Dualities and Symmetries in String Theory

Albrecht Klemm
Physikalisches Institut
Universität Bonn
Nussallee 12
53115 Bonn

Mathematische Arbeitstagung June 5-11, 2009
Max-Planck-Institut für Mathematik, Bonn, Germany

1 Exposition of the problem

In string theory one considers maps

$$X : \Sigma_g \to \hat{M} \tag{1}$$

from a Riemann surface Σ_g to a target space $\hat{M}$. For simplicity we focus on orientable closed Riemann surface of genus g. The standard supersymmetric string theory, called type II string, has desirable symmetries at quantum level if $\dim_R(\hat{M}) = 10$. This is called the critical dimension and to describe a four dimensional gravity theory, or more precisely a four dimensional $N = 2$ supergravity theory, one considers $\hat{M} = M \times M_4$. Here M_4 is a large space of signature $(3, 1)$, which is to be identified with our universe, while M is a three complex dimensional Calabi-Yau manifold and its typical radii are so small that according to Heisenbergs uncertainty principle one needs higher energy scales then presently explored to detect it directly in experiments. Physical amplitudes are given by variational integrals, the simplest one is the vacuum amplitude

$$Z(M) = \int \mathcal{D}X\mathcal{D}\chi e^{-S(X,\chi,M)} \ , \tag{2}$$

where the action S is schematically

$$S = \int_\sigma G^{\mu\nu}\partial_\alpha X_\mu \partial^\alpha X_\nu + iB^{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X_\mu \partial_\beta X_\nu + \text{supersymmetric completion} \ . \tag{3}$$

Here χ stands for fermionic partners of the bosonic coordinate X, which occur in the supersymmetric completion.

Note that the variational integral over the worldsheet metric does not appear since it trivializes due to the special symmetries in the critical dimension.

On the other hand the metric $G^{\mu\nu}$ and the antisymmetric 2-form field $B^{\mu\nu}$ on M are not varied over, so that Z depends on them as well as on other properies of M, which determines the nature of physics in M_4. The main interest in this talk are the invariances of Z if we modify its argument M. These are called spacetime dualities.

Note that the first term in S is equivalent to area of the image curve and the critical sets of S can be identified with the holomorphic maps.

Due to supersymmetric localization there exists a truncation of the theory to these critical bosonic configurations. The truncated theory is called the topological A-model. In the truncated theory Z collapses to Z_A, which is given by infinite sum over topological sectors labelled by g and the class of the image curve $\beta \in H_2(M, \mathbb{Z})$. The variational integral collapses in each sector into a mathematically welldefined integral over the finite dimensional moduli space of the holomophic maps $\overline{\mathcal{M}_g(M, \beta)}$. The A-model truncation is best decribed by nilpotent BRST operators, which allow to define a cohomological theory whose finite dimensional Hilbert spaces is spanned by states, which are in one to one correspondence with the de Rahm cohomology groups $H^{i,i}(M)$, $i = 0, \ldots, 3$. Its correlators are the cassical intersections deformed by contributions of the holomorphic maps.

The decisive A-model quantity is the free energy

$$F(\lambda, t) = \log(Z_A) = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) \tag{4}$$

with

$$F_g(t) = \text{classical} + \sum_{\beta \in H_2(M\mathbb{Z})} r^g_\beta q^\beta \ . \tag{5}$$

Here

$$r^g_\beta = \int_{\overline{\mathcal{M}_g(M,\beta)}} c_b^{vir}(M, \beta) \in \mathbb{Q} \tag{6}$$

are the Gromov-Witten invariants. They are defined as the integral of a virtual fundamental class over the compactifications of the moduli space of the holomorphic maps. The virtual dimension of the moduli space follows from an index theorem

$$\text{vdim}_{\mathbb{C}} \overline{\mathcal{M}_g(M, \beta)} = \int_\beta c_1 + (\dim - 3)(1 - g) \ . \tag{7}$$

