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by

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Introduction

Every algebraic variety X can be regarded as a real symplectic manifold. Indeed, by the very definition X has to admit an embedding to the projective space of certain dimension N. This implies that one can restrict to the embedded X any standard Fubini - Study metric which gives the corresponding Kahler form ω on X which can be regarded as the symplectic form. Every such an embedding is characterized by the homological data — the corresponding class $[\omega] \in H^2(X, \mathbb{Z})$ or the first Chern class $c_1(L)$ of the corresponding very ample line bundle L, which admits a sufficiently big space of holomorphic sections.

Such a form and such a metric is not unique in general: possible X can admit several different classes of the symplectic forms, but if we restrict our study considering compact and simply connected smooth algebraic varieties then different symplectic structures correspond to the classes in $H^2(X, \mathbb{Z})$. For any such a structure it is reasonable to study lagrangian submanifolds and try to form certain moduli spaces of such lagrangian submanifolds.

The choice of such $[\omega]$ is well known in the theory of (semi) stable holomprhic vector bundles: the choice of a principal polarization leads to the definition of finite dimensional moduli spaces, see [1]. In contrast, if we would like to activate symplectic geometry constructions it should be choosen a form ω, not a class $[\omega]$. But if we are interested in lagrangian geometry, the geometry of lagrangian submanifolds and subcycles and their classifications, it is not hard to see, that for different particular choices of the homologically equivalent forms ω, ω' in the case of simply connected X the submanifolds, lagrangian with respect to ω and ω' are related by certain natural transformations. Therefore if our aim is to derive certain finite dimensional moduli spaces from the lagrangian geometry, then for different choices ω or ω' the topology of the corresponding moduli spaces must be the same.

Suppose now that we consider a compact simply connected algebraic variety X of dimension n and fix the Kahler form ω of a Kahler metric of the Hodge type. From the real geometry point of view it corresponds to $2n$ dimensional real compact simply connected manifold M equipped with a symplectic form ω with integer cohomology class $[\omega] \in H^2(M,\mathbb{Z})$ and an integrable complex structure I, compatible with ω.

Then we say that

Definition 0. *A real submanifold $S \subset M$ is lagrangian iff $dim S = n$ and $\omega|_S \equiv 0$.*

Since the symplectic form is of the integer type, $[\omega] \in H^2(M,\mathbb{Z})$, one can impose a natural condition on a lagrangian submanifold $S \subset M$. To do this consider a prequantization pair (L, a), where $L \to M$ is a line bundle with a fixed hermitian structure h and $a \in \mathcal{A}_h(L)$ is a hermitian connection such that its curvature form $F_a = 2\pi i\omega$. This means in particular that $c_1(L) = [\omega]$.

Then

Definition 1. *A lagrangian submanifold $S \subset M$ is said to be Bohr - Sommerfeld iff the restriction $(L,a)|_S$ admits covariantly constant sections.*

Since we want to work with not only smoothly embedded lagrangian manifolds, but with cycles, submanifolds with simplest singularities, we reformulate the last definition in the following form

Definition 1'. *A lagrangian submanifold $S \subset M$ is said to be Bohr - Sommerfeld iff for any loop $\gamma \subset S$ and any disc $D \subset M$ bounded by this loop, one has $\int_D \omega \in \mathbb{Z}$.*

In the smooth situation the definitions are equivalent; at the same time the last one is applicable in non smooth case.

Recall, that in [2] one constructs the moduli space $\mathcal{B}_S$ of Bohr - Sommerfeld lagrangian cycles of fixed topological type. This is a smooth infinite dimensional real manifold; it depends on the topological data $([S], topS)$, where $[S] \in H_n(M,\mathbb{Z})$ is the corresponding homology class and $topS$ is the topological type of S as a "parametrization space". The details can be found in [2].

Above we were speaking about "finite dimensional moduli spaces", and the moduli space $\mathcal{B}_S$ is not of the desired type. The main example and pattern in our story is the celebrated SpLAG construction, proposed by N. Hitchin and J. MacLean, see [3]. Recall briefly the setup and main steps.

Consider a Kahler Calabi - Yau variety X of complex dimension n; by the very definition it can be equipped with a top holomorphic form θ, which is unique up to $\mathbb{C}^*$. The Kahler form is regarded as a symplectic form, and one says that a lagrangian submanifold $S \subset X$ is special iff it is a phase rotation $e^{i\phi}$ such that $\mathrm{Im} e^{i\phi}\theta|_S \equiv 0$ (in [3] one can find other equivalent definitions).

The main advantige of this specialty condition — it leads to finite dimensional moduli spaces of special lagrangian submanifolds, since as it was shown in [3], the local deformations of SpLag submanifolds are unobstructed forming $b_1(S)$ - dimensional space. Therefore the moduli space of SpLag submanifolds has dimension $b_1(S)$. Special lagrangian geometry was exploited in SYZ approach to Mirror Symmetry conjecture, see [4].

As it was pointed out by A. Tyurin, the Bohr - Sommerfeld condition (for compact simply connected algebraic Calabi - Yau's) is transversal to the specialty condition. This means that in the moduli space of special lagrangian submanifolds in an algebraic CY variety one has a finite set of isolated points which corresponds to special Bohr - Sommerfeld lagrangian submanifolds.

Our main goal is to extend this observation to much more wider class of algebraic varieties. This means that we propose a programme which combines

Special Lagrangian and Bohr - Sommerfeld lagrangian geometries, and this combination can be realized over any compact simply connected algebraic variety. As the main result we get certain moduli spaces of special Bohr - Sommerfeld lagrangian cycles, covering open subsets in projective spaces. Unfortunately our construction being applied to the Calabi - Yau case is not equivalent to the Hitchin - MacLean construction, but nevertheless it is applicable in this case.

The material of these notes is organized as follows. First, we present the general theory of Special Bohr - Sommerfeld lagrangian geometry (or SBS geometry for short) which arose in real symplectic case. Here we discuss how SBS - geometry can be applied in Geometric Qunatization. Then in Section 2 we specialize the story to pure algebraic geometrical setup. In particular we discuss natural problems arose in the framework of SBS - geometry in the simplest algebraic case — for algebraic curves (thus it is a version of SBS - geometry for non simply connected case). Furthemore in Section 3 we discuss the problem of the construction of moduli spaces. Here the possible way for correct definition is based on the Eliashberg conjecture about exact largangian submanifolds. We modify the direct definition of the moduli space and prove that the modified moduli space is smooth open Kahler variety, (and formulate the main conjecture: the modified moduli space is an open part of an algerbaic variety).

I started this text in November 2016 at the Max- Planck- Institute hoping that all technical problems which appear in the way can be easily solved after discussions with experts in different parts of modern mathematics. But almost every discussion shows that the corresponding problem is widely open and this could not help in my way; at the same time every discussion helped me to find a good trick to avoid the corresponding problem and make a small step in the progamme. It is the reason why this text was not finished in time and the story is not completely finished yet. Nevertheless today I can say that the moduli space of Special Bohr - Sommerfeld cycles exists. And it is possible to say only after the expression of my cordial thank and gratitude to Anton Zorich, Ioan Marcut, Martin Schlichenmaier, Fedor Bogomolov, Andrei Shafarevich, Yakov Eliashberg, Leonid Polterovich, Jorgen Andersen, Dmitry Orlov, Stefan Nemirovsky, Aleksander Kuznetsov, Yuri Prokhorov. I would like to thank the Max - Planck - Institute for Mathematics where the first part of this work was done for excellent working condition and friendel athmosphere.

1 Special Bohr - Sommerfeld lagrangian cycles: general theory

Consider a compact simply connected $2n$ - dimnesional symplectic manifold (M, ω) with integer symplectic form, so $[\omega] \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R})$, and the prequantization pair (L, a) where $a \in \mathcal{A}_h(L)$ such that $F_a = 2\pi\omega$. Let $\Gamma(M, L)$ be the space of all smooth sections of L; it is a Hilbert space with the hermitian product

$$< s_1, s_2 >= \int_M (s_1, s_2)_h d\mu_L$$

where $(;)_h$ is our fixed hermitian structure on L which defines pointwise hermitian product for sections.

Fix topological data: an oriented closed source submanifold S_0 and a homology class $[S] \in H_n(M, \mathbb{Z})$, and apply ALAG programme from [2], giving us as the output the moduli space $\mathcal{B}_S$ of Bohr - Sommerfeld lagrangian cycles of fixed topological type (see [2]). Recall, that it is an infifnte dimensional Frechet - smooth manifold, locally modelled by the space $C^\infty(S_0, \mathbb{R})$ modulo constants. Strictly speaking the moduli space $\mathcal{B}_S$ consists of classes, but below we will take in mind the geometrical meaning, so we will understand unparameterized Bohr - Sommerfeld lagrangian submaniflods as points of $\mathcal{B}_S$ in the cases when it does not misslead.

Then

Definition 2. *We say that a Bohr - Sommerfeld lagrangian submanifold $S \subset M$ is special with respect to a smooth section $\alpha \in \Gamma(M, L)$ iff $\alpha|_S = e^{ic} f \sigma_S$ where σ_S is a covariantly constant section of $(L, a)|_S$, c is a real constant, and $f \in C^\infty(S, \mathbb{R}^+)$ is a real strictly positive function.*

As Definition 1 above the last definition is valid for smooth lagrangian submanifolds only, therefore we need to reformulate it for applications in singular cases.

To do this we need the following observation. For any smooth section $\alpha \in \Gamma(M, L)$ define complex 1-form

$$\rho(\alpha) = \frac{\nabla_a \alpha}{\alpha} = \frac{(\nabla_a \alpha, \alpha)_h}{(\alpha, \alpha)_h} \in \Omega^1_{M \backslash D_\alpha} \otimes \mathbb{C}.$$

Here D_α is the zeroset of α.

Simple computation shows that for this form $\rho(\alpha)$ we have

$$\mathrm{Re}\rho(\alpha) = d(ln|\alpha|), \qquad d(\mathrm{Im}\rho(\alpha)) = 2\pi\omega$$

on the complement $M \backslash D_\alpha$.

Then

Definition 2'. *An n - dimensional submanifold $S \subset M$ is special Bohr - Sommerfeld lagrangian w.r.t. a smooth section $\alpha \in \Gamma(M, L)$ iff the restriction* $\mathrm{Im}\rho(\alpha)|_S \equiv 0$.

The equivalence of Definition 2 and Definition 2' for smooth case is proved in [5]. Note however that Definition 2' is applicable in the case when S has singularities: the requirement in the last case is about the tangent vectors from the tangent cone at a singular point, and even in very complicated cases the vanishing of $\mathrm{Im}\rho(\alpha)$ at singular points is still a relevant condition of certain types. Moreover, the condition from Definition 2' is a type of calibration, which is quite known and important in Lagrangian geometry.

Therefore we will work with Definition 2'. Since the form $\rho(\alpha)$ does not change under rescaling of the section by complex constants, the specialty condition depends on the class of sections modulo $\mathbb{C}^*$ and therefore our new specialty condition cuts a subset in the direct product

$$\mathcal{U}_{SBS} \subset \mathcal{B}_S \times \mathbb{P}\Gamma(M, L),$$

which contains pairs (S, p) such that S is α - special Bohr - Sommerfel lagrangian cycle where α is a lift of point $p \in \mathbb{P}\Gamma(M, L)$ to the vector space $\Gamma(M, L)$.

The main property of $\mathcal{U}_{SBS}$ which will be exploited many times reads as follows:

Proposition 1. *Let $f \in C^\infty(M, \mathbb{R})$ be a smooth function which generates the corresponding Hamiltonian flow $\phi^t_{X_f}$. Then for any pair $(S, p) \in \mathcal{U}_{SBS}$ the transformed pair $(\phi^t_{X_f}(S), \phi^t_{X_f} p)$ again belongs to $\mathcal{U}_{SBS}$.*

In other words, Special Bohr - Sommerfeld condition is stable with respect to the Hamiltonian deformations. Here the Hamiltonain action X_f on $\mathbb{P}\Gamma(M, L)$ is a lift defined by our choosen hermitian connection a.

