On $-K^2$ for normal surface singularities II

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Abstract

In this paper we continue our study [1] of the set of nonzero $-K^2$ of normal surface singularities. The main results are

1) Every accumulation point of the set is the sum of $-K^2$ of several normal surface singularities;

2) There are accumulation points of the set which are not $-K^2$ of any one normal surface singularity, which means that the set $\{-K^2\}$ is not closed.

1 Introduction

In our previous paper [1] we studied the set $\{-K^2 | K$ is the numerical canonical divisor of a normal surface singularity$\}$ (for the numerical canonical divisor see Definition 2.1 below). It is proved that any nonzero $-K^2$ is bigger than or equal to $1/3$ (Theorem 1.4 in [1]) and that the set has no accumulation point from above (Theorem 1.10 in [1]). We know from our previous results that all accumulation points from below are rational numbers (Theorem 2.3 in [1]) and that there are indeed many accumulation points from Propositions 2.4, 2.6 and Corollary 2.6 in [1]. It is natural to ask if the set $\{-K^2\}$ is closed, i.e., is every accumulation point the $-K^2$ of a normal surface singularity? In this paper we answer this problem. Although counterexamples are given for the problem in section 4, we prove that every accumulation point is the sum of $-K^2$'s of several normal surface singularities.

2 Preliminaries

In this section we recall the basic definitions and results which are needed in the following sections.

Definition 2.1. Let $A = \sum A_i$ be a compact reduced divisor whose intersection matrix $\{A_iA_j\}_{i,j}$ is negative definite on a normal Q-factorial surface Y. One defines the numerical canonical divisor $K(A)$ as a Q-divisor with the support on A as

$$K(A)A_i = K_Y A_i \quad (2.1)$$

for every i. In particular, if Y is the minimal resolution of a normal surface singularity (X, x) and A is the exceptional divisor, then $K(A)$ is called the numerical canonical divisor of the singularity (X, x).

We call a curve C (in a surface) with $C \simeq P^1$ and $C^2 = -n$ a $(-n)$-curve. The following formula in 1) is clear from the Adjunction formula, and 2) is Lemma 1.3 of [1].

Lemma 2.2. *Notation as above. Suppose Y is the minimal resolution of a normal surface singularity.*

1) Let p_i be the arithmetic genus of A_i, $c_i = KA_i = 2p_i - 2 - (A_i)^2$ and $E = (A_iA_j)_{1 \leq i,j \leq n}$ be the intersection matrix. We have

$$-K^2 = -(c_1, ..., c_n)(E^{-1})^t(c_1, ..., c_n) \qquad (2.2)$$

2) $-K^2 \geq \sum (KA_i)^2/(-A_i^2)$.

3) $(KA_i)^2/(-A_i^2) = (2p_i - 2)^2/(-A_i^2) + 2(2p_i - 2) - A_i^2$ is not less than 1/3. It is 1/3 if and only if A_i is (-3)-curve. If it is not (-3)-curve, it is not less than 1. It is 1 if and only if A_i is (-4)-curve or (-1)-elliptic curve.

Lemma 2.3. *Let E be a negative definite matrix of size n with non-negative integer entries at non-diagonal positions and negative integer entries at diagonal positions. Suppose that we cannot make E to be a diagonal (blocked) matrix after renumbering the order of rows and columns, respectively. Then E is the intersection matrix of exceptional divisors on a resolution of a normal surface singularity. Moreover, the genera of irreducible components of the exceptional set can be arbitrarily pre-assigned $(g_1, .., g_n)$.*

Proof. Take n smooth curves $A_1, ..., A_n$ of genera $g_1, ..., g_n$ and tubular neighborhoods $U_1, ..., U_n$ of zero sections in the negative line bundle $L_1, ..., L_n$ over these curves. Let the degree of L_i be the i-th diagonal entry of E. Patch $U_1, ..., U_n$ together according to the non-diagonal entries of E, we get a complex

2-dimensional manifold Y. We know that Y is connected from our assumption of E. Thus, we get (X, x) from contracting $\cup A_i$ by Grauert's theorem. Q.E.D.

The following result is a refinement of Proposition 1.2 in [1].

