IDENTIFYING ASSEMBLY MAPS IN K- AND L-THEORY

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Abstract. In this paper we prove the equivalence of various algebraically or geometrically defined assembly maps used in formulating the main conjectures in K- and L-theory, and C^*-theory.

1. Introduction

Information about the algebraic and geometric topology of a manifold X is contained in the functors $K_n(\mathbb{Z}G)$ and $L_n(\mathbb{Z}G)$ of its fundamental group $G = \pi_1(X, x_0)$ defined by D. Quillen and C. T. C. Wall. These functors are algebraically defined, but are difficult to compute except in special cases, and one of the main problems in this area is to find effective ways to extend the range of these computations.

The most general approach is via *assembly maps*, based on ideas of Quinn [24], [25] and Waldhausen [30]. In this approach the functors above are reinterpreted as the homotopy groups of spectra $\mathbb{K}(\mathbb{Z}G)$ and $\mathbb{L}(\mathbb{Z}G)$ arising from homotopy invariant functors $F\colon \mathit{Spaces} \longrightarrow \mathit{Spectra}$. An assembly map is a natural transformation $\alpha\colon E \longrightarrow F$ where E is homotopy invariant and strongly excisive functor, with $E(\bullet) \simeq F(\bullet)$. These conditions imply that $\pi_*(E(X))$ satisfies the Eilenberg-Steenrod axioms, except for the dimension axiom, and thus is a generalized homology theory. In this sense the functor E is "known" and computable through spectral sequences. One can then hope to obtain information about F by studying the fibre of the (essentially unique) assembly map associated to the functor F.

This approach has been very successful: a lot of work in geometric topology has been motivated by the Novikov conjecture, which asserts that the assembly map for L-theory applied to the space $X = B\Gamma$, for any group Γ, induces a monomorphism on rational homotopy groups. This has been proved in many cases, suggesting that assembly maps might entirely determine the integral K- and L-theory of torsion-free discrete groups, as conjectured by W.-C. Hsiang [13] and A. Borel. See [9] for a detailed survey of results in this area.

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The method of "assembly" is very flexible because the K- and L-theory spectra have many different models, which can be chosen to suit the technique of computation. On the other hand, it usually isn't clear from the definition of the different models for the spectra that the associated assembly maps are equivalent, or induce the ones originally defined by explicit algebraic or geometric methods on the homotopy groups. This problem becomes even more critical in the setting of the Farrell-Jones Isomorphism Conjectures, and in the Baum–Connes conjecture, where G is allowed to have torsion. In this paper we present a unified way to show that the assembly maps arising from several different spectrum models are equivalent. From our point of view, the primary assembly map (see § 6) is constructed from functors on the continuously controlled categories introduced in [1]. See [3], [19] for applications to the Novikov conjectures and further developments of this theory. In Theorem 7.3 we establish the equivalence of the continously controlled assembly to the assembly maps of Davis-Lück [6]. See Corollary 3.3 for the corresponding equivalence to Loday assembly, and Corollary 9.2 for the equivalence to Farrell-Jones assembly.

Our results also apply to the Baum-Connes assembly map [14] for $\mathbb{K}^{Top}(C^*_r(G))$, where $C^*_r(G)$ denotes the reduced C^*-algebra of the group G (see Theorem 6.4 and Corollary 7.4). In particular we prove that the assembly map considered by Davis and Lück for topological K-theory induces the same map on homotopy groups as the Baum-Connes map.

Finally in § 10, we propose a modification of the Farrell-Jones Isomorphism Conjecture [7] applicable to "intermediate" L-theories, including $\mathbb{L}^s$ and $\mathbb{L}^h$.

2. Weiss-Williams assembly

We recall some of the definitions and results presented in Weiss-Williams [33]. We will assume that all spaces are homotopy equivalent to CW-spaces and all spectra are CW-spectra. A functor F from spaces to spectra is *homotopy invariant* if it takes homotopy equivalences to homotopy equivalences. A homotopy invariant functor F is *strongly excisive* if $F(\emptyset)$ is contractible and F preserves arbitrary coproducts, up to homotopy equivalence.

If F is strongly excisive, then the functor $\pi_*(F)$ from spaces to graded abelian groups is a generalized homology theory satisfying the strong wedge axiom. Moreover, $F \simeq E$ where $E(X) = X_+ \wedge F(\bullet)$ for all CW-spaces X.

Theorem 2.1. ([33, 1.1]) *For any homotopy invariant functor F from spaces to spectra, there exists a strongly excisive and homotopy invariant functor $F^\%$ from spaces to spectra and a natural transformation*

$$\alpha_F \colon F^\% \longrightarrow F \ ,$$

depending functorially on F, such that $\alpha_F \colon F^\%(\bullet) \to F(\bullet)$ is a homotopy equivalence.

Corollary 2.2. ([33, p. 336]) *Let F be a homotopy invariant functor from spaces to spectra. Suppose that $\alpha\colon E \to F$ is a natural transformation inducing a homotopy equivalence $E(\bullet) \simeq F(\bullet)$. If E is homotopy invariant and strongly excisive, then α may be naturally identified with α_F up to homotopy equivalence.*

Proof. For any space X, we have a commutative diagram

$$\begin{tikzcd} X_+ \wedge E(\bullet) \arrow[r, "\simeq"] \arrow[d, "1\wedge\alpha_\bullet"', "\simeq"] & E(X) \arrow[d, "\alpha_X"] \\ X_+ \wedge F(\bullet) \arrow[r] & F(X) \end{tikzcd}$$

Since $F^{\%}(X) \simeq X_+ \wedge F(\bullet)$ the result follows from Theorem 2.1. □

Example 2.3. The theory above applies directly to Waldhausen's functor $A(X)$, but not directly to K-theory (or to Wall's algebraic L-theory) because these functors were originally defined on the category of *Rings* (or rings with involution). As described in [33, § 2], it is possible to extend them to functors on the category of *Ringoids* (or ringoids with involution). This uses the Waldhausen or Quillen construction of K-theory [22], [30] as a functor on exact categories to spectra, or Ranicki's L-theory of additive categories with involution [26]. One can now construct homotopy invariant functors from *Spaces* to *Spectra* by composing with the functor $X \mapsto R\pi(X)$, where $\pi(X)$ denotes the fundamental groupoid of a space equipped with the standard involution and R is a ring with unit. For L-theory we will see in § 9 that this functor agrees on homotopy groups with Wall's geometric definition [31] of the surgery obstruction groups $L_n(\mathbb{Z}\pi(X))$, and with Quinn's construction of the geometric surgery spectra $\mathbb{L}(\mathbb{Z}\pi(X))$ (a full exposition of this construction has been given by Nicas [18]), and also with the algebraic surgery spectra of Ranicki [27]. We will also see in § 3 that the Loday assembly map for K-theory can be recovered by this process.

