Δ-Operator on Semidensities and Integral Invariants in the Batalin-Vilkovisky Geometry

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Abstract *The action of Batalin-Vilkovisky Δ-operator on semidensities in an odd symplectic superspace is defined. This is used for the construction of integral invariants on surfaces embedded in an odd symplectic superspace and for more clear interpretation of the Batalin-Vilkovisky formalism geometry.*

1.Introduction

Density of the weight σ is the function on a space (superspace) subject to the condition that under change of coordinates it is multiplied on the σ-th power of the determinant (Berezinian) of the transformation. The density of the weight $\sigma = 1$ is a volume form on the space (superspace).

In this paper we consider semidensities (densities of the weight $\sigma = 1/2$)in a superspace provided with an odd symplectic structure and define the action of Δ-operator on them. It turns out that these constructions has clear geometrical meaning in an odd symplectic superspace. On one hand they lead to the straightforward geometrical interpretation of Batalin-Vilkovisky master-equation. On the other hand using them, we elaborate a new outlook on the invariant semidensity defined on (1.1)-codimensional surfaces embedded in the odd symplectic superspace [1] and construct integral invariants on these surfaces.

The concept of an odd symplectic superspace and Δ-operator on it appeared in mathematical physics in the pioneer works of Batalin-Vilkovisky [2,3], where these objects were used for constructing covariant Lagrangian version of the BRST quantization. The geometrical meaning of these objects and interpretation of BV master-equation in its terms was studied in [4,5,6]. The complete analysis of these constructions and their meaning in BV formalism was performed by A.S.Schwarz in [7].

Let us shortly sketch these results.

We call a superspace provided with an odd symplectic structure *odd symplectic superspace*. An odd symplectic superspace which is provided in addition with a volume form will be called *special odd symplectic superspace.*

If $d\mathbf{v}$ is the volume form of the special odd symplectic superspace, then one can consider operator $\Delta_{d\mathbf{v}}$ whose action on a function on this superspace is equal to the divergence w.r.t. the volume form $d\mathbf{v}$ of the Hamiltonian vector field corresponding to this function. This second order differential operator is not trivial because transformations preserving odd symplectic structure do not preserve any volume form (Liouville theorem fails to be fulfilled in a case of odd symplectic structure).

We call a superspace *normal special odd symplectic superspace* if in the special odd symplectic superspace there exist locally such Darboux coordinates, that in these coordinates the density of the volume form is equal to one:

$$d\mathbf{v} = dx^1 \dots dx^n d\theta_1 \dots d\theta_n \,. \tag{1.1}$$

(Darboux coordinates in an odd symplectic superspace are coordinates $z^A = x^1, \dots, x^n$, $\theta_1, \dots, \theta_n$ in which the Poisson bracket corresponding to the symplectic structure has the canonical form: $\{x^i, \theta_j\} = \delta^i_j$, $\{x^i, x^j\} = 0$.)

The concept of normal special odd symplectic superspace, which is called also SP superspace (see [7]), is crucial in the geometrical interpretation of BV formalism.

If f is an even function on a normal special odd symplectic superspace and $d\mathbf{v}' = f d\mathbf{v}$ is a new volume form on it then the main essence of geometrical formulation of BV formalism can be shortly expressed in the following two statements [4–7]:

Statement 1.

The following conditions:

$$\begin{gathered} a) \quad \text{the volume form } d\mathbf{v}' \text{ provides} \\ \text{the odd symplectic superspace by the normal special structure} \\ \text{in the same way as the initial volume form } d\mathbf{v}, \\ \text{(i.e. there exist Darboux coordinates} \\ \text{in which the volume form } d\mathbf{v}' \text{ is equal to (1.1))}\,. \end{gathered} \tag{1.2a}$$

$$b) \quad \Delta_0 \sqrt{f} = \sum_{i=1}^{n} \frac{\partial^2 \sqrt{f}}{\partial x^i \partial \theta_i} = 0\,, \quad \text{(BV master-equation for the master-action } S = \log \sqrt{f})\,. \tag{1.2b}$$

$$c) \quad \Delta^2_{d\mathbf{v}'} = 0\,, \tag{1.2c}$$

obey to the relation

$$a) \Rightarrow b) \Rightarrow c)$$

and under some assumptions they are equivalent (see the details below).

Statement 2.

