a_i \deg(\eta \cup S_i) = \deg(\eta \cup \Sigma a_i S_i) = \deg \eta \cup E .$$

By the Lefschetz duality $H^4_c(V,\mathbb{Z}) \cong H_2(V,\mathbb{Z}) \cong H_2(E,\mathbb{Z})$, η can be regarded as a 2-cycle η' on E and we have

$$\deg \eta \cup E = \deg E|\eta' .$$

Since E is numerically trivial on E , $\deg E|\eta' = 0$ which proves the lemma.

□

(2.3) Corollary. $\ker\{H_1(V-E,\mathbb{Z}) \longrightarrow H_1(V,\mathbb{Z}\}$ has positive rank.

Proof. By the Lefschetz duality we have

$$\ker\{H_1(V-E,\mathbb{Z}) \longrightarrow H_1(V,\mathbb{Z})\} \cong \ker\{H_c^5(V,E;\mathbb{Z}) \longrightarrow H_c^5(V,\mathbb{Z})\}$$

$$\cong \operatorname{Coker}\{H_c^4(V,\mathbb{Z}) \longrightarrow H^4(E,\mathbb{Z})\} ,$$

and the third term has positive rank by (2.2).

□

(2.4) Definition. Let $L \subset Y$ be a compact effective divisor such that (2.4.a) L is projective with an ample divisor H and that

(2.4.b) $L|L$ is numerically trivial.

Let $L = \Sigma e_i L_i$ be the decomposition into irreducible components. L is said to be primitive if L is connected and G.C.D. $\{e_i\} = 1$.

(2.5) Lemma. Suppose that an effective divisor $L = \Sigma e_i L_i$ satisfies (2.4.a) and (2.4.b). Assume that L is connected. If $(\Sigma e_i' L_i)\cdot H|L$ is numerically trivial, then $e_i' = ce_i$ for some constant $c \in \mathbb{Q}$ independent of i . In particular, L can be uniquely decomposed into $\Sigma l_i L_i$, where L_i's are primitive and disjoint with each other.

The proof is easy and left to the reader. Applying this to our original situation, we have

(2.6) Corollary. E can be uniquely decomposed into $\Sigma b_i E_i$, where E_i's are primitive divisors which are mutually disjoint and b_i's are positive integers.

Thus the small neighbourhood $V \subset Y$ is a disjoint union of neighbourhoods V_i of E_i. Therefore, without loss of generality, we may assume that E is connected in the argument below. Let $E = e \sum_{i=1}^{s} a_i' S_i$ be the decomposition into irreducible components, where $e \in \mathbb{N}$, G.C.D. $\{a_i'\} = 1$.

(2.7) Lemma. Assume that E is connected. Then

$$\operatorname{Im} \delta \circ \rho \subset \{x_1, \ldots, x_s) \in \mathbb{Z}^s ; \Sigma a_i' x_i = 0\}$$

is a sublattice of finite index.

Proof. It suffices to show $\operatorname{Im}(\delta \circ \rho \otimes \mathbb{Q}) = \{(x_1, \ldots, x_s) \in \mathbb{Q}^s ; \Sigma a_i' x_i = 0\}$. Consider the $\mathbb{Q}$-vector subspace $\Pi \subset \operatorname{Im}(\delta \circ \rho \otimes \mathbb{Q})$ generated by $S_1 H|E, \ldots, S_s H|E$. (Note that $S_i \in H_c^2(V, \mathbb{Z})$, $H \in H^2(V, \mathbb{Z})$ so that $S_i \cdot H \in H_c^4(V, \mathbb{Z})$.) Then, by (2.5), the unique relation between the $S_i \cdot H|E \in H^4(E, \mathbb{Q})$ is

$$\Sigma(a_i' S_i \cdot H)|E = 0 .$$

Hence $\dim_{\mathbb{Q}} \operatorname{Im}(\delta \circ \rho \otimes \mathbb{Q}) = \dim_{\mathbb{Q}} \operatorname{Im}(\rho \otimes \mathbb{Q})$

$$\geq \dim_{\mathbb{Q}} \Pi = s-1 = \dim_{\mathbb{Q}} \{(x_1, \ldots, x_s) \in \mathbb{Q}^s ; \Sigma a_i' x_i = 0\} .$$

This shows the assertion.