Non–Example 2.4. The "universal" assembly construction given above does not apply to $\mathbb{K}^{Top}(C_r^*(G))$ since this spectrum is not a functor of G.

3. Loday assembly

Higher algebraic K-theory of a ring A was originally defined by Quillen as the homotopy groups $K_n(A) := \pi_n(BGL(A)^+)$, for $n \geq 1$, of the $(+)$-construction on $BGL(A) = \lim_{k\to\infty} BGL_k(A)$. Later Quillen [22] introduced the Q-construction and an algebraic K-theory functor $\mathbf{K}\colon \mathit{Exact} \to \mathit{Ab}$, where Exact denotes the category of exact categories, and Ab the category of abelian groups. In [11, p. 228], Quillen showed that when the category $\mathcal{P}(A)$ of projective A-modules is regarded as an exact category in a natural way, then $K_n(A) = \pi_n(\mathbf{K}(\mathcal{P}(A)))$, for $n \geq 1$. The extension of $\mathbf{K}$ to a spectrum-valued functor was done by J. P. May [16], and this gives the (connective) algebraic K-theory functor. An alternate (but homotopy equivalent) description of algebraic K-theory was provided by Waldhausen's

$S_\bullet$-construction [30, p. 375], by a method that automatically produces spectra. Finally, Pedersen and Weibel [20], generalizing work of Gersten [10] and Wagoner [29], produced a non-connective functor $\mathbb{K}^{alg}\colon \mathit{AddCat} \to \mathit{Spectra}$, which agrees with $\mathbf{K}$ in positive dimensions when restricted to $\mathit{AddCat} \subset \mathit{Exact}$.

The original K-theory assembly map $h_n(BG_+; K(R)) \to K_n(RG)$ was constructed by Loday [15, 4.1.1], using Quillen's (+)-construction definition of K-theory. It is the map on homotopy groups induced by a maps of spaces

$$\alpha_G\colon BG_+ \wedge BGL(R)^+ \to BGL(RG)^+$$

obtained by composing the map $BG \to BGL(\mathbb{Z}G)^+$, defined by sending $g \in G$ to a 1×1 matrix, with the Loday pairing

$$BGL(\mathbb{Z}G)^+ \wedge BGL(R)^+ \to BGL(RG)^+ \ . \tag{3.1}$$

This pairing can also be constructed from the Waldhausen definition of K-theory (see [32, § 4] for details and references). On the other hand, the general theory of § 2 gives an assembly map of spectra

$$\alpha_X\colon X_+ \wedge \mathbb{K}^{alg}(R) \to \mathbb{K}^{alg}(R(\pi(X))_\oplus) \ .$$

In this section, we relate α_X to the Loday assembly map.

Theorem 3.2. *For any groupoid $\mathcal{G}$ there is a natural transformation*

$$A_\mathcal{G}\colon B\mathcal{G}_+ \wedge \mathbb{K}^{alg}(R) \to \mathbb{K}^{alg}(R(\mathcal{G})_\oplus),$$

such that $A_{(\bullet)}$ is a homotopy equivalence.

Let X be a path connected space, and let $G = \pi_1(X, x_0)$. Then there exists commutative diagram:

$$\begin{tikzcd} BG_+ \wedge BGL(R)^+ \arrow[r] \arrow[d, "\alpha_G"] & B\pi(X)_+ \wedge \mathbb{K}^{alg}(R) \arrow[d, "A_X"] \\ BGL(RG)^+ \arrow[r] & \mathbb{K}^{alg}(R(\pi(X))_\oplus) \end{tikzcd}$$

where $A_X := A_{\pi(X)}$ and the horizontal maps induce isomorphisms on homotopy groups in positive dimensions.

Proof. Note that $G = \pi_1(X, x_0) \cong \pi(X)_{x_0}$, where $\pi(X)_{x_0} \subset \pi(X)$ denotes the subgroupoid with one object $\{x_0\}$. This inclusion induces an isomorphism on K-theory and a homotopy equivalence $BG \simeq B\pi(X)$. The maps $BGL(R)^+ \to \mathbb{K}^{alg}(R)$ and $BGL(RG)^+ \to \mathbb{K}^{alg}(R(\pi(X))_\oplus)$ are the ones inducing Quillen's equivalence between the (+)-construction and the Q-construction for K-theory.

The natural transformation A_X is constructed by generalizing Loday's method from groups to groupoids. This has been done in much greater generality by J. P. May [17, p. 332], as explained in [32, § 3]. If $\mathcal{A}$ is a symmetric monoidal category, consider the category $\mathbf{F}(\mathcal{A}) = \coprod_{n=0}^{\infty} GL_n(\mathcal{A})$. The group completion $E_0B\mathbf{F}(\mathcal{A}) = \mathbb{Z} \times BGL(\mathcal{A})^+$. If $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a pairing of symmetric monoidal categories (in the sense of [17, § 2]), then so is the induced pairing $\otimes\colon \mathbf{F}(\mathcal{A}) \times \mathbf{F}(\mathcal{B}) \to \mathbf{F}(\mathcal{C})$.

