MODULI SPACES OF HIGHER SPIN CURVES AND INTEGRABLE HIERARCHIES

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ABSTRACT. We prove the genus zero part of the generalized Witten conjecture relating moduli spaces of spin curves to Gelfand-Dickey hierarchies. That is, we show that intersection numbers on the moduli space of stable r-spin curves assemble into a generating function which yields a solution of the semiclassical limit of the KdV_r equations. We formulate axioms for a cohomology class on this moduli space which allow one to construct a cohomological field theory of rank $r-1$ in all genera. In genus zero it produces a Frobenius manifold which is isomorphic to the Frobenius manifold structure on the base of the versal deformation of the singularity A_{r-1}. We prove analogs of the puncture, dilaton, and topological recursion relations by drawing an analogy with the construction of Gromov-Witten invariants and quantum cohomology.

0. INTRODUCTION

The moduli space of stable curves of genus g with n marked points $\overline{\mathcal{M}}_{g,n}$ is a fascinating object. Mumford [37] introduced tautological cohomology classes associated to the universal curve $\mathcal{C}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$, and Witten conjectured [42] and Kontsevich [28] proved that the intersection numbers of certain tautological cohomology classes on the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ have a generating function which satisfies the equations of the Korteweg-de Vries hierarchy (more precisely that it is a τ function of the KdV hierarchy satisfying some additional equations). This remarkable result provided an unexpected link between the algebraic geometry of these moduli spaces and integrable systems.

The spaces $\overline{\mathcal{M}}_{g,n}$ can be generalized in two ways. The first is by choosing a smooth projective variety V and considering the moduli space of stable maps into V of curves of genus g with n marked points, $\overline{\mathcal{M}}_{g,n}(V)$. When V is a point, $\overline{\mathcal{M}}_{g,n}(V)$ reduces to $\overline{\mathcal{M}}_{g,n}$. The second is

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by considering moduli space of r-spin curves $\overline{\mathcal{M}}^{1/r}_{g,n}$, introduced in [21, 22], which, upon forgetting the r-spin structure, reduces to $\overline{\mathcal{M}}_{g,n}$. It is natural to ask if Kontsevich's theorem admits a generalization to either of these cases.

The case of $\overline{\mathcal{M}}_{g,n}(V)$ remains mysterious. It gives the Gromov-Witten invariants of V (and their so-called gravitational descendants), which assemble into a generating function whose exponential is an analog of a τ function. In the case where V is a point, one recovers the τ function of the KdV hierarchy by Kontsevich's theorem. More generally, there is a conjecture of Eguchi-Hori-Xiong [12] and of S.Katz which essentially states that this generating function is a highest weight vector for a particular representation of the Virasoro algebra. Presumably, there is some analog of an integrable system which gives rise to this Virasoro algebra action, should the conjecture hold.

On the other hand, the KdV hierarchy is known to be merely the first in a series of integrable hierarchies known as the KdV_r (or r-th Gelfand-Dickey) hierarchies, where $r = 2, 3, 4, \ldots$. In the case $r = 2$ this becomes the usual KdV hierarchy. Each of these hierarchies has a formal solution corresponding to the unique τ function which satisfies an equation known as the string (or puncture) equation. In [40, 41] Witten formulated a generalization of his original conjecture, which states that for each $r \geq 2$ there should exist moduli spaces and cohomology classes on them whose intersection numbers assemble into the τ function of the KdV_r hierarchy.

In this paper, we are motivated by analogy with the construction of Gromov-Witten invariants from the moduli space of stable maps to introduce axioms which must be satisfied by a cohomology class $c^{1/r}$ (called the *virtual class*) on the moduli space of r-spin curves $\overline{\mathcal{M}}^{1/r}_{g,n}$ in order to obtain a cohomological field theory (CohFT) of rank $(r-1)$ in the sense of Kontsevich-Manin [29]. This virtual class on $\overline{\mathcal{M}}^{1/r}_{g,n}$ should be regarded as an analog of the Gromov-Witten classes of a variety V, *i.e.* the pullbacks via the evaluation maps of elements in $H^\bullet(V)$.

We realize this virtual class in genus zero as the top Chern class of a tautological bundle over $\overline{\mathcal{M}}^{1/r}_{0,n}$ associated to the r-spin structure. This yields a Frobenius manifold structure [10, 20, 33] on the state space of the CohFT which is isomorphic to the Frobenius manifold associated to the versal deformation of the A_{r-1} singularity [10].

This is an indication of the existence of a kind of "mirror symmetry" between the moduli space of r-spin curves and singularities. According to Manin [34] "isomorphisms of Frobenius manifolds of different classes remain the most direct expression of various mirror phenomena." Proving the generalized Witten conjecture for all genera would provide further evidence of this relationship.

As in the case of Gromov-Witten invariants, one can construct a potential function from the integrals of $c^{1/r}$ on different components of $\overline{\mathcal{M}}^{1/r}_{g,n}$ to form the *small phase space* of the theory. The *large phase space* is constructed by introducing the tautological classes ψ associated to the universal curve $\mathcal{C}^{1/r}_{g,n} \longrightarrow \overline{\mathcal{M}}^{1/r}_{g,n}$. It can be regarded as a parameter space for a family of CohFTs. A *very large phase space* (see [25, 26, 13, 35]) parametrizing an even larger space of CohFTs is obtained by considering tautological classes λ, associated to the Hodge bundles, and tautological classes μ, associated to the universal spin structure.

We show that the corresponding potential function satisfies analogs of the puncture and dilaton equations and also a new differential equation obtained from a universal relation involving the class μ_1. These relations hold in all genera. Topological recursion relations are also obtained from presentations of these classes in terms of boundary classes in low genera.

Finally, using the new relation on μ_1, we show that the genus zero part of the large phase space potential $\Phi_0(\mathbf{t})$ is completely determined by the geometry, and this potential agrees with the generalized Witten conjecture in genus zero.

Some of our constructions were foreshadowed by Witten, who formulated his conjecture even before the relevant moduli spaces and cohomology classes had been constructed, just as he had done in the case of the topological sigma model and quantum cohomology. We prove that his conjecture has a precise algebro-geometric foundation, just as in the case of Gromov-Witten theory. Witten also outlined a formal argument to justify his conjecture in genus zero. Our work shows that the formulas that he ultimately obtained for the large phase space potential function in genus zero are indeed correct, provided that the geometric objects involved are suitably interpreted. This is nontrivial even in genus zero because of the stacky nature of the underlying moduli spaces. We proceed further to prove relations between various tautological classes associated to the r-spin structures and derive differential equations for the potential function associated to them.

Notice also that one can introduce moduli spaces $\overline{\mathcal{M}}^{1/r}_{g,n}(V)$ of stable r-spin maps into a variety V, where one combines the data of both the stable maps and the r-spin structures. The analogous construction on these spaces yields a Frobenius manifold which combines Gromov-Witten invariants (and quantum cohomology) with the KdV_r hierarchies. Work in this direction is in progress [24].

In the first section of this paper, we review the moduli space of genus g, stable r-spin curves, $\overline{\mathcal{M}}^{1/r}_{g,n}$, which was introduced in [21, 22]. We also discuss the stratification of the boundary of $\overline{\mathcal{M}}^{1/r}_{g,n}$. The boundary strata fall into two distinct categories—the so-called Neveu-Schwarz and Ramond types. Only the former has the factorization properties analogous to properties of the moduli space of stable maps.

In the second section, we introduce canonical morphisms, tautological bundles, tautological cohomology classes, and cohomology classes associated to the boundary strata of $\overline{\mathcal{M}}^{1/r}_{g,n}$ and derive a new relation involving the μ_1 class. In the third section, we define a cohomological field theory (CohFT) in the sense of Kontsevich-Manin, its small phase space potential function, and the associativity (WDVV) equation. We then review the construction of Gromov-Witten invariants for the moduli space of stable maps and define the large and very large phase spaces in the Gromov-Witten theory. Motivated by this example, we explain how to construct a CohFT and the various potential functions from analogous intersection numbers on $\overline{\mathcal{M}}^{1/r}_{g,n}$, assuming that the virtual class $c^{1/r}$ exists.

In the fourth section, we state axioms which $c^{1/r}$ must satisfy in order to obtain a CohFT. We show that these axioms give a complete CohFT with a flat identity, and we construct the class $c^{1/r}$ in genus zero as well as in the case $r = 2$. In the fifth section, we obtain analogs of the string and dilaton equations (which essentially yield the L_{-1} and L_0 Virasoro generators) for this r-spin CohFT, and a find a new equation based on the relation involving the μ_1 class. We also prove the analog of topological recursion relations in genus zero. Finally in the sixth section, we determine the large phase space potential of this theory in genus zero and show that it yields a solution to (the semiclassical limit of) the KdV_r hierarchy, thereby proving the genus zero generalized Witten conjecture. We conclude with our own W-algebra conjecture, a KdV_r analog of a refinement of the Virasoro conjecture [12].

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1. The Moduli Space of r-spin Curves

1.1. Definitions.

Definition 1.1. Let $(X, p_1, \dots, p_n)$ be a nodal, n-pointed algebraic curve, and let $\mathcal{K}$ be a rank-one, torsion-free sheaf on X. A *d-th root of $\mathcal{K}$ of type* $\mathbf{m} = (m_1, \dots, m_n)$ is a pair $(\mathcal{E}, b)$ of a rank-one, torsion-free sheaf $\mathcal{E}$, and an $\mathcal{O}_X$-module homomorphism

$$b : \mathcal{E}^{\otimes d} \longrightarrow \mathcal{K} \otimes \mathcal{O}_X(-\sum m_i p_i)$$

with the following properties:

- $d \cdot \deg \mathcal{E} = \deg \mathcal{K} - \sum m_i$
- b is an isomorphism on the locus of X where $\mathcal{E}$ is locally free
- for every point $p \in X$ where $\mathcal{E}$ is not free, the length of the cokernel of b at p is $d - 1$.

For any d-th root $(\mathcal{E}, b)$ of type $\mathbf{m}$, and for any $\mathbf{m}'$ congruent to $\mathbf{m}$ mod d, we can construct a unique d-th root $(\mathcal{E}', b')$ of type $\mathbf{m}'$ simply by taking $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}(1/d \sum (m_i - m'_i) p_i)$. Consequently, the moduli of curves with d-th roots of a bundle $\mathcal{K}$ of type $\mathbf{m}$ is canonically isomorphic to the moduli of curves with d-th roots of type $\mathbf{m}'$. Therefore, unless otherwise stated, we will always assume the type $\mathbf{m}$ of a d-th root has the property that $0 \leq m_i < d$ for all i.

Unfortunately, the moduli space of curves with d-th roots of a fixed sheaf $\mathcal{K}$ is not smooth when d is not prime, and so we must consider not just roots of a bundle, but rather coherent nets of roots [21]. This additional structure suffices to make the moduli space of curves with a coherent net of roots smooth.

Definition 1.2. Let $\mathcal{K}$ be a rank-one, torsion-free sheaf on a nodal n-pointed curve $(X, p_1, \dots, p_n)$. A *coherent net of r-th roots of $\mathcal{K}$ of type* $\mathbf{m} = (m_1, \dots, m_n)$ consists of the following data:

- For every divisor d of r, a rank-one torsion-free sheaf $\mathcal{E}_d$ on X;
- For every pair of divisors d', d of r, such that d' divides d, an $\mathcal{O}_X$-module homomorphism

$$c_{d,d'} : \mathcal{E}_d^{\otimes d/d'} \longrightarrow \mathcal{E}_{d'}.$$

These data are subject to the following restrictions.

1. $\mathcal{E}_1 = \mathcal{K}$ and $c_{1,1} = \mathbf{1}$
2. For each divisor d of r and each divisor d' of d, let $\mathbf{m}'' = (m''_1, \dots, m''_n)$ be such that m''_i is the unique non-negative integer less than d/d', and congruent to $m_i \mod d$. Then the homomorphism $c_{d,d'}$ makes $(\mathcal{E}_d, c_{d,d'})$ into a d/d' root of $\mathcal{E}_{d'}$ of type $\mathbf{m}''$.

3. The homomorphisms $\{c_{d,d'}\}$ are compatible. That is, the diagram

$$\begin{tikzcd}
(\mathcal{E}_d^{\otimes d/d'})^{\otimes d'/d''} \arrow[r, "(c_{d,d'})^{\otimes d'/d''}"] \arrow[dr, "c_{d,d''}"'] & \mathcal{E}_{d'}^{\otimes d'/d''} \arrow[d, "c_{d',d''}"] \\
& \mathcal{E}_{d''}
\end{tikzcd}$$

commutes for every $d''|d'|d|r$.

If r is prime, then a coherent net of r-th roots is simply an r-th root of $\mathcal{K}$. Moreover, if $\mathcal{E}_d$ is locally free of type $\mathbf{m}$, then for $d'|d$ if $\mathbf{m} \equiv \mathbf{m} \pmod{d}'$ and $0 \leq m_i' < d'$, then the sheaf $\mathcal{E}_{d'}$ is isomorphic to $\mathcal{E}_d^{\otimes d/d'} \otimes \mathcal{O}(\frac{1}{d^r}\sum(m_i - m_i')p_i)$ and $c_{d,d'}$ is uniquely determined up to an automorphism of $\mathcal{E}_{d'}$.

Definition 1.3. An *n-pointed r-spin curve of type* $\mathbf{m} = (m_1, \ldots, m_n)$ is an n-pointed, nodal curve $(X, p_1, \ldots, p_n)$ with a coherent net of r-th roots of ω_X of type $\mathbf{m}$, where ω_X is the canonical (dualizing) sheaf of X. An r-spin curve is called *smooth*, if X is smooth, and it is called *stable*, if X is stable.

Example 1.4. Smooth 2-spin curves of type $\mathbf{m} = (0, 0, \ldots, 0)$ correspond to classical spin curves (a curve and a theta-characteristic) *together with an explicit isomorphism* $\mathcal{E}_2^{\otimes 2} \xrightarrow{\sim} \omega$.

Definition 1.5. An isomorphism of r-spin curves

$$(X, p_1, \ldots, p_n, \{\mathcal{E}_d, c_{d,d'}\}) \xrightarrow{\sim} (X', p_1', \ldots, p_n', \{\mathcal{E}_d', c_{d,d'}'\})$$

of the same type $\mathbf{m}$ is an isomorphism of pointed curves

$$\tau : (X, p_1, \ldots, p_n) \xrightarrow{\sim} (X', p_1', \ldots, p_n')$$

and a family of isomorphisms $\{\beta_d : \tau^*\mathcal{E}_d' \xrightarrow{\sim} \mathcal{E}_d\}$, with β_1 being the canonical isomorphism $\tau^*\omega_{X'}(-\sum_i m_i {p'}_i) \xrightarrow{\sim} \omega_X(-\sum m_i p_i)$, and such that the β_d are compatible with all the maps $c_{d,d'}$ and $\tau^* c_{d,d'}'$.

Every r-spin structure on a smooth curve X is determined up to isomorphism by a choice of a line bundle $\mathcal{E}_r$, such that $\mathcal{E}_r^{\otimes r} \cong \omega_X(-\sum m_i p_i)$. In particular, if X has no automorphisms, then the set of isomorphism classes of r-spin structures of a fixed type $\mathbf{m}$ is a principal $\mathrm{Jac}_r X$-bundle, where $\mathrm{Jac}_r X$ is the group of r-torsion points of the Jacobian of X, thus there are r^{2g} such isomorphism classes.

Example 1.6. If $g = 1$ and $\mathbf{m} = \mathbf{0} := (0, \ldots, 0)$, then $\omega_X \cong \mathcal{O}_X$ and a smooth r-spin curve is just an r-torsion point of the elliptic curve X, again with an explicit isomorphism of $\mathcal{E}_r^{\otimes r} \xrightarrow{\sim} \mathcal{O}_X$. In particular, the stack of stable r-spin curves of genus one and type $\mathbf{0}$ forms a gerbe over the disjoint union of modular curves $\coprod_{d|r} X_1(d)$.

Definition 1.7. The moduli space of stable n-pointed, r-spin curves of genus g and type $\mathbf{m}$ is denoted $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$. The disjoint union $\coprod_{\substack{\mathbf{m} \\ 0 \le m_i < r}} \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ is denoted $\overline{\mathcal{M}}_{g,n}^{1/r}$.

Remark 1.8. As mentioned above, no information is lost by restricting $\mathbf{m}$ to the range $0 \le m_i \le r-1$, since when $\mathbf{m} \equiv \mathbf{m}' \mod r$ every r-spin curve of type $\mathbf{m}$ naturally gives an r-spin curve of type $\mathbf{m}'$ simply by

$$\mathcal{E}_d \mapsto \mathcal{E}_d \otimes \mathcal{O}(\sum \frac{m_i - m'_i}{d} p_i),$$

and thus $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ is canonically isomorphic to $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}'}$.

1.2. Basic properties of the moduli space.

In [21] it is shown that $\overline{\mathcal{M}}_{g,n}^{1/r}$ is a smooth Deligne-Mumford stack, finite over $\overline{\mathcal{M}}_{g,n}$, with a projective coarse moduli space.

Throughout this paper we will denote the forgetful morphism by $p : \overline{\mathcal{M}}_{g,n}^{1/r} \longrightarrow \overline{\mathcal{M}}_{g,n}$, and the universal curve by $\pi : \mathcal{C}_{g,n}^{1/r} \longrightarrow \overline{\mathcal{M}}_{g,n}^{1/r}$. As in the case of the moduli space of stable curves, the universal curve possesses canonical sections $\sigma_i : \overline{\mathcal{M}}_{g,n}^{1/r} \longrightarrow \mathcal{C}_{g,n}^{1/r}$ for $i = 1 \ldots n$.

There is one other canonical morphism associated to these spaces; namely, when d divides r, the morphism

$$[r/d] : \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}} \longrightarrow \overline{\mathcal{M}}_{g,n}^{1/d,\mathbf{m}''} \text{ and } [r/d] : \overline{\mathcal{M}}_{g,n}^{1/r} \longrightarrow \overline{\mathcal{M}}_{g,n}^{1/d},$$

which forgets all of the roots and homomorphisms in the net of r-th roots except those associated to divisors of d. Here, as above, $\mathbf{m}''$ is congruent to $\mathbf{m} \mod d$ and has $0 \le m''_i < d$. In the case that the underlying curve is smooth, this is equivalent to replacing the line bundle $\mathcal{E}_r$ by its r/d-th tensor power (and tensoring with $\mathcal{O}(1/d \sum(m_i - m''_i)p_i)$).

As is shown in [21], for $g > 1$ the spaces $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ are irreducible if $\gcd(r, m_1, \ldots, m_n)$ is odd, and they are the disjoint union of two irreducible components if $\gcd(r, m_1, \ldots, m_n)$ is even. In the case that $r = 2$, this is due to the well-known fact that even and odd theta characteristics on a curve cannot be deformed into one another [36]. In general, the two components that arise in the case that $\gcd(r, m_1, \ldots, m_n)$

is even are just the preimages of the even and odd components of $\overline{\mathcal{M}}_{g,n}^{1/2,\mathbf{0}}$ under the map $[r/2] : \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}} \longrightarrow \overline{\mathcal{M}}_{g,n}^{1/2,\mathbf{0}}$. These two components will be denoted $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},\mathrm{even}}$ and $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},\mathrm{odd}}$ respectively.

When $g = 0$ the moduli space $\overline{\mathcal{M}}_{0,n}^{1/r,\mathbf{m}}$ is either empty (if r does not divide $\sum m_i + 2$) or is canonically isomorphic to $\overline{\mathcal{M}}_{0,n}$. Note, however, that this isomorphism is not an isomorphism of stacks, since the automorphisms of elements of $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ vary differently from the way that automorphisms of the underlying curves vary. In any case, $\overline{\mathcal{M}}_{0,n}^{1/r,\mathbf{m}}$ is always irreducible.

When $g = 1$ the space $\overline{\mathcal{M}}_{1,n}^{1/r,\mathbf{m}}$ is the disjoint union of d irreducible components, where d is the number of divisors of $\gcd(r, m_1, \dots , m_n)$.