□

(2.8) Corollary. If E is connected, then

$$\ker\{H_1(V-E,\mathbb{Z}) \longrightarrow H_1(V,\mathbb{Z})\}/\mathrm{tor} \cong \mathrm{Coker}\{H^4_c(V,\mathbb{Z}) \longrightarrow H^4(E,\mathbb{Z})\}/\mathrm{tor}$$
$$\cong \delta^{-1}(\mathbb{Z}(a'_1,\ldots,a'_s)) \subset H^4(E,\mathbb{Z}) .$$

(2.9) Corollary. For each positive integer l , there exists a canonical μ_l-covering $\sigma_l : V_l \longrightarrow V$ branching along E whose branch index along S_i is exactly $l/(l,a'_i)$. If l is divisible by $a'_1,\ldots,a'_s$, then $(\sigma_l^* E)/l$ is a reduced Cartier divisor.

The normal analytic space V_l has toric singularities over the double curves of E_{red} . However, it is known that V_l has a nice resolution:

(2.10) Theorem (G. Kempf and al. [KKMS]). If l is sufficiently divisible, then V_l has a resolution $\pi = \pi_l : W = W_l \longrightarrow V_l$ such that $\pi^* \sigma_l^* E/l$ is a reduced divisor with only simple normal crossings.

(2.11) Remark. The integer l above is not L.C.M.$\{a'_i\}$ in general.

Putting things together, we obtain

(2.12) Theorem. There exists a proper, generically finite covering $\sigma : W \longrightarrow V$ such that

(2.12.a) W is non-singular and that

(2.12.b) $\sigma^{*}E$ is a multiple of e reduced divisor with only simple normal crossings.

To show (0.2), we apply (2.12) to a suitable resolution (Y,E) of the Gorenstein reduction of $(\bar{U},D)$. Since D comes from X , its total transform E is projective; H is easily constructed from the pull-back of an ample divisor on X and the exceptional divisors with respect to the resolution.

3. Minimal model

Let N be an analytic 3-manifold with an effective, projective, reduced divisor T on it. Assume the following two conditions:

a) $T|T$ is numerically trivial (on T);

b) There are positive integers m_i such that $K_N|T \approx (\Sigma m_i T_i)|T$, where T_i's stand for the irreducible components of T.

(3.1) Remark. In this situation, $K_N|T$ is nef $\Leftrightarrow K_N|T \approx 0 \Leftrightarrow m_i = m_j$ for every i,j. If we start with $D \in |mK_X|$ for a minimal 3-fold X and take a Gorenstein reduction $\gamma : V \longrightarrow U$ of a small neighbourhood U of D and then a semi-simple reduction $\sigma : W \longrightarrow V$, then the pair $(W, \sigma^* \gamma^* D/\deg \sigma)$ satisfies the conditions a) and b) above. (Without Gorenstein reduction, the coefficient m_i might be a rational number.) Furthermore, we have in this case

$$K_W \sim \Sigma m_i \tilde{D}_i, \ m_i \in \mathbb{N}$$

where $\tilde{D}_i$ is an irreducible component of $\tilde{D} = \sigma^* \gamma^* D/\deg \sigma$.

(3.2) Theorem (Kulikov [K1], Persson-Pinkham [PP]). Let N and T be as above. Then, after finitely many smooth contractions of components of T and/or Kulikov's elementary transformation (or "symmetric flops") we come across a minimal model (M,S) ; the pair (M,S) has the following properties:

(3.2.A) M is non-singular and $K_M|S \approxeq 0$;

(3.2.B) The proper transform S of T is a reduced divisor with only simple normal crossings and $S|S \approxeq 0$;

(3.2.C) If $K_N \sim \Sigma m_i' T_i$, then $K_M \sim (\min\{m_i'\}) \cdot S$.

The original papers deal with a degeneration of smooth surfaces, but their numerical proof works in our setting.

(3.3) Remark. The assumption that m_i is integral is essential. If we allow rational numbers as coefficients, certain quotient singularities appear on a minimal model. S is not necessarily projective; however, contractions of finitely many curves on S gives a normal 3-fold $\hat{M}$ in which the image $\hat{S}$ of S is projective.