The proof is based on the correspondence between smooth sections up to scale and complex 1 - forms on open subsets of M. Take two smooth sections α_1, α_2; on the complement $M \backslash D_{12}$ where $D_{12} = \{\alpha_1 = 0\} \cup \{\alpha_2 = 0\}$ the equality $\rho(\alpha_1) = \rho(\alpha_2)$ implies $\alpha_1 = C\alpha_2$. Indeed, the first equality means

$$\frac{\nabla_a \alpha_1}{\alpha_1} = \frac{\nabla_a \alpha_2}{\alpha_2} \quad \Longrightarrow \alpha_2 \otimes \nabla_a \alpha_1 - \alpha_1 \otimes \nabla_a \alpha_2 = 0,$$

but it is equivalent to the condition $d\ln(\frac{\alpha_2}{\alpha_1}) = 0$ on the complement $M \backslash D_{12}$. Here $\frac{\alpha_2}{\alpha_1}$ is a nonvanishing complex function, and the differentiation rules for its logarithm leads exactly to the second equation above.

Thus the hamiltonian action on the pair (S, p) can be expressed via the Hamiltonian action on the pair $(S, \rho(\alpha))$, and the calibration condition $\mathrm{Im}\rho(\alpha)|_S = 0$ is evidently equivariant with respect to the Hamiltonian flow.

There is another discription of special Bohr - Sommerfeld submanifolds in terms of *Liouville structures*. Recall that symplectic manifold $(\tilde{M}, \omega)$ is endowed with a Lioville structure if one fixes a vector field λ such that $\mathcal{L}_\lambda \omega \equiv \omega$. In the presence of the symplectic form equally one can fix 1- form ρ such that $d\rho = \omega$. From this it is clear that $\tilde{M}$ can not be compact and we present such an $\tilde{M}$ as an open part of a compact symplectic manifold M.

In [6] one shows that Lioville structures play important role in symplectic topology; the homotopical type of M in the presence of λ is encoded in the stable part of M with respect to the flow generated by λ. This stable part is called *the core* or *the secelton* of the Liouville structure; its main property is that it is homotopical to $M \backslash D$. Indeed, it is formed by finite integral lines of the Liouville vector field Z, and since every infinte integral line must approaches to D it follows that in $M \backslash D$ after shrinking of all infinite integral lines we get the sceleton. In general the sceleton of a Liouville structure can be sufficienlt big, of dimension greater than n.

Now we claim that the situation we have studied above can be translated to the language of Liouville structures of special type: as we have seen above any sufficiently regular smooth section $\alpha \in \Gamma(M, L)$ of the prequantization bundle defines a Liouville structure on $M \backslash D_\alpha$ (here sufficient regularity means that the zeroset D_α is of dimension $2n-2$). Indeed, every such a section defines the complex 1 -form $\rho(\alpha)$ and its imaginary part gives a Liouville structure since $d\mathrm{Im}\rho(\alpha) = \omega$. We denote the corresponding vector field λ as $\lambda(\alpha)$.

Then in the language of the Liouville structures our special Bohr - Sommerfeld condition reads as follows:

Proposition 2. *A lagrangian submanifold $S \subset M$ is special Bohr - Sommerfeld with respect to a smooth section α if and only if it is stable with respect to the Liouville vector field $\lambda(\alpha)$.*

In other words, S must be contained by the sceleton of the corresponding Liouville structure.

How generic is the Liouville field defined by a smooth section?

Let λ_0 is a given Liouville structure on $M\backslash D$ for a smooth $2n-2$ - dimensional submanifold $D \subset M$ which represents the homology class Poincare dual to $[\omega] \in H^2(M, \mathbb{Z})$. Take any smooth section $\alpha \in \Gamma(M, L)$ vanishing on D and consider the corresponding Liouville field $\lambda(\alpha)$. Then the difference $\omega(\lambda(\alpha)) - \omega(\lambda_0)$ is a closed 1 - form on $M\backslash$ which represents the corresponding class $H^1(M\backslash D, \mathbb{R})$. If this class is integer then λ_0 is generated by a section from $\Gamma(M, L)$. Indeed, if the difference is integer 1- form, then this form can be presented as the logartihmic differential of a map $\psi : M\backslash D \to \mathbb{C}^*$, and if this ψ is applied to the section α as a gauge transformation of the prequantization line bundle. The section which generates λ_0, is smooth if certain additional condition holds implied on the value of $|\int_\gamma \omega(\lambda_0)|$ where γ is a small loop surrounding a point $p \in D$ in the normal bundle $N_{M/D}$. Summing up, we get two conditions: certain integrality condition (a Bohr - Sommerfeld condition on the Liouville field) and condition on the residue near D (which is again of integer type).

Therefore we can see that the set of Liouville structures generated by sections of the prequantization bundle is a sery of affine subspaces of the affine space of all Liouville structures stratified by the integer lattice $H^1(M\backslash D, \mathbb{Z})$.

Can we recognize that a given Liouville structure λ is generated by a section of the prequantization bundle without references to sections and computation of the differences as above? The answer is positive: as we have seen above if a Liouville structure is given by a smooth section then the sceleton of this structure must satisfy the Bohr - Sommerfeld condition for singular submanifolds; and if additionally the residue condition near D holds then the structure is generated by a section of the prequantization bundle.

The Bohr - Sommerfeld condition is applicable for sceletons via formulation given in Definition 1': since we study smooth vector fields (note that the symplectically dual 1 -forms are smooth) the sceletons ad hoc are presented by the union of smooth pieces, and for any piecewise smooth loop γ which lies on the sceleton we can impose the integrality condition on the symplectic area of a curved polygon in M bounded by γ. The residue condition states that the integral of $\omega^{-1}(Z)$ over a small loop in the fiber of normal bundle $(N_{D/M})_p$, surrounded $p \in D$, equals to the symplectic area of a disc, bounded by the loop in M, modulo $\mathbb{Z}$.

Coming back to the main subject of our work, we would like to show, that the space $\mathcal{U}_{SBS}$ carries several geometrical structures, and we are going to describe some of them.

By the very definition this space is projected to the first and to the second direct summands, and we consider the last projection

$$p_2 : \mathcal{U}_{SBS} \to \mathbb{P}\Gamma(M, L).$$

Note that the last projective space is endowed with a Kahler structure since we do have a hermitian product on $\Gamma(M, L)$. The main properties of this projection were studied in [5]:

Fact 1. The fibers of this projections are discrete;

Fact 2. The image of this porjection is an open subset in the projective space;

Fact 3. The differential of this projection has trivial kernel at smooth points.

Summing up the facts, one gets that the subset $\mathcal{U}_{SBS}$ at smooth points admits natural Kahler structure, lifted from the Kahler structure on the projective space. Note that the smoothness condition for S is mentioned since before we have claimed that all the story can be extended to singular points of certain complition of the moduli space $\mathcal{B}_S$: as we will see it is important for the case of algebraic varieties since in certain cases for holomorphic sections smooth special Bohr - Sommerfeld lagrangian submanifolds do not exist, and the smoothness condition should lead to empty moduli space. But in general case we can speak about smooth Bohr - Sommerfeld lagrangian submanifolds only and for these "smooth points" of $\mathcal{B}_S$ we can say that no branching appears in the covering $p_2 : \mathcal{U}_{SBS} \to \mathbb{P}\Gamma(M, L)$. Indeed, the fact is pure local, so we can study the situation restricting it to a Darboux - Weinstein neighborhood $\mathcal{O}_{DW}(S_0)$ of a fixed smooth Bohr - Sommerfeld submanifold together with a complex 1- form $\rho(\alpha_0)$. Since every Bohr - Sommerfeld lagrangian submanifold, sufficiently close to S_0 is presented by a Hamiltonain deformation of S_0, near the branching point we should have two different Hamiltonian vector fields X_{f_1}, X_{f_2} corresponding to different leaves of covering which transport $\rho(\alpha_0)$ to the same 1 - form $\rho(\alpha_t)$ such that $\mathrm{Im}\rho(\alpha_t)|_{S_1} = \mathrm{Im}\rho(\alpha_t)|_{S_2} = 0$ for Hamiltonian trasnformations $S_i = \phi^t_{X_{f_i}}(S_0)$. But S_1 and S_2 must have nontrivial intersection since they are Bohr - Sommerfeld consists of at least two points — they correspond to minimal and maximal points of function $f_1 - f_2$. Join these points by two pathes $\gamma_i \subset S_i$ and calculate the integral $\int_{\gamma_1 \cup \gamma_2} \rho(\alpha_t)$. It must be trivial since the real part of $\rho(\alpha_t)$ is exact and the imaginary part identically vanishes on both S_1 and S_2. But this integral is essentially the difference between $\max(f_1 - f_2) - \min(f_1 - f_2)$, therefore it is possible if and only if $f_1 = f_2 + \mathrm{const}$, and it implies that $S_1 = S_2$.

These arguments are similiar to the arguments from [5] where one establishes the discretness of the fibers.

Thus the space $\mathcal{U}_{SBS}$ admits a Kahler structure, lifted from the projective space. Why it is interesting ifself? We start with pure symplectic situation with (M, ω) only and get a Kahler manifold. From one world, symplectic, we arrive to another world, complex Kahler.

Another geometrical structure, naturally suggested by the very definition of $\mathcal{U}_{SBS}$, gives us a $U(1)$ bundle on the space. The construction is as follows.

Recall, see [2], that the moduli space $\mathcal{B}_S$ carries a natural $U(1)$ - bundle $\mathcal{P}_S$ which is called the Berry bundle. Over a point $S \in \mathcal{B}_S$ the fiber is given by the covariantly constant sections $\sigma_S \in \Gamma(S, L|_S)$ of unit hermitian norm at each point (since it is covariantly constant the norm is the same at each point of S). The corresponding $U(1)$ structure is inherited from the structure fixed on L at the very begining of our story. Note that simultenous rotations on the fibers of $S^1(\mathcal{P}_S)$ are correctly defined by the rotations on L.

On the other hand on the projective space $\mathbb{P}\Gamma(M, L)$ one has the (anti) tautological line bundle endowed with the corresponding hermitian structure. The associated principal $U(1)$ - bundles $S^1(\mathcal{O}(\pm 1))$ with rotations induced by the same $U(1)$ as for $S^1(\mathcal{P}_S)$ but by different rules: a point $x \in S^1(\mathcal{O}(-1)|_p$ in the fiber over $p \in \mathbb{P}\Gamma(M, L)$ is presented by a normalized smooth section $s_x \in \Gamma(M, L)$ such that $\int_M < s_x, s_x > d\mu_L = 1$, and the rotation e^{ic} maps it to $e^{ic} s_x$.

Now consider the direct product $S^1(calP_S) \times S^1(\mathcal{O}(-1))$ over the ambient space $\mathcal{B}_S \times \mathbb{P}\Gamma(M, L)$; if we restrict this bundle on the space $\mathcal{U}_{SBS}$ then the first product contains a S^1 - subbundle defined by the specialty condition. Namely, over point $(S, p) \in \mathcal{U}_{SBS}$ take an element in the product $S^1(\mathcal{P}_S) \times$

$S^1(\mathcal{O}(-1))|_{(S,p)}$ presented by pair σ_S, s_x and impose the condition $s_x|_S = f\sigma_S$ where f is strictly positive real function on S. In the fiber this condition cuts a circle S^1, and the induced $U(1)$ - action on the product rotates this cicrcle to itself. Therefore it is defined a $U(1)$ - bundle which we denote as $\tilde{\mathcal{P}}_S \to \mathcal{U}_{SBS}$. This bundle can be called *twisted Berry bundle*.