1.3. **Boundary behavior.**

1.3.1. *Neveu-Schwarz and Ramond nodes.* At any node q of a nodal curve X, an r-th root $(\mathcal{E}_r, b_r)$ of $\omega_X(-\sum m_i p_i)$ is either locally free at q, with b_r an isomorphism at q, or it is torsion-free but not locally free at q. The locally free case is what Witten [40] calls the *Ramond sector*, and the non-locally free case he calls a generalized *Neveu-Schwarz sector*.

In the case that $\mathcal{E}_r$ is Ramond, all of the terms in the spin structure are completely determined (up to isomorphism) by the root $(\mathcal{E}_r, c_{r,1})$, simply by the rule $\mathcal{E}_d = \mathcal{E}_r^{\otimes r/d} \otimes \mathcal{O}(q/d \sum(m_i - m_i')p_i)$, and $c_{d,1} = c_{r,1}$ tensored with the identity homomorphism

$$\mathcal{O}(\sum(m_i - m_i'')p_i) \longrightarrow \mathcal{O}(\sum(m_i - m_i'')p - i).$$

The Neveu-Schwarz sector is more complicated, in some sense, but this is compensated for by the fact that for the Neveu-Schwarz sector, the cohomology classes defined by boundary strata factor in a nice combinatorial way. Moreover, the hope of constructing a cohomological field theory from $\overline{\mathcal{M}}_{g,n}^{1/r}$ is based on the belief that one can construct a canonical cohomology class which vanishes on the Ramond sector. (This would follow from Axiom 4 in Section 4.1.)

Recall from [40] or [21, 22] that in the Neveu-Schwarz case, an r-th root $(\mathcal{E}_r, b_r)$ of $\omega_X(-\sum m_i p_i)$ is uniquely determined by an r-th root $(\tilde{\mathcal{E}}_r, \tilde{b}_r)$ of $\omega_{\tilde{X}}(-\sum m_i p_i - m^+ q^+ - m^- q^-)$ on the normalization $\nu : \tilde{X} \longrightarrow X$ of X at the node q. Here q^+ and q^- are the two points of $\tilde{X}$ lying over the point q of X, and m^+ and m^- are non-negative integers* which sum to $r - 2$. The sheaf $\tilde{\mathcal{E}}_r$ is simply $\nu^*\mathcal{E}_r$ modulo torsion; and $\nu_*\tilde{\mathcal{E}}_r = \mathcal{E}_r$, with b_r induced from $\tilde{b}_r$ by adjointness. In the case that $m^+ +$

*Note that u and v of the papers [21] and [22] are $m^+ + 1$ and $m^- + 1$, respectively.

1 and m^-+1 are relatively prime, one can show (see [21]) that $\tilde{\mathcal{E}}$ and $\tilde{b}$ uniquely determine the entire net. But if $\gcd(m^++1, m^-+1) = d$, then although $\mathcal{E}_r$ still completely determines the Neveu-Schwarz roots, d divides r, and the d-th root $(\mathcal{E}_d, c_{d,1})$ of the net is locally free (Ramond), as are all roots $(\mathcal{E}_{d'}, c_{d',1})$ such that $d'|d$. In particular, $\mathcal{E}_{d'}$ is simply $\mathcal{E}_d^{\otimes d/d'}$, tensored by the appropriate bundle $\mathcal{O}(1/d' \sum m'_i - m''_i)p_i)$. If the normalization $\tilde{X}$ at q has two connected components, then the d-th root $(\mathcal{E}_d, c_{d,1})$ is determined up to isomorphism by $(\mathcal{E}_r, b_r)$, but if $\tilde{X}$ is connected, then $(\mathcal{E}_d, c_{d,1})$ is not determined by $(\mathcal{E}_r, b_r)$, since an additional gluing datum of a d-th root of unity is required to construct $\mathcal{E}_d$ from $\mathcal{E}_r^{\otimes r/d}$ (see Section 1.5).

1.3.2. *More detailed study of the Ramond sector (after Witten [40]).* Let $(\mathcal{E}, b)$ be an r-th root of $\omega_X(-\sum m_i p_i)$ which is Ramond at a node q of X. The restriction of $\mathcal{E}$ to q gives an exact sequence

$$0 \longrightarrow \mathfrak{m}_q \otimes \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}|_q \longrightarrow 0,$$

where $\mathfrak{m}$ is the maximal ideal of the point q. The sheaf $\mathcal{E}' := \mathcal{E} \otimes \mathfrak{m}_q$ is a rank-one, torsion-free sheaf of degree $(2g-2-\sum m_i)/r - 1$ on X and pulling $\mathcal{E}'$ back to the normalization $\nu : \tilde{X} \longrightarrow X$ of X at q gives, after modding out torsion, $\mathcal{E}'' := \nu^*\mathcal{E}'/\text{torsion}$ with $\nu_*\mathcal{E}'' = \mathcal{E}'$.

If x and y are local coordinates on $\tilde{X}$ near q^+ and q^- respectively, then $\mathcal{E}''$ is locally generated by $x(\frac{dx}{x})^{1/r}$ (respectively $y(\frac{dy}{y})^{1/r}$). Therefore, the homomorphism

$$b'' : \mathcal{E}''^{\otimes r} \longrightarrow \nu^*\omega_X(-\sum m_i p_i) = \omega_{\tilde{X}}(-\sum m_i p_i + q^+ + q^-),$$

induced by $b : \mathcal{E}^{\otimes r} \longrightarrow \omega_{\tilde{X}}(-\sum m_i p_i)$ factors through

$$\omega_{\tilde{X}}(-\sum m_i p_i - (r-1)q^+ - (r-1)q^-) \longrightarrow \omega_{\tilde{X}}(-\sum m_i p_i + q^+ + q^-).$$

Thus the Ramond case can occur if and only if $m^+ = r-1$ and $m^- = r-1$ satisfy the degree conditions:

$$\deg_{X^{(j)}} \omega_{\tilde{X}} - \sum m_i - m^+ - m^- \equiv 0 \mod r$$

on every connected component $X^{(j)}$ of $\tilde{X}$.

Similarly,

$$(\nu^*\mathcal{E})^{\otimes r} \xrightarrow{\nu^* b} \nu^*\omega_X(-\sum m_i p_i) = \omega_{\tilde{X}}(-\sum m_i p_i + q^+ + q^-),$$

corresponds to a choice of $m^+ = m^- = -1$. In this special case there is a residue map that canonically identifies $\mathcal{E}_r|_{q^+}$ and $\mathcal{E}_r|_{q^-}$ with $\mathbb{C}$.

Proposition 1.9. *If $(\mathcal{E}_r, b)$ is an r-th root of $\omega_X(-\sum m_i p_i)$ with $m_i = -1$ for some i, then there is a canonical isomorphism*

$$R_{p_i} : \mathcal{E}_r|_{p_i} \xrightarrow{\sim} \mathbb{C}. \tag{1}$$

An immediate consequence is the following corollary:

Corollary 1.10. *If σ_i is the i-th section of the universal curve*

$$\pi : \mathcal{C}^{1/r,\mathbf{m}}_{g,n} \longrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$$

with $\mathbf{m} = (m_1, \ldots, m_n)$ and $m_i = -1$ for some i, then the pullback $\sigma_i^*(\mathcal{E}_r)$ of the universal r-th root $\mathcal{E}_r$ is the trivial line bundle $\mathcal{O}_{\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}}$.

Proof. (of the proposition) Let z be a local parameter on X near p, so that the sheaf $\mathcal{E}_r$ is locally generated by an element $(\frac{dz}{z})^{1/r}$. Define the map $R_p : \mathcal{E}_r|_p \xrightarrow{\sim} \mathbb{C}$ by $R_p((a_0 + a_1 z + \ldots)(\frac{dz}{z})^{1/r}) := a_0$. It is easy to check that this definition is independent of local parameter, and hence defines a canonical isomorphism. □

1.4. **Dual graphs.** Much of the information about the structure of the boundary can be encoded in terms of decorated dual graphs.

Recall that the *dual graph* of an n-pointed nodal curve X consists of the following elements:

- *Vertices* corresponding to the irreducible components of X; a vertex v is labeled with $g(v)$, the (geometric) genus of the component;
- *Edges* corresponding to the nodes of the curve: an edge connects two vertices if and only if the corresponding node lies on the associated irreducible components;
- *Tails*, corresponding to marked points on a component, protrude from the corresponding vertex, and an *ordering* of the marked points corresponds to a labeling of the tails by the integers $1, \ldots, n$.

Definition 1.11. Let Γ be a graph with n tails and $V(\Gamma)$ be its set of vertices. The *genus* of the graph Γ is defined as

$$g(\Gamma) = \sum_{v \in V(\Gamma)} g(v) + \dim H^1(\Gamma).$$

A graph Γ is called *stable* if $2g(v) - 2 + n(v) > 0$ for every $v \in V(\Gamma)$ (in particular, it satisfies $2g(\Gamma) - 2 + n > 0$). Here $n(v)$ is the total number of *half-edges* at the vertex v (i.e., the ends of true edges as well as tails of Γ at v).

In the case of spin curves we also decorate the graph with additional data coming from the spin structure. In particular the type $\mathbf{m} = (m_1, \ldots, m_n)$ gives a marking to each of the tails.

Definition 1.12. Fix a positive integer r. A *decorated stable graph* is a stable graph together with a marking of each half-edge (i.e. tail or half of a "real" edge) with a non-negative integer $m < r$, such that for each edge e the marks m^+ and m^- of the two half-edges of e satisfy

$$m^+ + m^- \equiv r - 2 \mod r.$$

As discussed in Section 1.3, decorated stable graphs with n tails correspond to boundary strata in $\overline{\mathcal{M}}_{g,n}^{1/r}$.

Definition 1.13. Let Γ be a connected stable graph (resp. a decorated stable graph) with n tails and of genus g . We denote by $\overline{\mathcal{M}}_\Gamma$ (resp. by $\overline{\mathcal{M}}_\Gamma^{1/r}$) the closure in $\overline{\mathcal{M}}_{g,n}$ (resp. in $\overline{\mathcal{M}}_{g,n}^{1/r}$) of the moduli space of stable curves (resp. r-spin curves) with the dual graph Γ.

If Γ is the disjoint union of connected subgraphs $\Gamma^{(i)}$ for some index set $\{ i \}$ then $\overline{\mathcal{M}}_\Gamma := \prod_i \overline{\mathcal{M}}_{\Gamma^{(i)}}$ and similarly for $\overline{\mathcal{M}}_\Gamma^{1/r}$.

1.5. **Gluing.**

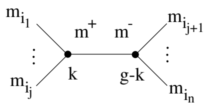
In the case of moduli spaces of ordinary stable curves, if a graph Γ is obtained from another graph $\tilde{\Gamma}$ (not necessarily connected) by gluing together two tails of $\tilde{\Gamma}$ (thus producing a new edge), there is a natural *gluing map*

$$\rho : \overline{\mathcal{M}}_{\tilde{\Gamma}} \longrightarrow \overline{\mathcal{M}}_\Gamma \hookrightarrow \overline{\mathcal{M}}_{g,n}. \tag{2}$$

It glues together the corresponding punctures on a curve $X \in \overline{\mathcal{M}}_{\tilde{\Gamma}}$ and produces an additional node on $\rho(X) \in \overline{\mathcal{M}}_\Gamma$.

A similar gluing operation sometimes exists for r-spin curves, but in this case we sometimes need to add some extra data.

1.5.1. *Two irreducible components.* Consider the case of a spin curve X with a single Neveu-Schwarz node whose normalization has two connected components: X^+ of genus k, and X^- of genus $g - k$. The dual graph of X looks like this

$\Gamma =$ 

.

The r-th root bundle $\mathcal{E}_r$ factors as $\mathcal{E}_r = \mathcal{E}_r^+ \oplus \mathcal{E}_r^-$ with $\mathcal{E}_r^+$ an r-th root of $\omega_{X^+}(-\sum_{l=1}^{j} m_{i_l} p_{i_l} - m^+ q^+)$ on X^+ and $\mathcal{E}^-$ an r-th root of $\omega_{X^-}(-\sum_{l=j+1}^{n} m_{i_l} p_{i_l} - m^- q^-)$ on X^-. Here m^+ and m^- can easily be calculated (for degree reasons) as the unique non-negative integers summing to $r-2$ which make their respective canonical bundles divisible by r; namely,

$$m^+ \equiv 2k-2-\sum_{l=1}^{j} m_{i_l} \mod r \qquad \text{and} \tag{3}$$

$$m^- \equiv 2(g-k)-2-\sum_{l=j+1}^{n} m_{i_l} \mod r. \tag{4}$$

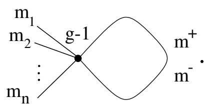
Of course, since the degree of the original bundle was divisible by r, each of the two relations implies the other. The Neveu-Schwarz case occurs exactly when the congruences (3) and (4) can be solved for m^+ and m^- in $\{0,1,\dots,r-2\}$ (see [22]).

If $\gcd(m^+ + 1, m^- + 1) = 1$ then these data completely determine the spin structure on X and in a canonical way, so that in this case we have a well-defined gluing morphism

$$\rho : \overline{\mathcal{M}}^{1/r}_{\Gamma_1 \sqcup \Gamma_2} = \overline{\mathcal{M}}^{1/r}_{\Gamma_1} \times \overline{\mathcal{M}}^{1/r}_{\Gamma_2} \longrightarrow \overline{\mathcal{M}}^{1/r}_{\Gamma} \hookrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}. \tag{5}$$

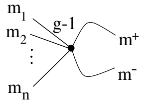
If, however, $\gcd(m^+ + 1, m^- + 1) = d > 1$, there is no longer a morphism of stacks as in (5). To construct a gluing morphism would require an isomorphism $\phi : \mathcal{E}_d^+|_{q^+} \xrightarrow{\sim} \mathcal{E}_d^-|_{q^-}$ that makes $\mathcal{E}_d$ Ramond at the node. Since the graph Γ is a tree, the root $(\mathcal{E}_d, c_{d,1})$ is determined *up to (non-canonical) isomorphism* by $(\mathcal{E}_r, c_{r,1})$. Still, choosing one gluing datum $\phi : \mathcal{E}_d^+|_{q^+} \xrightarrow{\sim} \mathcal{E}_d^-|_{q^-}$ does not give a morphism of stacks because an automorphism of the spin structure on X^+ or on X^- changes the gluing datum ϕ and thus induces a different (but isomorphic) spin structure on the curve X

1.5.2. *Irreducible curve with one node.* In the case of an r-spin curve X with a single component and one node we have the dual graph

$\Gamma_{irr} =$ 

.

This determines a spin structure on the normalized curve $\tilde{X}$, with the dual graph

$\tilde{\Gamma}_{irr} =$ 

.

If $d = \gcd(m^+ + 1, m^- + 1) = \gcd(m^+ + 1, r) = 1$, then all roots $\mathcal{E}_d$ in the spin structure are of the Neveu-Schwarz type and we can define the gluing map

$$\rho : \overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}_{irr}} \longrightarrow \overline{\mathcal{M}}^{1/r}_{\Gamma_{irr}} \hookrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n} \tag{6}$$

in the obvious way.

But if $d = \gcd(m^+ + 1, r) > 1$, then, as mentioned above, an additional gluing datum is required to construct $(\mathcal{E}_d, c_{d,1})$ from $(\mathcal{E}_r^{\otimes r/d}, c_{r,1})$. In this case, set $u^+ = \frac{m^+ + 1}{d}$ and $u^- = \frac{m^- + 1}{d}$ and define

$$\mathcal{E}'_d := \mathcal{E}_r^{\otimes r/d} \otimes \mathcal{O}(u^+ p^+ + u^- p^-)$$

where p^+ and p^- are the inverse images under normalization of the node. This shows that $\mathcal{E}'_d$ is a d-th root of $\omega_{\tilde{X}}(-\sum m_i p_i + p^+ + p^-)$, that is, m^+ and m^- are both replaced with -1. To construct $\mathcal{E}_d$ from $\mathcal{E}'_d$ we need to choose an isomorphism

$$\phi : \mathcal{E}'_d|_{p^+} \longrightarrow \mathcal{E}'_d|_{p^-}$$

compatible with the canonical isomorphism

$$\mathcal{E}'^{\otimes d}_d|_{p^+} \xrightarrow{\sim} \omega_{\tilde{X}}(p^+ + p^-) \xrightarrow{\sim} \mathcal{E}'^{\otimes d}_d|_{p^-},$$

and there are exactly d such isomorphisms.

Unlike in the case of the tree, now an automorphism of the spin structure on the normalized curve induces the same automorphism on both sides of the gluing datum ϕ, and thus it preserves ϕ. Consequently, we expect $d = \gcd(m^+ + 1, r)$ gluing morphisms

$$\rho_\phi : \overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}_{irr}} \longrightarrow \overline{\mathcal{M}}^{1/r}_{\Gamma_{irr}} \hookrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}.$$

indexed by the set of different choices of ϕ.

But in order to construct such a morphism, one needs to be able to define the glue ϕ in families, that is, we need to define isomorphisms

$$\phi : \mathcal{E}'_d|_{D^+} = \sigma^{+*}(\mathcal{E}'_d) \xrightarrow{\sim} \sigma^{-*}(\mathcal{E}'_d) = \mathcal{E}'_d|_{D^-}$$

where D^+ and D^- are the zero loci of the two sections σ^+ and σ^- to be glued. Such an isomorphism exists precisely because $\mathcal{E}'_d$ corresponds to the graph $\tilde{\Gamma}_{irr}$ with $m^+ = m^- = -1$. By Corollary 1.10, both $\mathcal{E}'_d|_{D^+}$ and $\mathcal{E}'_d|_{D^-}$ are trivial bundles on $\overline{\mathcal{M}}^{1/r,(\mathbf{m},-1,-1)}_{g-1,n+2}$, and hence are isomorphic. Thus if $\mathfrak{g}$ is the set of isomorphisms $\mathcal{E}'_d|_{D^+} \xrightarrow{\sim} \mathcal{E}'_d|_{D^-}$ of the universal sheaf $\mathcal{E}'_d = \mathcal{E}_r^{\otimes r/d} \otimes \mathcal{O}(u^+ D^+ + u^- D^-)$, we have a morphism

$$\rho_{\Gamma_{irr}} : \overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}_{irr}} \times \mathfrak{g} \longrightarrow \overline{\mathcal{M}}^{1/r}_{\Gamma_{irr}} \hookrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}.$$

2. Tautological cohomology classes

2.1. **Definitions.**

There are many natural cohomology classes in $H^\bullet(\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}, \mathbb{Q})$; these include the classes induced by pullback from $\overline{\mathcal{M}}_{g,n}$, as well as classes induced by replacing the canonical (relative dualizing) sheaf ω_π with $\mathcal{E}_r$ in the usual constructions of tautological cohomology classes on $\overline{\mathcal{M}}_{g,n}$.