It is not too difficult to classify S as an analytic space; the result is essentially given in Friedman-Morrison [FM, p.15 ff.].

(3.4) Theorem. S is isomorphic to one of the following surfaces:

(0) A smooth surface (S is either a $K3$, Enriques, abelian of hyperelliptic surface);

$(1)_s$ A cycle of (relatively) minimal elliptic ruled surfaces S_i $(i \in \mathbb{Z}/s\mathbb{Z},\ s \geq 2)$ and S_i meets only $S_{i\pm1}$ along two disjoint sections;

$(1')_s$ A chain of minimal elliptic ruled surfaces $S_1,\ldots,S_s$ $(s \geq 2)$ such that

(α) S_i meets only $S_{i\pm1}$ along two disjoint sections for $1 < i < s$,

(β) S_1 [resp. S_r] meets only S_2 [resp. S_{r-1}] along an étale double section;

$(2)_s$ A chain of surfaces $S_1,\ldots,S_s$ $(s \geq 2)$ such that

(α) S_i is a minimal elliptic ruled surface and meets only $S_{i\pm1}$ along two disjoint sections for $1 < i < s$,

(β) S_1 [resp. S_s] is a rational surface and $S_2|S_1$ [resp. $S_{s-1}|S_s$] is a smooth elliptic curve $\sim -K_{S_1}$ [resp. $-K_{S_s}$] ;

$(2')_s$ A chain of surfaces $S_1,\ldots,S_s$ $(s \geq 2)$ such that

(α) S_i is a minimal elliptic ruled surface and meets only $S_{i\pm1}$ along two disjoint sections for $1 < i < s$,

(β) S_1 is a minimal elliptic ruled surface with $S_2|S_1$ being an étale double section,

(γ) S_s is a rational surface with $S_{s-1}|S_s$ being a smooth elliptic curve $\sim -K_{S_s}$;

(3) Configuration of rational surfaces whose dual graph is a triangulation of either a 2-sphere S^2, a real projective plane $\mathbb{P}^2(\mathbb{R})$, a torus $S^1 \times S^1$ or a Klein bottle.

(3.5) Remark. A surface of type $(1')_s$ [resp. $(2')_s$] is an étale μ_2-quotient of that of type $(1)_{2s-2}$ [resp. $(2)_{2s-1}$].

(3.6) Proposition. If S is of type (0) or $(1)_s$ or $(1')_s$ [resp. $(2)_s$ or $(2')_s$ or (3)], then $4K_S$ or $6K_S \sim 0$ [resp. $2K_S \sim 0$]. Hence, by adjunction,

$$12(K_M + S)|_S \sim 0 .$$

(3.7) Corollary. If $K_M \sim nS$, $n \in \mathbb{Z} \setminus \{-1\}$, then $S|S$ is a torsion. For a tubular neighbourhood $M' \subset M$ of S, there is an étale covering $\varepsilon : \widetilde{M}' \longrightarrow M'$ such that $\varepsilon^* S|\varepsilon^* S \sim K_{\widetilde{M}'}|\varepsilon^* S \sim 0$.

(3.8) Theorem (Friedman [F]). Under the notation and assumption as in (3.7), $\widetilde{S}$ has a versal deformation

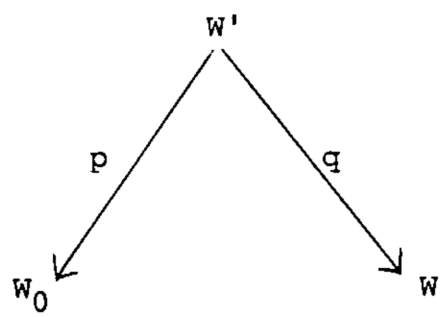
$$\phi : (X, \widetilde{S}) \longrightarrow (Y, 0) .$$

Here X and Y are complex manifolds, $0 \in Y$ is a reference point, and ϕ is a proper flat morphism with central fibre $\widetilde{S} = \phi^{-1}(0)$. The relative canonical sheaf $\omega_{X/Y} = \omega_X \otimes \phi^* \omega_Y^{-1}$ is trivial around $\widetilde{S}$.