Note that from certain point of view the space $\mathcal{U}_{SBS}$ can be regarded as a "complexification" of the moduli space $\mathcal{B}_S$ of Bohr - Sommerfeld cycles. In a lagrangian approach to Geometric Quantization one establishes that $\mathcal{B}_S$ is a very natural object looking as a phase space of quantized system; the dynamical properties are presented but it remains unknown how to present the measurment process as it stated in the standard Quantum Mechanics. In [7] one uses the moduli space $\mathcal{B}_S^{hw,r}$ of half weighted Bohr - Sommerfeld cycles which is an almost Kahler manifold and realizes the programme of ALGA - quantization for classical mechanical systems with compact phase space. The problem appeared in that programme was with the fact, that it is only almost Kahler, but for complete realization one needs a fair Kahler moduli space. Now we do have a Kahler space $\mathcal{U}_{SBS}$, fibered over $\mathcal{B}_S$. It is not a complexification of $\mathcal{B}_S$, but nevertheless it is a version of complexification, so one could try to exploit it in the lagrangian approach to Geometric Quantization.

2 Special Bohr - Sommerfeld lagrangian cycles: algebraic case

Suppose now that the symplectic manifold (M, ω) admits integrable complex structure I compatible with ω. This means (M, ω, I) is an algebraic variety; since the pair (ω, I) defines the corresponding Riemann metric g, and the hermitian triple (ω, I, g) gives a Kahler structure on M, and since the cohomology class of the Kahler form ω is integer, this Kahler metric is of the Hodge type. At the same time our connection $a \in \mathcal{A}_h(L)$ induces a holomorphic structure on L since its curvature F_a has type (1,1) w.r.t. complex structure I being proportional to ω. From this point we regard (M, ω, I) as a compact simply connected algebraic variety X with an ample line bundle $L \to X$ equipped with a hermitian structure h. Then the corresponding Kahler form ω can be reconstructed as follows: for any holomorphic section $\alpha \in H^0(X, L)$ the real function

$$\Psi_\alpha = -\frac{1}{2\pi} ln|\alpha|$$

is a Kahler potential on the complement $X \backslash D$, where D is the zeroset of α; and as L is very ample for any point $x \in X$ it exists a holomorphic section which doesn't vanish at a neighborhood of this point, therefore the Kahler (= symplectic form) can be completely recovered.

In this case we can specialize the projection p above, restricting it to the finite dimensional subspace $\mathbb{P}H^0(X, L) \subset \mathbb{P}\Gamma(X, L)$ of holomorphic sections; we denote the restricted map as

$$p_I : \mathcal{M}_{SBS} \to \mathbb{P}H^0(X, L),$$

where $\mathcal{M}_{SBS} \subset \mathcal{U}_{SBS}$ is the preimage $p^{-1}(\mathbb{P}H^0(X, L)$; this subset is clearly finite dimensional. Our main goal is to study this $\mathcal{M}_{SBS}$, and first of all we would like to attach the corresponding data which define such a set. Apart of the fixed above an ample line bundle L with a hermitian structure it depends on the data which one needs for the definition of the moduli space $\mathcal{B}_S$, therefore

$$\mathcal{M}_{SBS} = \mathcal{M}_{SBS}(c_1(L) \in H^2(X, \mathbb{Z}), h, [S] \in H_n(X, \mathbb{Z}), \mathrm{top} S).$$

Since the dependence on h does not change the geometry of $\mathcal{M}_{SBS}$ we will often omitt it.

To study the set $\mathcal{M}_{SBS}$ first let us specialize the general theory of special Bohr - Sommerfeld cycles to the algebraic case.

For this pure algebraic situation we have the folowing reformulation of Defenition 2'. It is based on the fact, that if α is holomorphic then the form $\rho(\alpha)$ by the very definition has type (1,0) w.r.t. I. This implies that the real and the imaginary parts of $\rho(\alpha)$ are related therefore

$$\mathrm{Im}\rho(\alpha) = I(d\Psi_\alpha),$$

and the vanishing condition in Definition 2' can be reformulated as

Definition 2". *A lagrangian submanifold $S \subset X$ is special Bohr - Sommerfeld w.r.t. a holomorphic form $\alpha \in H^0(X, L)$ iff $S \cap D_\alpha = \emptyset$ and at each point $p \in S$ one has $\mathrm{grad}\Psi_\alpha \in T_p S$.*

A direct corollary from the last definition says that S must be stable with respect to the gradient flow $\phi^t_{\mathrm{grad}\Psi_\alpha}$ for any t; singular points for S can appear at the singular points of the gradient vector field $\mathrm{grad}\Psi_\alpha$.

Essentially this is the same requirement, which we have discussed in the previous section: the gradient vector field for the function Ψ_α is a Liouville vector field, therefore in the holomorphic situation we get a specialization of the Lioville structure theory; it is known as the theory of Weinstein structures, see [6], [8], and this theory will be very important for us in the next section.

Recall, see [8], that the Weinstein structure is given by a Liouville structure Z plus a real function Ψ such that the vector field Z is *gradient like* for the function. This means that it exists a compatible Riemannian metric g such that $d\Psi(Z) \geq c|Z|^2$ for a constant $c \in \mathbb{R}$. In [6] it was shown that for Weinstein structure the sceleton, defined as for the Liouville structure as the stable subset of $M \backslash D$, is stratified by isotropical submanifolds. Thus the main difference with the case of a Liouville structure is the strong restriction on the dimensions of the sceleton components.

Holomorphic sections give us the corresponding Weinstein structures: as we have established above the Liouville vector field in this case is exactly the gradient vector field for the function Ψ_α. In the theory of Weinstein structures one requires for this function to be Morse or generalized Morse, and for our holomorphic situation it is indeed the case.

The prove of the fact that the Weinstein sceleton contains isotropical components comes with the properties of the function Ψ_α: as it was shown by J. Milnor, it can have isolated critical points of the Morse index not greater than n being a Kahler potential. Moreover, at each such a point the negative incoming tangent subspace in the whole tangent space must be isotropical (so in the case when the index equals to n this subspace must be lagrangian).

In any case the function Ψ_α is correctly defined on the complement $X \backslash D_\alpha$; and the algebraic nature of the situation leads to the following

Fact 1. for generic holomorphic section the function Ψ_α is Morse function on the complement $X \backslash D_\alpha$, so on the complement it has non degenerated isloated critical points only;

Fact 2. for generic holomorphic section the number of "finite" critical point is finite;

Fact 3. the set of "finite" trajectories joining the finite critical points splits in isotropical cells of finite number.

We denote the union of finite trajectories as B_α.

At this point it is very natural come back to the notion of sceleton of the Liouville structure, but now the same definition for the Weinstein structure, generated by a holomorphic section, gives us a correspondence "divisors — isotropical sceletons", and for the last ones we have:

Proposition 3. *An n - dimensional compact submanifold (possible, singular) is special Bohr - Sommerfeld w.r.t. a holomorphic section $\alpha \in H^0(X, L)$ iff it is contained by the sceleton B_α.*

Proof. The key observation here is that the sceleton B_α is a CW - complex, which is formed by isotropical cells therefore each cell has dimension less or equal to n.

If $S \subset X$ is special Bohr - Sommerfeld then according to Definition 2" it must be parallel to the gradient vector field $\mathrm{grad}\Psi_\alpha$. If S intersects an "infinite" unbounded trajectory of the gradient flow then due to this property it must have non trivial intersection with D_α which contradicts the definition. On the other hand if it intersects a "finite" trajectory it must contain it. Therefore S must lie in B_α. On the other hand, if a compact S lies in B_α it is decomposed as well in cells but from the properties of the cells one can see that S is lagrangian, and at the same time it is by the construction is parallel to the gradient vector field. Thus it is special Bohr - Sommerfeld.

Digression: algebraic curves. Let Σ be a Riemann surface of genus $g > 1$ equipped with a fixed complex structure I. Then this complex structure can be extended to the Kahler triple (G, I, Ω), where G is a riemannian metric of constant negative curvature compatible with I. Up to constant this riemannian metric is unique, and we normalize it by the condition $\int_\Sigma \Omega = 2g - 2$. It is well known that this metric is a solution of the Kahler - Einstein equation (see, f.e. [9]).

The Kahler structure (G, I, Ω) induces the corresponding hermitian structure on the complex line bundle $T^*\Sigma$, which we denote as K_Σ and call the canonical bundle following the algebro - geometrical traditions. In the presence of the complex structure one has a finite dimensional subspace $H^0(\Sigma, K_\Sigma)$ of holomorphic section in the big space of all smooth sections; we call the sections holomorphic differentials.

Any holomorphic differential $\rho \in H^0(\Sigma, K_\Sigma)$ has generically $2g - 2$ zeros $P_1, ..., P_m$ but in particular cases certain P_i are multiple zeros, in which case $m < 2g - 2$. Consider the following smooth real function

$$\Psi(\rho) = -\ln|\rho|_{\mathrm{h}},$$

which is correctly defined on the punctured surface $\Sigma \backslash \{P_1, ..., P_m\}$, where P_i — zeros of holomorphic differential ρ, and the norm is taken with respect to the hermitian structure on the canonical bundle. Since the function $\Psi(\rho)$ is strictly convex w.r.t. the complex structure I therefore all its "finite" isolated critical points have Morse indecies 0 or 1 (details see f.e. in [6]). Thus we have only minima and saddle points.

It's not hard to see, that for a generic holomorphic differential the corresponding function $\Psi(\rho)$ has non degenerated isolated critical points only, and the number of these points is finite. This remark implies the first natural question: for generic holomorphic differential ρ estimate (or find) the number of minima for the function $\Psi(\rho)$ on $\Sigma \backslash \{p_1, ..., p_m\}$.

Then the number of saddle points is given from the formula

$$\sharp \min - \sharp \text{saddle} + 2g - 2 = 2 - 2g$$

for the generic case $m = 2g-2$. Of course, the number varies along the projective space $\mathbb{CP}^{g-1} = \mathbb{P}H^0(\Sigma, K_\sigma)$, since for certain holomorphic differentials zeros are mutiple etc., the but what are the possible values?

Furthemore, following the SBS strategy we are interested in the base set $B_\rho \subset \Sigma$. It can be defined in our present situation: take all finite critical points and all finite trajectories of the gradient flow of $\Psi(\rho)$, this union is the base set B_ρ.

Note that despite of the fact that for certain ρ the finite critical points of $\Psi(\rho)$ may form 1 - dimensional subsets in Σ, the base set B_ρ is still 1 - dimensional. Indeed, the critical subsets cann't be 2 - dimensional since $\Psi(\rho)$ is Kahler potential for Ω, on the other hand B_ρ by the very definition must be stable w.r.t. to the gradient flow, and if it contains 2 - dimensional components then this component must be transported by the gradient flow to a saddle point, but then this saddle point should have 2 - dimensional negative subspace in the tangent space which is impossible.

Moreover, SBS geometry can help to prove here that the number of finite gradient trajectories for generic ρ is **finite**. Indeed, we just modify the arguments from [5]: if two finite gradient trajectories with the same ends are homotopical one to each other then they must bound a domain whose symplectic area equals to an integer multiple of $2g-2$, but since the total sysmplectic area of Σ is $2g-2$ it is impossible. On the other hand two gradient trajectories cann't intersect each other, consequently we can get only finite number of finite trajectories.

Note that if we oversee the picture on Σ we get that B_ρ looks like a graph where all minima present the vertices and the saddle points are not visible, being just marked points on the edges of the graph. Indeed, two trajectories meet at a saddle point p such that they a tangent to the negative subspace $T_p^- \Sigma \subset T_p \Sigma$, but since $T_p^- \Sigma$ is 1 - dimensional these two trajectories give us one smooth path with one marked point p. Going along an edge we can meet several marked points since in general it is possible to have several saddle points on the same line combinig the finite trajectories. Moreover, each part of each edge is naturally oriented since the function $\Psi(\rho)$ changes along these segments, and we can attach to each segment the approriate sign. Note that every edge must have at least one marked point.

Therefore for a generic holomorphic differential we get a finite graph $\Gamma(\rho) \subset \Sigma$ which we formally distinguish from the base set B_ρ (however as subsets in Σ they are the same). This graph carries additional equipments — fixed points on each edge and the corresponding orinetation for each segment ended at pairs of the closest marked points.