The integrand of BV partition function is the semidensity $\sqrt{f d\mathbf{v}}$ which is the natural integration object over Lagrangian surfaces in an special odd symplectic superspace. In the case if conditions (1.2) *are fulfilled, the corresponding integral does not change under small variations of Lagrangian surface (gauge-independence condition). To the semidensity $\sqrt{f d\mathbf{v}}$ corresponds the cohomological class of Lagrangian surfaces.*

The analysis of these statements in [4–7] is particularly founded on the following geometrical observations. It is well-known that there is a natural correspondence between polyvectorial antisymmetrical fields on the usual manifolds M and functions on the superspace $\Pi T^* M$ which is associated to the cotangent bundle of this manifold. This correspondence transforms the Schoutten bracket of polyvectorial fields on M to the canonical odd Poison bracket on $\Pi T^* M$. If the volume form $d\mathbf{v}$ on $\Pi T^* M$ corresponds to the square of some volume form dv on M, then $\Pi T^* M$ becomes normal odd symplectic superspace and the action of operator $\Delta_{d\mathbf{v}}$ on it ($d\mathbf{v} = dv^2$) corresponds to the action of divergence operator on polyvectorial fields w.r.t. the volume form dv. In the [7] was also observed that square root of a volume form in an odd symplectic superspace is a natural integration object over Lagrangian surfaces in this superspace, and it was established relations between differential forms on M and volume forms in an odd symplectic superspace ΠT^*M. This was crucial for complete analysis of the statements 1 and 2. (See in details 2-nd Section).

In this paper we pay attention on the fact that it is semidensities (square root of volume forms) in ΠT^*M, but not densities (volume forms) which are naturally related with differential forms on M. One can naturally define the action of Δ operator on semidensities in an odd symplectic superspace as well as on the functions. Moreover one does not need a volume form for defining the action of Δ operator on semidensities and this operator corresponds to the exterior differential of differential forms.

This leads to more clear interpretation of the relations in the Statements 1 and 2. In particular the condition (1.2b) in terms of semidensities receives its invariant formulation and the difference between conditions (1.2a,b,c) is formulated exactly.

But the main purpose of this paper is following. Our considerations provide us with a geometrical interpretation of the odd invariant semidensity on (1.1)-codimensional surfaces embedded in an special odd symplectic superspace[1,8], and this leads to construction of new integral invariants on surfaces embedded in an odd symplectic superspace.

We recall shortly the problem of invariant densities construction in an odd sympelctic superspace.

In the case if we consider the volume form not only on the space (superspace) but on arbitrary embedded surfaces we come to the concept of densities on embedded surfaces. The density of weight σ and rank k on embedded surfaces is a function $A(z, \frac{\partial z}{\partial \zeta}, \ldots, \frac{\partial^k z}{\partial \zeta \ldots \partial \zeta})$ which is defined on parametrized surfaces $z(\zeta)$, depends on first k derivatives of $z(\zeta)$ and is multiplied on the σ-th power of the determinant (Berezinian) of surface reparametrization. On the every given surface it defines the σ-th power of volume form. Such a concept of density is very useful in supermathematics where the notion of differential forms as integration objects is ill-defined. It was elaborated by A.S.Schwarz, particularly for analyzing supergravity Lagrangians [9,10,11].

In usual mathematics, for every $2k$-dimensional surface C^{2k} embedded in a symplectic space, so called Poincaré-Cartan integral invariants (invariant volume forms on embedded surfaces) are given by the formula

$$\int_{C^{2k}} \underbrace{w \wedge \ldots \wedge w}_{k-\text{times}} = \int \sqrt{\det \left(\frac{\partial x^\mu(\xi)}{\partial \xi^i} w_{\mu\nu} \frac{\partial x^\nu(\xi)}{\partial \xi^j} \right)} d^{2k}\xi \,, \tag{1.3}$$

where the non-degenerated closed two-form $w = w_{\mu\nu} dx^\mu \wedge dx^\nu$ defines symplectic structure, and the functions $x^\mu = x^\mu(\xi^i)$ define some parametrization of the surface C^{2k}.