(3.9) Remark. Since contractions and elementary transformations commute with étale covering, we can replace the semistable-Gorenstein reduction $\gamma\circ\sigma : W \longrightarrow U$ by a suitable étale covering of W so that the image $\tilde{D}_0$ of $\tilde{D} = (\gamma\circ\sigma)^*D/\deg\sigma$ on a minimal model W_0 satisfies

$$\tilde{D}_0|\tilde{D}_0 \sim K_{W_0}|\tilde{D}_0 \sim K_{\tilde{D}_0} \sim 0 .$$

It goes without saying that $\tilde{D}_0$ is a degeneration of K3 or abelian surfaces. As an immediate consequence of the construction of the minimal model W_0 , there exists a diagram of proper bimeromorphic morphisms



such that $p^*\tilde{D}_0 = q^*\tilde{D}$.

4. Formal neighbourhoods

In this section, we give the proof of Main Theorem. Let us start with an elementary observation.

(4.1) Lemma. Let S be a compact analytic space with the underlying reduced structure $T = S_{red}$. Let L be an invertible sheaf on S. If $L \otimes \mathcal{O}_T \cong \mathcal{O}_T$ and $L^{\otimes n} \cong \mathcal{O}_S$ for some positive integer n, then $L \cong \mathcal{O}_S$. In other words, $\ker\{\mathrm{Pic}(S) \longrightarrow \mathrm{Pic}(T)\}$ has no torsion.

Proof. Without loss of generality, we may assume that S is connected. Since T is compact and reduced,

$$H^0(T.\mathcal{O}_T) = \mathbb{C},\ H^0(T,\mathcal{O}_T^*) = \mathbb{C}^* .$$

Hence the exponential exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$ gives rise to a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & H^1(S,\mathbb{Z}) & \xrightarrow{i} & H^1(S,\mathcal{O}) & \longrightarrow & \mathrm{Pic}(S) & \longrightarrow & H^2(S,\mathbb{Z}) \\ & & \downarrow \wr & & \downarrow & & \downarrow & & \downarrow \wr \\ 0 & \longrightarrow & H^1(T,\mathbb{Z}) & \xrightarrow{j} & H^1(T,\mathcal{O}) & \longrightarrow & \mathrm{Pic}(T) & \longrightarrow & H^2(T,\mathbb{Z}) . \end{array}$$

Since j is injective, so is i and we see that

$$\ker\{\mathrm{Pic}(S) \longrightarrow \mathrm{Pic}(T)\} \cong \ker\{H^1(S,\mathcal{O}) \longrightarrow H^1(T,\mathcal{O})\}$$

is a $\mathbb{C}$-vector space. □

The main ingredient of this section is the following

(4.2) Theorem. Let S be a connected, compact, reduced analytic subspace of pure codimension 1 (hence an effective Cartier divisor) on an analytic manifold M . Assume the following three conditions:

(4.2.a) $\mathcal{O}_S(S) \cong \mathcal{O}_S$;

(4.2.b) $\mathcal{O}_M(aK_M) \cong \mathcal{O}_M(bS)$ for some $a,b \in \mathbb{Z}$, $a > 0$,
$b \neq -2a, -3a, -4a, \ldots$

(4.2.c) There exists a versal deformation

$$\phi : (\mathcal{X},S) \longrightarrow (\mathcal{Y},0)$$

of S such that $\mathcal{X}$ is smooth and $\omega_{\mathcal{X}/\mathcal{Y}} \cong \mathcal{O}_{\mathcal{X}}$ around S .

Then, for every positive integer n , we have

$$(4.2.1)_n \quad \mathcal{O}_{nS}(S) \cong \mathcal{O}_{nS}$$

and there exists a natural morphism

$$\phi_n : \mathrm{Spec}\, \mathbb{C}[\varepsilon]/(\varepsilon^n) \longrightarrow (\mathcal{Y},0)$$

which induces an isomorphism

$$(4.2.2)_n \quad nS \cong \mathrm{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^n)) \times_{\mathcal{Y}} \mathcal{X} .$$

Moreover,

$(4.2.3)_n$ $H^0(nD,\mathcal{O}(mD)) \longrightarrow H^0(n'D,\mathcal{O}(mD))$ is surjective for every $n' < n$ and $m \in \mathbb{Z}$.