Any such pairing functorially determines a pairing $\otimes\colon E_0B\mathbf{F}(\mathcal{A}) \times E_0B\mathbf{F}(\mathcal{B}) \to E_0B\mathbf{F}(\mathcal{C})$ of the group completions. To generalize the Loday pairing to groupoids, we use the pairing

$$\otimes\colon \mathbb{Z}(\mathcal{G})_\oplus \times \mathcal{P}(R) \to R(\mathcal{G})_\oplus$$

defined by tensor product. The machinery of [17] (see the remarks on [32, p. 502]) now produces the required assembly map $A_{\mathcal{G}}$ and the compatibility with the Loday assembly. $\square$

Corollary 3.3. *Let X be a connected space and $G = \pi_1(X, x_0)$. Then the Loday assembly map α_G and the generalized assembly maps $\alpha_{B\pi(X)}$ are naturally isomorphic in positive dimensions.*

Proof. The generalized assembly map $\alpha_X\colon X_+ \wedge \mathbf{K}(R) \to \mathbb{K}^{alg}(R(\pi(X))_\oplus)$ is defined for any space X. It factors through $\alpha_{B\pi(X)}$ by naturality under the composition with the classifying map $X \mapsto B\pi(X)$. By construction, there is a natural transformation $A_X\colon B\pi(X)_+ \wedge \mathbb{K}^{alg}(R) \to \mathbb{K}^{alg}(R(\pi(X))_\oplus)$ defined for any space X such that $A_{(\bullet)}$ is a homotopy equivalence. By Corollary 2.2, A_X may be naturally identified with the generalized assembly $\alpha_{B\pi(X)}$. However, since X is connected, the assembly map A_X is equivalent to Loday's assembly α_G in positive dimensions. $\square$

4. Davis-Lück assembly

In [6], Davis and Lück generalize the construction of the last section. Let G be any discrete group. The goal now is to find functors

$$H\colon G{-}Spaces \longrightarrow Spectra \tag{4.1}$$

which are G-homotopy invariant and G-excisive. The main features of this theory are:

(i) For any such functor, an assembly map $\alpha_X\colon H(X) \to H(\bullet)$ is defined for any G-CW complex X.

(ii) Any functor $\mathbf{Or}(G) \to Spectra$ out of the orbit category of G may be extended uniquely (up to homotopy) to a functor $H\colon G{-}Spaces \to Spectra$ which is G-homotopy invariant and G-excisive [6, 5.1]. The *orbit category* $\mathbf{Or}(G)$ is the category with objects G/K, for K any subgroup of G, and the morphisms are G-maps.

(iii) Any functor $F\colon Spaces \to Spectra$ gives a functor $\mathbf{Or}(G) \to Spectra$ by composition with the functor $E\colon \mathbf{Or}(G) \to Spaces$ defined on objects by $G/K \mapsto G/K \times_G EG$. We have a commutative diagram

$$\begin{tikzcd}
\mathbf{Or}(G) \arrow[r, "E"] \arrow[d, hook] & Spaces \arrow[r, "F"] & Spectra \\
G{-}Spaces \arrow[urr, "\bar{F}"'] & &
\end{tikzcd}$$

(iv) If $F\colon$ *Spaces* $\to$ *Spectra* is a homotopy invariant functor and $\bar{F}$ denotes its unique extension to $G-$*Spaces*, then the natural transformation $\bar{F}(EG) \to \bar{F}(\bullet)$ is homotopy equivalent to the Weiss-Williams assembly map $F^{\%}(BG) \to F(BG)$. This follows immediately from the formula

$$\bar{F}(EG) = \operatorname*{hocolim}_{\Delta\subset BG} \bar{F}(G\times\Delta) = \operatorname*{hocolim}_{\Delta\subset BG} \bar{F}(G) = BG_+ \wedge F(\bullet)$$

since $\bar{F}(G) = F(G\times_G EG)$ and $G\times_G EG \simeq \bullet$.

(v) For any such functor H as in (4.1) and any family $\mathcal{F}$ of subgroups of G, closed under taking subgroups and conjugation, there is an assembly map

$$\alpha_{\mathcal{F}}\colon H(\mathcal{E}_{\mathcal{F}}(G)) \to H(\bullet)$$

obtained by substituting the classifying space $\mathcal{E}_{\mathcal{F}}(G)$ associated to the family $\mathcal{F}$.

In the Davis-Lück theory, the interesting examples are all constructed from functors out of *Groupoids* (or a variant of this category called $\mathit{Groupoids}^{inj}$). Recall that a groupoid is a small category with all morphisms invertible. Then *Groupoids* is the category with groupoids as objects and functors of groupoids as morphisms. The subcategory $\mathit{Groupoids}^{inj}$ (see [6, p. 210]) has the same objects but only allows *faithful* functors $F\colon \mathcal{G}_1 \to \mathcal{G}_2$ as morphisms. A functor F is faithful if for any two objects $x,\ y\in\mathcal{G}_1$, the induced map $\mathrm{mor}_{\mathcal{G}_1}(x,y) \to \mathrm{mor}_{\mathcal{G}_2}(F(x),F(y))$ is injective. This subcategory is needed for applications of the theory to C^*_r-algebras.

The following construction (see [6, § 2]) gives a functor

$$Gr\colon \mathbf{Or}(G) \to \mathit{Groupoids}^{inj}\ .$$

A left G-set G/K defines a groupoid $\overline{G/K}$ whose objects are the cosets xK, for $x\in G$, and morphisms $\mathrm{mor}(xK,yK) = \{g\in G\,|\,y^{-1}gx\in K\}$. Composition is given by group multiplication and a map of left G-sets defines a functor on the associated groupoids. Now any spectrum valued functor out of *Groupoids* or $\mathit{Groupoids}^{inj}$ can be composed with Gr and the theory above will apply. The most useful applications so far involve those functors which factor through *AddCat*, the category whose objects are additive categories (with involution) or $C^*_r Cat$, the category whose objects are C^*_r-categories. Here are the three main examples of Davis-Lück [6, § 2]:

4.2. *Algebraic K-theory*: This uses a functor from *Groupoids* to *AddCat*, the category of additive categories. If $\mathcal{C}$ is a category and R is a commutative ring with unit, then $R\mathcal{C}$ denotes the associated R-category to $\mathcal{C}$. The objects of $R\mathcal{C}$ are the same as those of $\mathcal{C}$, and the morphism set $\mathrm{mor}_{R\mathcal{C}}(x,y)$ is the free R-module generated by $\mathrm{mor}_{\mathcal{C}}(x,y)$. For any groupoid $\mathcal{G}$, we let $R(\mathcal{G})_\oplus$ denote the free additive R-category generated by $R\mathcal{G}$. This is the symmetric, monoidal R-category described in [6, p. 214]. We get a K-theory functor

$$\mathbf{K}^{alg}\colon \mathbf{Or}(G) \to \mathit{Groupoids} \to \mathit{Spectra}$$

sending a groupoid $\mathcal{G}$ to the spectrum $\mathbb{K}^{alg}(R(\mathcal{G})_{\oplus})$, by composing with the (non-connective) algebraic K-theory functor

$$\mathbb{K}^{alg}\colon AddCat \to Spectra$$

of Pedersen and Weibel [20].