We note that Calabi-Yau threefolds are the critical cases as $\text{vdim}_{\mathbb{C}} \overline{\mathcal{M}_g(M, \beta)} = 0$. This implies that generically a point counting problem in a moduli stack yields $r^g_\beta \neq 0$. The variable $q^\beta = \exp(t_\beta)$, where $t_\beta = 2\pi i \int_\beta (b + \omega)$ is the complexified Kähler parameter. It is a complex variable build from linear deformation of the 2-form field $b = \delta B$ and the real Kähler form $\omega = i \delta G_{i\bar{\jmath}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}}$. Both take values in $H^{1,1}(M, \mathbb{R})$. We note that $q^\beta \to 0$ in the limit of large volume. I.e. the large volume limit suppresses the contributions of the holomorphic maps. The classical terms are constant map contributiions which are of course independed of the volume. An important feature is, that the A-model, does not depend on the pure deformations of the metric δG_{ij} and $\delta G_{\bar{\imath}\bar{\jmath}}$, which parametrize the complex structure deformations of M.

$F(\lambda, t)$ is a generating function for Gromov-Witten invariants. The problem that we pose here is how to calculate it and the main point of this lecture is to explain how $F(\lambda, t)$ can be reconstructed using dualities and symmetries of (2).

2 Other symplectic invariants and integrality conjectures

Before we focus on the main topic we notice that the mathematically well defined rational Gromov-Witten invariants r^g_β are conjecturally related to integral BPS invariants n^g_β, which are physically motivated to be an index on the cohomology of the moduli space of $D2-D0$ branes. The relation between the $n^g_\beta \in \mathbb{Z}$ and the r^g_β are defined by

$$Z'_A(Q,q) = \prod_\beta \left[\left(\prod_{r=1}^{\infty} (1-Q^r q^\beta)^{r n^0_\beta} \right) \prod_{g=1}^{\infty} \prod_{l=0}^{2g-2} (1-Q^{g-l-1} q^\beta)^{(-1)^{g+r}\binom{2g-2}{l} n^g_\beta} \right] , \tag{8}$$

where $Q = e^{i\lambda}$ and the prime indicates that we are omitting the constant map contributions.

To get an impression about the key properties of the BPS invariants we listed the complete information up degree $d = 11$ in table 1 for M the quintic hypersurface in $\mathbb{P}^4$. $d \in \mathbb{Z}$ represents β, in the one dimensional $H_2(M,\mathbb{Z})$ lattice. One important property is that within a fixed class d there is a bound g_{max} on g so that $n^g_d = 0$ for $g \geq g_{max}(d)$. The bound g_{max} growth assymtotically like $g_{max}(d) \propto d^2$. This a simple consequence of the adjunction formula, which implies that there are no embedded curves of genus g if the degree is not high enough. The important difference between r^g_β and n^g_β is that the latter is a property of the embedded curve in m rather then a property of the map to M. Puting it differently all information about the multi covering of the map into a given curve class is encoded in (8).

A simple example of the index definition of n^g_β can be stated for smooth curves C, where $n^g_\beta = (-1)^{\dim \mathcal{M}_C} e(\mathcal{M}_C)$. Here $\mathcal{M}_C$ is the deformation space. For $d = 5$ and $d = 10$ and maximal genus those smooth curves are complete intersections and a simple calculation of their moduli space yields $n^6_5 = 10$ and $n^{16}_{10} = -50$.