The proof of (4.2) is by induction on n . $(4.2.1)_1$ is nothing but (4.2.a), while $(4.2.3)_1$ is vacuous. The morphism $\phi_1 : \operatorname{Spec} \mathbb{C} \longrightarrow (V,0)$ is trivially defined as the constant map to 0 , which establishes $(4.2.2)_1$.

Let us fix the notation. Let $\{U_i\}$ be an open Stein covering of M and $f_i \in \Gamma(U_i,\mathcal{O}_M)$ a local defining equation of S . On $U_i \cap U_j$, there is a non-vanishing function $\varphi_{ij} \in \Gamma(U_i \cap U_j,\mathcal{O}_M^*)$ such that

$$f_i = \varphi_{ij} f_j \ .$$

Thus $\{f_i\}$ defines a global section of the invertible sheaf $\mathcal{O}_M(S)$ associated with the transition functions $\{\varphi_{ij}\}$.

(4.3) Proof of (4.2) for $n = 2$. Take an everywhere non-vanishing section $s = \{s_i\} \in H^0(S,\mathcal{O}_S(S))$, where

$$s_i \in \Gamma(U_i \cap S,\mathcal{O}_S^*), \ s_i = \varphi_{ij} s_j \ .$$

Let $\tilde{s}_i \in \Gamma(U_i,\mathcal{O}_M)$ be a local lifting of s_i and $\tilde{S}_i$ the divisor on $\operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2)\times U_i$ defined by

$$f_i - \varepsilon\tilde{s}_i = 0 \ .$$

Then we have $\tilde{S}_i = \tilde{S}_j$ on Spec $\mathbb{C}[\varepsilon]/(\varepsilon^2) \times (U_i \cap U_j)$. Indeed,

$$\begin{aligned} I_{\tilde{S}_j} &= (f_j - \varepsilon\tilde{s}_j)\mathcal{O}_M[\varepsilon] = \varphi_{ij}(f_j - \varepsilon\tilde{s}_j)\mathcal{O}_M[\varepsilon] \\ &= (f_i - \varepsilon\varphi_{ij}\tilde{s}_j)\mathcal{O}_M[\varepsilon] = \{(f_i - \varepsilon\tilde{s}_i) + \varepsilon(\tilde{s}_i - \varphi_{ij}\tilde{s}_j)\}\mathcal{O}_M[\varepsilon] \\ &\subset I_{\tilde{S}_i} + \varepsilon(\tilde{s}_i - \varphi_{ij}\tilde{s}_j)\mathcal{O}_M . \end{aligned}$$

On the other hand, since $\{\tilde{s}_i\}$ is a lift of $\{s_i\}$,

$$\tilde{s}_i - \varphi_{ij}\tilde{s}_j \in I_S = f_i\mathcal{O}_M ,$$

so that

$$\begin{aligned} I_{\tilde{S}_j} &\subset I_{\tilde{S}_i} + \varepsilon f_i \mathcal{O}_M \\ &= I_{\tilde{S}_i} + \varepsilon(f_i + \varepsilon\tilde{s}_i)\mathcal{O}_M \\ &= I_{\tilde{S}_i} \end{aligned}$$

thanks to $\varepsilon^2 = 0$. By the symmetry between i and j , we have $I_{\tilde{S}_i} = I_{\tilde{S}_j}$ on Spec $\mathbb{C}[\varepsilon]/(\varepsilon^2) \times (U_i \cap U_j)$. Thus $\{\tilde{S}_i\}$ defines an effective divisor $\tilde{S}$ on Spec $\mathbb{C}[\varepsilon]/(\varepsilon^2) \times M$. There are natural projections $p : \tilde{S} \longrightarrow$ Spec $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ and $q : \tilde{S} \longrightarrow M$. The ring homomorphism

$$q^{-1} : \mathcal{O}_M \longrightarrow \mathcal{O}_{\tilde{S}}$$

is surjective. In fact, noting $\tilde{s}_i \in \mathcal{O}_M^*$, we have $\varepsilon = f_i\tilde{s}_i^{-1}$.