In particular, we have the Chern classes λ_i of the Hodge bundle $\pi_*\omega$. However, $\pi_*\mathcal{E}_r$ is not especially well-behaved. Instead, we prefer to use $\pi_!\mathcal{E}_r$ (also called $R\pi_*\mathcal{E}$). Recall that for any coherent sheaf $\mathcal{F}$ on the universal curve $\pi : \mathcal{C}^{1/r,\mathbf{m}}_{g,n} \longrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ the element $\pi_!\mathcal{F}$ of $K_0(\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n})$ is the difference $\pi_*\mathcal{F} - R^1\pi_*\mathcal{F}$ (π has relative dimension 1). Here $R^1\pi_*\mathcal{F}$ is the sheaf whose fibre over a point p of the base $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ is the vector space $H^1(\pi^{-1}(p), \mathcal{F}|_{\pi^{-1}(p)})$. Serre duality shows that $H^1(X,\omega)$ is canonically isomorphic to $\mathbb{C}$, and so $R^1\pi_*\omega$ is a trivial line bundle. Hence $\pi_*\omega = \pi_!\omega + \mathcal{O}$ in $K_0(\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n})$. Therefore

$$c_t\pi_*\omega_\pi = c_t\pi_!\omega_\pi = 1 + \lambda_1 t + \lambda_2 t^2 + \dots$$

Tautological classes ν_i defined as components of the Chern character of the Hodge bundle

$$ch_t\pi_*\omega = 1 + ch_t\pi_!\omega = g + \nu_1 t + \nu_2 t^3 + \nu_3 t^5 + \dots \tag{7}$$

(The even components of $ch_t\pi_*\omega$ vanish by Mumford's theorem [37].) Similarly, we define classes μ_i as components of the Chern character of $\pi_!\mathcal{E}_r$:

$$ch_t\pi_!\mathcal{E}_r = D + \mu_1 t + \mu_2 t^2 + \dots. \tag{8}$$

Here $-D$ is the Euler characteristic $\chi(\mathcal{E}_r|_{\mathcal{C}_{\overline{s}}})$ of $\mathcal{E}_r$ on any geometric fibre $\mathcal{C}_{\overline{s}}$ of π, and by Riemann-Roch

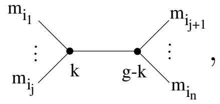
$$D = \frac{1}{r}\left((r-2)(g-1) + \sum_i m_i\right). \tag{9}$$

Serre duality shows that for any $\mathcal{F}$ we have $\pi_!(\mathcal{H}om(\mathcal{F},\omega_\pi)) = \pi_!\mathcal{F}$, so $\pi_!\mathcal{O} = \pi_!\omega$. More importantly, for purposes of comparison with Witten's calculations of [40], in the special case that $R^1\pi_*\mathcal{H}om(\mathcal{E}_r,\omega) = 0$ (or equivalently, $\pi_*\mathcal{E}_r = 0$), the bundle $\mathcal{V} := \pi_*\mathcal{H}om(\mathcal{E}_r,\omega)$ of [40] corresponds to $\pi_!\mathcal{E}_r$.

In addition to the Hodge-like classes, there are those induced by the canonical sections σ_i of $\pi : \mathcal{C}^{1/r,\mathbf{m}}_{g,n} \longrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$. These are classes

$$\psi_i := c_1\sigma_i^*(\omega_\pi) \quad \text{and} \quad \tilde{\psi}_i := c_1\sigma_i^*(\mathcal{E}_r)$$

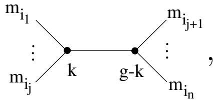
(and also class $\tilde{\psi}_i^{(d)}$ for each divisor d of r). When working in $\operatorname{Pic} \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$, we will abuse notation and use ψ_i to indicate the line bundle $\sigma_i^*(\omega_\pi)$, and $\tilde{\psi}_i$ the line bundle $\sigma_i^*(\mathcal{E}_r)$.

Finally, there are the boundary classes. In particular, if $A \sqcup B$ is a partition of $\{1, \ldots, n\}$ into two subsets, we denote by $\alpha_{k;A}$ the class of the divisor associated to spin curves with the dual graph of the form 

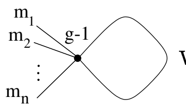
, with $\{i_1, \ldots, i_j\} = A$ and $\{i_{j+1}, \ldots, i_n\} = B$. Of course,

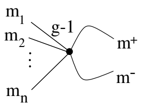
$$\alpha_{k;A} = \alpha_{g-k;B}$$

.

If g is greater than 1 and r and all of the m_i are even, then the moduli space has two components, $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},\text{even}}$ and $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},\text{odd}}$ If $2 \leq k \leq g-2$, then $\alpha_{k;A}$ is the sum of four divisors — two on each irreducible component of the moduli space. In this case, $\alpha_{k;A}^{(2,2)}$ denotes the class in $\operatorname{Pic} \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},\text{even}}$ of the divisor 

, where both components are endowed with an even spin structure; and $\alpha_{k;A}^{(1,1)}$ corresponds to both components carrying an odd spin structure. Similarly in $\operatorname{Pic} \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},\text{odd}}$, we use $\alpha_{k;A}^{(1,2)}$ to denote the class where the component of genus k carries an odd spin structure, and the component of genus $(g-k)$ carries an even one. The class $\alpha_{k;A}^{(2,1)}$ corresponds to the case where the component of genus k is even and the component of genus $(g-k)$ is odd. Thus $\alpha_{k;A}^{(1,2)} = \alpha_{g-k;B}^{(2,1)}$.

Finally, denote by $\tilde{\delta}_{irr,m^+}$ the divisor associated to 

with the spin structure inducing 

on the normalization. and the divisor corresponding to the Ramond root will be denoted by $\tilde{\delta}_{irr,r-1}$.†
$\tilde{\delta}_{irr,m^+}$ is not necessarily irreducible, since different gluings may induce distinct (even disjoint) divisors, all in $\tilde{\delta}_{irr,m^+}$. And even or odd spin structures on the normalization will also give disjoint divisors. The classes $\tilde{\delta}_{irr,m^+}^{even}$ and $\tilde{\delta}_{irr,m^+}^{odd}$ are defined in the obvious way, provided they

†Beware that the integer m of $\tilde{\delta}_{irr,m}$ here differs by one from that used with γ in [23].

make sense. Of course,

$$\tilde{\delta}_{irr,m} = \tilde{\delta}_{irr,r-2-m}.$$

2.2. **Basic properties of the tautological classes.**

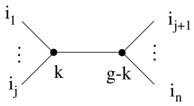
The following Proposition describes relations between various elements in the Picard group of $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$. It is a straightforward generalization of the corresponding result for the case $\mathbf{m} = \mathbf{0}$ proved in [23].

Proposition 2.1.

- *The forgetful map* $p : \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$ *induces an injection*

$$p^* : Pic\,\overline{\mathcal{M}}_{g,n} \otimes \mathbb{Q} \longrightarrow Pic\,\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n} \otimes \mathbb{Q}$$

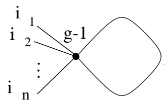
- *Let* $\delta_{k;A}$ *denote the pullback to* $Pic\,\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ *of the class in* $Pic\,\overline{\mathcal{M}}_{g,n}$ *associated to the union of all strata in* $\overline{\mathcal{M}}_{g,n}$ *with the dual graph*

$\Gamma =$ 

, with $A = \{i_1, \ldots, i_j\}$*, then*

$$\delta_{k;A} = \frac{r}{\gcd(m^+ + 1, r)} \alpha_{k;A}$$

where m^+ *is determined by* $k, \mathbf{m}$*, and* A*, as in Section 1.5.1.*

- *Let* δ_{irr} *be the pullback of the divisor of all curves in* $\overline{\mathcal{M}}_{g,n}$ *with the dual graph* $\Gamma_{\text{irr}} =$ 

, then

$$\delta_{irr} = \sum_{r-2/2 \leq m < r} \frac{r}{\gcd(r, m+1)} \tilde{\delta}_{irr,m}.$$

The fact that, for an r-th root $(\mathcal{E}, b)$ of $\omega(-\sum m_i p_i)$, the map b is almost an isomorphism means that $\tilde{\psi}_i$ and ψ_i are closely related. Indeed, the map $b : \mathcal{E}^{\otimes r} \longrightarrow \omega(-\sum m_j p_j)$ pulls back, via σ_i^* to give

$$\sigma_i^* b : (\sigma_i^* \mathcal{E})^{\otimes r} \longrightarrow \sigma_i^* \omega(-\sum m_j p_j).$$

Since $\mathrm{im}(\sigma_i)$ is disjoint from the nodes of X, and $\sigma_i^* b$ is an isomorphism even on the boundary strata where $\mathcal{E}$ fails to be locally free, we have the following result.

Proposition 2.2. *The line bundles* $\sigma_i^* \omega$ *and* $\sigma_i^*(\mathcal{E}_d)$ *on the stack* $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ *are related by*

$$r\sigma_i^*(\mathcal{E}_r) = (m_i + 1)\sigma_i^*(\omega)$$

and

$$d\sigma_i^*(\mathcal{E}_d) = (m_i'' + 1)\sigma_i^*(\omega),$$

where m_i'' *is the smallest non-negative integer congruent to* $m_i \mod d$. *Therefore, in* $Pic\,\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}} \otimes \mathbb{Q}$ *we have*

$$\tilde{\psi}_i = \frac{m_i + 1}{r}\psi_i$$

and

$$\tilde{\psi}_i^{(d)} = \left(\frac{m_i'' + 1}{m_i + 1}\right)\frac{r}{d}\tilde{\psi}_i.$$

The proof follows immediately from the fact that $\sigma_i^* c_{r,1}$ is an isomorphism and from the following well-known fact.

Lemma 2.3. *Let* $\pi : \mathcal{C}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$ *be the universal* n*-pointed curve, and* σ_i *the* i*-th tautological section of* π*. If* D_i *is the divisor of* $\mathcal{C}_{g,n}$ *associated to* σ_i*, and if* $\omega = \omega_\pi$ *is the canonical (relative dualizing) sheaf, then*

$$\sigma_i^*(\mathcal{O}(-D_j)) \cong \begin{cases} \sigma_i^*(\omega|_{D_i}) = \psi_i & \text{if } i = j \\ \mathcal{O} & \text{if } i \neq j \end{cases}$$

Proof. When $i \neq j$ the bundle $\sigma_i^*(\mathcal{O}(-D_j))$ is trivial because the sections are disjoint. In the case $i = j$ the result follows from the fact that taking residues gives an isomorphism between $\omega_\pi(D_i)|_{D_i}$ and $\mathcal{O}_{D_i}$. □

2.3. Non-trivial relations involving the class μ_1.

Proposition 2.4. *Define the boundary divisors* $<\tilde{\mathcal{E}}_r, \mathfrak{E}>$ *and* δ *by*

$$\begin{aligned} <\tilde{\mathcal{E}}_r, \mathfrak{E}> = \quad & \sum_{k,A} \frac{(m^+ + 1)(m^- + 1)}{u_{k;A}}\alpha_{k;A} \\ & + \sum_{\frac{r}{2}-1 \leq m < r-1} \frac{(m+1)(r-m-1)}{v_m}\tilde{\delta}_{irr,m} \end{aligned}$$

and

$$\delta = \sum \delta_{k;A} = \sum_{\frac{r}{2}-1 \leq m < r} \frac{r}{v_m}\tilde{\delta}_{irr,m} + \sum_{k,A} \frac{r}{u_{k;A}}\alpha_{k;A},$$

where $v_m = \gcd(m+1, r)$, $u_{k;A} = \gcd(m^+ + 1, r)$, *and* m^+ *is determined by* k, A, *and* $\mathbf{m}$ *via relation (3).*

Then the following relation holds in $Pic\,\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$:

$$r <\tilde{\mathcal{E}}_r, \mathfrak{E}> = (2r^2 - 12r + 12)\lambda_1 - 2r^2\mu_1 + (r-1)\delta + \sum_{1 \leq i \leq n} m_i(r - 2 - m_i)\psi_i.$$

The proof of the proposition is almost identical to its counterpart in [23, Theorem 4.3.3] except that $\mathcal{E}_r$ is not an r-th root of ω, but rather of $\omega(-\sum m_i p_i)$. The only extra information necessary to prove the proposition is the content of the following two lemmas.

Lemma 2.5. *Let $\pi : \mathcal{C}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$ be the universal curve and*

$$<,>: Pic\,\mathcal{C}_{g,n} \times Pic\,\mathcal{C}_{g,n} \longrightarrow Pic\,\overline{\mathcal{M}}_{g,n},$$

be Deligne's bilinear product defined by

$$< \mathcal{L}, \mathcal{M} > := \det(\pi_!(\mathcal{L} \otimes \mathcal{M}) - \pi_!\mathcal{L} - \pi_!\mathcal{M} + \pi_!\mathcal{O}).$$

If D is the image of a section $\sigma : \overline{\mathcal{M}}_{g,n} \longrightarrow \mathcal{C}_{g,n}$, then for any line bundle $\mathcal{L}$ on $\mathcal{C}_{g,n}$ the product $< \mathcal{L}, \mathcal{O}_{\mathcal{C}}(D) >$ is equal to the restriction of $\mathcal{L}$ to D; that is, it is just $\sigma^*\mathcal{L}$.

This is proven in [8, Prop. 6.1.3], but it also follows from the fact that when the base is a smooth curve B, then $\deg < \mathcal{L}, \mathcal{M} >$ is exactly the usual intersection number $(\mathcal{L}.\mathcal{M})$. Since line bundles on $\overline{\mathcal{M}}_{g,n}$ are completely determined by their degree on smooth curves in $\overline{\mathcal{M}}_{g,n}$ [4], and since in the case of a smooth, one-parameter base the degrees agree, then the lemma is true in general.

Lemma 2.6. *We have*

$$< \mathcal{O}(D_i), \mathcal{O}(D_j) > = \begin{cases} -\sigma_i^*\omega_\pi = -\psi_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

This follows immediately from Lemmas 2.3 and 2.5.

Corollary 2.7. *If $g = 0$ then classes λ_1 and $\tilde{\delta}_{irr,m}$ vanish (for all m), and*

$$\begin{aligned} 2r^2\mu_1 &= (r-1)\delta + \sum_i m_i(r-2-m_i)\psi_i - r(< \tilde{\mathcal{E}}, \mathfrak{E} >) \\ &= \sum_{A \in [n]} \frac{r}{u_{0;A}}[r - 1 - (m^+ + 1)(m^- + 1)]\alpha_{0;A} \\ &\quad + \sum_{1 \leq i \leq n} m_i(r-2-m_i)\psi_i. \end{aligned}$$

3. Cohomological field theory

In this section we begin with a review of the notion of cohomological field theory (CohFT) in the sense of Kontsevich and Manin [29]. This is an object which formalizes the factorization properties of the theory of topological gravity coupled to topological matter. The Gromov-Witten invariants associated to a smooth, projective variety V correspond to the physical situation where the matter sector arises from the topological sigma model [29, 38]. The analogous intersection numbers associated to the moduli space of r-spin curves have their physical origins in a different choice of the matter sector. Our goal in this section is to give a precise formulation of these notions in terms of the moduli spaces described above.

3.1. **Axioms of CohFT.**

Definition 3.1. A *(complete) cohomological field theory (CohFT) of rank* d (denoted by $(\mathcal{H}, \eta, \Lambda)$ or just $(\mathcal{H}, \eta)$) is a d-dimensional vector space $\mathcal{H}$ with a metric η and a collection $\Lambda := \{\Lambda_{g,n}\}$ of n-linear $H^\bullet(\overline{\mathcal{M}}_{g,n})$-valued forms on $\mathcal{H}$

$$\Lambda_{g,n} \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes \mathcal{H}^{*\otimes n} = Hom(\mathcal{H}^{\otimes n}, H^\bullet(\overline{\mathcal{M}}_{g,n})) \tag{10}$$

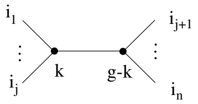
defined for stable pairs (g, n) and satisfying the following axioms **C1**–**C3** (where $\{e_0, \dots, e_{d-1}\}$ is a fixed basis of $\mathcal{H}$, $\eta^{\mu\nu}$ is the inverse of the matrix of the metric η in this basis, and the summation convention has been used).

C1. The element $\Lambda_{g,n}$ is invariant under the action of the symmetric group S_n.

C2. Let

$$\rho_{\text{tree}} : \overline{\mathcal{M}}_{\Gamma_1 \sqcup \Gamma_2} = \overline{\mathcal{M}}_{\Gamma_1} \times \overline{\mathcal{M}}_{\Gamma_2} \longrightarrow \overline{\mathcal{M}}_{\Gamma_{\text{tree}}} \hookrightarrow \overline{\mathcal{M}}_{g,n} \tag{11}$$

be the gluing map (2) corresponding to the stable graph

$\Gamma_{\text{tree}} =$ 

and the two graphs Γ_1 and Γ_2 obtained by cutting the edge of Γ_{tree}. Then the forms $\Lambda_{g,n}$ satisfy the composition property

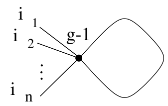
$$\begin{aligned} \rho^*_{\text{tree}}\Lambda_{g,n}(\gamma_1, \gamma_2 \dots, \gamma_n) &= \\ \Lambda_{k,j}(\gamma_{i_1}, \dots, \gamma_{i_j}, e_\mu)\eta^{\mu\nu} &\otimes \Lambda_{g-k,n-j}(e_\nu\, \gamma_{i_{j+1}}, \dots, \gamma_{i_n}) \end{aligned} \tag{12}$$

for all $\gamma_i \in \mathcal{H}$.

C3. Let

$$\rho_{\text{loop}} : \overline{\mathcal{M}}_{\tilde{\Gamma}} = \overline{\mathcal{M}}_{g-1,n+2} \longrightarrow \overline{\mathcal{M}}_{\Gamma_{\text{loop}}} \hookrightarrow \overline{\mathcal{M}}_{g,n} \tag{13}$$

be the gluing map (2) corresponding to the stable graph

(14) $\Gamma_{\text{loop}} =$ 

and the graph $\tilde{\Gamma}$ obtained by cutting the loop of Γ_{loop}. Then

$$\rho_{\mathrm{loop}}^{*} \Lambda_{g,n}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \Lambda_{g-1,n+2}(\gamma_1, \gamma_2, \ldots, \gamma_n, e_\mu, e_\nu)\, \eta^{\mu\nu}. \tag{15}$$

The pair $(\mathcal{H}, \eta)$ is called the *state space* of the CohFT.

An element $e_0 \in \mathcal{H}$ is called *a flat identity* of the CohFT, if in addition, the following equations hold.

C4a. For all γ_i in $\mathcal{H}$ we have

$$\Lambda_{g,n+1}(\gamma_1, \ldots, \gamma_n, e_0) = \pi^* \Lambda_{g,n}(\gamma_1, \ldots, \gamma_n), \tag{16}$$

where $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal curve on $\overline{\mathcal{M}}_{g,n}$ and

C4b.

$$\int_{\overline{\mathcal{M}}_{0,3}} \Lambda_{0,3}(\gamma_1, \gamma_2, e_0) = \eta(\gamma_1, \gamma_2) \tag{17}$$

A CohFT with flat identity is denoted by $(\mathcal{H}, \eta, \Lambda, e_0)$.

A *genus $\tilde{g}$ CohFT* on the state space $(\mathcal{H}, \eta)$ is the collection of forms $\{ \Lambda_{g,n} \}_{g \le \tilde{g}}$ that satisfy only those of the equations (12), (15), (16), and (17), where $g \le \tilde{g}$.

Remarks 3.2.

1. In general, the state space $\mathcal{H}$ of CohFT is $\mathbb{Z}_2$-graded, but here for simplicity, we are assuming that $\mathcal{H}$ contains only even elements, since this is the only case that will arise in this paper.

2. The definition of a CohFT given above has an equivalent dual description in terms of homology. Consider the maps $H_\bullet(\overline{\mathcal{M}}_{g,n}) \to T^n\mathcal{H}^*$ given by $[c] \mapsto \int_{[c]} \Lambda_{g,n}$. These maps are called the *correlators of the CohFT*. A structure of a (complete) CohFT on $(\mathcal{H}, \eta)$ is equivalent to the requirement that these correlators endow $(\mathcal{H}, \eta)$ with the structure of an algebra over the modular operad $\{ H_\bullet(\overline{\mathcal{M}}_{g,n}) \}$ in the sense of Getzler and Kapranov [18].

3. Clearly, the definition of a cohomological field theory extends from $\mathbb{C}$ to more general ground rings $\mathcal{K}$.

Let Γ be a stable graph, then there is a canonical composition map ρ_Γ

$$\rho_\Gamma : \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n} \tag{18}$$

where $V(\Gamma)$ denotes the set of vertices of Γ. Since the map ρ_Γ can be constructed from gluing morphisms (11) and (13), the forms $\Lambda_{g,n}$

satisfy a restriction property

$$\rho_\Gamma^* \Lambda_{g,n} \;=\; \rho_\Gamma^{-1}\big(\bigotimes_{v \in V(\Gamma)} \Lambda_{g(v),n(v)} \big) \tag{19}$$

where

$$\rho_\Gamma^{-1} \;:\; \bigotimes_{v \in V(\Gamma)} T^{n(v)}\,\mathcal{H}^* \;\to\; T^n\,\mathcal{H}^*$$

contracts the factors $T^n\mathcal{H}^*$ by means of the inverse of the metric η and successive application of equations (12) and (15) .