Proposition 2.4. *Let A be a compact connected divisor on a smooth surface whose intersection matrix is negative definite. Suppose that A does not contain (-1)-curves, and $A = A_1 + A_2$ is a nontrivial decomposition with no common components on A_1 and A_2. Then we have $-K(A_1)^2 - K(A_2)^2 \leq -K(A)^2$, and the equality holds if and only if A consists of (-2)-curves.*

Proof. First we show that $K(A) - K(A_1) - K(A_2)$ is nef on A. For an irreducible curve $C \subset A_1$, $(K(A) - K(A_1) - K(A_2))C = (K(A) - K(A_1))C - K(A_2)C$. From Definition 2.1, $(K(A) - K(A_1))C = 0$. $-K(A_2)C \geq 0$ since $-K(A_2)$ is an effective divisor. Similarly, we know $(K(A) - K(A_1) - K(A_2))C \geq 0$ for any irreducible curve $C \subset A_2$. Thus, $(K(A) - K(A_1) - K(A_2))K(A) = K(A)^2 - K(A_1)^2 - K(A_2)^2 \leq 0$. It is clear that the equality holds if A consists of (-2)-curves. Moreover, if A contains a non (-2)-curve, we can assume that this curve is in A_2 without loss of generality. Thus, $-K(A_2)$ is strictly bigger than 0 and $-K(A_2)C > 0$ for some irreducible curves C in A_1 and the strict inequality follows. Q.E.D.

We refer to both the concept of (-2)-insertion and the proof of the following proposition to [1].

Proposition 2.5. *Let A be a (-2)-insertion of A', then we have $-K(A')^2 \leq -K(A)^2$.*

The following lemma is similar to 1) of Lemma 2.2, which will be needed in the proof of our main result in section 3.

Lemma 2.6. *Let X be a nonsingular surface and $E = \sum_{i=1}^{r} E_i + \sum_{j=1}^{s} F_j$ be a compact divisor on X with negative definite intersection matrix, where $F_j, j = 1, ..., s$, are (-2)-curves. Suppose $\varphi : X \to X'$ is the contraction of $\sum_{j=1}^{s} F_j$, E_i' is the image of E_i under φ, $E' = \sum E_i'$ and M' is the intersection matrix of E_i's on X'. Then $-K(E)^2 = -c(M'^{-1})^t c$, where $c = (c_1, ..., c_r), c_i = K(E)E_i$.*

Proof. We write $K(E) = \sum m_i E_i + \sum n_j F_j$, then $K(E') = \sum m_i E_i'$. Since all singular points of X' are RDP's, thus $\varphi^*(K(E')) = K(E)$. Hence, $-K(E)^2 = -K(E')^2 = -\sum_{i=1}^{r} m_i K(E') E_i'$, where $K(E')E_i' = \varphi^*(K(E'))E_i = K(E)E_i = c_i$ by projection formula. We have $(M')^t m = {}^t c$, where $m = (m_1, ..., m_r)$ from Definition 2.1 of the numerical canonical divisor. So $-K(E)^2 = -m^t c = -c(M'^{-1})^t c$. Q.E.D.

3 An accumulation point is the sum of $-K^2$'s

In this section we prove the following statement.

Theorem 3.1. *Let W be an accumulation point of the set $\{-K^2\}$. Then finitely many normal surface singularities $(Y_1, y_1), ..., (Y_t, y_t)$ exist such that*

$$W = -K^2_{(Y_1,y_1)} - ... - K^2_{(Y_t.y_t)} \quad (3.1)$$

Proof. Let $\{A_{(n)}\}$ be the sequence of exceptional divisors of minimal resolutions $\{X_n\}$ of normal surface singularities with the corresponding sequence of $\{-K^2_{(n)}\}$ tending to W. First, as in the proof of Theorem 2.3 of [1], we can assume that $A_{(1)} \to A_{(n)}$ is a composite of (-2)-insertions: $A_{(1)} = A_{(0...0)} = \sum E_i$ and $A_{(n)} = A_{(n_1...n_k)} = \sum E'_i + \sum_{j=1}^{k} S_j$, where $S_j = \sum_{s=1}^{n_j} F_{js}$ is the insertion string with the props E'_{2j-1}, E'_{2j}. One may assume that if $n \to \infty$, then $n_j \to \infty$ for all $j = 1, ..., k$.