In the case of an even symplectic superspace, the l.h.s. of (1.3) is ill-defined but the r.h.s. of this formula can be straightforwardly generalized, by changing determinant on the Berezinian (superdeterminant). The properties of the integral invariant do not change drastically. In particular one can prove that the integrand in the (1.3) (the density of the weight $\sigma = 1$ and of the rank $k = 1$) in the case of even symplectic structure in the superspace is total derivative and all invariant densities on surfaces are exhausted by (1.3) as well as in the case of usual symplectic structure [12,13,14].

The situation is less trivial in the case of an odd symplectic superspace. As it was mentioned above transformations preserving an odd symplectic structure do not preserve any volume form. One can consider invariant densities only in an odd symplectic superspace provided with a volume form, i.e. in a special odd symplectic superspace.

The problem of the existence of invariant densities on non-degenerate surfaces embedded in an special odd symplectic superspace was studied in [1,8]. In particularly it was proved that there are no invariant densities of the rank $k = 1$ (except of the volume form itself), and it was constructed the invariant semidensity of the rank $k = 2$ which is defined on non-degenerate surfaces of the codimension (1.1) (see formulae (3.1, 3.4) below or [1,8] for details).

This semidensity takes odd values. It is an exotic analogue of Poincaré–Cartan invariant: the corresponding density (the square of this odd semidensity) is equal to zero, so it cannot be integrated nontrivially over supersurfaces. On the other hand this semidensity can be considered as an analog of the mean curvature of hypersurfaces in the Riemanian space [1]. It was also proved that at least in an normal special odd symplectic superspace this odd semidenstiy is unique (up to multiplication by a constant) in the class of densities of the rank $k = 2$ which are defined on non-degenerated surfaces of arbitrary dimension [8].

These results indicate that one have to search non-trivial integral invariants (invariant densities of the weight $\sigma = 1$) in higher ($k \geq 2$) order derivatives. The tedious calculations which lead to the construction of this odd semidensity in the papers [1,8] did not give hope to go further for finding them, using the technique which was used in these papers.

The analysis performed in this paper shows the important role of semidensity in an odd symplectic superspace, revealing its meaning in terms of differential forms on underlying space. It is semidensity in the ambient odd symplectic space, not the volume form, which naturally induces invariant densities on embedded surfaces (see the Theorem in the Section 3). This makes the fact that the simplest invariant density on surfaces is the odd semidensity less surprising. On the other hand our approach allows to construct new densities depending on fourth order derivatives on surfaces embedded in an special odd symplectic superspace.

In the second section we recall the basic definitions of an odd symplectic and special odd symplectic superspace, the properties of Δ operator acting on functions, and we give the definition of the Δ-operator acting on semidensities in an odd symplectic superspace. We analyze these objects in terms of underlying space geometry considering as the basic example of an odd symplectic superspace, the superspace associated to the cotangent bundle of the usual space. In the end of the section we carry out considerations similar to those in the paper [7], but on the base of our construcions, and we come to more clear interpretation of BV formalism geometry.

In the third section we come to natural interpretation of the odd invariant semidensity on (1.1)-codimensional surfaces, expressing this semidensity straightforwardly in terms of semidensity in the ambient symplectic space and using this construction we construct another semidenstiy and two densities (integral invariants), even and odd, of the rank $k = 4$ on (1.1)-codimensional surfaces. This is the main result of our paper.

Maybe these densities are the simplest (having the lowest rank) non-trivial integral invariants on surfaces in an special odd symplectic surfaces.

2. $\Delta^{\#}$-Operator on Semidensities

Let $E^{n.n}$ be a superspace with coordinates $z^A = x^1, \dots, x^n,\ \theta_1, \dots, \theta_n;\ p(x^i) = 0,\ p(\theta_j) = 1$, where p is a parity *.

We say that this superspace is an odd symplectic superspace if it is endowed with an odd symplectic structure, i.e., if an odd closed non-degenerate 2-form: $\Omega = \Omega_{AB}(z) dz^A dz^B$, $p(\Omega) = 1$, $d\Omega = 0$ is defined on it [16, 17].

In the same way as in the standard symplectic calculus one can relate to the odd symplectic structure the odd Poisson bracket (Buttin bracket) [16,17,15]:

$$\{f, g\} = \frac{\partial f}{\partial z^A} (-1)^{fA+A} \Omega^{AB} \frac{\partial g}{\partial z^B} ,$$

where $\Omega^{AB} = \{z^A, z^B\}$ is the inverse matrix to $\Omega_{AB} : \Omega^{AC} \Omega_{CB} = \delta^A_B$, $(\Omega^{AB} = -\Omega^{BA} (-1)^{(A+1)(B+1)})$.