There is a parameter which can be introduced into the definition of a CohFT. This parameter can be regarded as a coupling constant in the theory.

Lemma 3.3. *Let $(\mathcal{H}, \widetilde{\eta}, \widetilde{\Lambda}, e_0)$ be a CohFT with flat identity e_0 and let λ be a nonzero parameter. If $\Lambda = \{\, \Lambda_{g,n} \,\}$ where*

$$\Lambda_{g,n} \;:=\; \lambda^{2g-2}\,\widetilde{\Lambda}_{g,n}$$

and

$$\eta \;:=\; \lambda^{-2}\,\widetilde{\eta}$$

then $(\mathcal{H}, \eta, \Lambda, e_0)$ is a CohFT with flat identity.

The proof is obvious.

Definition 3.4. The *small phase space potential function* of the CohFT $(\mathcal{H}, \widetilde{\eta}, \widetilde{\Lambda})$ is a formal series $\Phi \in \mathbb{C}[[\mathcal{H}]]$ given by

$$\Phi(\mathbf{x}) \;:=\; \sum_{g=0}^{\infty} \Phi_g(\mathbf{x}), \tag{20}$$

where

$$\Phi_g(\mathbf{x}) \;:=\; \sum_n \frac{1}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Big\langle\, \widetilde{\Lambda}_{g,n}\,,\, \mathbf{x}^{\otimes n} \,\Big\rangle\,.$$

Here $\langle \cdots \rangle$ denotes evaluation, the sum over n is understood to be over the stable range, and $\mathbf{x} \;=\; \sum_\alpha x^\alpha\, e_\alpha$, where $\{\, e_\alpha \,\}$ is a basis of $\mathcal{H}$.

Remark 3.5. The small phase space potential function of the CohFT $(\mathcal{H}, \eta, \Lambda, e_0)$ associated to $(\mathcal{H}, \widetilde{\eta}, \widetilde{\Lambda}, e_0)$ as in Lemma 3.3 may be regarded as an element in $\lambda^{-2}\mathbb{C}[[\mathcal{H}, \lambda^2]]$.

All of the information of a genus zero CohFT is encoded in this potential.

Theorem 3.6. [29, 33] *An element Φ_0 in $\mathbb{C}[[\mathcal{H}]]$ is the potential of a rank d, genus zero CohFT $(\mathcal{H}, \eta)$ if and only if it contains only terms which are of cubic and higher order in the coordinates $x^0, \ldots, x^{d-1}$ (corresponding to a basis $\{ e_0, \ldots e_{d-1} \}$ of $\mathcal{H}$) and it satisfies the* associativity, or WDVV *(Witten-Dijkgraaf-Verlinde[2]) equation*

$$\partial_a \partial_b \partial_e \Phi_0 \, \eta^{ef} \, \partial_f \partial_c \partial_d \Phi_0 \;=\; \partial_b \partial_c \partial_e \Phi_0 \, \eta^{ef} \, \partial_f \partial_a \partial_d \Phi_0,$$

where η^{ab} is the inverse matrix of the matrix of η in the basis $\{e_a\}$, ∂_a is derivative with respect to x^a, and the summation convention has been used.

Conversely, a genus zero CohFT structure on $(\mathcal{H}, \eta)$ is uniquely determined by its potential Φ_0, which must satisfy the WDVV equation.

A genus zero CohFT with flat identity is essentially equivalent to endowing the state space $(\mathcal{H}, \eta)$ with the structure of a formal Frobenius manifold [10, 20, 33]. The theorem follows from the work of Keel [27] who proved that $H^\bullet(\overline{\mathcal{M}}_{0,n})$ is generated by boundary classes and that all relations between boundary divisors arise from lifting the basic codimension one relation on $\overline{\mathcal{M}}_{0,4}$.

Higher genus CohFTs are not completely determined by their potentials because $H^\bullet(\overline{\mathcal{M}}_{g,n})$ is not generated by its boundary classes. Nonetheless, Getzler [16] proved that the potential function of a genus one CohFT must satisfy an analogous (but much more complicated) equation arising from a codimension-two relation in $H^\bullet(\overline{\mathcal{M}}_{1,4})$.

3.2. Gromov-Witten invariants and their potentials.

Our construction of CohFTs from the moduli space of stable spin curves is guided by analogy with the moduli space of stable maps and Gromov-Witten invariants. Let us briefly review this construction. Let V be a smooth projective variety, $\mathcal{H} = H^\bullet(V, \mathbb{C})$, and η the Poincaré pairing. Let $\overline{\mathcal{M}}_{g,n}(V)$ be the moduli stack of stable maps into V of genus g with n marked points. The *Gromov-Witten invariants of V* are multilinear maps $\mathcal{H}^{\otimes n} \to \mathbb{C}$ given by

$$\langle \tau_0(\gamma_1) \cdots \tau_0(\gamma_n) \rangle_g \;=\; \lambda^{2g-2} \int_{[\overline{\mathcal{M}}_{g,n}(V)]^{\mathrm{vir}}} \mathrm{ev}_1^* \gamma_1 \cup \cdots \cup \mathrm{ev}_n^* \gamma_n \tag{21}$$

where $[\overline{\mathcal{M}}_{g,n}(V)]^{\mathrm{vir}}$ is the virtual fundamental class of the moduli stack $\overline{\mathcal{M}}_{g,n}(V)$ and λ is a formal parameter.

The corresponding *small phase space potential* $\Phi(\mathbf{x})$ is defined by (20) where the genus g part is given by

$$\Phi_g(\mathbf{x}) \;=\; \sum_n \lambda^{2g-2} \int_{[\overline{\mathcal{M}}_{g,n}(V)]^{\mathrm{vir}}} \mathrm{ev}_1^* \mathbf{x} \cup \cdots \cup \mathrm{ev}_n^* \mathbf{x},$$

$\mathbf{x} = \sum_a x^a e_a$, and $\{ e_0, \ldots, e_r \}$ is a basis for $\mathcal{H}$ such that e_0 is the identity element.

If V is a convex variety then $\overline{\mathcal{M}}_{0,n}(V)$ is a smooth stack and its virtual fundamental class coincides with its topological fundamental class. In this situation, [15] shows that $(\mathcal{H}, \eta)$ forms a genus zero CohFT with potential Φ_0. This result can be generalized to higher genera and to more general varieties, as well.

Remark 3.7. In the usual definition of Gromov-Witten invariants, there is no factor of λ^{2g-2} but, rather, this factor is inserted into the potential function by hand. We have chosen our conventions so that this factor appears in the correlator but is not explicitly inserted into the potential function.

The *gravitational descendants* are defined by twisting the Gromov-Witten classes with the tautological ψ classes as follows:

(22)
$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_g := \lambda^{2g-2} \int_{[\overline{\mathcal{M}}_{g,n}(V)]^{\mathrm{vir}}} \mathrm{ev}_1^* \gamma_1 \cup \psi_1^{a_1} \cup \cdots \cup \mathrm{ev}_n^* \gamma_n \cup \psi_n^{a_n}$$

for all $a_1, \ldots, a_n = 0, 1, 2, \ldots$ and $\gamma_1, \ldots, \gamma_n$ in $H^\bullet(V)$. This gives rise to the *large phase space potential* $\Phi(\mathbf{t}) \in \lambda^{-2}\mathbb{C}[[\mathbf{t}, \lambda^2]]$ where $\mathbf{t} = (\mathbf{t}_0, \mathbf{t}_1, \ldots)$ and $\mathbf{t}_n = (t_n^0, \ldots, t_n^r)$, which is defined by

$$\Phi(\mathbf{t}) := \sum_g \Phi_g(\mathbf{t}) \tag{23}$$

and

$$\Phi_g(\mathbf{t}) := \sum \langle \tau_{a_1}(e_{\alpha_1}) \cdots \tau_{a_n}(e_{\alpha_n}) \rangle_g \, t_{a_1}^{\alpha_1} \cdots t_{a_n}^{\alpha_n} \frac{1}{n!}. \tag{24}$$

Setting $t_n^\alpha = 0$ for $n \geq 1$ and $x^\alpha = t_0^\alpha$ reduces the large phase space potential $\Phi(\mathbf{t})$ to the small phase space potential $\Phi(\mathbf{x})$.

When V is a point, Kontsevich's theorem gives that

$$Z(\mathbf{t}) := \exp(\Phi(\mathbf{t}))$$

is a τ function of the KdV hierarchy. In addition Kontsevich showed that $Z(\mathbf{t})$ is a highest weight vector for the Virasoro Lie algebra, a condition which completely allows one to solve for these intersection numbers. The existence of a similar Virasoro algebra action has been conjectured by Eguchi, Hori, and Xiong [12] in the case where V is not a point. Evidence for this conjecture is mounting [19, 14].

A *very large phase space* has recently been introduced in [25, 35] for the case where V is a point and for more general varieties in [13] by including variables corresponding to the Hodge classes ν_i as well. These additional variables parametrize an even larger family of CohFTs than just the large phase space coordinates [25]. We will shortly introduce, in addition, variables associated to the r-spin structure (see (48)).

3.3. r-spin CohFT.

We perform an analogous construction of a very large phase space where the role of the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(V)$ is played by the moduli space of stable r-spin curves $\overline{\mathcal{M}}^{1/r}_{g,n}$. Unlike the moduli space of general stable maps, the moduli space of stable r-spin curves is a smooth stack. Intersection theory is therefore simpler in this case than for the case of stable maps. However, the difficulty lies instead in the construction of the analogs of the Gromov-Witten classes.

In the next section, we shall introduce axioms which a collection of cohomology classes (called a *virtual class*) $c^{1/r}_{g,n}(\mathbf{m})$ in $H^\bullet(\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n})$ must satisfy in order to insure that the following result holds.

Theorem 3.8. *Let $(\mathcal{H}^{(r)}, \widetilde{\eta})$ be a vector space of dimension $r-1$ with a basis $\{ e_0, \dots, e_{r-2} \}$ and metric $\widetilde{\eta}$ given by*

$$\widetilde{\eta}(e_\mu, e_\nu) := \widetilde{\eta}_{\mu\nu} := \frac{1}{r}\,\delta_{\mu+\nu, r-2}. \tag{25}$$

Let $c^{1/r}_{g,n}(\mathbf{m})$ be a virtual class in $H^\bullet(\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n})$ satisfying Axioms 1 through 5 from the next section, and let $p : \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be the map which forgets the r-spin structure. Let $\widetilde{\Lambda} := \{ \widetilde{\Lambda}_{g,n} \}$ be defined by

$$\widetilde{\Lambda}^{(\mathbf{s},\mathbf{u})}_{g,n}(e_{m_1}, \dots, e_{m_n}) := p_*\left(c^{1/r}_{g,n}(\mathbf{m}) \exp(\mathbf{s}\cdot\boldsymbol{\mu} + \mathbf{u}\cdot\boldsymbol{\nu})\right) \tag{26}$$

where these forms have values in the ring $\mathbb{C}[[\mathbf{s}, \mathbf{u}]]$, then $(\mathcal{H}, \widetilde{\eta}, \widetilde{\Lambda}^{(\mathbf{s},\mathbf{u})}, e_0)$ is a CohFT satisfying Axiom C4a. Furthermore, let

$$\widetilde{\Lambda}_{g,n} := \widetilde{\Lambda}^{(\mathbf{0},\mathbf{0})}_{g,n} \tag{27}$$

then $(\mathcal{H}, \widetilde{\eta}, \widetilde{\Lambda}, e_0)$ is a CohFT with flat identity. The latter will be called the r-spin CohFT.

Restricting the r-spin CohFT to genus zero shows that $(\mathcal{H}^{(r)}, \eta)$ is endowed with the structure of a Frobenius manifold.

This theorem is proven in Section 4.2.

Corollary 3.9. *Let $(\mathcal{H}, \eta, \Lambda, e_0)$ be constructed from $(\mathcal{H}, \widetilde{\eta}, \widetilde{\Lambda}, e_0)$ above by setting*

$$\eta(e_\mu, e_\nu) := \eta_{\mu\nu} := \frac{1}{r\lambda^2}\,\delta_{\mu+\nu, r-2}, \tag{28}$$

$$\Lambda^{(\mathbf{s},\mathbf{u})}_{g,n}(e_{m_1}, \dots, e_{m_n}) := \lambda^{2g-2}\,\widetilde{\Lambda}^{(\mathbf{s},\mathbf{u})}_{g,n}(e_{m_1}, \dots, e_{m_n}), \tag{29}$$

and

$$\Lambda_{g,n} := \Lambda^{(\mathbf{0},\mathbf{0})}_{g,n}, \tag{30}$$

then $(\mathcal{H}, \eta, \Lambda^{(\mathbf{s},\mathbf{u})}, e_0)$ *is a CohFT satisfying Axiom C4a and* $(\mathcal{H}, \eta, \Lambda, e_0)$ *is a CohFT with flat identity.*

Proof. This is a direct consequence of Lemma 3.3 and the previous theorem. □

The classes $c^{1/r}_{g,n}(\mathbf{m})$ are analogs of the Gromov-Witten classes in this theory. The analogs of the gravitational descendants (22) are given by

$$\langle \tau_{a_1}(e_{m_1}) \cdots \tau_{a_n}(e_{m_n}) \rangle_g := \lambda^{2g-2} \int_{\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \, c^{1/r}_{g,n}(\mathbf{m}) \tag{31}$$

and the large phase space potential function is defined by the equations (23) and (24). The small phase space potential function is defined by restricting (31) to correlators with $a_i = 0$.

We will see that the case where $\lambda = \frac{1}{\sqrt{r}}$ *corresponds to the generalized Witten conjecture. This corresponds to the metric*

$$\eta(e_{m_1}, e_{m_2}) = \delta_{m_1+m_2, r-2} \tag{32}$$

and the forms

$$\Lambda_{g,n}(e_{m_1}, \dots, e_{m_n}) := r^{1-g} \, p_* \, c^{1/r}_{g,n}(\mathbf{m}). \tag{33}$$

Remark 3.10. The astute reader will notice in the sequel that, strictly speaking, the state space of this r-spin CohFT (33) should be instead, $(\hat{\mathcal{H}}^{(r)}, \hat{\eta})$ where $\hat{\mathcal{H}}^{(r)}$ is an r-dimensional vector space with basis $\{e_0, \dots, e_{r-1}\}$ and a metric given by

$$\hat{\eta}(e_a, e_b) = \begin{cases} 1 & \text{if } a + b \equiv (r-2) \mod r \\ 0 & \text{otherwise.} \end{cases}$$

However, it follows from the axioms for $c^{1/r}$ in Section 4.1 that the obvious orthogonal decomposition $\hat{\mathcal{H}}^{(r)} = \mathcal{H}^{(r)} \oplus \mathcal{H}'$, where $\mathcal{H}'$ is the trivial one-dimensional CohFT with basis $\{ e_{r-1} \}$, is a direct sum of CohFTs. For this reason, we can (and will) restrict ourselves to the state space $(\mathcal{H}^{(r)}, \eta)$.

4. Virtual classes

To endow the pair $(\mathcal{H}^{(r)}, \eta)$ from Theorem 3.8 with the structure of a CohFT by equation (33), we must define cohomology classes $c^{1/r}_{g,n}(\mathbf{m})$. We will call this collection of classes an *r-spin virtual class*. It should satisfy the axioms described below.

Throughout this section, we will restrict ourselves to the case where the coupling constant $\lambda = \frac{1}{\sqrt{r}}$ *unless otherwise stated.* This is done purely for convenience as analogous results hold for general λ as well.

4.1. Axioms for the virtual class.

Definition 4.1. An *r-spin virtual class* is an assignment of a cohomology class to every genus g, stable, decorated graph with n-tails marked with the n-tuple $\mathbf{m} = (m_1, \dots, m_n)$ with $0 \leq m_i \leq r-1$

$$c^{1/r}_{\Gamma} \in H^{2D}(\overline{\mathcal{M}}^{1/r}_{\Gamma}), \tag{34}$$

where

$$D = \frac{1}{r}\big((r-2)(g-1) + \sum_{i=1}^{n} m_i\big) \tag{35}$$

satisfying the axioms below. In the special case where Γ has one vertex and no edges, we denote $c^{1/r}_{\Gamma}$ by $c^{1/r}_{g,n}(\mathbf{m})$.

Axiom 1a (Connected Graphs) Let Γ be a connected, genus g, stable decorated graph with n tails. Let $E(\Gamma)$ denote the set of edges of Γ. For each edge e of Γ, let $l_e := \gcd(m^+_e + 1, r)$ where m^+_e is an integer decorating a half-edge of e. The classes $c^{1/r}_{\Gamma}$ and $c^{1/r}_{g,n}(\mathbf{m})$ are related by

$$c^{1/r}_{\Gamma} = \Big(\prod_{e \in E(\Gamma)} \frac{r}{l_e}\Big)\, \tilde{i}^*\, c^{1/r}_{g,n}(\mathbf{m}) \in H^{2D}(\overline{\mathcal{M}}^{1/r}_{\Gamma}) \tag{36}$$

where $\tilde{i} : \overline{\mathcal{M}}^{1/r}_{\Gamma} \hookrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ is the canonical inclusion map.

Axiom 1b (Disconnected Graphs) Let Γ be stable, decorated graph which is the disjoint union of connected graphs $\Gamma^{(d)}$ then

$$c^{1/r}_{\Gamma} = \bigotimes_{d} c^{1/r}_{\Gamma^{(d)}} \in H^{\bullet}(\overline{\mathcal{M}}^{1/r}_{\Gamma}).$$

Axiom 2 (Convexity): Consider the universal r-spin structure $\{\mathcal{E}_d, c_{d,d'}\}$ on the universal curve $\pi : \mathcal{C}^{1/r,\mathbf{m}}_{g,n} \longrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$. For each irreducible (and connected) component of $\overline{\mathcal{M}}^{1/r}_{g,n}$ (denoted here by $\overline{\mathcal{M}}^{1/r,(d)}_{g,n}$ for some index d), if $\pi_*\mathcal{E}_r = 0$ on $\overline{\mathcal{M}}^{1/r,(d)}_{g,n}$, then $c^{1/r}_{g,n}(\mathbf{m})$ restricted to

$\overline{\mathcal{M}}_{g,n}^{1/r(d)}$ is $c_D(-R^1\pi_*\mathcal{E}_r)$, the top Chern class of the bundle with fibers $H^1(\mathcal{E}_r)^*$.

Axiom 3 (Cutting edges): Given any genus g stable decorated graph with n tails marked with $\mathbf{m}$, we have a diagram

$$\begin{tikzcd}
 & \overline{\mathcal{M}}_{\tilde{\Gamma}} \times_{\overline{\mathcal{M}}_{\Gamma}} \overline{\mathcal{M}}_{\Gamma}^{1/r} \arrow[r, "\tilde{\mu}"] \arrow[dl, "p_1"'] \arrow[dd, "p"] & \overline{\mathcal{M}}_{\Gamma}^{1/r} \arrow[r, hook, "\tilde{i}"] \arrow[dd, "p"] & \overline{\mathcal{M}}_{g,n}^{1/r} \arrow[dd, "p"] \\
\overline{\mathcal{M}}_{\tilde{\Gamma}}^{1/r} \arrow[dr, "p_2"'] & & & \\
 & \overline{\mathcal{M}}_{\tilde{\Gamma}} \arrow[r, "\mu"] & \overline{\mathcal{M}}_{\Gamma} \arrow[r, hook, "i"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd} \tag{37}$$

where $\overline{\mathcal{M}}_{\tilde{\Gamma}}$ is the stack of stable curves with graph $\tilde{\Gamma}$, the graph obtained by cutting all edges of Γ, and $\overline{\mathcal{M}}_{\tilde{\Gamma}}^{1/r}$ is the stack of stable spin curves with graph $\tilde{\Gamma}$ (still marked with $m^{\pm}$ on each half edge).

p_1 is the following morphism: The fibre product consists of triples of a spin curve $(X/T, \{\mathcal{E}_d, c_{d,d'}\})$, a stable curve $\tilde{X}/T$, and a morphism $\nu : \tilde{X} \longrightarrow X$ making $\tilde{X}$ into the normalization of X. Also the dual graphs of X and $\tilde{X}$ are Γ and $\tilde{\Gamma}$, respectively. The associated spin curve in $\overline{\mathcal{M}}_{\tilde{\Gamma}}^{1/r}$ is simply $(\tilde{X}/T, \nu^*\{\mathcal{E}_d, c_{d,d'}\})$.