4.3. *L-theory*: This uses a functor from *Groupoids* to $AddCat^{-}$, the category of additive categories with involution. Suppose that R has an involution $r \mapsto \bar{r}$, and $\mathcal{G}$ is a groupoid. Then there is an involution on $R\mathcal{G}$ defined by the R-linear extension the map $f \mapsto f^{-1}$ for each $f \in \mathrm{mor}_{\mathcal{C}}(x, y)$. This gives $R\mathcal{G}$ the structure of an R-category with involution, and $R(\mathcal{G})_{\oplus}$ becomes an additive category with involution. We get an L-theory functor

$$\mathbf{L}^{-\infty}\colon Groupoids \to Spectra$$

sending a groupoid $\mathcal{G}$ to the spectrum $\mathbb{L}^{-\infty}(R(\mathcal{G})_{\oplus})$, by composing with the $L^{-\infty}$ version of the periodic algebraic L-theory functor

$$\mathbb{L}^{alg}\colon AddCat^{-} \to Spectra$$

constructed by Ranicki [26], [27, p. 139]. There are versions of this construction for other forms of L-theory, including L^s, L^h and L^p. In this paper we don't consider assembly for non-oriented involutions, or more general antistructures.

4.4. C^*_r*-theory*: This uses a functor from $Groupoids^{inj}$ to C^*_r-Categories. Here $R = \mathbb{C}$ is the complex numbers, and we must complete the morphism sets of $\mathbb{C}\mathcal{G}$ as described on [6, pp. 212-213] to get a C^*_r-category $C^*_r\mathcal{G}$. Then we get a functor

$$\mathbf{K}^{Top}\colon Groupoids^{inj} \to Spectra$$

sending a groupoid $\mathcal{G}$ to $\mathbb{K}^{Top}(C^*_r\mathcal{G})$ by composing with topological K-theory functor

$$\mathbb{K}^{Top}\colon C^*_rCat \to Spectra$$

constructed by Davis-Lück [6, pp. 216-217]. We remark that there is a minor error in the construction in [6, p. 217, line 8], where it is asserted that the exterior tensor product extends functorially to the group completion before passing to the classifying spaces.[1] On the other hand, what is really needed for the construction is the pairing μ further down. Just as in our proof of Theorem 3.2, this pairing can be constructed by the categorical machinery of J. P. May [17, p. 332], as explained in [32, § 3].

[1]We are indebted to W. Lück and M. Joachim for pointing this out.

5. Quinn assembly

In [24], F. Quinn introduced an even more general form of assembly (more details are given also in [25]). Let $p\colon Y \to Z$ be a simplicial map of simplicial complexes. If $F\colon \mathit{Spaces} \to \mathit{Spectra}$ is a homotopy invariant functor, then one can define an assembly map

$$\alpha_Q\colon \operatorname*{hocolim}_{\sigma \subset Z} F(p^{-1}(\sigma)) \to F(Y)$$

induced by naturality and the homotopy equivalence $\operatorname{hocolim}_{\sigma\subset Z}(\sigma) \simeq Z$. The assembly map α_Q is independent of the choices for sufficiently fine triangulations $\{\sigma\}$ of Z. Notice that for the special case $p\colon X \to X$ is the *identity* map, $\alpha_Q\colon X_+ \wedge F(\bullet) \to F(X)$ is homotopy equivalent to α_X (by Corollary 2.2).

This construction can also be applied to any G-CW complex X and the fibration

$$\begin{array}{c} X \times_G EG \\ \downarrow{\scriptstyle p} \\ X/G \end{array}$$

to produce a functor $F_Q\colon G - \mathit{Spaces} \to \mathit{Spectra}$ which is G-homotopy invariant and G-excisive. By definition,

$$F_Q(X) = \operatorname*{hocolim}_{\Delta \subset X/G} F(p^{-1}(\Delta))$$

so by restricting to $\mathbf{Or}(G)$ we get $F_Q(G/K) = F(G/K \times_G EG)$. By uniqueness, we have $F_Q \simeq \bar{F}$.

Theorem 5.1. *For X a contractible G-space, and $F\colon \mathit{Spaces} \to \mathit{Spectra}$ a homotopy invariant functor, the generalized Quinn assembly map $\alpha_Q\colon F_Q(X) \to F(X \times_G EG)$ is homotopy equivalent to the Davis-Lück assembly $\bar{F}(X) \to \bar{F}(\bullet)$.*

Proof. The homotopy colimit definitions of F_Q and $\bar{F}$ give a commutative diagram

$$\begin{tikzcd}
F_Q(X) \arrow[r, "\simeq"] \arrow[d, "\alpha_Q"'] & \bar{F}(X) \arrow[r, "\alpha_X"] \arrow[d] & \bar{F}(\bullet) \arrow[d, "\simeq"] \\
F(X \times_G EG) \arrow[r, equal] & F(X \times_G EG) \arrow[r] & F(BG)
\end{tikzcd}$$

for any G-space X. If X is contractible, the homotopy invariance of F implies that $F(X \times_G EG) = F(BG)$ (induced by the projection $X \to \bullet$). □

An important example of this construction is $X = \mathcal{E}_{\mathcal{F}}(G)$, where $\mathcal{F}$ denotes the family of virtually infinite cyclic subgroups of G. The associated assembly map for $F = \mathbb{L}^{alg}$ will be identified in Section 9 with one used in the Farrell-Jones Isomorphism Conjectures [7].