A further relation links the above invariants to the Donaldson-Thomas invariants, which are integrals over the moduli space of ideal sheafs on M. Let

$$Z_{DT}(Q,q) = \sum_{\beta,k\in\mathbb{Z}} m^k_\beta Q^k q^\beta \tag{9}$$

define a generating series for the Donaldson-Thomas invariants $m^k_\beta \in \mathbb{Z}$ then the relation is given by

$$Z_{DT}(-Q,q) = Z'_A(Q,q) M(-Q)^{e(M)} , \tag{10}$$

where

$$M(Q) = \prod_{n\geq 1} \frac{1}{(1-q^n)^n} \tag{11}$$

is the McMahon function.

g	d=1	d=2	d=3	d=4	d=5	d=6
0	2875	609250	317206375	242467530000	229305888887625	248249742118022000
1	0	0	609250	3721431625	12129909700200	31147299733286500
2	0	0	0	534750	75478987900	871708139638250
3	0	0	0	8625	-15663750	3156446162875
4	0	0	0	0	49250	-7529331750
5	0	0	0	0	1100	-3079125
6	0	0	0	0	10	-34500
7	0	0	0	0	0	0

g	d=7	d=8	d=9
0	295091050570845659250	375632160937476603550000	503840510416985243645106250
1	715784060228807617 50	1549905417529615684181 25	3240644643102795856570087 50
2	5185462556617269625	2251684106310591776675 0	8146492178683956650256012 5
3	111468926053022750	1303464598408583455000	952321365916921756899150 0
4	245477430615250	25517502254834226750	5077234965144335614982 50
5	-1917984531500	465698896195706 25	10280743594493108319750
6	1300955250	-471852100909500	308841641958702172 50
7	4874000	2876330661125	-1351975081774407 50
8	0	-1670397000	1937652290971125
9	0	-6092500	-12735865055000
10	0	0	18763368375
11	0	0	5502750
12	0	0	60375
13	0	0	0

g	d=10	d=11
0	704288164978454686113488249750	1017913203569692432490203659468875
1	662863774391414096742406576300	1336442091735463067608016312923750
2	261910639528673259095545137450	775720627148503750199049691449750
3	52939966189791662442040406825	245749672908222069996115276347 50
4	5646690223118638682929856600	44847555720065830716840300475375
5	302653046360802682731297875	4695086609484491386537177620000
6	6948750094748611384962730	267789764216841760168691381625
7	40179519996158239076800	7357099242952070238708870000
8	-25301032766083303150	72742651599368002897701250
9	1155593062739271425	140965985795732693440000
10	-1797620952942470 0	72285071203117009200 0
11	150444095741780	-18998955257482171250
12	-454092663150	353650228902738500
13	50530375	-4041708780324500
14	-286650	22562306494375
15	-5700	-29938013250
16	-50	-7357125
17	0	-86250
18	0	0

Table 1: BPS invariants n^g_β on the Quintic hypersurface in $\mathbb{P}^4$

3 The duality symmetries

3.1 Mirror symmetry

Mirror symmetry can be summarized by the statement that

$$\begin{array}{ll} Z_A(M,\lambda,t) & = Z_B(W,\lambda,\hat{t}), \\ Z_A(W,\lambda,t) & = Z_B(M,\lambda,\hat{t}), \end{array} \tag{12}$$

here (W, M) are mirror pairs of manifolds with

$$H^{3-k,p}(M) = H^{k,p}(W), \tag{13}$$

for $k, p = 0, \ldots, 3$. B stands for the topological B-model. It emerges by a different localisation of the full variational integral $Z(M)$ to constant maps albeit with a more complicated measure. Mirror symmetry identifies the A-model on M with the B-model on W and vice versa. The topological states of the B model are in correspondence with the cohomology groups dual (13) to ones which define the states of the A-model. The B-model depends only on the complex structure variations $\hat{t}$ of the corresponding manifold. The latter are encoded in period integrals over the holomorphic $(3,0)$-form. Studying the latter at a point of maximal degeneration yields also a concrete expression for the mirror map $\hat{t}(t)$ in (12). It should be noted that (12) is a specialized version of mirror symmetry, which is designed to be mathematically controllable. The physical expectation is simply that string theory on M and on W are indistiguishable.