Thus q is a closed immersion. In the mean time

$$\ker q^{-1} = \mathcal{O}_M \cap \{(f_i - \varepsilon\tilde{s}_i)(\mathcal{O}_M \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2))\}$$

$$= f_i^2\mathcal{O}_M = I_{2S} \; ,$$

so that q gives an isomorphism $\tilde{S} \cong 2S$. On the other hand, since $\varepsilon\mathcal{O}_{\tilde{S}} = f_i\tilde{s}_i^{-1}\mathcal{O}_{\tilde{S}} = f_i\mathcal{O}_{\tilde{S}} \neq 0$, $\tilde{S}$ is flat over $\mathrm{Spec}\ \mathbb{C}[\varepsilon]/(\varepsilon^2)$, with central fibre S . Hence there exists a natural morphism

$$\phi_2 : \mathrm{Spec}\ \mathbb{C}[\varepsilon]/(\varepsilon^2) \longrightarrow (Y,0)$$

such that

$$(4.2.2)_2 \quad 2S \cong \tilde{S} \cong \mathrm{Spec}\ \mathbb{C}[\varepsilon]/(\varepsilon^2) \times_Y X \; .$$

In particular, it gives isomorphisms of dualizing sheaves:

$$\omega_{2S} \cong \omega_{\tilde{S}} \cong p^*\omega_{\mathrm{Spec}\ \mathbb{C}[\varepsilon]/(\varepsilon^2)} \otimes \phi_2^*\,\omega_{X/Y}$$

$$\cong \mathcal{O}_{\tilde{S}} \otimes_{\mathcal{O}_{\tilde{S}}} \mathcal{O}_{\tilde{S}} \cong \mathcal{O}_{\tilde{S}} \cong \mathcal{O}_{2S} \; ;$$

while the adjunction formula shows

$$\omega_{2S} \cong \mathcal{O}_{2S}(K_M + 2S)$$

whence follows

$$\mathcal{O}_{2S} \cong \omega_{2S}^{\otimes a} \cong \mathcal{O}_{2S}(aK_M + 2aS) \cong \mathcal{O}_{2S}((2a + b)S) \ .$$

Since $b \neq -2a$, this implies that $\mathcal{O}_{2S}(S)$ is a torsion in $\mathrm{Pic}(2S)$. Now, by (4.1) and (4.2.a) we conclude:

$$(4.2.1)_2 \ \mathcal{O}_{2S}(S) \cong \mathcal{O}_{2S} \ .$$

$(4.2.3)_2$ is easy. In fact, a non-vanishing section of $\mathcal{O}_{2S}(mS) \cong \mathcal{O}_{2S}$ gives a $\mathbb{C}$-basis of $H^0(S,\mathcal{O}_S(mS)) \cong \mathbb{C}$.

(4.4) Proof of (4.2) for $n \geq 3$. Suppose that $(4.2.2)_{n-1}$, $(4.2.2)_{n-1}$ and $(4.2.3)_{n-1}$ hold $(n \geq 3)$. By $(4.2.2)_{n-1}$, we can identify $\mathcal{O}_{(n-1)S}$ with the flat $\mathbb{C}[\varepsilon]/(\varepsilon^{n-1})$ - algebra

$$\mathbb{C}[\varepsilon]/(\varepsilon^{n-1}) \underset{\mathcal{O}_Y}{\otimes} \mathcal{O}_X$$

via ϕ_{n-1} . Note that $\varepsilon\mathcal{O}_{(n-1)S} = f_i\mathcal{O}_{(n-1)S} \subset \mathcal{O}_{(n-1)S}$ on $U_i \cap (n-1)S$:

$$\varepsilon \equiv f_i\alpha_i \bmod f_i^{n-1}\mathcal{O}_M \ ,$$

where $\alpha_i \in \Gamma(U_i,\mathcal{O}_M^*)$. Then

$$f_i(\alpha_i-\varphi_{ij}^{-1}\alpha_j) = f_i\alpha_i-f_j\alpha_j \equiv \varepsilon-\varepsilon = 0 \bmod f_i^{n-1}\mathcal{O}_M \ ;$$

or, equivalently

$$\alpha_i \equiv \varphi_{ij}^{-1}\alpha_j \bmod f_i^{n-2}\mathcal{O}_M$$

so that $\{\alpha_i\}$ gives rise to a global section $\alpha \in H^0((n-2)S, \mathcal{O}(-S))$. (We need here the hypothesis $n \geq 3$). By $(4.2.3)_{n-1}$, α can be lifted to $\tilde{\alpha} \in H^0((n-1)S, \mathcal{O}(-S))$. $\tilde{\alpha}$ is represented by $\tilde{\alpha}_i \in \Gamma(U_i, \mathcal{O}_M)$ such that

$$\tilde{\alpha}_i \equiv \varphi_{ij}^{-1}\tilde{\alpha}_j \mod f_i^{n-1}\mathcal{O}_M .$$