6. Assembly via Controlled Categories

The boundedly controlled categories of Pedersen [19], Carlsson-Pedersen [3], [5] are our main tool for identifying various different assembly maps. We will recall the definition of these categories and then describe three important examples of functors $H\colon \mathrm{G}{-}\mathit{Spaces} \to \mathit{Spectra}$ as in (4.1).

Let G be any discrete group, and let Z be a locally compact Hausdorff G-space. If $X \subset Z$ is a closed G-invariant subspace, we will use the notation $Y = Z - X$. The isotropy subgroup of $y \in Y$ is denoted G_y. The category $\mathcal{B}_G(Z, X; R)$ has objects (A, f), where A is a free RG-module, and $f\colon A \to \mathit{Fin}(Y)$ is a G-equivariant map to the finite subsets of Y, satisfying:

(i) $f(a + b) \subseteq f(a) \cup f(b)$.
(ii) $A_y = \{a \in A \mid f(a) \subseteq \{y\}\}$ is a finitely generated RG_y-module, for each $y \in Y$.
(iii) $A = \bigoplus_{y\in Y} A_y$ as an R-module.
(iv) $\{y \in Y \mid A_y \neq 0\}$ is locally finite in Y, and relatively compact in Z.

A morphism $\phi\colon (A, f) \to (B, g)$ consists of an RG-homomorphism $\phi\colon A \to B$ commuting with the reference maps f, g, such that ϕ is *continuously controlled* at $X \subset Z$. In terms of the components $\phi^z_y\colon A_y \to B_z$ of the map (for y, $z \in Y$), this means that for every $x \in X$, and for every neighbourhood U of x in Z, there is a smaller neighbourhood $V \subset U$ of x in Z so that $\phi^z_y = 0$ and $\phi^y_z = 0$ whenever $y \in Y - U$ and $z \in V \cap Y$.

Remark 6.1. This category $\mathcal{B}_G(Z, X; R)$ is fairly close to the G-fixed point set $\mathcal{D}_G(Z, X; R)$ of the category defined in [19, 3.1]. Our reference space is just $Y = Z - X$ instead of $Y \times G$ and we have replaced the "relatively G-compact condition" by assuming that "A_y is a finitely generated RG_y module" plus a relative compactness condition in Z. This gives us a continuously controlled version of the bounded equivariant categories introduced in [12]. On the other hand, the category $\mathcal{B}_G(Z, X; R)$ is not itself a fixed-point category in any obvious way, as required for the applications to the Novikov conjecture in [3], [5].

In our applications to assembly maps, we will take X to be a G-CW complex and let $Z = X \times [0, 1]$ with the subspace $X = X \times 1 \subset Z$. To simplify the notation, let

$$\mathcal{B}_G(X \times [0, 1); R) := \mathcal{B}_G(X \times [0, 1], X \times 1; R)\ .$$

Let $\mathcal{U} = \mathcal{B}_G(X \times [0, 1); R)_\emptyset$ denote the full subcategory of $\mathcal{A} = \mathcal{B}_G(X \times [0, 1); R)$ with objects (A, f) such that the closure of the intersection

$$\operatorname{supp}(A, f) = \overline{\{(x, t) \in X \times [0, 1) \mid A_{(x,t)} \neq 0\}} \cap (X \times 1)$$

is the empty set. This category is equivalent to the category of finitely generated free RG-modules.

The quotient category will be denoted $\mathcal{B}_G(X\times[0,1);R)^{>0}$, and we remark that this is a germ category since $\mathcal{A}$ is $\mathcal{U}$-filtered [21], [2]. The objects are the same as in $\mathcal{B}_G(X\times[0,1);R)$ but morphisms are identified if they agree close to $X = X\times 1$ (i.e. on the complement of a neighbourhood of $X\times 0$).

The category $\mathcal{B}_G(X\times[0,1);R)^{>0}$ is an additive category with involution, and we obtain a functor $G{-}spaces \to AddCat^{-}$. The results of [2, 1.28, 4.2] now show that the functors $H_C\colon G{-}Spaces \to Spectra$ defined by

$$H_C(X) := F(\mathcal{B}_G(X\times[0,1);R)^{>0}), \tag{6.2}$$

where $F = \mathbb{K}^{alg}$ or $F = \mathbb{L}^{-\infty}$, are G-homotopy invariant and G-excisive. In [4, § 3], the definition of $\mathcal{B}_G(X\times[0,1);R)^{>0}$ is modified for $R=\mathbb{C}$ to give a functor

$$C^*_r\mathcal{B}_G\colon G{-}Spaces \to C^*_rCat$$

such that by composing with $\mathbb{K}^{Top}$ we again get a functor H_C with the properties of (4.1).

Applying $F = \mathbb{K}^{alg}$ or $F = \mathbb{L}^{-\infty}$ to the sequence of additive categories (with involution):

$$\mathcal{B}_G(X\times[0,1);R)_\emptyset \to \mathcal{B}_G(X\times[0,1);R) \to \mathcal{B}_G(X\times[0,1);R)^{>0}$$

gives a fibration of spectra.

Theorem 6.3. ([19, 4.1]) *The generalized assembly map $H_C(X) \to H_C(\bullet)$ is homotopy equivalent to the connecting map*

$$F(\mathcal{B}_G(X\times[0,1);R)^{>0}) \to \Omega^{-1}F(\mathcal{B}_G(X\times[0,1);R)_\emptyset$$

for $F = \mathbb{K}^{alg}$ or $F = \mathbb{L}^{-\infty}$.

