The Fundamental Lemma

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The fundamental lemma is a celebrated result in the theory of automorphic forms, it is the key result that is needed for the stabilization of the trace formula. It has been formulated by Langlands and Diana Shelstad and it has been proved recently by Ngo Bau Chau, the final proof is based on the work of many other mathematicians.

My aim in this talk was to explain the meaning of the fundamental lemma to general audience.

In a certain sense the fundamental lemma was explained by Langlands in a talk at the Arbeitstagung in the early 1970-th. In this talk Langlands reported on his joint paper with Labesse with title "L-indistinguishability for SL(2)." They discovered the phenomenon that two automorphic representations which have the same L function may occur with different multiplicities in the space of automorphic forms.

At the same Arbeitstagung Hirzebruch proved the following theorem:

Let p be a prime which is $3 \bmod 4$ let $F = \mathbb{Q}[\sqrt{p}]$. Let $\mathcal{O}$ be its ring of integers, then the group $Sl_2(\mathcal{O})$ acts on the product of two upper half planes $H^+ \times H^+$, the compactification of the quotient $Sl_2(\mathcal{O})\backslash H^+ \times H^+$ yields a Hilbert-Blumenthal surface S^{++}. But it also acts on $H^+ \times H^-$ and we get a second such surface S^{+-}. Then we have a discrepancy between the spaces of holomorphic 2-forms:

$$\dim H^0(S^{++}, \Omega^2) - \dim H^0(S^{+-}, \Omega^2)) = h(\sqrt{-p})$$

where the number on the right is the class number of $\mathbb{Q}(\sqrt{-p})$.

The elements in these spaces provide L-indistinguishabie automorphic forms.

In their paper Labesse and Langlands used a baby version of the fundamental lemma for the group Sl_2. This was not so difficult to prove, but after that it turned out to be incredibly difficult to prove generalizations of this fundamental lemma for other reductive groups.

The fundamental lemma was formulated as a very precise conjecture and looks like that (here G is a reductive group over $\mathbb{Q}$)

$$\sum_{\xi_p} \int_{Z_{\gamma\cdot\xi_p}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} f_p(x_p^{-1}\gamma \cdot \xi_p x_p)\kappa_p(\xi_p)\epsilon(\xi_p)dx_p =$$

$$\Delta_p(\gamma,\kappa) \int_{Z_{\gamma H^\kappa}(\mathbb{Q}_p)\backslash H^\kappa(\mathbb{Q}_p)} f_p^{H^\kappa}(y_p^{-1}\gamma y_p)dy_p$$

here κ is -under certain conditions a character on a finite abelian group from which the ξ_p are taken. The H^κ is the so called endoscopic group attached to κ, the factor in front is the transfer factor. The integrals on the left are the κ orbital integrals, the integral for the trivial character $\kappa = 1$ is called the stable orbital integral for the group G and the fundamental lemma says that a κ orbital integral -which now is unstable if $\kappa \neq 1$- is up to the tranfer factor equal to a stable orbital integral on the endoscopic group.

The fundamental lemma enters the stage if we apply the trace formula (Arthur-Selberg or topological trace formula) to compute the trace of a Hecke operator T_h on a certain space X of functions or on some cohomology groups. Then we encounter certain sums of orbital integrals which can be manipulated to become a sum of κ -orbital integrals. Eventually we get

$$tr(T_h|H) = \sum_{\kappa} tr(T_{h^\kappa}^{H^\kappa}|X^{H^\kappa}) = T_h^G|X + \sum_{\kappa\neq 1} tr(T_{h^\kappa}^{H^\kappa}|X^{H^\kappa})$$

where the terms on the right hand side are stable.

If our group is $G/\mathbb{Q} = R_{F/\mathbb{Q}}(Sl_2/F)$ as above then the torus $H/\mathbb{Q}$ of norm one elements in $F^\times$ is endoscopic and the resulting term on the right side explains the class number in the difference of dimensions.

Ngo Bao Chau: Le lemme fondamental pour les algebres de Lie, arXiv:0801.0446