The topological structure of the graph $\Gamma(\rho)$ is given by the following observation: by the very construction Γ_ρ is homotopic to $\Sigma \backslash \{p_1, ..., p_m\}$. Indeed, the base set is the union of critical points and finite trajectories, therefore the punctured surface consists of the base set plus infinite trajectories, and we can shrink every infinite trajectory to point. We can compute the Euler characteristic $h_1(\Gamma(\rho)) - h_0(\Gamma(\rho))$ equals to $4g-4$ in the general case for holomorphic differential without multiple zeros using the Mayer–Vietoris exact sequence, which is essensially the same as the formula for the numbers of minima and saddle points above. For holomorphic differentials with multiple zeros one gets the bound of the same type. Thus the presence of $\Gamma(\rho)$ from the topological point of view doesn't distinguish different holomorphic differentials unless it has multiple zeros.

But we can reduce some geometrical objects, f.e. vector bundles, from whole Σ to the graphs which correspond to holomorphic differentials, and al least we can define certain functions on the bundle $\mathcal{H} \to \mathcal{M}_g$ where $\mathcal{M}_g$ is the moduli space of Riemann surfaces and $\mathcal{H}$ is the bundle whose fiber is $H^0(\Sigma, K_\Sigma)$.

As a toy example let us present a complex valued function on the total space $\mathcal{H}$. To do this we first take holomorphic differentials such that the corresponding functions $\Psi(\rho)$ have isolated finite critical points. In this situation each ρ defines the graph $\Gamma(\rho)$ and since the number of edges is finite one can correctly define the integral

$$A_{SBS} : \mathcal{H} \to \mathbb{C}, \quad A_{SBS}(\rho) = \int_{\Gamma(\rho)} \rho \in \mathbb{C},$$

where the integration is done first on the oriented segments of $\Gamma(\rho)$ and then by summing up these numbers.

If $\Psi(\rho)$ admits degenerated critical points which form critical subsets then we still get the corresponding graph $\Gamma(\rho)$ admits certian edges with "zero" orientation being the components of the critical subset. Nevertheless the expression A_{SBS} is correctly defined in this situation: one just excludes the edges with zero orientation and integrates along the oriented segments only.

Then the natural probelm arises: what are the properties of the function A_{SBS} on $\mathcal{H}$? Is it contineous? At least one thing is clear from the defintion: it is linear along the fibers of $\mathcal{H} \to \mathcal{M}_g$. This means that it corresponds to a section of the dual bundle $\mathcal{H}^* \to \mathcal{M}_g$. Take the zeroset of this section and denote it as $D_{SBS} \subset \mathcal{M}_g$. What can be said about this subset? Is it a real submanifold?

More serious questions arise when we see that all the story looks somehow related to a subject which is based on the same geometrical data: theory of flat surfaces, see [10] and references therein. Every holomorphic differential ρ defines a flat metric on Σ with conical singularities at zeros $P_1, ..., P_m$. One defines there certain dynamics on the total space $\mathcal{H}$ and on the components $\mathcal{H}(d_1, ..., d_m)$ where d_i are multiplicities of zeros $P_1, ..., P_m$. It is natural to ask: what is the meaning of the function A_{SBS} from the point of view of flat surfaces?

On the other hand the construction of finite graph $\Gamma(\rho)$ works for any positive degree of the canonical bundle: the Kahler structure (G, I, Ω) induces the corresponding hermitian structure on K^k_Σ for any $k \in \mathbb{Z}$, and the distinguished prequantization connection a_{LC} induces the corresponding connection $a_k \in \mathcal{A}_h(K^k_\Sigma$. The construction of $\Gamma(\rho_k)$ where $\rho_k \in H^0(\Sigma, K^k_\Sigma)$ is holomorphic k - differential is the same as in the case of K_Σ, therefore we get certain new ingredients of any theory based on the geometry of the bundle $\mathcal{H}_k \to \mathcal{M}_g$, but it is already non toy level.

Now come back to the general situation.

In the theory of Weinstein structures the property used in Definition 2“ is very well known: this means that S is *regular* with respect to the Weinstein structure given by our holomorphic section, see [8].

Thus the search of special Bohr - Sommerfeld lagrangian submanifolds is reduced to the study of the sceleton B_α. As a corollary we get the following result

Proposition 4. *For any divisor $D_\alpha \in |L| = \mathbb{P}H^0(X, L)$ the fiber $p_I^{-1}(D_\alpha)$ is finite.*

Proof. Since we have just a finite number of cells in the cell decomposition of the sceleton B_α it is possible to construct only finite number of single compact submanifolds (note that we consider lagrangian submanifolds geometrically without multiplicities).

Furthemore, using this desription we can establish an important heuristic result about the set $\mathcal{M}_{SBS}$.

Proposition 5. *Near a generic point the set $\mathcal{M}_{SBS}$ is a smooth variety locally isomorphic to the projective space. Moreover, it is fibered over an open part in the projective space $\mathbb{P}H^0(X, L)$. In particular it carries a Kahler structure.*

The arguments are as follows: the topological structure of the sceleton B_α by the definition is the same for small neighborhood of a generic point in $|L|$, and it is the same since the homotopy type of B_α is the same as $X \backslash D_\alpha$ therefore it does not depend on small variation of the holomorphic section. Thus if we have a compact n - dimensional cycle in B_{α_0} for a fixed generic section α_0 and then slightly vary the section we get essentially the same picture for any sufficiently close section, which implies the result.

Thus if we consider singular lagrangian submanifolds it is possible to speak about the moduli space $\mathcal{M}_{SBS}$, but it should require very hard analysis of these singular components of the sceletons. Even in the simplest cases as we will see in the next section this is very hard task; moreover in geometrically more interesting situations the sceletons are always very complicated, and the direct way for definition of the moduli space looks not quite reasonable. Instead of doing this we will try to avoid the main problem and work with smooth lagrangian submanifolds only. The idea is based on a conjecture which relates regular smooth lagrangian submanifolds and *exact* lagrangian submanifolds. Below we will modify the definition of the moduli space of special Bohr - Sommerfeld lagrangian submanifolds and show that if *Eliashberg conjectures* are true then these modified moduli spaces are naturally isomorphic to the old ones.

3 Moduli spaces, old and modified

We begin with examples.

The simplest algebraic variety is $\mathbb{CP}^1$ — complex projective line with the standard Fubini - Study metric, see [9]. Then the corresponding Kahler form ω_{FS} gives an integer symplectic form. Then every smooth loop $\gamma \subset \mathbb{CP}^1$ is a smooth lagrangian submanifold, and no other types of smooth lagrangian submanifolds exist.

Consider $L = \mathcal{O}(1)$ as the prequantization bundle which corresponds to $\omega_{FS} = \omega_1$. Then as it is well known no Bohr - Sommerfled loop exists: since every $\gamma \subset \mathbb{CP}^1$ divides the ambient manifold into two pieces, each of them can be taken as the disc bounded by γ, and since the total symplectic area of $\mathbb{CP}^1$ equals to 1, then it is impossible to have a smooth Bohr - Sommerfeld loop. We can say that in this case the moduli space is empty.

Shift the *level of quantization* and consider $L = \mathcal{O}(2)$; then a loop γ is Bohr - Sommerfeld if it cuts a disc of symplectic area ± 1 with respect to the symplectic form $\omega_2 = 2\omega_1$. Then the space of holomorphic sections $H^0(\mathbb{CP}^1, \mathcal{O}(2))$ is 3 - dimensional; every holomorphic section is given by a homogenious polynomial of degree 2, and up to scaling the section is defined by two points — zeros of the polynomial.

Consider first the case when the polynomial is irreducible therefore the corresponding divisor consists of two distinct points p_1, p_2. Take the function $\Psi_\alpha = -\ln|\alpha|$ and study its finite critical points. At p_1, p_2 the function goes to infinity, and it is not hard to see that it admits one minimal point $x_{min} \in \mathbb{CP}^1 \backslash \{p_1, p_2\}$. But then it must have one saddle point: infinite trajectories of the gradient flow go either to p_1 or to p_2 and therefore it must be a separatrix line γ joining the minimal and the saddle points. This separatrix line must be stable with respect to the gradient flow which implies that γ is special Bohr - Sommerfeld lagrangian submanifold with respect to the section vanishing at p_1, p_2.

Now if the section is reducible, $p_1 = p_2$, then the function Ψ_α has one infinite maximum, only one minimal point — and no other critical points. This means that no special Bohr - Sommerfeld lagrangian submanifolds exist for reducible sections.

The subset of reducible sections form a curve $C \subset \mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(2)) = \mathbb{CP}^2$ which is essentially the same $\mathbb{CP}^1$ embedded by Veronese map to the projective plane. From this we see that the moduli space $\mathcal{M}_{SBS}$ is naturally isomorphic to $\mathbb{CP}^2 \backslash C$, an affine algebraic variety.

Shift again the level of quantization and consider the case $L = \mathcal{O}(3)$; the corresponding symplectic form is $\omega_3 = 3\omega_{FS}$. What happens in this situation?

An irreducible section $\alpha \in H^0(\mathbb{CP}^1, \mathcal{O}(3))$ is defined by three point where it vanishes, thus the corresponding Kahler potential Ψ_α has three infinities there, but what are the finite critical points? For certain sections there are two local minima and three saddle points: the corresponding separatrices join the local minima passing through the saddle points, therefore one has three segments and no smooth loops which belong to the sceleton B_α.

Indeed, suppose we take the section α given by homogenious polynomial $P_3 = z_0^3 - z_1^3$ with simple roots; the function Ψ_α has exactly 5 finite critical points:

— at $[1:0], [0:1]$ it admits local minima;

— at $[1:-1], [1:\rho], [1:\bar{\rho}]$ it admits saddle points (ρ here is the standard notation of the cubic root of -1).

One has exactly 6 finite trajectories of the gradient vector field of Ψ_α which form three segments with ends at the North and the South poles, passing through the saddle points. Thus the sceleton B_α is a graph of type Θ with two vertices and three edges; and no smooth loops lie on B_α!

On the other hand there are the sections for which two of the three segments meet each other smoothly at the end points, and for this configuration we may say that the set of special Bohr - Sommerfeld lagrangian submanifolds is non empty; the subset of such sections in $\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(3))$ form non algebraic piece of very strange nature, and it is clear that at this point we face an obstruction in the theory of moduli space of SBS lagrangian submanifolds.

As it was claimed above all the old definitions are still work in the case of singular lagrangian submanifold, thus we may study the previous example from this point of view. Take again the graph B_α of type Θ and derive singular lagrangian submanifolds from it. Since we are still in very simple situation it is possible to do: we get exactly three piecwise smooth loops: take two segments and consider it as a loop with two corners at the North and the South poles. But what would happen in a bit more interesting geometrical situation? The theory of Weinstein manifolds says us (see [8]) that even in the best cases the sceletons are very complicated, but the same theory hints how we can avoid this main difficulty in the construction.

Recall that in the Weinstein manifold terminology our SBS lagrangian submanifolds are regular being components of the sceletons. On the other hand in the same theory one studies another type of lagrangian submanifolds: *exact* lagrangian submanifolds. The definition can be found f.e. in [8]: a lagrangian submanifold $S \subset M$ is said to be exact if the restriction of the 1 - form $\lambda|_S$ is an exact form (note that the restriction must be closed since $d\lambda = \omega$).

In our special situation when the Weinstein structure on the complement $X \backslash D_\alpha$ is defined by the holomorphic section α via formula $\lambda_\alpha = Id\Psi_\alpha$ we can show that the exactness property is related to the Bohr - Sommerfeld condition:

Proposition 6. *In the situation described above a lagrangian submanifold $S \subset X \backslash D_\alpha$ is exact with respect to λ_α if and only if for any loop $\gamma \subset S$ and any disc $B_2 \subset M$ bounded by γ the topological intersection $D_\alpha \cap B_2$ equals to the symplectic area $\int_{B_2} \omega$.*

In particular S must be Bohr - Sommerfeld lagrangian submanifold. Here we understand the topological intersection as follows: if B_2 intersects D non transversally then we take a small deformation of B_2 which is already transversal and then calculate the intersection index.

The proof is given by fair computation: the Stocks formula for $\int_\gamma \lambda_\alpha$ relates the residues of λ_α on B_2 and the symplectic area of B_2, and the sum of the residues is given by the intersection number.