Proof. Following [4], we compare the functors under the maps induced by the projection $X\to\bullet$ to a basepoint. Consider the diagram:

$$\begin{tikzcd}
\mathcal{B}_G(X\times[0,1);R)_\emptyset \arrow[r] \arrow[d, "a"] & \mathcal{B}_G(X\times[0,1);R)] \arrow[r] \arrow[d, "b"] & \mathcal{B}_G(X\times[0,1);R)^{>0} \arrow[d, "c"] \\
\mathcal{B}_G(\bullet\times[0,1);R)_\emptyset \arrow[r] & \mathcal{B}_G(\bullet\times[0,1);R) \arrow[r] & \mathcal{B}_G(\bullet\times[0,1);R)^{>0}
\end{tikzcd}$$

The functor a is an equivalence of categories, and the K-theory or L-theory of $\mathcal{B}_G(\bullet\times[0,1);R)$ is trivial. Comparing the fibration sequences we obtain an identification of $c\colon H_C(X)\to H_C(\bullet)$, which is just

$$c\colon F(\mathcal{B}_G(X\times[0,1);R)^{>0}) \to F(\mathcal{B}_G(\bullet\times[0,1);R)^{>0})$$

to the connecting map $F(\mathcal{B}_G(X\times[0,1);R)^{>0}) \to \Omega^{-1}F(\mathcal{B}_G(X\times[0,1);R)_\emptyset$, up to homotopy equivalence. ☐

For the C^*_r-category version, there is an assembly map

$$\alpha_X\colon \mathbb{K}^{Top}(C^*_r\mathcal{B}_G(X\times[0,1);\mathbb{C})^{>0}) \to \mathbb{K}^{Top}(C^*_r\mathcal{B}_G(\bullet\times[0,1);\mathbb{C})^{>0})$$

and the corresponding result of Carlsson, Pedersen and Roe [4, Thm. A] identifies this with the Baum-Connes assembly map [14].

Theorem 6.4. *Let $X = \mathcal{E}_{\mathcal{F}}G$ denote the classifying space for G-actions with finite isotropy. Then the continuously controlled assembly map α_X induces the Baum-Connes assembly map*

$$KK_i^G(C_0(X);\mathbb{C}) \to K_i(C_r^*G)$$

on homotopy groups.

7. Groupoids to Controlled Categories

In this section we will relate the Davis-Lück assembly maps constructed from functors $F\colon \mathbf{Or}(G) \to \mathit{AddCat}$ via groupoids to those constructed on G-Spaces using the continuously controlled categories. Since the extension of functors $\mathbf{Or}(G) \to \mathit{Spectra}$ to G-Spaces is unique, given the properties (4.1), it is enough to compare these functors for $X = G/H$.

We define a subcategory $\mathcal{B}_G^{(c)}(X\times[0,1);R)$ of $\mathcal{B}_G(X\times[0,1);R)$, with the same objects but we allow only morphisms $\phi\colon A\to B$ such that the components $\phi_{(x,s)}^{(y,t)} = 0$ whenever $s\neq t$, for $0\le s,t<1$.

Proposition 7.1. *There is a functor*

$$I\colon R(\overline{G/H})_{\oplus} \to \mathcal{B}_G^{(c)}(G/H\times[0,1);R)_{\emptyset}$$

inducing isomorphisms in $\mathbb{K}^{alg}$-theory, $\mathbb{L}^{-\infty}$-theory, and similarly for $\mathbb{K}^{Top}$ applied to $C_r^*(\overline{G/H})$.

Proof. We will discuss the functor on $R(\overline{G/H})_{\oplus}$ and then point out the modifications necesary to handle the C_r^*-category version. If $S = G/H$, then objects in $\overline{G/H}$ are just cosets $s\in S$. Let $H_s = \{g\in G\,|\,gs = s\}$. This is a conjugate subgroup of H in G. We define I on objects by sending $s\mapsto RH_s$ over $(s,0)$, and then extend by equivariance to $RG\otimes_{H_s} RH_s$. A morphism $g\in \mathrm{mor}(s,t)$ has the property that $g_2^{-1}gg_1^{-1} = h\in H$, where $s = g_1H$ and $t = g_2H$. Then $I(g)(a) := ag^{-1}$ for all $a\in RH_s$, and

$$I(g)\in \mathrm{Hom}_{RH_s}(R[g_1Hg_1^{-1}], R[g_1Hg_2^{-1}])$$

where both $R[g_1Hg_1^{-1}] = RH_s$ and $R[g_1Hg_2^{-1}]$ are left RH_s-modules, since $H_s = g_1Hg_1^{-1}$. The module $R[g_1Hg_2^{-1}]$ is isomorphic (as a left RH_s-module) to $I(H_t)_s = (g_1^{-1}g_2)\otimes_{H_t} RH_t$, so we have defined a component

$$I(g)_s^s\colon I(s)_s\to I(t)_s$$

of the morphism. The components $I(g)_s^t = 0$ when $s\neq t$.

This is again extended by equivariance to get an element in $\mathrm{mor}(I(s),I(t))$. Notice that these morphisms preserve the levels $\{s\}\times[0,1)$, and have empty support at $G/H\times 1$, so we get a functor into $\mathcal{B}_G^{(c)}(X\times[0,1);R)_{\emptyset}$.

To complete the proof we need to show that I induces isomorphisms on homotopy groups. For any coset $s \in G/H$, let $FMod(RH_s)$ denote the category of finitely generated free left RH_s-modules. Consider the commutative diagram:

$$\begin{tikzcd} R(\overline{G/H})_\oplus \arrow[r, "\approx", "I"'] & \mathcal{B}_G^{(c)}(G/H \times [0,1); R)_\emptyset \\ FMod(RH_s) \arrow[u] \arrow[r] & \mathcal{B}^{(c)}([0,1); RH_s)_\emptyset \arrow[u, hook] \end{tikzcd}$$

where the horizontal arrows give equivalences of categories (by our stronger control condition on morphisms). The left vertical arrow is an inclusion of a full cofinal subcategory (since $\overline{G/H}$ is a connected groupoid, in the sense that any two objects are related by an isomorphism in the category). By the Waldhausen Cofinality Theorem [28, 2.1] the left vertical arrow induces isomorphisms in K-theory or L-theory.

The right vertical arrow is also an inclusion of a full subcategory. Since each object in $\mathcal{B}_G^{(c)}(G/H \times [0,1); R)_\emptyset$ is isomorphic to an induced object $RG \otimes_{RH_s} RH_s$, this map again induces isomorphisms on K-theory and L-theory.