The construction of mirror manifolds is understood conceptually in symplectic geometry, by the SYZ conjecture, which states that every Calabi-Yau manifold is a (degenerate) Lagrangian T^3 fibration over a 3-dim base and that the mirror can be constructed by dualizing the T^3 torus fibrewise. Pragmatically thousands of mirror pairs can be easily constructed within the framework of algebraic geometry as anticanonical hypersurfaces in pairs of toric varities defined by pairs of reflexive polyhedra as pointed out by Batyrev

3.2 Periods and monodromy

We discuss now the monodromy of one paramter family of mirror quintics $W(\hat{t})$,

$$W(\hat{t}) = \left\{ p = \sum_{i=1}^{5} x_i^5 - 5e^{-\frac{\hat{t}}{5}} \prod_{i=1}^{5} x_i = 0 \text{ in } \mathbb{P}^4 \right\} . \tag{14}$$

It can be obtained as orbifold $M/\mathbb{Z}_5^3$ of the original quintic M, where the $\mathbb{Z}_5$'s are generated by phase rotations on the homogeneous coordinates $\mathbb{P}^4$

$$x_i \to \exp(2\pi i g_i^{(\alpha)}/5)x_i, \quad \alpha = 1,2,3, \quad i = 1, \ldots, 5 \ , \tag{15}$$

with $g^{(1)} = (1,4,0,0,0)$, $g^{(2)} = (1,0,4,0,0)$ and $g^{(3)} = (1,0,0,4,0)$. We identify $z = e^{\hat{t}}$ and notice that the complex moduli space is parametrized by z as $\mathcal{M} = \mathbb{P} \setminus \{z = 0, 1, \infty\}$.

The holomorphic $(3,0)$-form is locally $\Omega = \frac{z^{-\frac{1}{5}} x_i \wedge_{k\neq i,j} \mathrm{d}x_k}{\partial_j p}$. There is a flat connection on the period vector

$$\Pi = \begin{pmatrix} \int_{A^I} \Omega = X^I \\ \int_{B_I} \Omega = P_I = \frac{\partial F_0}{\partial X^I} \end{pmatrix}, \qquad , I = 0, \ldots, 3 \tag{16}$$

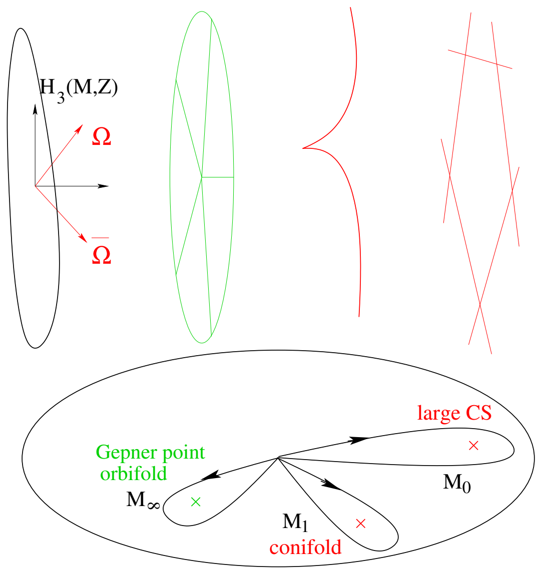
expressed by the PIcard-Fuchs equation

$$[\theta^4 - 5z \prod_{k=1}^{4} (\theta + k)]\Pi(z) = 0, \qquad \theta = z \frac{\mathrm{d}}{\mathrm{d}z}, \tag{17}$$

which undergoes the monodromies $\Pi \mapsto M_i \Pi$ with $M_{z=z_i} \in SP(4, \mathbb{Z})$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 5 & -3 & 1 & -1 \\ -8 & -5 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{18}$$

generate the monodromy group Γ_M, where the loops are schematically



Mirror quintic family

3.3 $g = 0$

The first sucess of mirror symmetry is that

$$F_0(t) = class. + \sum_{d=1} n_d^0 \mathrm{Li}_3(q^d), \tag{19}$$

where the mirror map at large complex strcuture (CS) $z = 0$ is

$$t = \frac{X^1}{X^0}(z) \ , \tag{20}$$

where $X^0 = 1 + holom$ and $\frac{1}{2\pi i}(X^0 \log(z) + holom.)$ are completly determined from (17).