The property that we used in the last proposition can be separately presented as a new notion:

Definition 3. Let (X, ω, D) is a simply connected symplectic manifold with integer symplectic form ω and $D \subset X$ be $2n-2$ submanifold whose homology class is Poincare dual to $[\omega]$. Then we say that a lagrangian submanifold $S \subset X$ is D - exact iff for any loop $\gamma \subset S$ and any disc $B_2 \subset X$ bounded by γ the topological intersection $B_2 \cap D$ equals to $\int_{B_2} \omega$.

Note that this property does not depend on any additional structures — just on the mutual arrangement of S and D. On the other hand this defintion can be extended even to the case when $S \cap D \neq \emptyset$.

With the definition of D - exactness in hands we are ready to present a new definition of the moduli space, which we call *modified* moduli space of special Bohr - Sommerfeld cycles and denote as $\tilde{\mathcal{M}}_{SBS}$.

Let X be a compact simply connected algebraic variety of complex dimension n, $L \to X$ be a very ample line bundle and h be an appropriate hermitian structure on L such that it gives a Kahler structure on X with the Kahler form ω_h. Let $|L|$ be the projective space corresponding to holomorphic section space for L. Fix a topological type top$S$ and a middle class $[S] \in H_n(X, \mathbb{Z})$ where S is a real n - dimensional orientable manifold.

Consider the space of pairs $(\{S\}, D)$ where

— $D \in |L|$ is a zero divisor of a holomorphic section from $H^0(X, L)$;

— $\{S\}$ is a class of smooth homologically non trivial D- exact with respect to D lagrangian submanifolds from the complement $X \backslash D$ modulo Hamiltonian isotopies in $X \backslash D$, which are of topological type top$S$ and represent the class $[S]$ being considered in X.

Step by step, first we take a divisor $D \in |L|$ and consider the complement $X \backslash D$. In this complement we have the group $H_n(X \backslash D, \mathbb{Z})$ with the natural epimorphism $\pi : H_n(X \backslash D, \mathbb{Z}) \to H_n(X, \mathbb{Z})$. Take the preimage $\pi^{-1}([S])$ and for each class from $\pi^{-1}([S]) = \{\chi_1, ... \chi_m\}, \chi_i \in H_n(X \backslash D, \mathbb{Z})$, realizable by smooth D - exact lagrangian submanifolds, find such smooth D- exact with respect to D lagrangian submanifolds which represent in the complement $X \backslash D$ the classes χ_i. These lagrangian submanifolds form spaces of solutions $\mathcal{L}^{\chi_i}_{D-ex}$ for each i. The last (but not least) step is the factorization of each $\mathcal{L}^{\chi_i}_{D-ex}$ by the natural action of Hamiltonian isotopies on the complement $X \backslash D$. Each element of the factorized space gives a point $(\{S\}, D)$ of our modified moduli space $\tilde{\mathcal{M}}_{SBS}$.

Our first aim is to prove that without any references to our old definitions and notions such a modified moduli space is correctly defined smooth complex manifold of dimension $h^0(L) - 1$ (if it is non empty). To show this we use the natural projection to the second component which gives

$$p : \tilde{\mathcal{M}}_{SBS} \to |L|.$$

The first claim is

Proposition 7. *For each $D \in |L|$ the fiber $p^{-1}(D)$ is descrite.*

Consider the moduli space $\mathcal{B}_S(\text{top}S, [S])$ of all Bohr - Sommerfeld lagrangian submanifolds of fixed topological type and cut the determinantal subspace $\Delta(D) \subset \mathcal{B}_S$ consists of submanifolds with non trivial intersection with D. For each connected component $\mathcal{B}^j_S \subset \mathcal{B}_S \backslash \Delta(D)$ it is defined a class w_j from $H^1(S, \mathbb{Z})$ given by the map $\frac{\alpha}{\sigma_S} : S \to \mathbb{C}^*$. If $S_0 \subset X \backslash D$ is a represetative of our moduli space then it must lie in the corresponding component $\mathcal{B}^{j_0}_S$ such that the class w_{j_0} is trivial (note that it is equivalent to the D - exactness condition in the case when D is the zero divisor of holomorphic section α). Then whole the connected component $\mathcal{B}^{j_0}_S$ consists of exact lagrangian submanifolds with the same non trivial homology class in $H_n(X \backslash D, \mathbb{Z})$; but the Hamiltonian deformations locally form $\mathcal{B}^{j_0}_S$ near S_0, therefore the quotient space of $\mathcal{B}^{j_0}_S$ modulo Hamiltonian isotopies must be just a point. Summing up we get a discrete set of points, corresponding to levels j_i for which the classes w_{j_i} are trivial.

The next fact is

Proposition 8. *Every point $(\{S\}, D) \in \tilde{\mathcal{M}}_{SBS}$ admits a neighborhood naturally isomorphic to a neighborhood of $D \in |L|$ in the projective space.*

If S_0 is a representative for a divisor D from the class $\{S\}$ of exact smooth homologically non trivial lagrangian submanifolds then S_0 is D - exact for D and for all D's from a neighborhood of D in the projective space $|L|$. Indeed, if D' is sufficiently close to D in $|L|$ then it still does not intersect S_0; since S_0 is Bohr - Sommerfeld universally with out any dependence on the choice of D, then the restriction of $\lambda(\alpha')$ to S_0 must present an integer valued class in

$H^1(S_0, \mathbb{Z})$ but this class must be the same for D and D', consequently it must be trivial. Here α' is a holomorphic section with zero divisor D'.

Therefore we can present the neighborhood of the point $(\{S\}, D)$ just by the constant first element and varying the second element in a suitible neighborhood of D in the projective space.

In particular we get that the differential dp of map p is an isomorphism for each non trivial point.

As a consequence we get

Theorem A. *The modified moduli space $\tilde{\mathcal{M}}_{SBS}$ is a Kahler manifold of complex dimension $h^0(X, L) - 1$.*

We would like to emphersize that the last Theorem is an independent result about an independent object — the modified moduli space $\tilde{\mathcal{M}}_{SBS}$. But we must explain why we denote it using the abbreviation SBS thus why it is somehow related to special Bohr - Sommerfeld cycles. The explanations lead to much more strong fact which we first illustarte by an example which has been presented above.

Come back to the case $X = \mathbb{CP}^1$ and $L = \mathcal{O}(3)$ when as we have seen above no smooth special Bohr - Sommerfeld lagrangian submanifolds for generic holomorphic section.

On the other hand we can easily construct the modified moduli space $\tilde{\mathcal{M}}_{SBS}$ for this case. Irreducible divisor D is defined by three distinct points $p_1, p_2, p_3 \in \mathbb{CP}^1$; a smooth loop $\gamma \subset \mathbb{CP}^1 \backslash \{p_i\}$ is D - exact iff it bounds disc B_2 of symplectic area $1/3$ with respect to ω_{FS} containing exactly 1 points p_i (the complimentary case when B_2 of symplectic area $2/3$ containing exactly 2 points is essentially the same). Up to Hamiltonian equivalence in $\mathbb{CP}^1 \backslash \{p_i\}$ we have exactly three possibilities: for each point p_i take a small smooth loop surrounding this point and then blow it to have the right symplectic area for the bounded region — and such γ_i is exactly the required one. Therefore for each irreducible divisor $D = \{p_i\}$ we have exactly three classes of exact lagrangian submanifolds.

To describe the global structure of $\tilde{\mathcal{M}}_{SBS}$ we have to study what happens over points of $|L|$ representing reducible divisors. The set of such points is given by the discriminant equation for cubic equation, so they form a surface $\mathcal{D}_4$ of degree 4 in $\mathbb{CP}^3 = |\mathcal{O}(3)|$. This surface includes a rational curve $C \subset \mathcal{D}_4$ which corresponds to cubic polynomial with only one triple root. Consider first the points in $\mathcal{D}_4 \backslash C$. Over such a point we have only one smooth loop which divide the surface of $\mathbb{CP}^1$ in the ratio $1:2$, and the first part contains the single root while the complement part contains the double root. Furthemore for a divisor which is presented by triple point we don't have D - exact loops at all. Therefore we have the following grading over $|L| \supset \mathcal{D}_4 \subset C$: nothing over C, one element over $\mathcal{D}_4 \backslash C$ and three elements over generic point.

Very interesting fact appears in this picture: no ramification takes place! When we start with an irreducible divisor and then two points p_1 and p_2 are merging together to one double point the picture is as follows: loops γ_1 and γ_2 after their fusion under the process $p_1 \mapsto p_2$ give us the same γ_3 which remains to present the corresponding class in $H_1(\mathbb{CP}^1 \backslash \{p_i\}, \mathbb{Z})$. At the same time the non trivial class $[\gamma_1], [\gamma_2] \in H_1(\mathbb{CP}^1 \backslash \{p_i\})$ either vanishes or turns to be equal to $[\gamma_3]$.

Summing up, the modified moduli space $\tilde{\mathcal{M}}_{SBS}$ is isomorphic to an algebraic variety minus very ample divisor. Indeed, take in the direct product $\mathbb{CP}^1 \times \mathbb{CP}^3$ a subvariety Y given by the equation $\alpha_0 z_0^3 + \alpha_1 z_0^2 z_1 + \alpha_2 z_0 z_1^2 + \alpha_3 z_1^3 = 0$ where

$[z_i], [\alpha_j]$ are homogenious coordinates on the fisrt and the second summand correspondingly. In this algebraic variety Y cut the ramification divisor $\Delta \subset Y$ with respect to the natural projection on the second summand, — then

Proposition 9. *The modified moduli space $\tilde{\mathcal{M}}_{SBS}$ for $X = \mathbb{CP}^1$ and $L = \mathcal{O}(3)$ is naturally isomorphic to affine algebraic variety $Y \backslash \Delta$.*

The answer is amazing: we get algebraic variety with a fixed very ample divisor! On the other hand we can take the Weinstein sceleton of $Y \backslash \Delta$, study D- exact smooth lagrangian submanifolds in $Y \backslash \Delta$ *et cetera*, thus some new horizons are opened.

In particular we would like first to answer the following natural question: coming back to the starting point of our discussion we must study the dependence of the constructed moduli space $\tilde{\mathcal{M}}_{SBS}$ on the choice of hermitian structure h on the very ample line bundle $L \to X$. To do this recall that h on L makes it possible to assciate a Kahler potential Ψ_α to a holomorphic section α. If we vary $h = h_0 \to h_t$ in the class of admissible hermitian structures (note that not each hermitian structure gives a Kahler form on X) then the difference of h and h_t is presented by a real non vanishing function $e^{f_t} = \frac{|\alpha|_{h_t}}{|\alpha|_h}$ defined first locally on $X \backslash D_\alpha$; then taking different α's we get a global non vanishing function of X. Then the Kahler structure changes in a simple manner: $\omega_t = \omega + dIdf_t$. We claim that for small changes of h_t the geometrical structure of the modified moduli space is the same.

Indeed, let for a divisor D_α we have a smooth representative S_0 for the point $(\{S\}, D)$ where S_0 is a D - exact lagrangian with respect to $\omega = \omega_0$ submanifold. Consider the family of Weinstein structures $(\ln|\alpha|_{h_t}, \lambda_t = Id\ln|\alpha|_{h_t})$ with *different* symplectic forms ω_t. Now if we take the time dependent vector field $v_t = \omega_t^{-1}(\lambda_t)$ and generate the corresponding flow $\phi^t_{v_t}$ on a neighborhood of S_0 then every $S_t = \phi^t_{v_t}(S_0)$ must be exact for 1 - form $\phi^t_{v_t}(\lambda_t)$ since it is a family of local diffeomorphisms. On the other hand 1 - form $\phi^t_{v_t}(\lambda_t)$ equals to λ_t plus an exact form therefore S_t must be exact as well with respect to λ_t. Note that exactness in this setup is not related to symplectic or lagrangian properties — just to smoothness of the transformations.