We require that

$$p_{1*}\tilde{\mu}^* c_{\Gamma}^{1/r} = r^{|E(\Gamma)|} c_{\tilde{\Gamma}}^{1/r}$$

where $E(\Gamma)$ is the set of edges of Γ that are cut in $\tilde{\Gamma}$.

Axiom 4 (Vanishing): If Γ contains a tail marked with $m_i = r-1$, then $c_{\Gamma}^{1/r} = 0$.

Axiom 5 (Forgetting tails): Let $\tilde{\Gamma}$ be a stable graph with a tail t_i marked by $m_i = 0$, and Γ the stable graph obtained by removing the tail t_i, and

$$\pi : \overline{\mathcal{M}}_{\tilde{\Gamma}}^{1/r} \longrightarrow \overline{\mathcal{M}}_{\Gamma}^{1/r}$$

the forgetful morphism, then

$$c_{\tilde{\Gamma}}^{1/r} = \pi^* c_{\Gamma}^{1/r}.$$

Remarks 4.2.

1. The factor of $\prod_e \frac{r}{l_e}$ in Axiom 1 arises from the fact that the right hand square in equation (37) is not quite Cartesian. Rather, because of ramification of p over $\overline{\mathcal{M}}_\Gamma$, we have that for any cohomology class c on $\overline{\mathcal{M}}^{1/r}_{g,n}$,

$$i^* p_* c = (\prod_e \frac{r}{l_e})\, p_* \tilde{i}^* c. \tag{38}$$

2. The astute reader will notice in Axiom 3 that, unlike the case of a tree, if the spin structure is Ramond and Γ contains a loop, the dimensions D_Γ and $D_{\tilde{\Gamma}}$ of the virtual classes $c^{1/r}_\Gamma$ and $c^{1/r}_{\tilde{\Gamma}}$ are different. Thus Axiom 3 actually requires the vanishing of both $p_{1*}\tilde{\mu}^* c^{1/r}_\Gamma$ and $c^{1/r}_{\tilde{\Gamma}}$ in this case. Of course for the Ramond case, Axiom 4 already requires the vanishing of $c^{1/r}_{\tilde{\Gamma}}$, since the cut half-edges are both marked with $r-1$. Thus for any graph (tree or otherwise) in the Ramond case Axiom 3 amounts essentially to the requirement that $p_{1*}\tilde{\mu}^* c^{1/r}_\Gamma$ vanish.
3. Although the vanishing of H^0 is often called concavity, the Serre dual $\mathcal{H}om(\mathcal{E}_r, \omega)$, corresponding to Witten's sheaf $\mathcal{V}$ in [40] is convex (H^1 vanishes) exactly when $\mathcal{E}_r$ is concave. Moreover, $\pi_!\mathcal{V} = \pi_!\mathcal{E}_r$. Therefore, we use the term convex to describe the case when $\pi_*\mathcal{E}_r = 0$.
4. One might think that the class $c_D\pi_!\mathcal{E}_r$ would be a good candidate for a virtual class, since it coincides with $c^{1/r}$ in the convex case. However, this is not the case (see Section 4.4).
5. Witten has described in [40, §1.3] an analytic construction of a class that he calls the "top Chern class", but it is not clear that this class satisfies the above axioms. Witten's index-like construction is reminiscent of the analytic construction of a virtual fundamental class of the moduli space of stable maps in the theory of Gromov-Witten invariants. Ideally one should be able to construct $c^{1/r}$ by methods similar to those used in algebraic constructions of the fundamental class.
6. Although, as explained in Remark 1.8, the restriction $0 \le m_i < r$ does not change the moduli space $\overline{\mathcal{M}}^{1/r}_{g,n}$, it does give a different choice of $c^{1/r}$. Indeed, replacing m_i by $m_i + r$ changes the dimension of $c^{1/r}$ by 1. Thus on a given moduli space $\overline{\mathcal{M}}^{1/r}_{g,n}$ there are potentially several (but still only finitely many, for dimension reasons) choices of $c^{1/r}$ and the corresponding $CohFT$. However, without the restriction $0 \le m_i < r$ the corresponding metric η

is not necessarily invertible, and several other unusual considerations also arise. These issues will be treated in a forthcoming paper [24]. In the remainder of this paper we will assume $0 \leq m_i < r$ except where explicitly stated otherwise.

4.2. Verification of the CohFT axioms for $(\mathcal{H}^{(r)}, \eta)$.

In this section we give a proof of Theorem 3.8, first for the case where $\mathbf{s} = \mathbf{u} = \mathbf{0}$, and then in general.

4.2.1. *The case* $\mathbf{s} = \mathbf{u} = \mathbf{0}$. Let $c^{1/r}_{g,n}$ be a cohomology class on $\overline{\mathcal{M}}^{1/r}_{g,n}$ satisfying the axioms of Section 4.1. We will show that the collection of forms $\{\Lambda\}$ given by (33) satisfies the CohFT axioms C1–C4. Axiom C1 clearly holds by the definition of $\{\Lambda\}$.

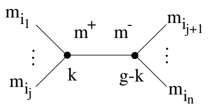
Let $\rho = \rho_{tree}$ be the gluing map (11). Condition (12) of Axiom C2 is equivalent

$$\rho^* \, p_* c^{1/r}_{g,n}(\mathbf{m}) r^{1-g} \; = \tag{39}$$

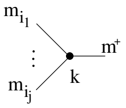
$$\sum_{a,b=0}^{r-2} r^{1-k} \, p_* c^{1/r}_{k,j+1}(m_{i_1}, \ldots, m_{i_j}, a) \otimes r^{1-(g-k)} \, p_* \, c^{1/r}_{g-k,n-j+1}(b, m_{i_{j+1}}, \ldots, m_{i_n j}) \eta^{ab}$$

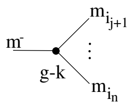
for all $0 \leq m_i \leq r-1$.

Consider the stable decorated graph

$\Gamma =$ 

and denote the graph obtained from cutting its edge by $\tilde{\Gamma} = \Gamma_1 \sqcup \Gamma_2$ where

$\Gamma_1 =$ 

and $\Gamma_2 =$ 

.

Since the spaces $\overline{\mathcal{M}}^{1/r}_{\Gamma_i}$ are non-empty for $0 \leq m^{\pm} \leq r-1$ only when $m^{\pm}$ are determined by the conditions (3) and (4), the sum in the right-hand side of equation (39) has only one non-vanishing term. By the definition of the metric η (32), Axiom C2 reduces to the following

$$\rho^* p_* c^{1/r}_{g,n}(\mathbf{m}) r^{1-g} = \begin{cases} (p \times p)_*(c^{1/r}_{\Gamma_1} r^{1-k} \otimes c^{1/r}_{\Gamma_2} r^{1-(g-k)}), & \text{if } 0 \leq m^+ \leq r-2 \\ 0 & \text{if } m^+ = r-1 \end{cases} \tag{40}$$

with $m^- = r - 2 - m^+$.

In other words, we must show that

$$\rho^* p_* c^{1/r}_{g,n}(\mathbf{m}) = r(p_* c^{1/r}_{\Gamma_1} \otimes p_* c^{1/r}_{\Gamma_2}) = r \cdot p_{2*}(c^{1/r}_{\tilde{\Gamma}}) \tag{41}$$

where we have the diagram

$$\begin{tikzcd}
& \overline{\mathcal{M}}_{\tilde{\Gamma}} \times_{\overline{\mathcal{M}}_{\Gamma}} \overline{\mathcal{M}}_{\Gamma}^{1/r} \arrow[r, "\tilde{\chi}"] \arrow[dl, "p_1"'] \arrow[dd, "p"'] & \overline{\mathcal{M}}_{\Gamma}^{1/r} \arrow[r, "\tilde{i}"] \arrow[dd, "p"'] & \overline{\mathcal{M}}_{g,n}^{1/r} \arrow[dd, "p"'] \\
\overline{\mathcal{M}}_{\tilde{\Gamma}}^{1/r} \arrow[dr, "p_2"'] & & & \\
& \overline{\mathcal{M}}_{\tilde{\Gamma}} \arrow[r, "\chi"'] & \overline{\mathcal{M}}_{\Gamma} \arrow[r, "i"'] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd} .$$

But if $l = \gcd(m^+ + 1, r)$ then

$$\begin{aligned}
\rho^* p_* c_{g,n}^{1/r}(\mathbf{m}) &= \chi^* i^* p_* c_{g,n}^{1/r} \\
&= \chi^* p_* \tilde{i}^* (r/l) c_{g,n}^{1/r} \\
\text{(by the def. of } c_{\Gamma}^{1/r}) &= \chi^* p_* c_{\Gamma}^{1/r} \\
&= p_* \tilde{\chi}^* c_{\Gamma}^{1/r} \\
&= p_{2*} p_{1*} \tilde{\chi}^* c_{\Gamma}^{1/r} \\
\text{(by Axiom 3)} &= (p_{2*} c_{\tilde{\Gamma}}^{1/r}) r
\end{aligned}$$

This gives (39) and, therefore, Axiom C2 is verified.

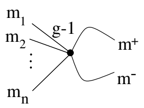
The statement (15) of Axiom C3 is equivalent by (10) and (32) to

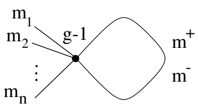
$$\rho^* p_* c_{g,n}^{1/r}(\mathbf{m}) = r \sum_{m^+=0}^{r-2} p_* \, c_{g-1,n+2}^{1/r}(m_1, \ldots, m_n, m^+, m^-) , \tag{42}$$

where $\rho = \rho_{\mathrm{loop}}$ is the gluing map (13) and

$$m^- = \begin{cases} r - 2 - m^+ & \text{if } 0 \le m^+ \le r-2 \\ r-1 & \text{if } m^+ = r-1. \end{cases}$$

Let

$\tilde{\Gamma}_{m^+} =$ 

and $\Gamma_{m^+} =$ 

be stable decorated graphs and $\tilde{\Gamma}$ and $\Gamma = \Gamma_{\text{loop}}$ be the corresponding underlying graphs. We have the commuting diagram

$$\begin{tikzcd}
 & \displaystyle\bigsqcup_{(r-2)/2\le m^+<r} F_{\Gamma,m^+} \arrow[r, "\tilde{\chi}"] \arrow[dl, "p_1"'] \arrow[dd, "p"] & \displaystyle\bigsqcup_{(r-2)/2\le m^+<r} \overline{\mathcal{M}}^{1/r}_{\Gamma,m^+} \arrow[r, "\tilde{i}"] \arrow[dd, "p"] & \overline{\mathcal{M}}^{1/r}_{g,n} \arrow[dd, "p"] \\
\displaystyle\bigsqcup_{(r-2)/2\le m^+<r} \overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}} \arrow[dr, "p_2"'] & & & \\
 & \overline{\mathcal{M}}_{\tilde{\Gamma}} \arrow[r, "\chi"'] & \overline{\mathcal{M}}_{\Gamma} \arrow[r, "i"'] & \overline{\mathcal{M}}_{g,n}.
\end{tikzcd}$$

where $F_{\Gamma,m^+} := \overline{\mathcal{M}}_{\tilde{\Gamma}} \times_{\overline{\mathcal{M}}_{\Gamma}} \overline{\mathcal{M}}^{1/r}_{\Gamma,m^+}$. Therefore, if $l_e = \gcd(m^+ + 1, r)$ then

$$\begin{aligned}
\rho^* p_* c^{1/r}_{g,n} &= \chi^* p_* \sum_{m^+} (\tilde{i}^* c_{g,n}) r / l_e \\
&= \chi^* p_* \sum_{m^+} c^{1/r}_{\Gamma,m^+} \\
&= p_* \tilde{\chi}^* \sum_{m^+} c^{1/r}_{\Gamma,m^+} \\
&= p_{2*} p_{1*} \tilde{\chi}^* \sum_{m^+} c^{1/r}_{\Gamma,m^+} \\
&= r \cdot p_{2*} \sum_{m^+} c^{1/r}_{\tilde{\Gamma},m^+}.
\end{aligned}$$

This proves axiom C3.

To prove Axiom C4a, consider the Cartesian square

$$\begin{tikzcd}
\overline{\mathcal{M}}^{1/r,(m_1,\dots,m_n,0)}_{g,n+1} \arrow[r, "\tilde{\pi}"] \arrow[d, "p"] & \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n} \arrow[d, "p"] \\
\overline{\mathcal{M}}_{g,n+1} \arrow[r, "\pi"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

Then by (10) and Axiom 5 (forgetting tails) we have the required equation (16)

$$\begin{aligned}\pi^*\Lambda_{g,n}(e_{\mathbf{m}}) &= r^{1-g}\,\pi^* p_* c^{1/r}_{g,n}(\mathbf{m}) \;=\; r^{1-g}\; p_*\tilde{\pi}^* c^{1/r}_{g,n}(\mathbf{m})\\ &= r^{1-g}\, c^{1/r}_{g,n+1}(m_1,\dots,m_n,0) = \Lambda_{g,n+1}(e_{m_1},\dots,e_{m_n},e_0).\end{aligned}$$

Finally, a direct calculation yields Axiom 4b (see Proposition 6.1).

4.2.2. *The general case.* The proof of Axioms C1 and C4a are the same. We need only prove axioms C2 and C3. First consider the Cartesian square

$$\begin{tikzcd} \mathcal{C}_\Gamma \arrow[r, "\hat{i}"] \arrow[d, "\pi"'] & \mathcal{C} \arrow[d, "\pi"'] \\ \overline{\mathcal{M}}^{1/r}_{\Gamma} \arrow[r, "\tilde{i}"'] & \overline{\mathcal{M}}^{1/r}_{g,n} \end{tikzcd}$$

where π is the universal curve. If $\mathcal{E}_r$ is the rth root from the universal spin structure on $\mathcal{C}$, since π is projective, and $\overline{\mathcal{M}}^{1/r}_{g,n}$ carries an ample invertible line bundle [21, 3.1.1], Lemma 4.3, below, on regular imbeddings holds, so that $\mathrm{ch}_i(\pi_!\hat{i}^*\mathcal{E}_r) = \mathrm{ch}_i(\tilde{i}^*\pi_!\mathcal{E}_r)$, but $\hat{i}^*\mathcal{E}_r$ is the rth root from the universal spin structure $\mathcal{C}_\Gamma$. Now, if F is the fibre product

$$F := \overline{\mathcal{M}}_{\tilde{\Gamma}} \times_{\overline{\mathcal{M}}_\Gamma} \overline{\mathcal{M}}^{1/r}_{\Gamma}, \tag{43}$$

then we have

$$\begin{tikzcd} \mathcal{C}_{\tilde{\Gamma}} \arrow[dd] & \hat{\mathcal{C}}_F \arrow[l] \arrow[d, "\theta"'] & \\ & \mathcal{C}_F \arrow[r] \arrow[d, "j"'] & \mathcal{C}_\Gamma \arrow[d] \\ \overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}} & F \arrow[l, "p_1"'] \arrow[r] & \overline{\mathcal{M}}^{1/r}_{\Gamma} \end{tikzcd}$$

where $\mathcal{C}_{\tilde{\Gamma}}$ is defined to be the universal curve over $\overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}}$ and $\hat{\mathcal{C}}_F$ and $\mathcal{C}_F$ are the fibred products $F \times_{\overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}}} \mathcal{C}_{\tilde{\Gamma}}$ and $F \times_{\overline{\mathcal{M}}^{1/r}_{\Gamma}} \mathcal{C}_\Gamma$, respectively. The morphism θ is just normalization, and in fact, if $\hat{\mathcal{C}}_\Gamma \xrightarrow{\theta_\Gamma} \mathcal{C}_\Gamma$ is the normalization of $\mathcal{C}_\Gamma$, then we have the following fibred diagram (all rectangles are Cartesian). ‡

$$\begin{tikzcd}
\mathcal{C}_{\tilde{\Gamma}} \arrow[dd, "\pi"'] & \hat{\mathcal{C}}_F \arrow[l, "\hat{p}_1"] \arrow[r, "\hat{\chi}"] \arrow[d, "\theta"] & \hat{\mathcal{C}}_\Gamma \arrow[d, "\theta_\Gamma"] \\
 & \mathcal{C}_F \arrow[r] \arrow[d, "\pi"] & \mathcal{C}_\Gamma \arrow[d, "\pi"] \\
\overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}} & F \arrow[l, "p_1"'] \arrow[r, "\tilde{\chi}"] & \overline{\mathcal{M}}^{1/r}_{\Gamma}
\end{tikzcd}$$

Moreover, if $\mathcal{E}_{\tilde{\Gamma}}$, $\mathcal{E}_\Gamma$, $\hat{\mathcal{E}}_\Gamma$, $\mathcal{E}_F$, and $\hat{\mathcal{E}}_F$ are the rth roots of the universal spin structures on $\mathcal{C}_{\tilde{\Gamma}}$, $\mathcal{C}_\Gamma$, $\hat{\mathcal{C}}_\Gamma$, $\mathcal{C}_F$, and $\hat{\mathcal{C}}_F$, respectively, we have $\hat{\mathcal{E}}_F = \hat{p}_1^*\mathcal{E}_{\tilde{\Gamma}}$. Also, since θ is finite, $(\pi\circ\theta)_!\hat{\mathcal{E}}_F = \pi_!\mathcal{E}_F$. Since $\tilde{\chi}$ is flat, we have $\tilde{\chi}^*\pi_!\mathcal{E}_\Gamma = \pi_!\tilde{\chi}^*\mathcal{E}_\Gamma = \pi_!\mathcal{E}_F = (\pi\circ\theta)_!\hat{\mathcal{E}}_F$. Also, since p_1 is flat $p_1^*\pi_!\mathcal{E}_{\tilde{\Gamma}} = (\pi\circ\theta)_!p_1^*\mathcal{E}_{\tilde{\Gamma}} = (\pi\circ\theta)_!\hat{\mathcal{E}}_F$. Furthermore, $p_1^*\,\pi_! = (\pi\circ\theta)_!\,p_1^*$. If $\tilde{i} : \overline{\mathcal{M}}_\Gamma \longrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ is the inclusion map then the above implies that

$$\begin{aligned}
(44)\qquad \tilde{\chi}^*\,\tilde{i}^*\mathrm{ch}_i(\pi_!\,\mathcal{E}) \;&=\; \tilde{\chi}^*\,\mathrm{ch}_i(\pi_!\,\mathcal{E}_\Gamma) \;=\; \mathrm{ch}_i(\pi_!\,\hat{\chi}^*\,\mathcal{E}_\Gamma) \;=\\
\mathrm{ch}_i((\pi\circ\theta)_!\,\hat{p}_1^*\,\mathcal{E}_{\tilde{\Gamma}}) \;&=\; \mathrm{ch}_i(p_1^*\,\pi_!\,\mathcal{E}_{\tilde{\Gamma}}) \;=\; p_1^*\,\mathrm{ch}_i(\pi_!\,\mathcal{E}_{\tilde{\Gamma}}).
\end{aligned}$$

The previous equation, the fact that ν_i on $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ is the lift of ν_i on $\overline{\mathcal{M}}_{g,n}$ and the projection formula yields the desired result.

It remains to prove the lemma about regular imbeddings.