To a function f there corresponds a Hamiltonian vector field

$$\mathbf{D}_f = \{f, z^A\} \frac{\partial}{\partial z^A} , \quad \mathbf{D}_f(g) = \{f, g\}, \quad \Omega(\mathbf{D}_f, \mathbf{D}_g) = -\{f, g\} . \tag{2.1}$$

(See for the details e.g. [1].)

The condition of the closedness of the form defining symplectic structure leads to the Jacoby identities:

$$\{f, \{g, h\}\} (-1)^{(f+1)(h+1)} + \text{cycl.} \quad \text{permutations} = 0 . \tag{2.2}$$

* We use the definition , when a point of superspace $E^{n.n}$ is Λ-point— $2n$–plet $(a^1, \dots, a^n, \alpha^1, \dots, \alpha^n)$, where $(a^1, \dots, a^n)$ are even and $(\alpha^1, \dots, \alpha^n)$ are odd elements of an arbitrary Grassmann algebra Λ. It is the most general definition of superspace as the functor on the category of Grassmann algebras, suggested by D.A.Leites and A.S.Schwarz [15,11].

Using the analog of Darboux Theorem [18] one can consider the coordinates in which the symplectic structure and the corresponding Buttin bracket have locally the canonical expressions: $\Omega = \sum dx^i d\theta_i$ and respectively

$$\{x^i, x^j\} = 0,\ \{\theta_i, \theta_j\} = 0,\ \{x^i, \theta_j\} = \delta^i_j\,,\ \{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial \theta_i} + (-1)^f \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial x^i} \right)\,. \quad (2.3)$$

We call these coordinates *Darboux coordinates*.

The odd symplectic space provided with a volume form

$$d\mathbf{v} = \rho(z) dz^1 \dots dz^{2n} \quad (2.4)$$

will be called *special odd symplectic superspace*. We suppose that the volume form is non-degenerated, i.e. for the every point z_0 the number (non-nilpotent) part of $\rho(z_0)$ is not equal to zero.

In the case if in the special odd symplectic superspace there exist Darboux coordinates such that the volume form in these coordinates is given by (1.1), then this space will be called *normal special odd symplectic superspace*.

The action of Δ operator on an arbitrary functions in the special odd symplectic superspace is equal (up to coefficient) to the divergence w.r.t. volume form (2.4) of the Hamiltonian vector field corresponding to this function [4,5]. Using (2.1) we come to the formula

$$\Delta_{d\mathbf{v}} f = \frac{1}{2} (-1)^f \mathrm{div}_{d\mathbf{v}} \mathbf{D}_f = \frac{1}{2} (-1)^f \left((-1)^{\mathbf{D}_f A + A} \frac{\partial}{\partial z^A} \{f, z^A\} + D^A_f \frac{\partial \log \rho(z)}{\partial z^A} \right)\,. \quad (2.5)$$

In Darboux coordinates:

$$\Delta_{d\mathbf{v}} f = \Delta_0 f + \frac{1}{2} \{\log \rho, f\}\,, \quad (2.6)$$

where $\rho(z)$ is given by (2.4), and

$$\Delta_0 f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^i \partial \theta_i}\,. \quad (2.7)$$

(Later on we use only Darboux coordinates.)

This operator obeys to the relations [3,5]:

$$\Delta_{d\mathbf{v}} \{f, g\} = \{\Delta_{d\mathbf{v}} f, g\} + (-1)^{f+1} \{f, \Delta_{d\mathbf{v}} g\}\,,$$

$$\Delta_{d\mathbf{v}} (f \cdot g) = \Delta_{d\mathbf{v}} f \cdot g + (-1)^f f \cdot \Delta_{d\mathbf{v}} g + (-1)^f \{f, g\}\,. \quad (2.8)$$

Now we define the action of operator $\Delta^\#$ on semidensties in an odd symplectic superspace.