Now we get X'_n by contracting (-2)-curve strings $S_j, j = 1, ..., k$. Let the intersection matrix of the image of E'_i in X'_n be $M_{(n_1...n_k)}$. This takes the following form as in the proof of Theorem 2.3 of [1].

$$\begin{pmatrix} -\frac{n_1+2}{n_1+1} & \frac{1}{n_1+1} & ... & ... & 0 & 0 & {}^t a_1 \\ \frac{1}{n_1+1} & -\frac{n_1+2}{n_1+1} & ... & ... & 0 & 0 & {}^t a_2 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & -\frac{n_k+2}{n_k+1} & \frac{1}{n_k+1} & {}^t a_{2k-1} \\ 0 & 0 & ... & ... & \frac{1}{n_k+1} & -\frac{n_k+2}{n_k+1} & {}^t a_{2k} \\ a_1 & a_2 & ... & ... & a_{2k-1} & a_{2k} & N \end{pmatrix} \quad (3.2)$$

and $-K^2_{(n)} = -cM^{-1}_{(n_1...n_k)}{}^t c$, where $c = (c_1, ..., c_h), c_i = K(A_{(0...0)})E_i$ from Lemma 2.6.

Since it is proved that $\lim M_{(n_1,...,n_k)} = M_{(\infty...\infty)}$ (see (3.3)) is a nonsingular (thus negative definite) matrix in the proof of Theorem 2.3 of [1], we have $W = -cM^{-1}_{(\infty,...,\infty)}{}^t c$, where

$$M_{(\infty...\infty)} = \begin{pmatrix} -1 & 0 & ... & 0 & {}^t a_1 \\ 0 & -1 & ... & 0 & {}^t a_2 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & -1 & {}^t a_{2k} \\ a_1 & a_2 & ... & a_{2k} & N \end{pmatrix} \quad (3.3)$$

Let D_l be the following configuration:

$$\begin{array}{ccccccc} o_1 - \cdots - o_{l-3} - & o_{l-2} & - o_{l-1} \\ & | & \\ & o_l & \end{array} \qquad (3.4)$$

of (-2)-curves corresponding to RDP D_l. Consider the configuration $A_{(\infty\ldots\infty)} = \sum_{i=1}^{r} E'_i + \sum_{j=1}^{k}(D_l^j + D_l'^j)$, where D_l^j, $D_l'^j$ for $j = 1, ..., k$ are copies of D_l. Each D_l^j (respectively, $D_l'^j$) intersects the other part only at o_1 (respectively, o'_1), which has one intersection point with E'_{2j-1} (respectively, E'_{2j}). This is equivalent to replacing each string S_j in $A_{(n_1\ldots n_k)}$ with two copies of configuration D_l. Here we should note that $A_{(\infty\ldots\infty)}$ may not be connected. We construct a nonsingular surface X containing $A_{(\infty\ldots\infty)}$ as a compact divisor (but might be non connected) as in Lemma 2.3. Now we contract $D_l's$ in X, by $\varphi : X \to X'$. Let $A'_{(\infty\ldots\infty)}$, E''_{2j-1}, E''_{2j} be the images of $A_{(\infty\ldots\infty)}$, E'_{2j-1}, E'_{2j} for $j = 1, ..., k$ under the contraction. Since $\varphi^*(E'_{2j-1})o_t = 0$, for $t = 1, ..., l$, we have $\varphi^*(E''_{2j-1}) = E'_{2j-1} + o_1 + ... + o_{l-2} + (o_{l-1} + o_l)/2$. Thus, ${E''_{2j-1}}^2 = -1$. Similarly, ${E''_{2j}}^2 = -1$. Thus, the intersection matrix of $A'_{(\infty\ldots\infty)}$ in X' is just the same as $M_{(\infty\ldots\infty)}$ in (3.3). From Lemma 2.6 we have $-K(A'_{(\infty\ldots\infty)})^2 = -cM^{-1}_{(\infty\ldots\infty)}{}^t c = W$. On the other hand, by contracting $A'_{(\infty\ldots\infty)}$ in X' (since the intersection matrix $M_{(\infty\ldots\infty)}$ is negative definite) we get several normal surface singularities $(Y_1, y_1), ..., (Y_t, y_t)$. It is clear that $-K^2_{(Y_1,y_1)} - ... - K^2_{(Y_t,y_t)} = -K(A'_{(\infty\ldots\infty)})^2 = W$. The conclusion is proved.