The argument for the C_r^* version follows exactly the same steps. We only need to use the C_r^*-category versions of $R(\overline{G/H})_\oplus$ (see [6, p. 212-213]) and $\mathcal{B}_G^{(c)}(G/H \times [0,1); R)$ (see [4, § 3]), and the rest goes through with minor modifications. ☐

Theorem 7.2. *Let F denote one of the functors $\mathbb{K}^{alg}$ or $\mathbb{L}^{-\infty}$. The functors $F_i\colon \mathbf{Or}(G) \to Spectra$, for $i = 1, 2$, defined by $F_1(G/H) = F(R(\overline{G/H})_\oplus)$, and $F_2(G/H) = \Omega F(\mathcal{B}_G(G/H \times [0,1); R)^{>0})$ are naturally isomorphic, and similarly for $F = \mathbb{K}^{Top}$ applied to the C_r^*-category versions.*

Proof. In this argument, no special modifications are needed for the $\mathbb{K}^{Top}$-version beyond taking the C_r^*-categories in the definition of F_1 and F_2. We will construct a natural transformation between the functors F_1 and F_2 which induces isomorphisms on the homotopy groups.

Consider the commutative diagram of fibrations:

$$\begin{tikzcd} \mathcal{B}_G^{(c)}(G/H \times [0,1); R)_\emptyset \arrow[r] \arrow[d, "a"'] & \mathcal{B}_G^{(c)}(G/H \times [0,1); R) \arrow[r] \arrow[d, "b"'] & \mathcal{B}_G^{(c)}(G/H \times [0,1); R)^{>0} \arrow[d, "c"'] \\ \mathcal{B}_G(G/H \times [0,1); R)_\emptyset \arrow[r] & \mathcal{B}_G(G/H \times [0,1); R) \arrow[r] & \mathcal{B}_G(G/H \times [0,1); R)^{>0} \end{tikzcd}$$

where the spectrum $F(\mathcal{B}_G^{(c)}(G/H \times [0,1); R))$ has trivial homotopy groups, and the map c induces an equivalence of categories. It follows that the connecting map from the upper fibration, together with c, induce a homotopy equivalence

$$\Omega F(\mathcal{B}_G(G/H \times [0,1); R)^{>0}) \xrightarrow{\simeq} F(\mathcal{B}_G^{(c)}(G/H \times [0,1); R)_\emptyset)$$

of spectra. Now the composite of the natural isomorphism I from Lemma 7.1, and the inverse of this homotopy equivalence gives the required natural isomorphism between F_1 and F_2. □

We have now reached one of our main goals.

Theorem 7.3. *The Davis-Lück assembly maps arising from $\mathbf{K}^{alg}$, $\mathbf{L}^{-\infty}$, and $\mathbf{K}^{Top}$ are naturally isomorphic to the continuously controlled assembly maps.*

Proof. The description in (4.2)-(4.4) of the Davis-Lück functors shows that they all factor through the functor Gr (or its C_r^* variant) out of the orbit categrory $\mathbf{Or}(G)$. On the other hand, the continuously controlled functors H_C from (6.2) are all determined by K-theory or L-theory applied to $\mathcal{B}_G(G/H \times [0,1); R)^{>0}$. The result now follows from Theorem 7.2. □

Corollary 7.4. *The Davis-Lück assembly map arising from $\mathbf{K}^{Top}$ applied to $\mathcal{E}_{\mathcal{F}}(G)$, for $\mathcal{F}$ the family of finite subgroups of G, is naturally isomorphic to the Baum-Connes assembly map.*

Proof. The result of Carlsson, Pedersen and Roe given in Theorem 6.4 (see [4, Thm. A]) identifies the Baum-Connes assembly map with the one induced by $\mathbb{K}^{Top}$ and the continuously controlled categories (applied to the G-space $\mathcal{E}_{\mathcal{F}}(G)$). The final step is provided by Theorem 7.3. □

8. $\mathbf{Or}(G)$ to fundamental groupoids

In this section we factor the functor $Gr\colon \mathbf{Or}(G) \to$ *Groupoids* of Davis-Lück through the functor $E\colon \mathbf{Or}(G) \to$ *Spaces* defined on objects by $G/H \mapsto G/H \times_G EG$, where EG denotes the universal contractible G-space on which G acts freely. The passage $\pi\colon$ *Spaces* $\to$ *Groupoids* is just $X \mapsto \pi(X)$, where $\pi(X)$ denotes the fundamental groupoid of X.

Proposition 8.1. *There is a functor $\Phi\colon \overline{G/H} \to \pi(G/H \times_G EG)$ inducing isomorphisms in algebraic K-theory and L-theory.*

Proof. We fix a basepoint $\bullet \in EG$, and let $(*) \subset \pi(G/H \times_G EG)$ denote the full subcategory with one object $[eH, \bullet]$. Similarly, let $(\bullet) \subset \overline{G/H}$ denote the full subcategory with one object eH. For any object $s \in G/H$ we set $\Phi(s) = [s, \bullet] \in G/H \times_G EG$. For a morphism $g \in \mathrm{mor}(s,t)$, we set $\Phi(g) = [s, \gamma(u)]$, using a path $\gamma(u)$, $0 \le u \le 1$, in EG joining $\bullet$ and $g^{-1} \cdot \bullet$. The path is unique up to homotopy, and $[s, g^{-1} \cdot \bullet] = [t, \bullet]$. Since $\Phi(eH) = [eH, \bullet]$, we get an equivalence of the subcategories $(\bullet)$.

Consider the commutative diagram:

$$\begin{tikzcd} R(\bullet)_\oplus \arrow[r] \arrow[d] & R(\bullet)_\oplus \arrow[d] \\ R(\overline{G/H})_\oplus \arrow[r, "\Phi"] & R(\pi(G/H \times_G EG))_\oplus \end{tikzcd}$$

where the inclusion $R(\bullet)_\oplus \subset R(\overline{G/H})_\oplus$ induces isomorphisms on $\mathbb{K}^{alg}$ or $\mathbb{L}^{-\infty}$, Since $\overline{G/H}$ is a connected groupoid [28, 2.1], [6, 2.4(ii)]. Similarly, the inclusion $R(\bullet)_\oplus \subset \pi(G/H \times_G EG)$ induces isomorphisms on $\mathbb{K}^{alg}$ or $\mathbb{L}^{-\infty}$, and the result for Φ follows. ☐

9. Assembly in L-theory

In this section we show that the Davis-Lück assembly maps are compatible with the original geometric L-theory assembly maps

$$X_+ \wedge \mathbf{L}(\mathbb{Z}) \to \mathbf{L}(\mathbb{Z}\pi_1(X, x_0))$$

of Quinn [23], or with the subsequent algebraic L-spectrum construction of Ranicki [27], and those used in the Farrell-Jones Isomorphism Conjectures.