Definition *If* $\mathbf{s}$ *is a semidensity in an odd symplectic superspace and* $s(z)|dz|^{1/2}$ *is local expression for this semidensity in Darboux coordinates* $z^A = (x^1, \dots, x^n, \theta_1, \dots, \theta_n)$ *then the local expression for the density* $\Delta^\#\mathbf{s}$ *in these coordinates is given by the following formula:*

$$\Delta^\#\mathbf{s} = (\Delta_0 s(z))|dz|^{1/2} = \sum_{i=1}^n \frac{\partial^2 s}{\partial x^i \partial \theta_i}|dxd\theta|^{1/2}\,. \tag{2.9}$$

One can prove that the r.h.s. of this formula defines the semidensity also, i.e. if $\tilde z = (\tilde x^1, \dots, \tilde x^n, \tilde\theta_1, \dots, \tilde\theta_n)$ are another Darboux coordinates then

$$\left(\sum_{i=1}^n \frac{\partial^2}{\partial x^i \partial\theta_i} s(z)\right)_{z(\tilde z)} \cdot \left(\mathrm{Ber}\frac{\partial z(\tilde z)}{\partial \tilde z}\right)^{1/2} = \sum_{i=1}^n \frac{\partial^2}{\partial \tilde x^i \partial \tilde\theta_i}\left(s(z(\tilde z)) \cdot \mathrm{Ber}\frac{\partial z(\tilde z)}{\partial \tilde z}^{1/2}\right)\,. \tag{2.10}$$

Canonical transformations from some Darboux coordinates to another Darboux coordinates infinitezimally are generated by an odd function (Hamiltonian) via corresponding Hamiltonian vector filed (2.1). To an odd function $Q(z)$ corresponds transformation $\tilde z^a = z^A + \varepsilon\{Q, z^A\}$. To the action of this transformation on the semidenity $\mathbf{s}$ corresponds differential $\delta_Q s = \Delta_0 Q \cdot s - \{Q, s\}$, because $\delta s = -\varepsilon\{Q, s\}$ and $\delta|dz| = \varepsilon\delta\mathrm{Ber}(\partial z/\partial\tilde z)|dz| = 2\Delta_0 Q|dz|$. Using that $\Delta_0^2 = 0$ and (2.8) we come to commutation relations $\Delta_0\delta_Q = \delta_Q\Delta_0$. This leads to relation (2.10), which proves the correctness of definition (2.9). One can check the relations (2.10) by straightforward computations also, using the properties of the operator Δ_0 which were investigated in details in [3].

The action of differential δ_Q on semidensities can be rewritten in a explicitly invariant way:

$$\delta_Q\mathbf{s} = Q \cdot \Delta^\#\mathbf{s} + \Delta^\#(Q\mathbf{s}) = [Q, \Delta^\#]_+\mathbf{s}\,. \tag{2.11}$$

Contrary to the operator $\Delta_{d\mathbf{v}}$ on functions, the operator $\Delta^\#$ on semidensities does not need volume structure.

On a special odd symplectic superspace we can construct new invariant objects, expressing them via the semidensity related with volume form ($\mathbf{s} = \sqrt{d\mathbf{v}}$) and operator $\Delta^\#$:

$$\mathbf{s} = \sqrt{d\mathbf{v}} \text{—semidensity } (\sigma = \tfrac{1}{2})\,, \tag{2.12a}$$

$$\Delta^\#\mathbf{s} = \Delta^\#\sqrt{d\mathbf{v}} \text{—semidensity } (\sigma = \tfrac{1}{2})\,, \tag{2.12b}$$

$$\mathbf{s}\Delta^\#\mathbf{s} = \sqrt{d\mathbf{v}}\Delta^\#\sqrt{d\mathbf{v}} \text{—density } (\sigma = 1)\,, \tag{2.12c}$$

$$\frac{1}{\mathbf{s}}\Delta^\#\mathbf{s} = \frac{1}{\sqrt{d\mathbf{v}}}\Delta^\#\sqrt{d\mathbf{v}} \text{—function } (\sigma = 0)\,. \tag{2.12d}$$

Using (2.6)—(2.9) we can see that operator $\Delta^{\#}$ obeys to the following properties:

$$(\Delta^{\#})^2 = 0\,,$$
$$\Delta^{\#}(f \cdot \sqrt{d\mathbf{v}}) = (\Delta_{d\mathbf{v}} f) \cdot \sqrt{d\mathbf{v}} + (-1)^f f \cdot \Delta^{\#}\sqrt{d\mathbf{v}}\,, \tag{2.13}$$

and

$$\Delta^2_{d\mathbf{v}} f = \{\frac{1}{\sqrt{d\mathbf{v}}}\Delta^{\#}\sqrt{d\mathbf{v}}, f\}\,. \tag{2.14}$$

To clarify the geometrical meaning of the definition (2.9) and of its correctness we consider the following example

Example 1 Let M be a n-dimensional space and T^*M be its cotangent bundle. Let ΠT^*M be a superspace associated with T^*M. To local coordinates $(x^1, \ldots, x^n)$ on M there correspond local coordinates $z^A = (x^1, \ldots, x^n, \theta_1, \ldots, \theta_n)$ on ΠT^*M.