Corollary 3.2. *Let W be an accumulation point of the set $\{-K^2\}$. Suppose $\{A_{(n)}\}$ is the sequence of exceptional divisors on the minimal resolutions of normal surface singularities with $\{-K^2_{(n)}\}$ tending to W. Moreover, the non (-2)-curve part (which can be assumed to be independent of n after choosing a suitable subsequence as showed in [1]) of $\{A_{(n)}\}$ is connected. Then we can find a normal surface singularity (Y, y), such that $W = -K^2_{(Y,y)}$.*

Proof. We note that in this case, except for one connected component of $A_{(\infty\ldots\infty)}$, the other connected components are exceptional divisors of rational double points whose $-K^2$'s are zero.

4 $\{-K^2\}$ is not closed

In this section, we give a sequence of accumulation points of $\{-K^2\}$ which are not in this set.

Proposition 4.1. *Let $T_{n,m}$ be normal surface singularities with the following weighted dual graph (the existence of such singularities is clear from Lemma*

2.3)

$$T_{n,m} : x - o - \cdots - o - x - o - \cdots - o \ ,$$

where x and o denote a (-3)-curve and a (-2)-curve respectively, there are n o's between the two x's and m o's at the right side of the second x. We have

$$-K(T_{n,m})^2 = (3nm + 4n + 7m + 8)/(2nm + 4n + 5m + 8) \ \ (4.1)$$

Proof. This is a direct computation.

The following statement is the main result of this section.

Theorem 4.2. *Let*

$$v_m = \lim_{n \to \infty}(-K(T_{n,m})^2) = (3m + 4)/(2m + 4) \ \ (4.2)$$

Suppose m is a sufficiently large prime number of the form $4k + 1$. Then v_m is not in the set $\{-K^2\}$.

From Theorem 4.2 we obtain the following conclusion immediately.

Corollary 4.3. *The set $\{-K^2\}$ is not closed.*

Proof. We know there are infinitely many prime numbers of the form $4k+1$ from Dirichlet theorem (see Chapter VI in [2]). Thus, from Theorem 4.2 there are infinitely many accumulation points of the set $\{-K^2\}$ which are not in this set. Q.E.D.

Assume that the conclusion of Theorem 4.2 is not true, we can then get a sequence of normal surface singularities $\{ \ (Y_n, y_n)\}$ with $-K^2_{(Y_n,y_n)} = v_{m_n}$, where m_n is a prime number of the form $4k+1$, and hence $\{ \ -K^2_{(Y_n,y_n)} \ \}$ tends to $3/2$ from below. The main part of the proof for Theorem 4.2 is to study all possible such singularities. Let $\{(Y_n, y_n)\}$ be a sequence of normal surface singularities with their $\{-K^2_{(Y_n,y_n)}\}$ tending to $3/2$ from below, and $\{A_n\}$ be the sequence of exceptional divisors on the minimal resolutions of $\{(Y_n, y_n)\}$. As showed in [1] we can assume that the non (-2)-curve part $\{A\}$ of $\{A_n\}$ is independent of n after taking a suitable subsequence.

Lemma 4.4. *Let X be a normal surface singularity whose exceptional set contains a D_l. We have $-K_X^2 \geq 1/2$. Moreover, if the D_l intersects at a non (-3)-curve, we have $-K_X^2 > 1$.*

Proof. Let E_0 be a non (-2)-curve which connects to D_l. The exceptional set contains either the following configurations ($\bullet$ denotes the non (-2)-curve E_0):

$$\begin{array}{ccccccccccccc} \bullet & - & o & - & \cdots & - & o & - & o & - & o \\ & & & & & & & & | & & \\ & & & & & & & & o & & \end{array}$$

or

$$\begin{array}{ccccc} & & o & & \\ & & | & & \\ \bullet & - & o & - & o \\ & & | & & \\ & & o & & \end{array}$$