The L-theory assembly maps of Farrell-Jones uses Quinn's construction (described in Section 5) applied to the functor $\mathbb{L}^{geom}\colon \mathit{Spaces} \to \mathit{Spectra}$. This is the connective geometric surgery spectrum defined by Quinn [23] whose homotopy groups $\pi_n(\mathbb{L}_0^{geom}(X)) \cong L_n(\mathbb{Z}\pi_1(X, x_0))$ are the geometric surgery obstruction groups of C. T. C. Wall [31]. In particular, $\pi_n(\mathbb{L}_0^{geom}(\bullet)) = L_n(\mathbb{Z})$ are the surgery obstruction groups for the trivial group. The K-theory decorations will be suppressed in this section, and we mean the $L^{-\infty}$ versions throughout (see Yamasaki [34]).

We have already shown in Theorem 5.1 that the generalized Quinn assembly for the functor $F = \mathbb{L}^{alg}$ or $F = \mathbb{L}^{geom}$ is naturally isomorphic to the corresponding Davis-Lück assembly map. We compare $(\alpha_X, \mathbb{L}^{alg})$ to $(\alpha_X, \mathbb{L}^{geom})$ by checking what happens over the orbit category $\mathbf{Or}(G)$.

The first step is Proposition 8.1, where a functor $\Phi\colon \overline{G/H} \to \pi(G/H \times_G EG)$ is defined inducing equivalences $\mathbb{L}^{alg}(\mathbb{Z}(\overline{G/H})_\oplus) \simeq \mathbb{L}^{alg}(\mathbb{Z}(\pi(G/H \times_G EG)_\oplus)$. The remaining step is the following result.

Proposition 9.1. *There a natural equivalence*

$$\Theta\colon \mathbb{L}^{geom}(G/H \times_G EG) \simeq \mathbb{L}^{alg}(\mathbb{Z}(\pi(G/H \times_G EG)_\oplus)$$

of spectrum-valued functors on $\mathbf{Or}(G)$.

Proof. There is a homotopy equivalence $\mathbb{L}^{geom}(G/H \times_G EG) \simeq \mathbb{L}^{alg}(\mathbb{Z}H)$ by one of the main results of geometric surgery theory [31, Ch. 9]. The equivalence depends on a choice of base-point in $G/H \times_G EG \simeq BH$, so this isn't natural.

Similarly, there is a homotopy equivalence

$$\mathbb{L}^{alg}(\mathbb{Z}(\pi(G/H \times_G EG)_\oplus)) \simeq \mathbb{L}^{alg}(\mathbb{Z}H),$$

since the category $\mathbb{Z}(\overline{G/H})_\oplus$ is equivalent to the category of finitely generated free $\mathbb{Z}H$-modules (again by picking a base point as in § 7).

However, the algebraic chain complex theory of Ranicki [26], [27] over the additive category $\mathbb{Z}(\pi(G/H \times_G EG)_\oplus$, provides a natural setting for the passage from geometry to algebra without choosing base points. We triangulate EG and apply the construction of the algebraic quadratic Poincaré complexes over the semi-simplicial Δ-set models for $\mathbb{L}^{geom}$ and $\mathbb{L}^{alg}$. This gives a natural transformation Θ and a homotopy commutative diagram

$$\begin{tikzcd} \mathbb{L}^{geom}(G/H \times_G EG) \arrow[r, "\simeq"] \arrow[d, "\Theta"'] & \mathbb{L}^{alg}(\mathbb{Z}H) \\ \mathbb{L}^{alg}(\mathbb{Z}(\pi(G/H \times_G EG)_\oplus) \arrow[ur, "\simeq"'] & \end{tikzcd}$$

showing that Θ is a natural equivalence. $\square$

Corollary 9.2. *The Farrell-Jones assembly map for geometric $L^{-\infty}$-theory is naturally isomorphic to the Davis-Lück assembly map $(\alpha_X, \mathbb{L}^{-\infty})$ for $X = \mathcal{E}_{\mathcal{F}}(G)$, where $\mathcal{F}$ denotes the family of virtually infinite cyclic subgroups of G.*

10. Isomorphism conjectures in L-theory

The Farrell-Jones Isomorphism Conjecture in L-theory is stated only for $\mathbb{L}^{-\infty}$. Recently it was noticed that the corresponding assertion for the functor $\mathbb{L}^h$ is *not* valid for $G = \mathbf{Z}^2 \times \mathbf{Z}/5$ [8]. From the conceptual point of view, this is not surprising since the computation of the domain $F^{\%}(\mathcal{E}_{\mathcal{F}}(G))$ for $F = \mathbb{L}^h$ will involve a stratified system of splitting obstructions in various K-groups. However, neither the Davis-Lück theory nor the Quinn generalized assembly framework seems to have a suitable "intermediate" functor with which to formulate an $\mathbb{L}^h$ or $\mathbb{L}^s$ conjecture. We would like to propose the following:

Conjecture 10.1. *For any discrete group G and any commutative ring R, the assembly map*

$$\mathbb{L}^s(\mathcal{B}_G(X \times [0,1); R)^{>0}) \to \mathbb{L}^s(\mathcal{B}_G(\bullet \times [0,1); R)^{>0}$$

is a homotopy equivalence for $X = \mathcal{E}_{\mathcal{F}}(G)$, where $\mathcal{F}$ denotes the family of virtually infinite cyclic subgroups of G.

We remark that this conjecture holds for any group G over a ring R such that the Farrell-Jones Isomorphism Conjectures are valid for $\mathbb{L}^{-\infty}(RG)$, and for $\mathbb{K}^{alg}(RG)$ in dimensions ≤ 1. Since $\Omega\mathbb{L}^s(\mathcal{B}_G(\bullet \times [0,1); R)^{>0} \simeq \mathbb{L}^s(RG)$, the conjecture provides a possible description of the L-groups of RG with K-theory decorations in $Wh(RG)$, $\tilde{K}_0(RG)$, or $K_i(RG)$ for $i < 0$.

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