Lemma 4.3. *Let i be a regular imbedding, and let*

$$\begin{tikzcd}
Y' \arrow[r, "\tilde{i}"] \arrow[d, "\pi'"'] & X' \arrow[d, "\pi"] \\
Y \arrow[r, "i"] & X
\end{tikzcd}$$

be a fibre (Cartesian) square with π and π' proper and flat, and $\mathcal{E}$ a coherent sheaf on X', flat over X. If X and Y both carry an ample invertible sheaf, then the Chern character behaves well with respect to

‡The maps $\tilde{\chi}$ and θ were called $\tilde{\mu}$ and ν respectively elsewhere in the paper but we have renamed them here in order to avoid confusion with the cohomology classes μ_i and ν_i.

base change, that is $i^* ch \pi_! \mathcal{E} = ch \pi'_! \tilde{i}^* \mathcal{E}$. Moreover, if the class of $\pi_! \mathcal{E}$ in the Grothendieck group is represented by a locally free sheaf, then the Chern classes behave well with respect to base change; that is, $i^* c_j \pi_! \mathcal{E} = c_j \pi'_! \tilde{i}^* \mathcal{E}$ for every j.

Proof. Proposition 5.13 of [39] states that in this case if i has finite Tor dimension, and if $\mathcal{O}_{X'}$ and $\mathcal{O}_Y$ are Tor-independent over X, then $Li^* \pi_! = \pi'_! L\tilde{i}^*$, where Li^* and $L\tilde{i}^*$ are the left derived functors of i^* and $\tilde{i}^*$, respectively. But since $\mathcal{E}$ is flat over X, we have $L\tilde{i}^* \mathcal{E} := \sum_{j=0}^{\infty} (-1)^j Tor_j^X(\mathcal{E}, \mathcal{O}_Y) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_Y = \tilde{i}^* \mathcal{E}_j$ and since Li^* commutes with the Chern character in general and with Chern classes for locally free sheaves, the lemma will hold provided the conditions of finite Tor dimension and Tor independence hold. Tor independence means simply that $Tor_j^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y) = 0$ for all $j > 0$, but that follows from the flatness of π. Finite Tor dimension means that there is an integer N such that for every coherent $\mathcal{O}_X$-module $\mathcal{F}$, the $\mathcal{O}_Y$-modules $\mathcal{T}or_j^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{F})$ vanish for $j > N$. This follows from the fact that i is a regular imbedding. (e.g. one may assume $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} A/(x)$ for some regular element $x \in A$. This gives the free resolution $0 \longrightarrow A \xrightarrow{(x)} A \longrightarrow A/(x) \longrightarrow 0$ of $\mathcal{O}_Y$, and shows i has finite Tor dimension).

□

This finishes the proof of Theorem 3.8.

4.3. **The genus-zero case.**

As we explained in Remark 4.2.3, we call the spin structure $\{\mathcal{E}_d, c_{d,d'}\}$ on the universal curve $\mathcal{C}_\Gamma^{1/r}$ *convex* if $\pi_* \mathcal{E}_r$ is identically zero. This occurs, for example, when $g = 0$, as is shown in the following proposition.

Proposition 4.4. *Let X be a nodal curve of genus zero with n punctures and markings $(m_1, \ldots, m_n)$, such that $-1 \leq m_i \leq r-1$ for all i and $m_i \geq 0$ for all i except at most one. Then, if $(\mathcal{E}, b)$ is an r-th root of $\omega_X(-\sum m_i p_i)$, we have $H^0(X, \mathcal{E}) = 0$.*

Proof. The degree of $\mathcal{E}$ is an integer and is equal to $-(2 + \sum m_i)/r$. Thus $\sum m_i \geq r-2$, and the degree of $\mathcal{E}$ is strictly negative. Therefore, when X is irreducible, $\mathcal{E}$ has no global sections. When X is not irreducible, but $\mathcal{E}$ is locally free (Ramond) at each node, the same argument holds. If $\mathcal{E}$ is Neveu-Schwarz at some nodes, then normalization $\nu : \tilde{X} \longrightarrow X$ at the nodes of X where $\mathcal{E}$ is not locally free gives $\mathcal{E} = \nu_* \mathcal{F}$, where $\mathcal{F}$ is locally free on $\tilde{X}$. Restricting $\mathcal{F}$ to $\tilde{X}$ we obtain an r-th root of $\omega_{\tilde{X}}(-\sum \tilde{m}_i \tilde{p}_i)$, where the points $\tilde{p}_i$ are either marked points or inverse images of nodes and thus the collection $\tilde{m}_i$

still meets the hypotheses of the proposition, but now $\mathcal{F}$ is locally free on each component, and hence has no global sections. Since ν is finite, $H^0(X,\mathcal{E}) = H^0(\tilde{X},\mathcal{F}) = 0$. □

The previous proposition shows that if a class $c^{1/r}$ on $\overline{\mathcal{M}}^{1/r}_{0,n}$ satisfying Axioms 1–5 exists, then by Axiom 2 it must be given by the top Chern class of the index bundle. In this case it does indeed satisfy the required properties.

Theorem 4.5. *Define cohomology classes on* $\overline{\mathcal{M}}^{1/r}_{0,n}$ *by*

$$c^{1/r}_{0,n}(\mathbf{m}) = c_D\pi_!\mathcal{E}_r = (-1)^D c_D(R^1\pi_*\mathcal{E}_r), \tag{45}$$

where

$$D = \frac{1}{r}(2 - r + \sum_{i=1}^{n} m_i)$$

Then the collection of classes $c^{1/r}_\Gamma$ *defined by (36) for decorated stable graphs of genus zero satisfies Axioms 1–5.*

Proof. It is clear from the construction of the classes $c^{1/r}_\Gamma$ that they satisfy Axiom 2 (convexity).

Axiom 1 follows from the fact that $\mathcal{E}_{r,\Gamma} = \mathcal{E}_{r,\Gamma_1} \oplus \mathcal{E}_{r,\Gamma_2}$ and $c_{D(\Gamma)} = c_{top}$, the top-dimensional Chern class of $\pi_!\mathcal{E}_{r,\Gamma}$, is simply the product of the top-dimensional classes of $\pi_!\mathcal{E}_{r,\Gamma_1}$ and $\pi_!\mathcal{E}_{r,\Gamma_2}$.

Now we will show that Axiom 3 holds. If $\mathcal{E}_r$ is the rth root from the universal spin structure on $\mathcal{C}^{1/r}_{g,n} \to \overline{\mathcal{M}}^{1/r}_{g,n}$ then since $g = 0$, $\mathcal{E}_r$ is convex and repeating the argument in the proof of Axiom 3 in Section 4.2.2 with the Chern character replaced by the top Chern class yields

$$c^{1/r}_\Gamma = (r^{|E|}/\prod_{e \in E} l_e)\, c_D(-R^1\pi_*\mathcal{E}_r)$$

where E denotes the edge set of Γ.

Also, we can compute the degree of $p_1 : F \longrightarrow \overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}}$ from the diagram

$$\begin{tikzcd}
 & F \arrow[r, "\tilde{\chi}"] \arrow[dl, "p_1"'] \arrow[dd, "\tilde{p}"] & \overline{\mathcal{M}}^{1/r}_{\Gamma} \arrow[dd, "p_\Gamma"] \\
\overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}} \arrow[dr, "p_2"'] & & \\
 & \overline{\mathcal{M}}_{\tilde{\Gamma}} \arrow[r] & \overline{\mathcal{M}}_{\Gamma}
\end{tikzcd}$$

The morphism $\overline{\mathcal{M}}_{\tilde{\Gamma}} \longrightarrow \overline{\mathcal{M}}_{\Gamma}$ has degree 1, since Γ is a tree, and therefore $\tilde{\chi}$ also has degree 1. The morphism p_Γ has degree $\prod_e l_e r^{-|V(\Gamma)|}$ as can be seen from the fact that the coarse moduli space map induced by p_Γ has degree 1, and there are $r^{|V(\Gamma)|}/\prod_e l_e$ automorphisms of a generic spin structure. Thus the map $\tilde{p}$ also has degree $\prod_e l_e r^{-|V(\Gamma)|}$, the map p_2 has degree $r^{-|V(\Gamma)|}$, so p_1 has degree exactly $\prod_e l_e$.

Now we can compute $p_{1*}\tilde{\chi}^* c^{1/r}_{\Gamma}$ using equation 44 (replacing ch_i with c_D) to obtain

$$\begin{aligned}
p_{1*}(r^{|E|}/\prod_e l_e)\tilde{\chi}^* c_D(-R^1\pi_*\mathcal{E}_\Gamma) &= (r^{|E|}/\prod_e l_e) p_{1*} c_D(p_1^*(-R^1\pi_*\mathcal{E}_{\tilde{\Gamma}})) \\
&= (r^{|E|}/\prod_e l_e) p_{1*} p_1^* c_D(-R^1\pi_*\mathcal{E}_{\tilde{\Gamma}}) \\
&= r^{|E|} c_D(-R^1\pi_*\mathcal{E}_{\tilde{\Gamma}}) \\
&= r^{|E|} c^{1/r}_{\tilde{\Gamma}}
\end{aligned}$$

as desired.

All that remains to check is Axiom 4 (vanishing). Let p be a point corresponding to a tail marked by $m = r-1$. Tensoring $\mathcal{E}_r$ with the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(p) \longrightarrow \mathcal{O}(p)|_p \longrightarrow 0$$

gives the exact sequence

$$0 \longrightarrow \mathcal{E}_r \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E}'|_p \longrightarrow 0,$$

where $\mathcal{E}' = \mathcal{E}_r \otimes \mathcal{O}(p)$. Thus $\mathcal{E}'$ corresponds to a root with p_i marked by $m' = -1$. Since $R^1\pi_*(\mathcal{E}'|_p) = 0$, and since the residue isomorphism (1)

$R_p : \pi_*(\mathcal{E}'|_p) \xrightarrow{\sim} \mathcal{O}$ shows that $\pi_*(\mathcal{E}'|_p)$ is a trivial bundle, we have $\pi_!(\mathcal{E}') = \mathcal{O} + \pi_!(\mathcal{E}_r)$. By Proposition 4.4 the sheaves $\mathcal{E}_r$ and $\mathcal{E}'$ are both convex; thus $\pi_!\mathcal{E}_r = -R^1\pi_*\mathcal{E}_r$ and $\pi_!\mathcal{E}' = -R^1\pi_*\mathcal{E}'$ are both locally free and have the same Chern classes in all dimensions. But the vector bundle $\pi_!\mathcal{E}'$ has dimension $D' = D - 1$, and so $c^{1/r} = c_D(\pi_!\mathcal{E}_r) = 0$. This gives Axiom 4.

□

4.4. **The case $\overline{\mathcal{M}}_{1,1}^{1/r,\mathbf{0}}$.** In this section, we will calculate the virtual class on $\overline{\mathcal{M}}_{1,1}^{1/r,\mathbf{0}}$ from the axioms. Let $\Gamma = \Gamma_{loop}$ be the graph with one tail, one node of genus zero, and one loop. Let Γ_{m^+} be Γ with one half edge of the loop marked with m^+, the other half-edge marked with m^-. The stack $\overline{\mathcal{M}}_{1,1}^{1/r,\mathbf{0}}$ is a disjoint union $\coprod_{d|r} \overline{\mathcal{M}}_{1,1}^{1/r,(d)}$, where the component indexed by d has a generic geometric point corresponding to a smooth spin curve $(X, \mathcal{E}_r, c_{r,1})$ with $\mathcal{E}_r^{\otimes d}$ isomorphic to $\mathcal{O}_X$. That is, $\mathcal{E}_r$ is a d-torsion point in the Jacobian of X.

Since $\mathcal{E}_r$ has global sections if and only if $d = 1$, the case of $d > 1$ is convex. $D = 0$, and so $c_{1,1}^{1/r,(d)} = 1$ for $d > 1$. Moreover, consider the graph Γ_{r-1}. By Axiom 3, $p_{1*}\tilde{\mu}^* = c_{\widetilde{\Gamma}_{r-1}}$ where $\widetilde{\Gamma}_{r-1}$ is Γ_{r-1} with its edge cut. Since one of the halves of the cut edge will be labeled by $r-1$, by Axiom 4, it must vanish. Therefore, the Ramond case of Γ_{r-1} with trivial glue (i.e. $\mathcal{E}_r = \mathcal{O}_X$) yields

$$c_{\Gamma_{r-1}}^{1/r,(1)} = i^* c_{1,1}^{1/r,(1)} = -(r-1), \tag{46}$$

since the remaining Ramonds all have $i^* c_{1,1}^{1/r,(d)} = +1$, and there are $r - 1$ of them. *Notice that this differs from the top Chern class of the index bundle $c_D(\pi_!\,\mathcal{E}_r) = c_0(\pi_!\mathcal{E}_r) = 1$.*

Since $D = 0$, this means $c_{1,1}^{1/r,(1)}$ is also $-(r-1)$. Now $p^{(1)} : \overline{\mathcal{M}}_{1,1}^{1/r,(1)} \to \overline{\mathcal{M}}_{1,1}$ has degree $1/r$, and $p^{(d)} : \overline{\mathcal{M}}_{1,1}^{1/r,(d)} \to \overline{\mathcal{M}}_{1,1}$ has degree $[d^2 \prod_{p|d}(1 - 1/p^2)]/r$. The latter is $1/r$ times the number of points of exact order d inside of the Jacobian of the underlying curve. Thus

$$\begin{aligned} p_* c_{1,1}^{1/r} &= 1/r \sum_{d>1} d^2 \prod_{p|d} (1 - 1/p^2) - (r-1)/r \\ &= \frac{r^2 - 1}{r} - \frac{r-1}{r} = r - 1. \end{aligned}$$

Therefore,

$$\begin{aligned}\langle\, \tau_{1,0}\,\rangle_1 &= \int_{\overline{\mathcal{M}}^{1/r,0}_{1,1}} \psi_1 c^{1/r,0}_{1,1} = \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 p_* c^{1/r,0}_{1,1} = (r-1)\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 \\ &= \frac{r-1}{24}.\end{aligned}$$

This is consistent with the prediction from CohFT stated in equation (57).

4.5. **The case $r = 2$.**

In this Section we will show that in the case of theta-characteristics (i.e. when $r = 2$) there exists a unique virtual class $c^{1/2}$ satisfying the axioms of Section 4.1.

Theorem 4.6. *If $r = 2$, then the collection of classes*

$$c^{1/2}_{g,n}(\mathbf{m}) \in H^\bullet(\overline{\mathcal{M}}^{1/2,\mathbf{m}}_{g,n}),$$

satisfies Axioms 1–5 of Section 4.1 if and only if $c^{1/2}_{g,n}(\mathbf{m}) = 0$ for $\mathbf{m} \neq \mathbf{0}$, and for $\mathbf{m} = \mathbf{0}$ the class $c^{1/2}_{g,n}(\mathbf{0})$ belongs to $H^0(\overline{\mathcal{M}}^{1/2,\mathbf{0}}_{g,n})$ and is given by

$$c^{1/2}_{g,n}(\mathbf{0}) = \begin{cases} 1 & \text{on } \overline{\mathcal{M}}^{1/2,\mathbf{0},\mathrm{even}}_{g,n} \\ -1 & \text{on } \overline{\mathcal{M}}^{1/2,\mathbf{0},\mathrm{odd}}_{g,n} \end{cases} \tag{47}$$

Proof. Let us show first that the conditions of the theorem are necessary. If $\mathbf{m} \neq \mathbf{0}$, then by Axiom 4, $c^{1/2}_{g,n}(\mathbf{m})$ must vanish, therefore we can assume that $\mathbf{m} = \mathbf{0}$. In this case the dimension (35) of the class $c^{1/2}_{g,n}$ is equal to zero, and since $\overline{\mathcal{M}}^{1/2,\mathbf{0}}_{g,n}$ has two connected components $\overline{\mathcal{M}}^{\mathrm{even}}_{g,n}$ and $\overline{\mathcal{M}}^{\mathrm{odd}}_{g,n}$, it will be sufficient to find $c^{1/2}_{\Gamma}$ for two single graphs Γ_0 and Γ_1, such that the intersections $\overline{\mathcal{M}}^{1/2}_{\Gamma_0} \cap \overline{\mathcal{M}}^{\mathrm{even}}_{g,n}$ and $\overline{\mathcal{M}}^{1/2}_{\Gamma_1} \cap \overline{\mathcal{M}}^{\mathrm{odd}}_{g,n}$ are non-empty.

Let Γ_0 be the graph with one genus-zero vertex, n tails, and g Neveu-Schwarz (i.e. all half-edges are decorated with zeroes) loops. In this case, $\overline{\mathcal{M}}^{1/2}_{\tilde{\Gamma}_0} = \overline{\mathcal{M}}_{\tilde{\Gamma}_0} \times_{\overline{\mathcal{M}}_{\Gamma_0}} \overline{\mathcal{M}}^{1/2}_{\Gamma_0}$ in equation (37), so by Axiom 3 (cutting edges) the class $c^{1/2}_{\Gamma_0}$ pulls back to $c^{1/2}_{\tilde{\Gamma}_0}$, where $\tilde{\Gamma}_0$ is the graph with one vertex of genus zero and $n+2g$ tails. Since $g = 0$ the universal square root $\mathcal{E}$ of ω on the universal curve over $\overline{\mathcal{M}}^{1/2,\mathbf{0}}_{0,n+2g}$ is convex by Proposition 4.4. Therefore, if E is the edge set of Γ_0

$$c^{1/2,\mathbf{0},\mathrm{even}}_{g,n} = \frac{1}{2^{|E|}} c^{1/2,\mathbf{0},\mathrm{even}}_{\Gamma_0} = c^{1/2}_{\tilde{\Gamma}_0} = c_0(-R^1\pi_*\mathcal{E}) = 1$$

where the first equality follows from Axiom 1, the second from Axiom 3, and the third from Axiom 2.

To find $c^{1/2}_{\text{odd}}$, consider the graph Γ_1 with a single vertex of genus one, n tails, and $g-1$ Neveu-Schwarz loops. Axiom 3 again shows that $c^{1/2}_{\Gamma_1}$ pulls back to $c^{1/2}_{\tilde{\Gamma}_1}$, where $\tilde{\Gamma}_1$ has a single vertex of genus one and $2g-2+n$ tails. Since $\mathbf{m}=\mathbf{0}$, Axiom 5 (forgetting tails) shows that $c^{1/2}_{\text{odd}}$ is a pullback from $\overline{\mathcal{M}}^{1/2,0,\text{odd}}_{1,1} = \overline{\mathcal{M}}^{1/2,(1)}_{1,1}$, and $c^{1/2,(1)}_{1,1} = -1$ by equation (46).

Now let us show that the classes $c^{1/2}_{g,n}(\mathbf{m})$ defined above for $r=2$ indeed satisfy Axioms 1–5.

Axiom 2 (convexity) holds when $\mathbf{m}=\mathbf{0}$, since in this case the class has dimension 0, and if $\mathcal{E}$ is convex (and therefore, even) $R^1\pi_*\mathcal{E}=0$ and $c^{1/2}=c_{top}=1$ as required.

If $\mathbf{m}\neq\mathbf{0}$ then $\mathcal{E}$ is not convex on the universal curve over $\overline{\mathcal{M}}^{1/2,\mathbf{m}}_{g,n}$ for any $g>0$. In particular, consider the degenerate curve of genus g which has all n marked points on a common component E of genus 0, and which has only one other irreducible component C of genus g. For degree reasons the node must be Neveu-Schwarz, and so $\mathcal{E}$ corresponds to $\mathcal{E}_E\oplus\mathcal{E}_C$, for $\mathcal{E}_E$ a square root of type $\mathbf{m}$ of $\omega_E=\mathcal{O}_E(-2-\sum m_i)$, and $\mathcal{E}_C$ is a theta-characteristic on C and hence in general does have global sections. Thus $\mathcal{E}$ has non-zero global sections, hence the universal square root is not convex.

When $g=0$ and $\mathbf{m}\neq 0$, $\mathcal{E}$ is convex by Proposition 4.4. By Theorem 4.5 (and Axiom 4) the class $c_D(-R^1\pi_*\mathcal{E})$ vanishes, and so agrees with our definition $c^{1/2}_{g,n}(\mathbf{m})$.