In the former case, the exceptional set contains a (-2)-insertion of the following graph H

$$\begin{array}{ccccccc} \bullet & - & o & - & o & - & o \\ & & & & | & & \\ & & & & o & & \end{array}$$

Because H is not negative definite when $E_0^2 = -1$, we can assume that $E_0^2 < -1$. Then $-K(H)^2 = \frac{(2p-2-E_0^2)^2}{-1-E_0^2} = \frac{(2p-1)^2}{-1-E_0^2} + 2(2p-1) - 1 - E_0^2$, where p is the arithmetic genus of E_0. It is easy to know $-K(H)^2 > 1/2$ generally and $-K(H)^2 > 1$ when E_0 is a non (-3)-curve. In the latter case, the non (-2)-curve E_0 has $E_0^2 \leq -3$, because the subgraph should be negative definite. Hence, the subgraph has $-K^2 = \frac{(2p-2-E_0^2)^2}{-2-E_0^2} = \frac{4p^2}{-2-E_0^2} + 4p + (-2-E_0^2)^2 > 1$, where p is the arithmetic genus of E_0. Then from Proposition 2.4 we obtain our conclusion.

Lemma 4.5. *The A has to be one of the following possibilties:*
1) one (-3)-curve;
2) the sum of two (-3)-curves.

Proof. We apply Theorem 3.1 and its proof here. There are serveral non RDP normal surface singularities $X_1, ...X_t$, such that $3/2 = -K_{X_1}^2 ... - K_{X_t}^2$. Moreover, each of the corresponding exceptional divisors $B_1, ..., B_t$ contains a divisor of the form D_l as showed in the proof of Theorem 3.1. From Lemma 4.4 we have $t \leq 3$.

If $t = 3$, $-K_{X_i}^2 = 1/2$ for each i. Again from Lemma 4.4 there is only one (-3)-curve in each B_i, and one B_i has at least two $D_l's$ from the proof of Theorem 3.1. From Propositon 2.4, $-K^2$ of this B_i is strictly bigger than $1/2$, which is a contradiction.

When $t = 2$, $-K_{X_i}^2 \leq 1$ from Lemma 4.4. If $-K_{X_1}^2 = 1$, consider the configuration B_1', which is the deletion of o_l and edges connecting to it from B_1, (where o_l is the irreducible component in D_l as in (3.4)), we have $-K(B_1')^2 < 1$ from Proposition 2.4. Hence, B_1' corresponds to a rational triple point as showed in Proposition 2.8 of [1], and there is at most one (-3)-curve in B_1' (hence in B_1). On the other hand, $-K_{X_2}^2 = 1/2$, the singularity is a rational triple point, and B_2 contains at most one (-3)-curve. Otherwise we know $-K_{X_i}^2 < 1$ for each i. So they are rational triple points, and there is at most one (-3)-curve in each B_i.

Assume $t = 1$, and B_1 contains at least two non (-2)- curves $T_i's$. By Proposition 2.4 we know $-K(B_1')^2 + (2p(T_i) - 2 - T_i^2)^2/(-T_i^2) < 3/2$, where $B_1' = B_1 - T_i$. Since $-K(B_1')^2 \geq 1/2$ from Lemma 4.4, we have $(2p(T_i) - 2 - T_i^2)^2/(-T_i^2) < 1$. Thus, T_i is a (-3)-curve. Since B_1 is connected, we know these (-3)-curves have to be connected or connected by RDP (of fixed length) in A. It is easy to see that the configuration is obtained by inserting a string of (-2)-curves into the following configurations (x and o denote a (-3)-curve and a (-2)-curve, respectively):

$$x - x - x$$

$$\begin{array}{c} x - o - x \\ | \\ x \end{array}$$

$$\begin{array}{cc} x & - \; x \\ & \backslash \; | \\ & x \end{array}$$

We can check that the $-K^2$ of the above configurations are bigger than $3/2$ by direct computation. Hence, by Propositions 2.4 and 2.5 we have at most two (-3)-curves in B_1. We get our conclusion from the relation of A and B_i as described in the proof of Theorem 3.1.