We define a $\mathbb{C}[\varepsilon]/(\varepsilon^n)$ - algebra structure on $\mathcal{O}_{nS}$ by the formula

$$\varepsilon g = (f_i\tilde{\alpha}_i)g \quad \text{for} \quad g \in \mathcal{O}_{nS} .$$

This is well-defined because

$$\begin{aligned} f_i\tilde{\alpha}_i - f_j\tilde{\alpha}_j &= (\varphi_{ij}f_j)(\varphi_{ij}^{-1}\tilde{\alpha}_j + \delta_{ij}) - f_j\tilde{\alpha}_j \\ &= \varphi_{ij}f_j\delta_{ij} \in f_j^n\mathcal{O}_M , \end{aligned}$$

where $\delta_{ij} = \tilde{\alpha}_i - \varphi_{ij}^{-1}\tilde{\alpha}_j \in f_j^{n-1}\mathcal{O}_M$. This extends the $\mathbb{C}[\varepsilon]/(\varepsilon^n)$ - algebra structure on $\mathcal{O}_{(n-1)S}$ to $\mathcal{O}_{nS}$. Moreover $\mathcal{O}_{nS}$ is flat over $\mathbb{C}[\varepsilon]/(\varepsilon^n)$ by

$$\varepsilon^{n-1}\mathcal{O}_{nS} = (\tilde{\alpha}_i f_i)^{n-1}\mathcal{O}_{nS} = f_i^{n-1}\mathcal{O}_{nS} \neq 0 ;$$

in other words, we have a proper flat morphism

$$nS \longrightarrow \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^n)$$

whence derives a morphism

$$\phi_n : \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^n) \longrightarrow (Y,0) ,$$

which extends ϕ_{n-1} and induces an isomorphism

$$(4.2.2)_n \quad nS \cong \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^n) \times_Y X .$$

Therefore, similarly as in (4.3),

$$\omega_{nS} \cong \mathcal{O}_{nS} \quad \text{by (4.2.c)} ,$$

$$\omega_{nS}^{\otimes a} \cong \mathcal{O}_{nS}(aK_M+anS) \quad \text{by adjunction}$$

$$\cong \mathcal{O}_{nS}(bS+anS) \quad \text{by (4.2.b)} .$$

Since $b \neq -an$, $\mathcal{O}_{nS}(S)$ is a torsion so that

$$(4.2.1)_n \quad \mathcal{O}_{nS}(S) \cong \mathcal{O}_{nS} \quad \text{by (4.1)} .$$

Finally $(4.2.3)_n$ is immediate from $(4.2.1)_n$ and $(4.2.2)_n$.

□

(4.5) Corollary. Under the same assumption as in (4.2), we have

$$\dim H^0(nS,\mathcal{O}_{nS}(kS)) = n$$

for $n \in \mathbb{N}$, $k \in \mathbb{Z}$.

(4.6) Corollary. Let M, N and U be three analytic spaces and $f : N \longrightarrow M$, $g : N \longrightarrow U$ proper, surjective, generically finite morphisms. Assume that there are compact, effective Cartier divisors $S \subset M$, $T \subset N$ and $D \subset U$ such that $f^* S = T$, $g^* D = kT$ $(k \in \mathbb{N})$. If (M,S) satisfies the hypotheses in (4.2), then

$$\dim H^0(nD, \mathcal{O}_{nD}(nD))$$

grows like n.