To show that the difference $\phi^t_{v_t}(\lambda_t) - \lambda_t$ is exact compute the Lie derivative

$$\mathcal{L}_{v_t}\lambda_t = d\lambda_t(v_t) + d(\lambda_t(v_t));$$

the first term is the difference between $\lambda_t - \lambda_0$ while the second term is exact. Without loss of generality we can suppose $f_t = tf$ and get the result.

Thus for the changing of the hermitian structure we get a one - to one correspondence for the classes $\{S\}$ of D - exact smooth submanifolds lagrangian with respect to different symplectic structures. Consequently we can show that for all hermitian structures the modified moduli spaces $\tilde{\mathcal{M}}_{SBS}$ are isomorphic.

In the last sentences we have repeated the arguments from [6] when one establishes relations between homotopic Liouville structures. Our theory of modified moduli space is related to the theory of Weinstein manifold as well, and to claim some important facts about the modified moduli spaces we need the statements known as *the Eliashberg conjectures.* They are presented as an open problem for the Weinstein manifolds from (see [8], Problem 5.1), but for us they are of different kinds being "stratified" into two levels.

The first level we would like to call **Soft Eliashberg conjecture.** It says that if $S \subset \tilde{M}$ is a smooth exact lagrangian submanifold in a Weistein manifold

$(\tilde{M}, \Psi, \lambda)$ then it must be homologically non trivial, so $H_n(\tilde{M}, \mathbb{Z}) \ni [S] \neq 0$. Moreover, one claims that each homology class in $H_n(\tilde{M}, \mathbb{Z})$ can contain at most one smooth exact lagrangian submanifold up to Hamiltonian isotopies; and the number of the homology classes realized by smooth exact lagrangian submanifolds is finite.

For our modified moduli spaces the Soft Eliashberg conjecture (SEC) would imply that

— the number of elements in *any* fiber $p^{-1}(D)$ is finite;

— no ramification happens near subspaces in the projective space $|L|$ where degenerations of the group $H_n(X \backslash D, \mathbb{Z})$ take place.

The first item is obvious, let us deduce the second item from SEC.

According to SEC the modified moduli space can be realized as a subset of the product space $\mathbb{Z}^N \times |L|$ where N is the rank of the group $H_n(X \backslash D, \mathbb{Z})$ where D is generic. Note that the group is the same for each generic D and it degenerates at the elements when the divisor D turns to be singular; but at the same time the smaller groups for singular divisors can be naturally embedded in the big group $\mathbb{Z}^N$. Our picture should depend on the trivialization: since the first stratum of singular divisors has complex codimension 1 then it is a monodromy in the group $\mathrm{Aut} H_n(X \backslash D, \mathbb{Z})$, but nevertheless a trivialization can be fixed globally on $|L|$. Therefore if SEC is true then we can mark the classes in $\mathbb{Z}^N$ for every $D \in |L|$ realizable by smooth D- exact lagrangian submanifolds with respect to D of fixed topological and homological types which shall lead to a finite covering of the projective space $|L|$. In this picture ramification could not appear: points in the lattice can not merge being integer, and every point $a \in \mathbb{Z}^N$ under the limiting process when D tends to $D_0 \in \mathcal{D}_{Sing} \subset |L|$ either disappears if it is not contained by the subgroup $H_n(X \backslash D_0, \mathbb{Z}) \subset \mathbb{Z}^N$ or just survives if it is contained.

Therefore if SEC is true then we get a Kahler structure on $\tilde{\mathcal{M}}_{SBS}$ realized as a subspace in the product $\mathbb{Z}^N \times |L|$: the Kahler structure can be easily lifted from the projective space $|L|$. However in the example presented above we got much more interesting picture: an algebraic variety of the form $Y \backslash \Delta$. Now our main goal in the story is to find a good realization of the modified moduli spaces so to find certain algebraic structure on these spaces.

On the other hand SEC implies certain relation between the moduli space $\mathcal{M}_{SBS}$ as they were defined above and the modified moduli space $\tilde{\mathcal{M}}_{SBS}$. As we have discussed above for a holomorphic section $\alpha \in H^0(M, L)$ the special Bohr - Sommerfeld cycles must be singular being components of the Weinstein sceleton of $M \backslash D_\alpha$. On the other hand every such a cycle must present a non trivial homology class in $H_n(M \backslash D_\alpha, \mathbb{Z})$. If this cycle is sufficiently primitive then according to SEC it should be a smooth realization by a smooth exact lagrangian submanifold. Moreover such a submanifold should be unique up to Hamiltonian isotopies if SEC is true. Therefore we can present a one - to one correspondence between sufficiently primitive cycles in the Weinstein sceleton and the classes of smooth exact lagrangian submanifolds in the complement. Again, SEC says that $\mathcal{M}_{SBS}$ is isomorphic to $\tilde{\mathcal{M}}_{SBS}$. And since we strongly belive in SEC, the modified moduli space was notated with the same abbreviation SBS.

Fairly, we have not defined above what are the points of $\mathcal{M}_{SBS}$, which cycles from the Weinstein sceleton we understand as these points; now we can formulate all the details following the standard ideas from ALAG, [2]. Idealogically we understand Bohr - Sommerfeld lagrangian cycles as limiting elements of families of *isodrastic deformations* of given smooth Bohr - Sommerfeld lagrangian submanifolds. Let $S_t \subset M$ be a family of Bohr - Sommerfeld lagrangian submanifolds such that for $t \in (0;1]$ every S_t is smooth, and under the limit $t \mapsto 0$ the deformation is continuous. Therefore the limiting S_0 can be singular, but for every smooth piece of S_0 it must be lagrangian or isotropical, and for every piecewise smooth loop $\gamma \subset S_0$ the symplectic area of a disc with corners $B \subset M$ such that $\partial B = \gamma$ is integer, — which is the refinement of the Bohr - Sommerfeld condition.

Now if we have a singular cycle S_0 inside of the Weinstein sceleton of $M \backslash D_\alpha$ such that there is a isodrastic lagrangian deformation S_t such that S_t are smooth Bohr - Sommerfeld for $t > 0$ then one claims

Proposition 10. *For sufficiently small $t > 0$ smooth lagrangian submanifolds S_t are exact.*

The proof follows from Propostion 6 above: since S_0 does not intersect D_α the same happens for every S_t for small $t > 0$. It implies that these S_t are D - exact with repsect to D_α; at the same time all S_t lie in the same orbit under the action of Hamiltonian isotopies on $M \backslash D_\alpha$. And if SEC is true then such an exact S_t is the unique representation for the corresponding class $[S_0] \in H_n(M \backslash D_\alpha, \mathbb{Z})$ which must be non trivial since S_0 belongs to the sceleton (is *regular*).

If we have a smooth exact lagrangian submanifold $S \subset M \backslash D_\alpha$ then according to SEC it must be homologically nontrivial therefore it is a cycle S_0 in the Weinstein sceleton which is homologically equivalent to S but it is a hard task to prove that S is given by an isodrastic lagrangian deformation. Another conjecture helps in this problem:

Hard Eliashberg Conjecture. *Every smooth exact lagrangian submanifold is regular.*

This conjecture is much more complicated; at the same time it should lead to the equivalence between the moduli space of special Bohr - Sommerfeld cycles $\mathcal{M}_{SBS}$ and the modified moduli space $\tilde{\mathcal{M}}_{SBS}$. We finalize the discussion with the statement:

Theorem B. *If Hard Eliashberg Conjecture (HEC) is true then $\mathcal{M}_{SBS}$ is naturally isomoprhic to $\tilde{\mathcal{M}}_{SBS}$.*

At the end we present more examples of the moduli spaces.

Let $X = Q \subset \mathbb{CP}^3$ be a smooth complex quadric. It is isomorphic to the direct product $\mathbb{CP}^1 \times \mathbb{CP}^1$, see [9]. The line bundle $\mathcal{O}(1,1)$, given by the tensor product of two copies of $\mathcal{O}(1)$ from each direct summand, is very ample, and it is well known that it is the same as the restriction $\mathcal{O}(1)|_Q$, therefore the zero divisors of its holomorphic sections are given by the hyperplane section in $\mathbb{CP}^3$. Thus the generic divisor is given by non tangent plane section of Q: in this case the complement $Q \backslash D$ is homotopic to 2 - sphere, therefore $H_2(Q \backslash D, \mathbb{Z}) = \mathbb{Z}$, and it is not hard to see that this class is realized by a lagrangian 2 - sphere, essentially isotopic to the sphere given by the antidiagonal embedding into the product $\mathbb{CP}^1 \times \mathbb{CP}^1$. If D is reducible it corresponds to a tangent plane and in this case the complement $Q \backslash D$ is isomorphic to $\mathbb{C} \times \mathbb{C}$ which imples non existence of exact lagrangian spheres in the complement. Summing up we get that $\tilde{\mathcal{M}}_{SBS}$ in this case is naturally isomoprhic to $\mathbb{CP}^3 \backslash Q^\vee$, where $Q^\vee$ is the dual quadric. Moreover, in this case it is possible to find the "old" moduli space $\mathcal{M}_{SBS}$ computing the critical points of the Kahler potentials Ψ_α defined by holomorphic sections, and this moduli space is the same as the modified one.

Now consider a bit more complicated situation: let $X = Q$ be again complex projective quadric, but take as L the bundle $L = \mathcal{O}(2,1)$ which is very ample too. The computations of critical points of the Kahler potentials for construction of the "old" moduli space" now is a hard task therefore we would like to describe the modified moduli space only. For this we fix homogenious coordinates $[x_0 : x_1], [y_0 : y_1]$ on the direct summands $\mathbb{CP}^1 \times \mathbb{CP}^1 = Q$; note that a section of the bundle $\mathcal{O}(2,1)$ is given by a polynomial

$$(a_0 x_0^2 + a_1 x_0 x_1 + a_2 x_1^2) y_0 + (b_0 x_0^2 + b_1 x_0 x_1 + b_2 x_1^2) y_1 = 0,$$

thus we have as the complete linear system the projective space $\mathbb{CP}^5$ with the coordinates $[a_0 : a_1 : a_2 : b_0 : b_1 : b_2]$; every couple (a_i, b_j) defines a section, whose zeroset corresponds to this couple up to scale.

Consider the canonical projection $\pi_x : Q \to \mathbb{CP}^1$ to the first projective line with coordinates $[x_0 : x_1]$. Then for a given section we have the following correspondence: the polynomial (*) defines a section of the projection π_x unless the points $[x_0 : x_1]$ where both "coefficient" polynomials $P_a(x) = a_0 x_0^2 + a_1 x_0 x_1 + a_2 x_1^2$ and $P_b = b_0 x_0^2 + b_1 x_0 x_1 + b_2 x_1^2)$ vanish. Indeed, if for certain $[x_0 : x_1]$ at least one "coefficient" polynomial does not vanish we have unique up to scale pair $[y_0 : y_1]$ such that $[x_0 : x_1], [y_0 : y_1]$ solve the equation (*); and this defines a section of the projection π_x.

It is clear that for generic choice of a_i and b_j the polynomials P_a and P_b do not have common roots, and therefore for generic divisor from $|L|$ we get that the complement $Q \backslash D$ is isomorphic to the product $\mathbb{CP}^1 \times \mathbb{C}$ therefore the group $\pi_2(Q \backslash D)$ is naturally isomorphic to $\mathbb{Z}$. It follows that there are no topological obstructions to the existence of 2 - sphere in $Q \backslash D$ but one more topological obstruction is presented here: our lagrangian 2 - spheres represent the homology class $(1; -1)$ and therefore it must intersect any divisor from the complete linear system $|\mathcal{O}(2,1)|$. Therefore despite of the fact that $\pi_2(Q \backslash D)$ is non trivial we do not have D - exact lagrangian spheres. At the same time if we ask about the existence of other types of lagrangian submanifolds with trivial homology class in $H_2(Q, \mathbb{Z})$ then HEC does not allow the existence of smooth exact lagrangian tori (note however that Q contains lagrangian tori with trivial homology class, f.e. — the product tori).