In the complex moduli space one has special geometry, with Kählerpotential $e^{-K} = i\int \Omega\wedge\bar\Omega$, $C_{ijk} = \int \Omega\partial_i\partial_j\partial_k\Omega = D_iD_jD_K F_0$ and the integrability condition

$$R^i_{k\bar l m} = \delta^i_k g_{\bar l m} + \delta^i_m g_{\bar l k} + C_{kmj}\bar C^{ij}_{\bar l} \tag{21}$$

with $\bar C^{ij}_{\bar l} = \bar C_{\bar l \bar k \bar l} g^{\bar m i} g^{\bar k j} e^{2K}$.

3.4 $g = 1$

The genus one amplitude is a Ray-Singer-Torsion family index over $\mathcal{M}$ and fullfills

$$\partial_i\bar\partial_{\bar j} F_1 = \frac{1}{2}\bar C^{mn}_{\bar j} C_{imn} - \left(\frac{e(m)}{24} - 1\right) g_{i\bar j}\ . \tag{22}$$

It can be fixed by the boundary behaviour $F_1 \sim \frac{1}{12}\log(t_c)$, where t_c is the flat coordinate near the conifold and $F_1 \sim 50\frac{t}{24}$ near large complex strcuture.

3.5 $g > 1$

For higher genus the F_g fullfill the holomorphic anomaly equation

$$\partial_{\bar i} F_g = \frac{1}{2}\bar C^{mn}_{\bar i}\left(D_m D_n F_{g-1} + \sum_{r=1}^{g-1} D_m F_r D_n F g - r\right) \tag{23}$$

It has an holomorphic function as an ambiguity. The latter can be fixed by the fact that F_g is modular invariant and physical boundary conditions. The first fact implies that the F_g are finetly generated by a ring which can be viewed as the generalization of the ring of almost holomorphic modular forms from elliptic curves to Calabi-Yau manifolds.

In local flat coordinates the leading behaviour at the boundaries is as follows

- Expansion around the conifold point $z = 1$:

$$\begin{aligned}
F_0^{\rm c} &= -\tfrac{5}{2}\log(\hat t_c)\hat t_c^2 + \tfrac{5}{12}\,(1-6b_1)\,\hat t_D^3 \\
&\quad + \left(\tfrac{5}{12}\,(b_1 - 3b_2) - \tfrac{89}{1440} - \tfrac{5}{4}\,b_1^2\right)\hat t_c^4 + \mathcal{O}(\hat t_c^5) \\
F_1^{\rm c} &= -\tfrac{\log(\hat t_c)}{12} + \left(\tfrac{233}{120} - \tfrac{113\,b_1}{12}\right)\hat t_c \\
&\quad + \left(\tfrac{233\,b_1}{120} - \tfrac{113\,{b_1}^2}{24} - \tfrac{107 b_2}{12} - \tfrac{2681}{7200}\right)\hat t_c^2 + \mathcal{O}(\hat t_c^3) \\
F_2^{\rm c} &= \tfrac{1}{240\hat t_c^2} - \left(\tfrac{120373}{72000} + \tfrac{11413 b_2}{144}\right) \\
&\quad + \left(\tfrac{107369}{150000} - \tfrac{120373\,b_1}{36000} + \tfrac{23533\,b_2}{720} - \tfrac{11413 b_1 b_2}{72}\right)\hat t_c + \mathcal{O}(\hat t_c^2) \\
F_3^{\rm c} &= \tfrac{1}{1008\,\hat t_c^4} - \left(\tfrac{178778753}{324000000} + \tfrac{2287087\,b_2}{43200} + \tfrac{1084235\,{b_2}^2}{864}\right) + \mathcal{O}(\hat t_c) \\
F_4^{\rm c} &= \tfrac{1}{1440\,\hat t_c^6} - \left(\tfrac{977520873701}{3402000000000} + \tfrac{162178069379\,b_2}{3888000000}\right. \\
&\quad \left. + \tfrac{5170381469\,{b_2}^2}{2592000} + \tfrac{490222589\,{b_2}^3}{15552}\right) + \mathcal{O}(\hat t_c)\ .
\end{aligned}$$