Applying this corollary to the original situation, we get

(4.7) Corollary. Let X be a minimal 3-fold with $\nu = 1$. Let D_i be a connected component of $D \in |mK_X|$, $m > 0$, $\mathrm{ind}(X) | m$. Then

$$\dim H^0(nD_i, \mathcal{O}_{nD_i}(nD_i)) = O(n) .$$

(4.8) Proof of Main Theorem. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(nD) \longrightarrow \mathcal{O}_{nD}(nD) \longrightarrow 0$$

and the associated cohomology exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X(nD)) \longrightarrow H^0(nD, \mathcal{O}_{nD}(nD)) \longrightarrow H^1(X, \mathcal{O}_X) .$$

The first and the last terms are independent of n and their dimensions are bounded, so $h^0(nD,\mathcal{O}(nD)) = \sum_i h^0(nDi,\mathcal{O}(nDi))$ $\sim O(n)$ implies $h^0(X,\mathcal{O}_X(nD)) \sim O(n)$, i.e. $\kappa(X) = 1$. Similarly, $h^0(X,\mathcal{O}_X(nD_i)) \sim O(n)$. D_i is a multiple of a primitive divisor E_i : $D_i = e_iE_i$. Noting that $D_i|E_i \approx 0$, we see that the moving part $|L_i^{(n)}|$ of $|nD_i|$ has no base points and of the form $|n_i^!E_i|$ $n_i^! > 0$. Hence $|n_i^!D_i| = |e_iL_i^{(n)}|$ is base point free; therefore, for $n_0 = \text{L.C.M.}\{n_i^!\}$, $|n_0D| = |n_0mK_X|$ is also base point free.

□

(4.9) Remark. In the assumption in (4.2), the strange condition $b \neq -2a, -3a, \ldots$ is actually necessary. For instance, let A be an abelian variety and consider an non-trivial extension

$$0 \longrightarrow \mathcal{O}_A \longrightarrow E \longrightarrow \mathcal{O}_A \longrightarrow 0 .$$

Let $M = \mathbb{P}(E)$. $\mathbb{P}(E)$ contains a unique section $S \cong A$. (M,S) satisfies all the hypotheses in (4.2) except that $K_M \sim -2S$. Moreover, $(4.2.2)_2$ holds, too. However, $\mathcal{O}_{2S}(S)$ is not isomorphic to $\mathcal{O}_{2S}$. In fact, since $S \sim \mathbf{1}_E$, the tautological line bundle, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(E)}(-\mathbf{1}_E) \longrightarrow \mathcal{O}_{\mathbb{P}(E)}(\mathbf{1}_E) \longrightarrow \mathcal{O}_{2S}(S) \longrightarrow 0$$

so that $H^0(2S,\mathcal{O}_{2S}(S)) \cong H^0(\mathbb{P}(E),\mathcal{O}(\mathbf{1}_E)) \cong H^0(A,E) \cong \mathbb{C}$, while $H^0(2S,\mathcal{O}_{2S}) \cong \mathbb{C}^2$. It is therefore impossible to extend the $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ - algebra structure on $\mathcal{O}_{2S}$ to a $\mathbb{C}[\varepsilon]/(\varepsilon^3)$ - algebra structure on $\mathcal{O}_{3S}$, i.e. the connected component of Chow(M) that contains $\{S\}$ is a non-reduced point $\cong \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2)$.

(4.10) Remark. Applying our argument to the minimal surface case, we can prove without complicated dichotomy that $\nu(\underset{\sim}{X}) = 1$ implies the existence of an elliptic fibration.

References

[F] R. Friedman, Global smoothing of varieties with normal crossings,

[FM] R. Friedman and D. Morrison, The birational geometry of degenerations, Birkhäuser, Boston-Basel-Stuttgart (1983).

[Kw] Y. Kawamata, Pluricanonical systems on minimal algebraic varieties, Inv. Math. 79 (1985), 567-588.

[KKMS] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, Toroidal embeddings I, Lect. Notes in Math. 339, Springer, Berlin-Heidelberg-New York (1973).

[Kl] V.S. Kulikov, Degeneration of K3 and Enriques surfaces, Math. USSR Izvestija 11 (1977), 957-989.

[Ml] J. Milnor, Singular points of complex hypersurfaces, Annals of Math. Studies 61, Princeton Univ. Press, Princeton (1968)

[My] Y. Miyaoka, Kodaira dimension of a minimal 3-fold, submitted to Math. Ann.

[Mr] S. Mori, Flip theorem and the existence of minimal models for 3-folds, preprint.

[PP] U. Persson and H. Pinkham, Degeneration of surfaces with trivial canonical bundle, Ann. of Math. 113 (1981), 45-66.