Axiom 1 holds because of the simple observation that the parity of a root $\mathcal{E}$ over a curve with the graph $\Gamma_1\sqcup\Gamma_2$ is equal to the sum modulo 2 of the parities of the restrictions of $\mathcal{E}$ to the components corresponding to Γ_1 and Γ_2

To prove Axiom 3, we may assume that $\mathbf{m}=\mathbf{0}$, and it is sufficient to check the case that Γ has only one edge. We have by definition $c^{1/2,even}_{\Gamma}=2$ and $c^{1/2,odd}_{\Gamma}=-2$ for a tree. Let F be defined as in equation (43). The canonical morphism $p_1: F\longrightarrow\overline{\mathcal{M}}^{1/2}_{\tilde{\Gamma}}$ is actually an isomorphism if Γ is a tree, and so we get $p_{1*}\tilde{\mu}^*c^{1/2}_{\Gamma}=2\cdot c^{1/2}_{\tilde{\Gamma}}$ since the parity of $\mathcal{E}_2$ does not change when restricting to the normalization.

In the case of a loop, there are two cases: First when $m^+=1$, F is isomorphic to two copies of $\overline{\mathcal{M}}^{1/2}_{\tilde{\Gamma}}$ (because of the two choices of gluing data—see Section 1.3.1). Consequently, p_1 has degree 2 if $m^+=1$.

In the second case $m^+ = 0$, and F is isomorphic to $\overline{\mathcal{M}}^{1/2}_{\tilde{\Gamma}}$, so p_1 has degree 1. Also, $c^{1/2}_{\Gamma}$ is $2\tilde{i}^* c^{1/2}_{g,n}$ if $m^+ = 0$, and $\tilde{i}^* c^{1/2}_{g,n}$ if $m^+ = 1$.

Thus, when $m^+ = 0$, $p_{1*}\tilde{\mu}^* c^{1/2}_{\Gamma} = 2c^{1/2}_{\tilde{\Gamma}}$ as desired. When $m^+ = 1$, the two gluings give different parities. Since parity is deformation invariant, this can be seen by degenerating to the special case of the curve X, whose partial normalization $\tilde{X}$ at one node q consists of two irreducible components, joined at a single node q. One component C is of genus $g-1$, and contains the marked points $p_1, \dots, p_n$. The other component $\tilde{E}$ is of genus zero and contains marked points q^+ and q^-. Degree reasons force the node to be Neveu-Schwarz, and so $\tilde{\mathcal{E}}$ is simply a direct sum $\mathcal{E}_C \oplus \mathcal{E}_{\tilde{E}}$. Moreover, since $\tilde{\mathcal{E}}_{\tilde{E}}$ is a square root of $\omega_{\tilde{E}} = \mathcal{O}_{\tilde{E}}(-2)$ of type $(0,-1,-1)$, $\mathcal{E}_{\tilde{E}}$ must be trivial ($\mathcal{E}_{\tilde{E}} \cong \mathcal{O}_{\tilde{E}}$). Gluing $\mathcal{E}_{\tilde{E}}$ via $+1$ and -1 yields the trivial bundle $\mathcal{O}_E = \mathcal{E}^+_E$ and another non-trivial bundle $\mathcal{E}^-_E$ of degree zero, respectively. Consequently, $h^0(\mathcal{E}^+) = h^0(\mathcal{E}_C \oplus \mathcal{O}) = 1 + h^0(\mathcal{E}_C \oplus \mathcal{E}^-_E)$. Since the parities of $\mathcal{E}^+$ and $\mathcal{E}^-$ are simply the parities of $h^0(\mathcal{E}^+)$ and $h^0(\mathcal{E}^-)$, respectively, $\mathcal{E}^+$ and $\mathcal{E}^-$ have different parities.

The different parities of the two gluings give $p_{1*}\tilde{\mu}^* c^{1/2}_{\Gamma} = 0 = c^{1/2}_{\tilde{\Gamma}}$, since $\tilde{\Gamma}$ has a tail marked with $+1 = m^+$. This completes the proof of Axiom 3.

Axiom 5 is true because the projection $\overline{\mathcal{M}}^{1/2,(m_1,\dots,m_n,0)}_{g,n+1} \longrightarrow \overline{\mathcal{M}}^{1/2,\mathbf{m}}_{g,n}$ respects the parity of components. Axiom 4 is satisfied by the definition of the classes $c^{1/2}$.

□

5. Intersection numbers and recursion relations

In this section, we use relations between boundary classes and tautological classes in order to derive recursion relations between intersection numbers on the moduli space of stable r-spin curves.

Throughout this section, we assume the existence of the virtual class $c^{1/r}$ satisfying Axioms 1–5 of Section 4.1

This class was shown to exist in genus zero for arbitrary r, and in arbitrary genus for $r = 2$.

Let $(\mathcal{H}^{(r)}, \eta, \Lambda, e_0)$ be the r-spin CohFT with the standard basis $\{ e_0 \dots, e_{r-2} \}$ described in Corollary 3.9.

We will find it convenient to introduce a very large phase potential in this section. For all $v_1, \dots, v_n$ in $\mathcal{H}^{(r)}$, define

$$\langle\langle \tau_{a_1}(v_1) \cdots \tau_{a_n}(v_n) \rangle\rangle_g := \langle \tau_{a_1}(v_1) \cdots \tau_{a_n}(v_n) \exp(\mathbf{t}\cdot\boldsymbol{\tau} + \mathbf{s}\cdot\boldsymbol{\mu} + \mathbf{u}\cdot\boldsymbol{\nu}) \rangle_g$$

where

$$\mathbf{t} \cdot \boldsymbol{\tau} = \sum_{\substack{0 \le m \le r-2 \\ a \ge 0}} \tau_{a,m}\, t_a^m$$

and $\tau_{a,m} = \tau_a(e_m)$. Here

$$\mathbf{u} \cdot \boldsymbol{\nu} = \sum_{i \ge 1} u_i\, \nu_i \qquad \mathbf{s} \cdot \boldsymbol{\mu} = \sum_{i \ge 1} s_i \mu_i,$$

where the classes ν_i and μ_i are defined by (7) and (8) as components of the Chern characters of the Hodge bundle and its spin analog. These expressions should be understood as formal power series in variables t_a^m, u_i, v_i. The correlators are defined by

$$\begin{aligned} &\langle\, \tau_{a_1}(e_{m_1}) \cdots \tau_{a_n}(e_{m_n})\, \rangle_g := \\ &\lambda^{2g-2} \int_{\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}} \psi_1^{a_1} \cdots \psi_n^{a_n}\, c_{g,n}^{1/r}(\mathbf{m}) \exp(\mathbf{s} \cdot \boldsymbol{\mu} + \mathbf{u} \cdot \boldsymbol{\nu}) \end{aligned}$$

where $\mathbf{m} = (m_1, \ldots, m_n)$. In particular, the *very large phase space potential* is

$$\Phi(\mathbf{t}, \mathbf{s}, \mathbf{u}) = \sum_{g=0}^{\infty} \Phi_g(\mathbf{t}, \mathbf{s}, \mathbf{u}), \tag{48}$$

where

$$\Phi_g(\mathbf{t}, \mathbf{s}, \mathbf{u}) = \langle\langle\ \rangle\rangle_g.$$

The other two potentials are restrictions of $\Phi(\mathbf{t}, \mathbf{s}, \mathbf{u})$. The large phase space potential $\Phi(\mathbf{t}) = \Phi(\mathbf{t}, \mathbf{0}, \mathbf{0})$ corresponds to setting all the $\mathbf{s}$ and $\mathbf{u}$ variables to zero. The small phase space potential is

$$\Phi(\mathbf{x}) := \Phi(x^0, \ldots, x^{r-2})$$

where $x^i := t_0^i$ and all other variables are set to zero. The function $\Phi(\mathbf{x})$ is the potential of the cohomological field theory $p_* c^{1/r}$. It will also be useful to define

$$Z(\mathbf{t}, \mathbf{s}, \mathbf{u}) := \exp(\Phi(\mathbf{t}, \mathbf{s}, \mathbf{u})). \tag{49}$$

5.1. **The Euler vector field.**

We begin with a differential equation arising from the grading. The dimensions of the moduli spaces and cohomology classes induce a grading on the potential function.

Definition 5.1. The *Euler vector field* E is the differential operator

$$E = \sum_{\substack{a \geq 0 \\ 0 \leq m \leq r-2}} (a - 1 + \frac{m}{r}) t_a^m \frac{\partial}{\partial t_a^m} + \sum_{a \geq 1} (a s_a \frac{\partial}{\partial s_a} + (2a-1) u_a \frac{\partial}{\partial u_a})$$

Proposition 5.2. *The very large phase space potential* $\Phi(\mathbf{t}, \mathbf{u}, \mathbf{s})$ *satisfies the* grading equation

$$E\,\Phi = (1 + \frac{1}{r}) \lambda \frac{\partial}{\partial \lambda} \Phi \tag{50}$$

Proof. This follows from the definition of the potential, the dimensions of the cohomology classes, and the dimensions of the moduli spaces $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$. It encodes the fact that intersection numbers between cohomology classes vanish if the classes do not have proper the dimension □

This equation encodes the fact that the potential function is invariant under the rescaling

$$t_a^m \mapsto \epsilon^{a-1+\frac{m}{r}} t_a^m, u_a \mapsto \epsilon^{2a-1} u_a, s_a \mapsto \epsilon^a s_a, \text{ and } \lambda \mapsto \epsilon^{-1-\frac{1}{r}} \lambda.$$

This grading shows that our small phase space potential function *cannot* arise as the small phase space potential associated to the Gromov-Witten invariants of a smooth, projective variety. This is because the elements in $\mathcal{H}^{(r)}$ have fractional dimension with respect to this Euler vector field, whereas cohomology classes of a space always have integral dimensions.

5.2. The string equation and its cousins.

We begin by proving the analog of the string equation, the dilaton equation, and a new equation arising from our identity on the μ_1 class.

Theorem 5.3. *The following identities are satisfied:*

$$\langle \tau_{0,0} \tau_{0,m_1} \tau_{0,m_2} \exp(\mathbf{s} \cdot \boldsymbol{\mu} + \mathbf{u} \cdot \boldsymbol{\nu}) \rangle_0 = \eta_{m_1,m_2} \tag{51}$$

and

$$\langle \tau_{0,0} \tau_{a_1,m_1} \cdots \tau_{a_n,m_n} \exp(\mathbf{s} \cdot \boldsymbol{\mu} + \mathbf{u} \cdot \boldsymbol{\nu}) \rangle_g = \sum_{i=1}^{n} \langle \tau_{a_1,m_1} \cdots \tau_{a_i-1,m_i} \cdots \tau_{a_n,m_n} \exp(\mathbf{s} \cdot \boldsymbol{\mu} + \mathbf{u} \cdot \boldsymbol{\nu}) \rangle_g,$$

where we assume that the terms in the sum containing $\tau_{a,m}$ *with* $a < 0$ *vanish.*

These two equations are equivalent to the string (or puncture) equation

$$\mathcal{L}_{-1} Z = 0 \tag{52}$$

where

$$\mathcal{L}_{-1} := -\frac{\partial}{\partial t_0^0} + \sum_{0 \le m_1, m_2 \le r-2} \frac{1}{2} \eta_{m_1, m_2} t_0^{m_1} t_0^{m_2} + \sum_{\substack{0 \le m \le r-2 \\ a \ge 0}} t_{a+1}^m \frac{\partial}{\partial t_a^m}$$

Similarly, the following identities are satisfied:

$$\langle \tau_{1,0} \exp(\mathbf{s} \cdot \boldsymbol{\mu} + \mathbf{u} \cdot \boldsymbol{\nu}) \rangle_1 = \frac{r-1}{24} \tag{53}$$

and

$$\langle \tau_{1,0} \tau_{a_1, m_1} \cdots \tau_{a_n, m_n} \exp(\mathbf{s} \cdot \boldsymbol{\mu} + \mathbf{u} \cdot \boldsymbol{\nu}) \rangle_g = (2g - 2 + n) \langle \tau_{a_1, m_1} \cdots \tau_{a_n, m_n} \exp(\mathbf{s} \cdot \boldsymbol{\mu} + \mathbf{u} \cdot \boldsymbol{\nu}) \rangle_g$$

These two equations are equivalent to the dilaton equation

$$\mathcal{D} Z = 0 \tag{54}$$

where

$$\mathcal{D} = -\frac{\partial}{\partial t_1^0} + \sum_{\substack{0 \le m \le r-2 \\ a \ge 0}} t_a^m \frac{\partial}{\partial t_a^m} + \lambda \frac{\partial}{\partial \lambda} + \frac{r-1}{24} \tag{55}$$

Finally, let $\mathcal{L}_0$ be the differential operator

$$\mathcal{L}_0 := -(1 + \frac{1}{r}) \frac{\partial}{\partial t_1^0} + \sum_{\substack{0 \le m \le r-2 \\ a \ge 0}} (a + \frac{m+1}{r}) t_a^m \frac{\partial}{\partial t_a^m} + \frac{r^2 - 1}{24r} + \sum_{a \ge 1} ((2a-1) u_a \frac{\partial}{\partial u_a} + a \frac{\partial}{\partial s_a})$$

then the following equation holds:

$$\mathcal{L}_0 Z = 0. \tag{56}$$

Furthermore, $[\mathcal{L}_0, \mathcal{L}_{-1}] = \mathcal{L}_{-1}$. When restricted to the large phase space ($\mathbf{u} = \mathbf{s} = \mathbf{0}$), the operators $\mathcal{L}_0$ and $\mathcal{L}_{-1}$ become the usual generators L_0 and L_1 in the Virasoro Lie algebra.

Proof. The arguments are lifts of analogous arguments on the moduli space of stable curves. Recall that on the moduli space of stable curves, $\overline{\mathcal{M}}_{g,n}$, the ψ_i classes (where $i = 1, \ldots, n$) satisfy the $\psi_i = \pi^* \psi_i + D_{i,n+1}$ where $D_{i,n+1}$ is the image of the i-th canonical section and $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. Let p be the forgetful morphism $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}} \to \overline{\mathcal{M}}_{g,n}$, then since the class ψ_i on $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ is the pullback via p of the ψ_i class on $\overline{\mathcal{M}}_{g,n}$, one can lift the same formula to $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ to obtain $\psi_i = \tilde{\pi}^* \psi_i + D_{i,n+1}$ where this equation is now regarded as being on $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$, $D_{i,n+1}$ is the pullback via p of the divisors with the same name on $\overline{\mathcal{M}}_{g,n}$, and $\tilde{\pi}$ is the forgetful morphism $\overline{\mathcal{M}}_{g,n+1}^{1/r,\mathbf{m}\sqcup 0} \to \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$.

Suppose that $(g, n+1) \neq (0,3), (1,1)$. Using the lifting formula and canceling trivial terms, we obtain

$$\psi_j^a = \tilde{\pi}^* \psi_j^a + D_{j,n+1} \tilde{\pi}^*(\psi_j^{a-1}).$$

Since $\tilde{\pi}^* c^{1/r} = c^{1/r}$, we have

$$\begin{aligned} \langle\, \tau_{0,0}\, \tau_{a_1,m_1} \cdots \tau_{a_n,m_n} \,\rangle_g &= \lambda^{2g-2} \int_{\overline{\mathcal{M}}_{g,n+1}^{1/r,\mathbf{m}}} \psi_1^{a_1} \ldots \psi_n^{a_n}\, c^{1/r} \\ &= \sum_{1 \le i \le n} \lambda^{2g-2} \int_{\overline{\mathcal{M}}_{g,n+1}^{1/r,\mathbf{m}}} D_{i,n+1}\, \psi_1^{a_1} \cdots \psi_i^{a_i-1} \cdots \psi_n^{a_n}\, c^{1/r} \end{aligned}$$

where the right hand side is understood to vanish if an exponent is negative. Integration over the fiber of $\overline{\mathcal{M}}_{g,n+1}^{1/r,\mathbf{m}\sqcup 0} \to \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ yields the desired result. Inclusion of the additional μ and ν classes into the correlators does not change the argument since $\tilde{\pi}^* \mu_i = \mu_i$ and similarly for ν_i.

Finally, the exceptional cases follow from dimensional considerations and the fact that on $\overline{\mathcal{M}}_{0,3}^{1/r,\mathbf{m}}$, $c^{1/r}$ is the identity element in cohomology provided that $m_1 + m_2 + m_3 = r - 2$,.

The dilaton equation is proven by a similar analysis, where the exceptional case can be computed by using the explicit presentation for ψ_1 on $\overline{\mathcal{M}}_{1,1}^{1/r,0}$ to obtain

$$\langle\, \tau_{1,0} \,\rangle_1 = \frac{1}{24}\, \eta^{m_+,m_-} \langle\, \tau_{0,0}\, \tau_{0,m_+}\, \tau_{0,m_-} \,\rangle_0 = \frac{r-1}{24}. \tag{57}$$

Finally, the equation $\mathcal{L}_{-1} Z = 0$ is obtained by combining the dilaton equation and the grading equation (50). ◻

The new relation for the μ_1 class yields new equations between correlators.

Theorem 5.4. *Let $\xi : \mathcal{H}^{(r)} \to \mathcal{H}^{(r)}$ be defined by*

$$\xi_{m_+}{}^{m_-} := \frac{m_+ m_-}{r^2}\,\delta_{m_+ + m_-, r-2}$$

relative to the standard basis and $\xi^{m_+,m_-} := \eta^{m_+,m}\,\xi_m{}^{m_-}$. The following differential equation holds:

$$\frac{\partial\Phi}{\partial s_1} = \frac{r^2-6r+6}{r}\frac{\partial\Phi}{\partial u_1} + \sum_{\substack{a\geq 0\\ 0\leq m_+,m_-\leq r-2}} \frac{1}{2} t_a^{m_+}\,\xi_{m_+}{}^{m_-}\,\frac{\partial\Phi}{\partial t_{a+1}^{m_-}}$$
$$- \sum_{m_+,m_-} \frac{1}{2}\,\frac{\partial\Phi}{\partial t_0^{m_+}}\,\xi^{m_+,m_-}\,\frac{\partial\Phi}{\partial t_0^{m_-}} - \sum_{m_+,m_-} \frac{1}{4}\,\frac{\partial^2\Phi}{\partial t_0^{m_+}\partial t_0^{m_-}}\,\xi^{m_+,m_-}$$

where the summation over m_+ and m_- runs over $0,\dots,r-2$. This is equivalent to the following relations between the correlators:

$$\langle\langle\, \tau_{a_1,m_1}\,\cdots\,\tau_{a_n,m_n}\,\mu_1\,\rangle\rangle_g = \frac{r^2-6r+6}{r^2}\,\langle\langle\, \tau_{a_1,m_1}\,\cdots\,\tau_{a_n,m_n}\,\nu_1\,\rangle\rangle_g$$
$$+ \sum_{\substack{1\leq i\leq n\\ m_i'}} \frac{1}{2}\,\xi_{m_i}{}^{m_i'}\,\langle\langle\, \tau_{a_1,m_1}\,\cdots\,\tau_{a_i+1,m_i'}\,\cdots\,\tau_{a_n,m_n}\,\rangle\rangle_g$$
$$- \sum_{\substack{I_+\sqcup I_-=[n]\\ g_++g_-=g\\ m_+,m_-}} \frac{1}{2}\,\langle\langle\, (\prod_{i\in I_+}\tau_{a_i,m_i})\tau_{0,m_+}\,\rangle\rangle_{g_+}\,\xi^{m_+,m_-}\,\langle\langle\, \tau_{0,m_-}\,(\prod_{j\in I_-}\tau_{a_j,m_j})\,\rangle\rangle_{g_-}$$
$$- \sum_{m_+,m_-} \frac{1}{4}\,\langle\langle\, \tau_{a_1,m_1}\,\cdots\,\tau_{a_n,m_n}\,\tau_{0,m_+}\tau_{0,m_-}\,\rangle\rangle_{g-1}\,\xi^{m_+,m_-},$$

where we use the notation

$$[n] = \{1,2,\dots,n\}.$$

The class ν_1 which appears above is precisely λ_1. This class vanishes on $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ for $g = 0$.