In the following, the configuration $D_l - x$ always means that x (where x denotes a (-3)-curve) connects to the o_1 in the D_l of (3.4).

Lemma 4.6. *Let X_i, B_i, $i = 1, .., t$ be as in Lemma 4.5. If $t = 1$ there is at most (hence exactly) one (-3)-curve in B_1.*

Proof. In this case, the weighted dual graph of B_1 contains a subgraph of the form: $D_l - x$.

If there are two (-3)-curves, we have the following two possibilities:

1) The two (-3)-curves intersect.

In this case they have to intersect only at one point from the fact that its $-K^2$ is not bigger than $3/2$ (direct computation and Proposition 2.4). Since

$-K^2$ of $D_l - x - x$ is less than $3/2$, B_1 has to contain one of the following subgraphs:

$$D_l - x - x - o$$

$$\begin{array}{ccccc} D_l & - & x & - & x \\ & & | & & \\ & & o & & \end{array}$$

or

$$\begin{array}{ccccc} D_l & - & x & - & x \\ & & & \backslash & | \\ & & & & o \end{array} .$$

In each case we can calculate its $-K^2 > 3/2$. From Propositions 2.4 and 2.5, 1) is impossible.

2) Two $x's$ are connected by (-2)-curves.

Then either B_1 contains a subgraph which is obtained by inserting a string of (-2)-curves into the three graphs as above, or B_1 is $D_l - x - RDP - x$. In the former case we have $-K^2(B_1) > 3/2$ by Propositions 2.4 and 2.5. For the latter case, by a simple calculation we know the only possibility of middle RDP has to be (-2)-curve string if we want $-K^2 \leq 3/2$. However, in that situation it is always less than $3/2$ (tending to $3/2$ from below when the length of string tends to infinity). This contradicts to the fact $-K(B_1)^2 = 3/2$. Therefore 2) is impossible. The conclusion is proved.

Lemma 4.7. *Let X_i, B_i $(i = 1, ..., t)$ be as above. If $t = 1$, B_1 and corresponding $\{A_n\}$ are one of the following forms:*

1) The B_1 is:

$$\begin{array}{ccccccccccccc} D_l & - & x & - & o & - & o & - & o & - & o & - & o \\ & & & & | & & & & & & & & \\ & & & & o & & & & & & & & \end{array}$$

and the A_n is

$$\begin{array}{ccccccccccccccccc} o & - & \cdots & - & o & - & x & - & o & - & o & - & o & - & o & - & o \\ & & & & & & & & | & & & & & & & & \\ & & & & & & & & o & & & & & & & & \end{array}$$

, where the left side of x is a string of (-2)-curves of length n;

2) The B_1 is

$$\begin{array}{ccccccccccc} D_l & - & x & - & o & - & o & - & o & - & o \\ & & & & | & & & & & & \\ & & & & o & & & & & & \end{array}$$

and the A_n is

$$\begin{array}{c} o - \cdots - o - x - o - o - o - o \\ \qquad\qquad\quad | \qquad\qquad \\ \qquad\qquad\quad o \qquad\qquad \end{array}$$

Moreover, $-K(A_n)^2$ is $(3n+3)/(2n+5)$ in both 1) and 2).

Proof. From Lemma 4.6 there is only one (-3)-curve in B_1 if $t = 1$. From direct computation we know that B_1 can only have the above two possibilities from the fact $-K(B_1)^2 = 3/2$. Thus we can know A_n from its relation to B_1. Q.E.D.

Lemma 4.8. *Let X_i, B_i $(i = 1, .., t)$ be as above and $t = 2$. If $-K(X_i)^2 < 1$ for $i = 1, 2$, B_i, A_n have to be one of the following forms.*

1) B_i :$o - o - x - D_l$, where $i = 1, 2$ and $A_n : o - o - x - o - \cdots - o - x - o - o$, where there is a (-2)-curve string of length n between two x's;

2) $B_1 : o - x - D_l$; $B_2 : D_l - x - o - o - o - o$ and $A_n : o - x - o - \cdots - o - x - o - o - o - o$.

Moreover, $-K(A_n)^2$ is $3/2$ in case 1) and $(27n + 67)/(18n + 45)$ in case 2).