To finish the example consider the case of non generic divisors from the complete linear system $|\mathcal{O}(2,1)|$: if the coefficient polynomials P_a and P_b have a common root, then the complement $Q \backslash D$ is isomorphic to the product $(\mathbb{CP}^1 \backslash \{pt\}) \times \mathbb{C}$ since the divisor D contains the fiber over this root; therefore in this case $Q \backslash D$ is homotopic to $\mathbb{CP}^1 \backslash \{pt\} = \mathbb{C}$, thus $H_2(Q \backslash D, \mathbb{Z}) = 0$. The same vanishing result happens when P_a and P_b are proportional.

Summing up we see that $\tilde{\mathcal{M}}_{SBS}$ is trivial for the case $X = Q, L = \mathcal{O}(2,1)$. But the computation above is not ineffective since we will exploit the arguments below in the most interesting example.

Let $X = F^3$ be the full flag variety in $\mathbb{C}^3$, realized as a hypersurface $X = \{\sum_{i=0}^2 x_i y_i = 0\}$ in the direct product $\mathbb{CP}^2 \times \mathbb{CP}^2$ where $[x_i], [y_j]$ are homogenious coordinates on each direct summand. The line bundle $\mathcal{O}(1,1)$ being restricted to X is very ample, and the corresponding symplectic form ω is given by the direct sum of the lifted standard Fubini - Study forms from the direct summands. Thus we consider $L = \mathcal{O}(1,1)|_X$.

It is known that the flag variety $F^3 = X$ with the standard symplectic form ω contains a lagrangian 3 - sphere which is called *the Gelfand - Zeytlin* sphere since it arose in the framework of the Gelfand - Zeytlin systems. It is explicitly presented in the homogenious coordinates by the condition

$$S_{GZ} = \{[x_0 : x_1 : x_2] \times [y_0 = \bar{x}_0 : y_1 = \bar{x}_1 : y_2 = -\bar{x}_2] | \sum_{i=0}^{2} x_i y_i = 0\} \subset X. \quad *$$

It is not hard to check that the condition above describes a lagrangian sphere: since the antidiagonal embedding of $\mathbb{CP}^2$ to the direct product $\mathbb{CP}^2 \times \mathbb{CP}^2$ is lagrangian, the subset $\{[x_0 : x_1 : x_2] \times [y_0 = \bar{x}_0 : y_1 = \bar{x}_1 : y_2 = -\bar{x}_2]\}$ is Hamitonian isotopic to the antidiagonal embedding thus is lagrangian as well, and the intersection of this subset and X is *cotransversal* one deduces that it is lagrangian. On the other hand S_{GZ} is described by the condition $|x_0|^2 + |x_1|^2 - |x_2|^2 = 0$ in $\mathbb{CP}^2$ therefore it is isomorphic to 3 - sphere.

Since the same is true for the case when minus in the formula (*) is placed not before $\bar{x}_0$ but before any other $\bar{x}_i$ we get another lagrangian sphere of the same type, namely

$$S_0 = \{[x_0 : x_1 : x_2] \times [-\bar{x}_0 : \bar{x}_1 : \bar{x}_2]\}, S_1 = [x_0 : x_1 : x_2] \times [\bar{x}_0 : -\bar{x}_1 : \bar{x}_2]\},$$

and $S_2 = S_{GZ}$ — all are lagrangian spheres, which are Hamiltonian isotopic in X. At the same time the homology class $[S_i] \in H_3(X, \mathbb{Z})$ is trivial since $H_3(X, \mathbb{Z})$ is trivial itself.

We would like to describe the moduli space $\tilde{\mathcal{M}}_{SBS}$ for the following topological data: our S is isomorphic to 3 - sphere homologically trivial in X. Note that in this case *every* lagrangian sphere must be exact having trivial fundamental group, which drastically simplify the analysis.

We start with an irreducible divisor $D \in |L|$ and our aim is to find the classes of lagrangian spheres in the complement $X \backslash D$ up to Hamiltonain isotopies. For this first study the homotopy type of the complement $X \backslash D$. Being a hypersurface in the direct product $\mathbb{CP}^2 \times \mathbb{CP}^2$ our flag variety admits the canonical projection to the first summand $\pi_x : X \to \mathbb{CP}^2$, whose fibers are projective lines in the second projective plane. Any irreducible $D \subset X$ gives a section of this projection plus three fibers, therefore the complement $X \backslash D$ is given by a complex line bundle over $\mathbb{CP}^2 \backslash \{p_1, p_2, p_3\}$ where p_i is a point in this projective plane. To illustrate it we can take say the following $D \subset X$:

$$D = \{x_0 y_0 - x_1 y_1 + i x_2 y_2 = 0, x_0 y_0 + x_1 y_1 + x_2 y_2 = 0\} \subset X.$$

It is equivalent to conditions $(1 - i) x_0 y_0 = (1 + i) x_1 y_1 = -x_2 y_2$ therefore for each $[x_0 : x_1 : x_2]$ we either have exactly one point in $\pi_x^{-1}([x_0 : x_1 : x_2])$ if at least two coodrinates are non zero or whole the fiber if two coordinates are zero. Therefore $X \backslash D$ is naturally isomorphic to the product

$$(\mathbb{CP}^2 \backslash \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}) \times \mathbb{C}.$$

Contracting the punctured fibers we get that $X \backslash D$ is homotopic to the complement $\mathbb{CP}^2 \backslash \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$. Note that for any irreducible divisor the picture is essentially the same.

Exclusion of three distinct points from $\mathbb{CP}^2$ generates non trivial homotopy group $\pi_3(\mathbb{CP}^2 \backslash \{p_1, p_2, p_3\})$ equals to $\mathbb{Z} \oplus \mathbb{Z}$; thus $H_3(X \backslash D, \mathbb{Z})$ is the same $\mathbb{Z} \oplus \mathbb{Z}$, and one can expect that certain classes from this lattice can be realized by smooth lagrangian spheres.

This is indeed the case: we claim that for each $i = 0, 1, 2$ lagrangian sphere S_i defined above:

i) does not intersect divisor D;

ii) lies in a non trivial class $a_i \in H_3(X \backslash D, \mathbb{Z})$ such that $a_i \neq a_j$ for $i \neq j$;

iii) no other classes of lagrangian spheres do exist.

First check item i): since our divisor is given by the conditions $(1-i)x_0y_0 = (1+i)x_1y_1 = -x_2y_2$ substitute there $y_0 = -\bar{x}_0, y_1 = \bar{x}_1, y_2 = \bar{x}_2$ as it is for S_0. This leads to the condition $(1-i)|x_0|^2 = (1+i)|x_1|^2 = -|x_2|^2$ which can take place if and only if $x_0 = x_1 = x_2 = 0$ but such a point does not exist on the projective plane. Therefore $D \cap S_0 = \emptyset$. The same arguments work for S_1 and S_2 as well.

Further, consider the projections of S_i under π_x to the first projective plane. Note that the projection realizes the contraction of X to $\mathbb{CP}^2$; the image of S_0 is given by the equation $\pi_x(S_0) = \{-|x_0|^2 + |x_1|^2 + |x_2| = 0\} \subset \mathbb{CP}^2$, it is a smooth 3 - dimensional sphere which divides the projective plane into two open parts. These parts are described by the sign of the value of the real function

$$F_0 = \frac{-|x_0|^2 + |x_1|^2 + |x_2|^2}{\sum_{i=0}^2 |x_i|^2},$$

which maps $\mathbb{CP}^2$ to the segment $[-1, 1]$. Thus function has non degenerated critical point at $[1:0:0]$ and degenerated critical set at the line $x_0 = 0$. The value $F_0 = 0$ is non critical, therefore S_0 is a smooth sphere. At the same time at points $[1:0:0]$ and $[0:1:0], [0:0:1]$ the function F_0 has different signs, consequently S_0 lies in a non trivial class in $\pi_3(X \backslash D)$ since π_x realizes a homotopy $X \backslash D \mapsto \mathbb{CP}^2 \backslash \{[1:0:0], [0:1:0], [0:0:1]\}$ and $\pi_x(S_0)$ represents a non trivial class.

The same arguments show that S_1 does represent a non trivial class as well, but it is a different class from $H_3(X \backslash D, \mathbb{Z})$ since it is projected to a different class for the punctured projective plane. Indeed, S_0 is "centered" in $[1:0:0]$ and the resting points lie "outside" of S_0; at the same time S_1 is "centered" in $[0:1:0]$ and the resting points $[1:0:0], [0:0:1]$ lie "outside" of it; the same is true for S_2. The basis in $H_3(X \backslash D, \mathbb{Z})$ can be choosen in such a way that S_0, S_1, S_2 represent the classes $(1,0), (0,1)$ and $(1,1)$ respectively. In particular this fact ensures us that S_i and S_j are not Hamiltonian isotopical for $i \neq j$.

For momentary prove of iii) we need the Soft Eliashberg Conjecture; however if one knows that S_{GZ} is essentially unique lagrangian sphere in F^3 up to Hamiltonian isotopy it were possible to establish the fact without references to SEC.

Thus we can say that for irreducible divisors the classes can be described by triple of points $p_1, p_2, p_3 \in \mathbb{CP}^2$; below we present an algebraic way to define this attachment. Now we consider reducible divisors from the geometrical poitn of view.

A reducible divisor $D \in |L|$ is presented by pair of sections of $\mathcal{O}(1)$ on each copy of the projective planes: therefore we can represent it by two projective lines $l_x \subset \mathbb{CP}^2_x, l_y \subset \mathbb{CP}^2_y$, and the divisor $D \subset X$ is given by the union of $\pi_x^{-1}(l_x) \cup \pi_y(l_y)$. These two components of D in X are isomorphic the Hirzebruch surface F_1, and if we study the projection $\pi_x : X \to \mathbb{CP}^2_x$ then the first component is given by the fibers over l_x while the second component presents a section of this projection plus the fiber over a point which corresponds to the porjective line l_y (thus it is isomorphic to the projective plane with one point blown up). Therefore we have two different situations: in more general case the point, corresponding to l_y does not lie on l_x in $\mathbb{CP}^2_x$, in more specific case the point lies on l_x.

In the first case the homotopy type of $X \backslash D$ is the same as of $\mathbb{C}^2 \backslash \{pt\}$: indeed, after the cancelation of the first component we get a projective bundle over $\mathbb{CP}^2_x \backslash l_x$ which means the projective bundle over $\mathbb{C}^2$; then the cancelation of the second component removes one point in each fiber and removes totally one fiber over the point which corresponds to l_y. Therefore $X \backslash D$ is isomorphic to $(\mathbb{C}^2 \backslash \{pt\}) \times \mathbb{C}$, and consequently $H_3(X \backslash D, \mathbb{Z}) = \mathbb{Z}$. Now we can again show that there exists unique up to Hamitonian isotopy smooth lagrangian sphere in the complement $X \backslash D$, which is projected by π_x to a 3 - sphere in $\mathbb{C}^2 \backslash \{pt\}$ "centered" exactly in this point. This point p characterizes the sphere, and we attach to this point the corresponding class of lagrangian spheres up to Hamiltonian isotopy.

The last case when l_x and l_y are related by the condition that the point in $\mathbb{CP}^2_x$, corresponding to l_y, lies on l_x, gives trivial set of lagrangian spheres since in this case $X \backslash D$ is isomorphic to $\mathbb{C}^2 \times \mathbb{C}$ therefore $H_3(X \backslash D, \mathbb{Z})$ is trivial. Here again we use SEC, but it is possible to verify by hands, that for this case every Gelfand - Zeytlin sphere intersects the divisor.

Summing up, we see that the moduli space $\tilde{\mathcal{M}}_{SBS}$ can be described as a subset of the direct product $|L| \times \mathbb{CP}^2_x$: for each element of $|L| \cong \mathbb{CP}^7$ the classes of smooth lagrangian spheres are given by the "centers" of their projections to $\mathbb{CP}^2_x$; and the complete linear system $|L|$ is stratified by the conditions "three points in $\mathbb{CP}^2_x$", "one point in $\mathbb{CP}^2_x$" and " no points in $\mathbb{CP}^2_x$. Note that the picture looks quite similiar to the first example of this section, for $X = \mathbb{CP}^1, L = \mathcal{O}(3)$, discussed above.