Proof. The proof follows from the facts that $p_*(c^{1/r}\exp(\mathbf{s}\boldsymbol{\mu}))$ forms a CohFT, ν and ψ are lifts of the analogous classes on the moduli space of stable curves, and Proposition 2.4. □

5.3. Topological recursion relations.

Topological recursion relations are relations between correlators which arise from presentations of tautological classes in terms of boundary classes in low genera.

Theorem 5.5. *The following topological recursion relations hold in genus zero:*

$$\langle\langle\, \tau_{a_1+1,m_1} \cdots \tau_{a_n,m_n} \,\rangle\rangle_0 =$$
$$\sum_{\substack{I_+ \sqcup I_- = [n-2] \\ m_+,m_-}} \langle\langle\, \tau_{a_1,m_1} (\prod_{i\in I_+} \tau_{a_i,m_i}) \tau_{0,m_+} \,\rangle\rangle_0\, \eta^{m_+,m_-} \,\langle\langle\, \tau_{0,m_-} \tau_{a_2,m_2} \tau_{a_3,m_3} \prod_{j\in I_-} \tau_{a_j,m_j} \,\rangle\rangle_0$$

which is equivalent to the differential equation

$$\frac{\partial^3 \Phi_0}{\partial t^{m_1}_{a_1+1} \partial t^{m_2}_{a_2} t^{m_3}_{a_3}} = \sum_{m_+,m_-} \frac{\partial^2 \Phi_0}{\partial t^{m_1}_{a_1} \partial t^{m_+}_0} \eta^{m_+,m_-} \frac{\partial^3 \Phi_0}{\partial t^{m_-}_0 \, \partial t^{m_2}_{a_2} \, \partial t^{m_3}_{a_3}}$$

Proof. On $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{0,n}$, the class ψ_1 can be written in terms of boundary classes as

$$\psi_1 = \sum_{\substack{I_+ \sqcup I_- = [n] \\ n-1, n \in I_+ \\ 1 \in I_-}} \delta_{0;I_+}.$$

This equation is obtained from lifting the analogous relation on $\overline{\mathcal{M}}_{0,n}$. The classes ψ_i can be written similarly by applying an element of the permutation group S_n. The recursion relation follows from this presentation and the restriction properties of the ψ_i to the boundary strata. □

Theorem 5.6. *The following topological recursion relation holds in genus one:*

$$\langle\langle\, \tau_{a_1+1,m_1} \tau_{a_2,m_2} \cdots \tau_{a_n,m_n} \,\rangle\rangle_1 = \frac{1}{24} \langle\langle\, \tau_{a_1,m_1} \tau_{a_2,m_2} \cdots \tau_{a_n,m_n} \tau_{0,m_+} \tau_{0,m_-} \,\rangle\rangle_0 \eta^{m_+,m_-}$$
$$+ \sum_{I_+ \sqcup I_- = \{2,\dots,n\}} \langle\langle\, \tau_{a_1,m_1} (\prod_{j\in I_+} \tau_{a_j,m_j}) \,\tau_{0,m_+} \,\rangle\rangle_0 \,\eta^{m_+,m_-} \,\langle\langle\, \tau_{0,m_-} (\prod_{i\in I_-} \tau_{a_i,m_i}) \,\rangle\rangle_1$$

It is equivalent to

$$\frac{\partial \Phi_1}{\partial t^{m_1}_{a_1+1}} = \frac{1}{24} \sum_{m_+\,,\,m_-} \frac{\partial^3 \Phi_0}{\partial t^{m_1}_{a_1} \partial t^{m_+}_0 \partial t^{m_-}_0} \eta^{m_+,m_-}$$
$$+ \sum_{m_+,m_-} \frac{\partial^2 \Phi_0}{\partial t^{m_1}_{a_1} \partial t^{m_+}_0} \eta^{m_+,m_-} \frac{\partial \Phi_1}{\partial t^{m_-}_0}$$

The topological recursion relation for $\nu_1 = \lambda_1$ *from* [25]

$$\langle\langle\, \tau_{a_1,m_1} \cdots \tau_{a_n,m_n} \nu_1 \,\rangle\rangle_1 =$$
$$\frac{1}{24} \sum_{m_+,m_-} \eta^{m_+,m_-} \langle\langle\, \tau_{a_1,m_1} \cdots \tau_{a_n,m_n} \tau_{0,m_+} \tau_{0,m_-} \,\rangle\rangle_0$$

can be written as

$$\frac{\partial \Phi_1}{\partial u_1} = \frac{1}{24} \sum_{m_+,m_-} \eta^{m_+,m_-} \frac{\partial^2 \Phi_0}{\partial t_0^{m_+} \partial t_0^{m_-}}.$$

Proof. The proof of the first topological recursion relation arises from the relation on $\overline{\mathcal{M}}_{1,n}^{1/r,\mathbf{m}}$

$$\psi_1 = \frac{1}{12}\delta_{irr} + \sum_{\substack{I_+ \sqcup I_- = [n] \\ n-1, n \in I_+ \\ 1 \in I_-}} \delta_{0;I_+}$$

which is obtained from lifting the analogous relation from $\overline{\mathcal{M}}_{1,n}$. The action of S_n yields ψ_i. This combined with the restriction properties of the ψ classes yields the desired result. The second comes from the presentation of λ_1 on $\overline{\mathcal{M}}_{1,n}^{1/r,\mathbf{m}}$

$$\lambda_1 = \frac{1}{12}\delta_{irr}$$

and the restriction properties of $\lambda_1 = \nu_1$. □

These two relations allow one to completely reduce the large phase space potential to the small phase space potential in genus zero and one. Combined with the previous equations, one can compute $\Phi(\mathbf{t}, \mathbf{s}, \mathbf{u})$ in genus zero and one when one sets $s_i = u_i = 0$ for all $i \geq 1$.

The genus one potential satisfies a generalization of the WDVV equation due to Getzler [16] which arises from relations between codimension-two boundary classes on $\overline{\mathcal{M}}_{1,4}$. Using this equation, Dubrovin-Zhang [11] showed that if the Frobenius manifold is semisimple, then the genus-one potential is determined by the Frobenius structure. Since the Frobenius structure on $(\mathcal{H}^{(r)}, \eta)$ associated to $\Phi_0(\mathbf{x})$ is isomorphic to the Frobenius structure on the base of versal deformation of the A_{r-1} singularity and, therefore, is semisimple [10], the genus 1 potential $\Phi_1(\mathbf{x})$ is completely determined by $\Phi_0(\mathbf{x})$. However, in this case, the small phase potential in genus greater than zero must vanish due to dimensional considerations.

There exist presentations of products of ψ classes in terms of boundary classes [17, 7] which give rise to topological recursion relations in genus 2, but in general, they do not allow one to reduce the computation of the large phase space potential to the computation of the small phase space potential.

6. The Relationship to the KdV_r Hierarchy

In this section, we show that the genus zero, large phase space potential in the previous section corresponds to the τ function of the KdV_r hierarchy.

Throughout this section, We shall restrict ourselves to the large phase space potential $\Phi(\mathbf{t})$ where the other variables u_a and s_a are all set to zero. We will also restrict ourselves to the case where $\lambda = \frac{1}{\sqrt{r}}$ since this is the value of the coupling constant which is relevant to the generalized Witten conjecture.

We follow the outline provided by Witten [40] and are able to rigorously prove the validity of his computations now that the relevant moduli spaces and classes have been constructed.

6.1. **The large phase space potential in genus zero.**

The following proposition rigorously demonstrates the formulas from [40] but the idea of our proof is quite different, as it uses our new relation for the μ_1 class.

Proposition 6.1. *The following identities hold:*

$$\langle\, \tau_{0,m_1}\, \tau_{0,m_2}\, \tau_{0,m_3}\, \rangle_0 \;=\; \delta_{m_1+m_2+m_3,r-2}$$

and

$$\langle\, \tau_{0,m_1}\, \ldots\, \tau_{0,m_4}\, \rangle_0 \;=\; \frac{1}{r}\,\mathrm{Min}_{1\le i\le 4}(m_i, r-1-m_i)$$

where Min *is minimum value.*

Proof. $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{0,n}$ is nonempty if and only if $(2+\sum_i m_i)/r \in \mathbb{Z}$ where $\mathbf{m} = (m_1,\ldots,m_n)$ and $m_i = 0\ldots r-1$ for all i. The genus zero correlators are given by

$$\langle\, \tau_{0,m_1}\, \ldots\, \tau_{0,m_n}\, \rangle_0 \;=\; r\int_{\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{0,n}} c^{1/r}$$

where $c^{1/r} = c_D(-R^1\pi_*\mathcal{E}_r)$ is the (top) Chern class of degree

$$D = -1 + \frac{2}{r} + \frac{1}{r}\sum_i m_i.$$

The class $c^{1/r}$ vanishes unless $m_i = 0, \dots r-2$ for all i by Theorem 4.5. Furthermore, the correlator can only be nonzero if $D = n-3$.

If $n = 3$ then the dimensionality condition becomes $m_1 + m_2 + m_3 = r-2$, in which case $c^{1/r}$ is the identity. This proves the first part of the proposition.

If $n = 4$ then the dimensionality condition becomes

$$m_1 + \dots + m_4 = 2r-2.$$

If this condition is satisfied then $c^{1/r} = \mu_1$. The correlator is

$$\langle\, \tau_{0,m_1} \dots \tau_{0,m_4} \,\rangle_0 = r \int_{\overline{\mathcal{M}}_{0,4}^{1/r,\mathbf{m}}} \mu_1.$$

The right hand side can be computed using the relation for the class μ_1 in Proposition 2.4, which becomes, in genus zero,

$$\begin{aligned} \mu_1 = & \sum_{1 \le i \le n} \frac{m_i(r-2-m_i)}{2r^2} \psi_i \\ & + \sum_{I_+ \subset [n]} \frac{r-1-(m_+ + 1)(m_- + 1)}{2r^2}\, \delta_{0;I_+} \end{aligned}$$

where m_+ and m_- are uniquely determined by the divisor $\delta_{0:I_+}$. Let $\delta_{ij,kl}$ denote the divisor $\delta_{0;\{\,i,j\,\}}$ on $\overline{\mathcal{M}}_{0,4}^{1/r,\mathbf{m}}$.

Plugging in this formula, one obtains (after doing a little case by case analysis to write m_+ and m_- in terms of $m_1, \dots, m_4$)

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{0,4}^{1/r,\mathbf{m}}} \mu_1 = & \sum_{i=1}^{4} \frac{m_i(r-2-m_i)}{2r^2} \int_{\overline{\mathcal{M}}_{0,4}^{1/r,\mathbf{m}}} \psi_i \\ & + \frac{r-1-(\chi_{12,34}+1)(r-1-\chi_{12,34})}{2r^2} \int_{\overline{\mathcal{M}}_{0,4}^{1/r,\mathbf{m}}} \delta_{12,34} \\ & + \frac{r-1-(\chi_{13,24}+1)(r-1-\chi_{13,24})}{2r^2} \int_{\overline{\mathcal{M}}_{0,4}^{1/r,\mathbf{m}}} \delta_{13,24} \\ & + \frac{r-1-(\chi_{14,23}+1)(r-1-\chi_{14,23})}{2r^2} \int_{\overline{\mathcal{M}}_{0,4}^{1/r,\mathbf{m}}} \delta_{14,23} \end{aligned}$$

where $\chi_{ij,kl} := Min(m_i + m_j, m_k + m_l)$. Since each $\delta_{ij,kl}$ is Poincaré dual to the (topological) homology class represented by a point, one has

$$r\int_{\overline{\mathcal{M}}_{0,4}^{1/r,\mathbf{m}}} \delta_{ij,kl} \;=\; 1.$$

Similarly, each class ψ_i can be represented by $\delta_{ij,kl}$ for some i, j, k, l. Therefore, one obtains

$$\begin{aligned} r\int_{\overline{\mathcal{M}}_{0,4}^{1/r,\mathbf{m}}} \mu_1 \;=\; & \sum_{i=1}^{4} \frac{m_i(r-2-m_i)}{2r^2} \\ &+ \frac{r-1-(\chi_{12,34}\;+1)(r-1-\chi_{12,34})}{2r^2} \\ &+ \frac{r-1-(\chi_{13,24}\;+1)(r-1-\chi_{13,24})}{2r^2} \\ &+ \frac{r-1-(\chi_{14,23}\;+1)(r-1-\chi_{14,23})}{2r^2}. \end{aligned}$$

The right hand side of this equation can be shown to be equal to $\frac{1}{r}\operatorname{Min}_{1\leq i\leq 4}(m_i, r-1-m_i)$, an elementary but not obvious identity.

□

Proposition 6.2. [40] *The genus zero potential $\Phi_0(\mathbf{t})$ is completely determined by $\langle\, \tau_{a_1,m_1}\, \tau_{a_2,m_2}\, \tau_{a_3,m_3}\, \tau_{a_4,m_4}\,\rangle_0$ and the fact that*

$$\langle\, \tau_{a_1,m_1}\, \tau_{a_2,m_2}\, \tau_{a_3,m_3}\,\rangle_0 \;=\; \delta_{m_1+m_2+m_3,r-2}.$$

Proof. The large phase space, genus zero potential $\Phi_0(\mathbf{t})$ is completely determined by its values on the small phase space by the topological recursion relations. Let $\Phi_0(\mathbf{x})$ denote the small phase space potential. $\Phi_0(\mathbf{x})$ must satisfy the WDVV equation since $c^{1/r}$ yields a CohFT. Furthermore, the grading equation (50) shows that the small phase space potential is a polynomial in the variables $\{\, x^0, \,\dots\,, x^{r-2}\,\}$ of degree of at most $r+1$. One then performs an induction on the degree of the polynomial to show that $\Phi_0(\mathbf{x})$ is uniquely determined by the above data and the WDVV equation. The proof is straightforward. □

6.2. The KdV_r hierarchy and the generalized Witten conjecture.

In this section, we state the generalized Witten conjecture in the language of Gelfand-Dickey (see [32] for a nice overview). Let $D := \frac{i}{\sqrt{r}}\frac{\partial}{\partial x}$ and consider the space of (formal) differential operators

$$V \;:=\; \{\, L \;:=\; D^r \;-\; \sum_{i=0}^{r-2} u_i(x)\, D^i\,\}$$

where $u_i(x)$ are formal functions of x. A differential operator $L \in V$ has an r-th root $L^{1/r}$ in the space of (formal) pseudodifferential operators, *i.e.* there exists a pseudodifferential operator

$$L^{1/r} = D + \sum_{i>,0} w_i D^{-i}$$

where each w_i is a differential polynomial in u_i such that $(L^{1/r})^r = L$.

A given pseudodifferential operator A has a maximal piece which is a differential operator, denoted by A_+. The KdV_r *(or the r-th Gelfand-Dickey) hierarchy* is an infinite family of formal commuting flows on V satisfying the Gelfand-Dickey equations

$$i \frac{\partial L}{\partial t_a^m} = [(L^{n+(m+1)/r})_+, L] \frac{k_{n,m}}{\sqrt{r}} \tag{58}$$

where

$$k_{n,m} := \frac{(-1)^n r^{n+1}}{(m+1)(r+m+1)\cdots(nr+m+1)}$$

and the brackets denotes the commutator in the ring of differential operators.

For all $m = 0, \ldots, r-2$, let

$$v_m := -\frac{r}{m+1} \operatorname{res}(L^{(m+1)/r}) \tag{59}$$

where $\operatorname{res} A$ picks off the coefficient of D^{-1} in the pseudodifferential operator A. The v_m can be written as differential polynomials in the u_i's and conversely.

The KdV_r equations reduce in genus zero to an integrable hierarchy which can be regarded as the semiclassical (or dispersionless) limit of the KdV_r hierarchy [10, 40]. This hierarchy is obtained by replacing all appearances of the differential operator D by a formal variable p and the commutator of differential operators $[L, M]$ is replaced by the Poisson bracket $\frac{i}{\sqrt{r}} \{ \tilde{L}, \tilde{M} \}$ where

$$\{ \tilde{L}, \tilde{M} \} := \frac{\partial \tilde{L}}{\partial p}\frac{\partial \tilde{M}}{\partial x} - \frac{\partial \tilde{L}}{\partial x}\frac{\partial \tilde{M}}{\partial p}.$$

The semiclassical limit of the KdV_r hierarchy is

$$\frac{\partial \tilde{L}}{\partial t_a^m} = \{(\tilde{L}^{n+(m+1)/r})_+, \tilde{L}\} \frac{k_{n,m}}{r}. \tag{60}$$

As before, one can define v_m using equation (59) and the u_i's can be written as differential polynomials in the v_m's and conversely. Putting everything together, we have proven the following.

Theorem 6.3. *(The generalized Witten conjecture in genus zero) Let*

$$x := t_0^0 \text{ and } v_m := \frac{\partial^2 \Phi_0(\mathbf{t})}{\partial t_0^0 \partial t_0^m} \tag{61}$$

for $m = 0, \ldots, r-2$ then it solves equation (60). This is the unique solution of the hierarchy satisfying the string equation [40].

Proof. Our large phase space potential in genus zero $\Phi_0(\mathbf{t})$ agrees with Witten's expression, and he proves [40] that this generating function is a solution of this hierarchy using a straightforward computation. □

Corollary 6.4. *The Frobenius manifold structure on $(\mathcal{H}^{(r)}, \eta)$, which exists by Theorems 3.8 and 4.5, is isomorphic to the Frobenius structure on the base of the versal deformation of the A_{r-1} singularity.*

The generalized Witten conjecture in all genera is the following.

Conjecture 6.5. *(The generalized Witten conjecture) The generalization of equation (61) to all genera solves the KdV_r hierarchy (equation (58)).*

This solution is the unique one which satisfies the string equation. The differential operator L constructed from these v_m is therefore computable and it determines the large phase space potential.

Remarks 6.6.

1. The generalized Witten conjecture as it is stated here should be viewed as a refinement of the original Witten's formulation [40], since it is not clear that his construction yields a class with the desired factorization properties.

2. The original Witten's conjecture proved by Kontsevich is a particular case of the Conjecture 6.5 corresponding to the case $r = 2$. Indeed, for $r = 2$ the KdV_r hierarchy is the original KdV hierarchy. On the other hand, our analysis in Section 4.5 shows that for $r = 2$ and $\mathbf{m} = \mathbf{0}$ we have $c^{1/r} = \pm 1$ depending on the parity of the theta characteristic (2-spin structure), and $c^{1/r} = 0$ when $\mathbf{m} \neq \mathbf{0}$. Therefore, the potential $\Phi(\mathbf{t})$ in this case coincides with the potential given by $\overline{\mathcal{M}}_{g,n}$ (cf. [40]).

3. According to general philosophy of mirror symmetry (cf. [34]), the coincidence of Frobenius structures given by Corollary 6.4 is a genus zero manifestation of some mirror phenomenon relating the moduli space of r-spin curves and singularities of type A_{r-1}. The generalized Witten conjecture should improve understanding of this relationship.

Finally, the unique τ function of the KdV_r hierarchy satisfying the string equation is annihilated by a series of differential operators (which forms a W_r^+-algebra) [3] part of which forms a subalgebra isomorphic to (half of) the Virasoro algebra. This yields an alternate formulation of the generalized Witten conjecture.

Conjecture 6.7. *(W-algebra conjecture) There exist a collection of differential operators forming a W_r^+ algebra (in which the generators $\{L_n\}_{n\geq -1}$ of the Virasoro algebra forms a subset) which annihilates and completely determines $Z(\mathbf{t})$.*

This can be regarded as the KdV_r analog of a refinement of the Virasoro conjecture [12]. When $r = 2$, this conjecture reduces to the usual Virasoro highest weight condition.

There is clearly much work to be done to be able to prove the conjecture in all genera.

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