Proof. If $-K(B_i)^2 < 1$ for $i = 1, 2$, $-K(B_i)^2$ is an accumulation point of the set $\{-K^2\}$ (of sequence of weighted dual graphs with D_l in B_i replaced by string of (-2)-curves as showed in Theorem 3.1) as described in Proposition 2.8 in [1]. Hence, they have to be the form $m/(m+1)$ from the conclusion of Proposition 2.8 of [1]. If we want $m/(m+1)+m'/(m'+1) = 3/2$, (m, m') has to be either $(3, 3)$ or $(2, 5)$. Thus $-K(B_i)^2 = 3/4$ for $i = 1, 2$ or $-K(B_1) = 2/3, -K(B_2)^2 = 5/6$. By direct computation we know that B_1, B_2 have to be one of 1) and 2) from these data. The conclusion is proved.

Lemma 4.9. *Let X_i, B_i $(i = 1, 2)$ be as above. If $-K(B_1)^2 = 1$ and $-K(B_2)^2 = 1/2$, B_2 is always of the form $x - D_l$ and B_1, A_n have to be one of the following forms.*

1) $B_1 : D_l - x - D_l$ and $A_n : D_l - x - o - \cdots - o - x$;

2) $B_1 : D_l - x - o - \cdots - o$ and $A_n : x - o - \cdots - o - x - o - \cdots - o$ (i.e., A_n is of the form $T_{t,w}$ of proposition 4.1);

3) B_1 is of the form

$$\begin{array}{ccccccc} o_{l-1} & - & o_{l-2} & - \cdots - & o_2 & - & o_1 \\ | & & & & | & & \\ o_l & & & & x & & \end{array}$$

and A_n is of the following form

$$
\begin{array}{ccccccccccc}
x & - & o & - & \cdots & - & o & - & o & - & o \\
 & & & & & & & & | & & \\
 & & & & & & & & x & &
\end{array}
$$

4) B_1 *is of the following form*

$$
\begin{array}{ccccc}
D_l & - & x & - & o \\
 & & | & & \\
 & & o & &
\end{array}
$$

and A_n *is of the following form*

$$
\begin{array}{ccccccccccc}
x & - & o & - & \cdots & - & o & - & x & - & o \\
 & & & & & & & & | & & \\
 & & & & & & & & o & &
\end{array}
$$

Moreover, $-K(A_n)^2$ *is*
$(3n+7)/(2n+5)$ *in case 1);*
$(3tw+4w+7t+8)/(2tw+4w+5t+8)$ *in case 2);*
$(3n+12)/(2n+11)$ *in case 3);*
$(3n+7)/(2n+5)$ *in case 4).*

Proof. We can get the conclusion of B_1, B_2 from the fact $-K(B_1)^2 = 1$ and $-K(B_2)^2 = 1/2$ by direct computation. The corresponding $A_n's$ can be obtained according to its relation with B_1 and B_2 described in the proof of Theorem 3.1. Q.E.D.

Proof of Theorem 4.2. We can easily check that if $v_m = (3m+4)/(2m+4)$ is one of $-K(A_n)^2$ of Lemma 4.7, Lemma 4.8 or 1), 3), 4) of Lemma 4.9, m has to be divided by 4. This is impossible from our hypothesis on m. For 2) of Lemma 4.9, suppose $v_m = -K(T_{t,w})^2$ for some (t, w). We have $m = t - t^2/(4w+t+8)$ from a direct computation. Suppose $\gcd(t, 4w+8) = d, t = dw'$ and $4w+8 = dw''$. Then $t^2 = d^2w'^2$ can be divided by $d(w'+w'')$, hence d can be divided by $w'+w''$. Let $d = u(w'+w'')$. Then $m = dw' - uw'^2 = uw'w''$. From our condition that m is prime we know that two of (u, w', w'') have to be 1. In the case $u = w'' = 1$, $m = w' = 4w+7$ since $d = u(w'+w'') = w'+1 = 4w+8$. This contradicts to the fact that m is of the form $4k+1$. Similarly, we have contradictions in other cases. Thus, the conclusion of Theorem 4.1 follows. Q.E.D.

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