The detailed analysis shows that the answer even closer to the result for $X = \mathbb{CP}^1, L = \mathcal{O}(3)$.

Study the situation algebracally: with respect to the fixed homogenious coordinates $[x_0 : x_1 : x_2], [y_0 : y_1 : y_2]$ every divisor D from $|L|$ is given by two equations: $\sum_{i=0}^2 x_i y_i = 0, \sum_{i,j} a_{ij} x_i y_j = 0$ where in the last expression i and j are taken from 0,1 or 2. The representation of D by the numbers $\{a_{ij}\}$ is not unique since we can add any λ to a_{00}, a_{11} and a_{22} without changing of the system and as well we can scale the numbers a_{ij} therefore we normalize the matrix

$$A = (a_{ij})$$

such that $\mathrm{tr} A = 0$ and consider it up to scaling.

For a given divisor D write the defining system as

$$\begin{gathered} x_0 y_0 + x_1 y_1 + x_2 y_2 = 0 \\ (a_{00} x_0 + a_{10} x_1 + a_{20} x_2) y_0 + (a_{01} x_0 + a_{11} x_1 + a_{21} x_2) y_1 \\ +(a_{02} x_0 + a_{12} x_1 + a_{22} x_2) y_2 = 0 \end{gathered}$$

For a particular choice of $[x_0 : x_1 : x_2]$ understood as parameters the last system looks like a system of linear equations in three variables y_0, y_1, y_2, which has a unique up to scale non zero solution if and only if the system

$$\begin{aligned} (a_{00} x_0 + a_{10} x_1 + a_{20} x_2) &= \lambda x_0 \\ (a_{01} x_0 + a_{11} x_1 + a_{21} x_2) &= \lambda x_1 \\ (a_{02} x_0 + a_{12} x_1 + a_{22} x_2) &= \lambda x_2 \end{aligned} \qquad **$$

does not admit non zero solution for any λ. It implies that any divisor D represents a section of the projection $\pi_x : X \to \mathbb{CP}^2_x$ over the subset $N_A \subset \mathbb{CP}^2_x$ consists of such $[x_0 : x_1 : x_2]$ that the system (**) does not admit non zero solutions. But it is exactly the condition that vector (x_0, x_1, x_2) is not an eigenvector for the matrix A with an eigenvalue λ. It follows that the divisors from the complete linear system $|L|$ can be combined with respect to the following stratified conditions:

1) (generic case) the matrix A has three eigenvectors with different eigenvalues $\lambda_1, \lambda_2, \lambda_3$, $\lambda_i \neq \lambda_j$, therefore we have three distinct points $p_1, p_2, p_3 \in \mathbb{CP}^2_x$ — and thus three different classes of smooth lagrangian spheres;

2) the matrix A has one multiple eigenvalue, say $\lambda_1 = \lambda_2$, and admits one eigenvector and one eigenspace, therefore we have one distinct point p and the corresponding line l_x in the projective space $\mathbb{CP}^2_x$ such that p does not lie on l_x, — and for such a divisor we have one class of lagrangian spheres;

3) the matrix A has one multiple eigenvalue, say $\lambda_1 = \lambda_2$, but admits two eigenvectors and no eigenspace (the case of Jordan cell), therefore after the projection by π_x we get $\mathbb{CP}^2_x \backslash \{p_1, p_2\}$ but $\pi_3(\mathbb{CP}^2_x \backslash \{p_1, p_2\}) = \mathbb{Z}$, — therefore we again as in the case 2) get only one class of lagrangian spheres;

4) the matrix A has $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and is presented by one Jordan cell, so it admits a single eigenvector, — this corresponds to absence of lagrangian spheres in $X \backslash D$ since $\pi_3(\mathbb{CP}^2_x \backslash \{pt\}) = 0$;

5) the matrix A has $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and contains 2×2 Jordan cell, thus we have just an eigenspace of dimension 2 and no separate eigenvector, so this is the case discussed above when l_x and l_y are related, and — in this case no lagrangian spheres in the complement.

Summing up the cases 1) - 5) we see that the classes of lagrangian spheres correspond to *single* eigenvalues of the matrix A. Indeed, in the case 1) we have three single eigenvalues — and three classes of lagrangian spheres; in the cases 2) and 3) we have one single eigenvalue λ_3 — and unique class of lagrangian sphere; at finally in the cases 4) and 5) we do not have lagrangian spheres in the complement $X \backslash D$.

These arguments leads to the following realization of the modified moduli space $\tilde{\mathcal{M}}_{SBS}$: in the direct sum $H^0(X, \mathcal{O}(1,1)|_X) \oplus \mathbb{C} = \mathbb{C}^8 \oplus \mathbb{C}$ consider an affine hypersurface given in the affine coordinates $(a_{ij}, z) | a_{00} + a_{11} + a_{22} = 0$ by the cubic equation $\det(A - zI) = 0$ which is *homogenious* since the multiplication of matrix A by a constant leads to the multiplication of the eigenvalues by the same constant. Therefore this equation defines a projective cubic hypersurface $Y \subset \mathbb{P}(H^0(X, \mathcal{O}(1,1)|_X) \oplus \mathbb{C}) = \mathbb{CP}^8$ definig a finite covering of the complete linear system $|L|$. The ramification divisor $\Delta \subset Y$ is described by the following condition: since $\det(A - zI) = -z^3 - az + b$ where a has degree 2 in the homogenious coordinates in $|L|$ and $b = \det A$ has degree 3, then the multiple eigenvalues correspond to non trivial solutions of the system

$$\begin{aligned} 2z^3 + az - b &= 0, \\ (z^3 + az - b)'_z &= 3z^2 + a = 0. \end{aligned}$$

Therefore $\Delta \subset Y \subset \mathbb{CP}^8$ is a divisor from the complete linear system $|\mathcal{O}(2)|_Y|$ (note that $3z^2 + a = 0$ is homogenious of degree 2 in $H^0(X, \mathcal{O}(1,1)|_X) \oplus \mathbb{C}$).

Proposition 11. The modified moduli space for the lagrangian 3- spheres in the flag variety F^3 and the line bundle $K_{F^3}^{-\frac{1}{2}}$ is isomorphic to

$$\tilde{\mathcal{M}}_{SBS} = Y \backslash \Delta.$$

In particular it admits a natural compactification, isomorphic to Y.

It is very interesting result since we get again the geometrical data of the same form (algebraic variety, very ample divisor). In particular we can attach to our given pair $(F^3, K_{F^3}^{-\frac{1}{2}})$ either the Weinstein sceleton of $Y\backslash\Delta$ or thew finite set of the smooth exact lagrangian submanifolds of $Y\backslash\Delta$ up to Hamiltionian isotopies.

4 Problems

The studies of special Bohr - Sommerfeld submanifolds and cycles lead to certain natural questions about numerical invariants related to lagrangian geometry.

The first problem is stated for the simplest possible algebraic variety — complex projective line $\mathbb{CP}^1$. For this base variety consider the series of positive bundles $\mathcal{O}(k), k > 0$. Fixing an approriate hermitian structure $|.|_1$ on $\mathcal{O}(1)$ we automatically get the corresponding hermitian structures $|.|_k$ on its powers $\mathcal{O}(k)$. Then for any holomorphic section $\alpha \in H^0(\mathbb{CP}^1, \mathcal{O}(k))$ we can construct the function

$$\Psi_\alpha = -\ln|\alpha|_k,$$

which is subharmonic thus it admits critical points of two types: minima and saddle points (for special α this function admits critical subsets, but we consider generic sections). The numbers of minimal and saddle points, denoted as m and s correspondingly, are realted by the Euler characteristic relation since the number of infinite maxima equals to k so

$$m - s + k = 2,$$

but it is not clear what is the number $m = m(k)$ itself. It should be the same for generic sections therefore if one takes the generating function

$$W_0 = \sum_{k=1}^{\infty} m_k q^k,$$

it should reflect certain properties of our given base manifold $\mathbb{CP}^1$. At least asymptotic behavior of such W is interesting to know.

We can reformulate this problem using the realization of holomorphic sections by homogenious polynomials: consider in $\mathbb{C}^2$ with the standard hermitian structure $<,>$, take the quadratic function $F(\psi) =< \psi, \psi >$, given in some approriate coordinate system (z_1, z_2) as $F(z_1, z_2) = |z_1|^2 + |z_2|^2$, generic homogenious polynomial $P_k(z_1, z_2)$; take in $\mathbb{C}^2$ the surface $\Sigma_c = \{P_k(z_1, z_2) = c\}$ for certain $c \in \mathbb{C}^*$ and restrict the function F to Σ_c. The restriction $F|_{\Sigma_c}$ is not Morse: each critical set must be invariant under the phase cyrcle action, but it is possible to define the index of the critical subset distinguishing "minima" and "saddle" critical subsets.

There are other equivalent reformulations of this problem.

Next problem is related to the previous one: we can extend the question posted above to the case of Riemann surfaces of positive genus. For this extension we leave our main framework of simply connected symplectic manifolds.

Let Σ be a compact smooth Riemann surface of genus $g > 1$ and I be a complex structure on it. Then one can take the unique compatible riemannian metric G of constant negative curvature, normalizing the corresponding Kahler form Ω such that $\int_\sigma \Omega = 2g - 2$. Then the Kahler triple (G, I, Ω) induces a hermitian structure h_1 on the cotangent bundle $T^*\Sigma$ and thus on all the tensor powers of it. For a generic holomorphic k - differential $\alpha \in H^0(\Sigma, (T^*\Sigma)^k)$ take the function Ψ_α as above and ask the same question as above: what is the number of local minima of Ψ_α? It seems that this number should be independent on I which leads to the definition of generating function

$$W_g = \sum_{k=1}^{\infty} m_k q^k;$$

it is interesting to find it (even on the asympotitc level).

Now for any algebraic variety X, smooth compact and simply connected, one can formulate a natural problem related to the questions of lagrangian geometry.

For such an X take any very ample line bundle $L \to X$ which exists by the very definition. Fix an appropriate hermitian structure on L and for generic holomorphic section $\alpha \in H^0(X, L)$ take smooth exact lagrangian submanifolds modulo Hamiltonian isotopies. According to SEC, the number of classes is finite, and we denote it as $m_1 = m(L)$. Here it is possible to classifiy the submanifolds distinguishing the topological types, classes from $H_n(X, \mathbb{Z})$ etc, but the most generic way is to take the global number m_1 of the classes of smooth exact lagrangian submanifolds modulo Hamiltonian isotopies. This number should be independent on the choice of generic holomorphic section and even more — on the choice of an appropriate hermitian structure on L. Thus we get a number defined in the framework of lagrangian geometry but which does depend on the algebraic variety and very ample line bundle only. Note that it is not the rank of the group $H_n(X \backslash D_\alpha, \mathbb{Z})$, the number m_1 is different even in the simplest cases: when $X = \mathbb{CP}^1$ and $L = \mathcal{O}(3)$ as we have seen above (and for $L = \mathcal{O}(4)$ the situation is even more complicated). Therefore it is not just a topological question although non vanishing number m_1 implies non triviality of the rank.

Now we can repeat the question for any positivie power of L since the very amplness of L implies the same for L^k for positive integer k; applying the same procedure we get numbers m_2 for L^2, m_3 for L^3 *et cetera...* Totally it gives us a generating function

$$F(L) = \sum_{k=1}^{\infty} m_k q^k.$$

This function depends on X and L only. The coefficients m_k characterize our moduli spaces $\tilde{\mathcal{M}}_{SBS}$ since their dimensions can be calculated using the Riemann - Roch formula (since it is the same as $h^0(L^k) - 1$), and m_k is the covering degree at generic point over the projective space $|L^k|$. At the same time we can comletely forget about our moduli space, Bohr - Sommerfeld conditions and study this problem just as it is stated above. Moreover we can omitt the condition on the fundamental group of X since the definition of exact lagrangian submanifold needs a hermitian structure on L only.

Thus globally for a smooth compact algebraic variety X we can attach to any very ample line bundle the corrresponding generating function; the properties of the generating functions look quite interesting from the Mirror Symmetry point of view understood as a duality between complex and symplectic geometries.

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