$$F_g^{\rm conifold} = \frac{(-1)^{g-1}B_{2g}}{2g(2g-2)(\hat t_c)^{2g-2}} + \mathcal{O}(\hat t_c^0).$$

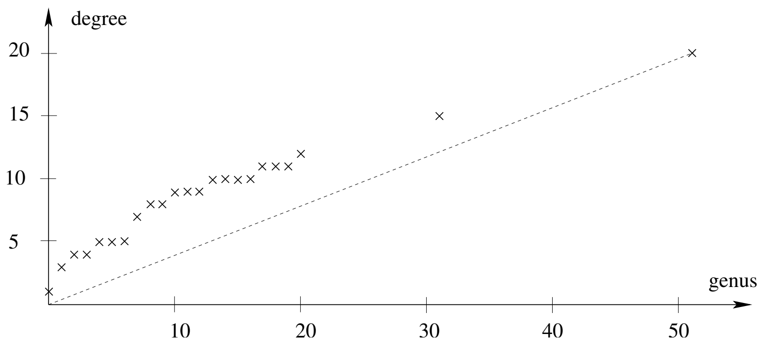
I.e. at the conifold we have the gap condition that the $2g-2$ subleading coefficients are absent.

- Expansions around the orbifold point $\frac{1}{z}=0$

$$
\begin{aligned}
F_0^{\rm o} &= \tfrac{5\,s^3}{6}+\tfrac{5\,s^8}{1008}+\tfrac{5975\,s^{13}}{10378368}+\tfrac{34521785\,s^{18}}{266765571072}+\ldots\\
F_1^{\rm o} &= -\tfrac{s^5}{9}-\tfrac{163\,s^{10}}{18144}-\tfrac{85031\,s^{15}}{46702656}-\tfrac{6909032915\,s^{20}}{20274183401472}+\ldots\\
F_2^{\rm o} &= \tfrac{155\,s^2}{18}-\tfrac{5\,s^7}{864}+\tfrac{585295\,s^{12}}{14370048}+\tfrac{1710167735\,s^{17}}{177843714048}+\ldots\\
F_3^{\rm o} &= \tfrac{488305\,s^4}{9072}-\tfrac{3634345\,s^9}{979776}-\tfrac{1612981445\,s^{14}}{7846046208}-\tfrac{2426211933305\,s^{19}}{116115777662976}+\ldots\\
F_4^{\rm o} &= \tfrac{48550\,s}{567}+\tfrac{36705385\,s^6}{163296}+\tfrac{16986429665\,s^{11}}{603542016}+\tfrac{341329887875\,s^{16}}{70614415872}+\ldots
\end{aligned}
$$

I.e. at the orbifold point we have the constion that F_g behaves regular. The coefficients of the expansion in the flat coordinate s are the orbifold Gromov-Witten invariants and some checks using direct computations of the latter have been made.

It can be shown that these boundary conditions fix $\left[\frac{2g-1}{5}\right]+2g-2$ constant in the holomorphic or modular ambiguity, which is parametrized by $3g-3$ coeffcients. If one uses the fact that $n^g_d=0$ for $g>g_{max}$ one can solve the equation (22) up to genus 51 as can be seen from the follwing figure



References

[HKQ] M. x. Huang, A. Klemm and S. Quackenbush, "Topological String Theory on Compact Calabi-Yau: Modularity and Boundary Conditions," Lect. Notes Phys. **757** (2009) 45 [arXiv:hep-th/0612125].