The odd coordinates θ_j transforms via the differential of corresponding transformation of the even coordinates x^i:

$$\tilde{x}^i = \tilde{x}^i(x), \quad \tilde{\theta}_i = \sum_{k=1}^{n} \frac{\partial x^k(\tilde{x})}{\partial \tilde{x}^i}\theta_k\,. \tag{2.15}$$

Therefore one can define the canonical odd symplectic structure on ΠT^*M in a such way that $z^A = (x^1, \ldots, x^n, \theta_1, \ldots, \theta_n)$ are Darboux coordinates for this symplectic structure. The pasting formulae (2.15) provide the correctness of the definition of this structure.

The relations between the cotangent bundle structure on T^*M and the odd canonical symplectic structure (2.3) on ΠT^*M reveal in the properties of the canonical map $\tau_{_M}$ between polyvectorial antisymmetric fields on M and functions on ΠT^*M:

$$\tau_{_M}\left(T^{i_1\ldots i_k}\frac{\partial}{\partial x^{i_1}}\wedge\ldots\wedge\frac{\partial}{\partial x^{i_k}}\right) = T^{i_1\ldots i_k}\theta_{i_1}\ldots\theta_{i_k}\,.$$

This map transforms the Schouten bracket of polyvectorial fields to the odd canonical Poisson (Buttin) bracket (2.3) of corresponding functions [15,19]:

$$\tau_{_M}([\mathbf{T_1}, \mathbf{T_2}]) = \{\tau_{_M}(\mathbf{T_1}), \tau_{_M}(\mathbf{T_2})\}\,. \tag{2.16}$$

Now we consider the map $\tau^{\#}_{_M}$ from the differential forms on M to the semidensities on ΠT^*M, defining it in the following way:

$$\begin{aligned}
\tau^{\#}_{_M}(1) &= \theta_1\ldots\theta_n|dxd\theta|^{1/2},\\
\tau^{\#}_{_M}(dx^i) &= (-1)^{i+1}\theta_1\ldots\widehat{\theta}_i\ldots\theta_n|dxd\theta|^{1/2}\,,\\
\tau^{\#}_{_M}(dx^i\wedge dx^j) &= (-1)^{i+j}\theta_1\ldots\widehat{\theta}_i\ldots\widehat{\theta}_j\ldots\theta_n|dxd\theta|^{1/2}\,,\ (i<j),\\
&\ldots\\
\tau^{\#}_{_M}(dx^{i_1}\wedge\ldots\wedge dx^{i_k}) &= (-1)^{i_1+\ldots+i_k+k}\theta_1\ldots\widehat{\theta}_{i_1}\ldots\widehat{\theta}_{i_k}\ldots\theta_n|dxd\theta|^{1/2},\ (i_1<\ldots<i_k),
\end{aligned}$$

$$\tau^{\#}_{M}(f(x)w) = f(x)\tau^{\#}_{M}(w)\,, \quad \text{for every function } f(x) \text{ on } M\,, \tag{2.17}$$

where the sign $\widehat{\ }$ means the omitting of corresponding term. For example if M is two-dimensional space, then $\tau^{\#}_{M}(f(x)) = f(x)\theta_1\theta_2$, $\tau^{\#}_{M}(w_1(x)dx^1 + w_2(x)dx^2) = w_1\theta_2 - w_2(x)\theta_1$, $\tau^{\#}_{M}(w(x)dx^1 \wedge dx^2) = -w(x)$ *.

We say that semidensity $\mathbf{s}$ corresponds to differential form w (to the linear combination of differential forms $\sum w_k$) if $\mathbf{s} = \tau^{\#}_{M}(w)$ ($\mathbf{s} = \tau^{\#}_{M}(\sum w_n)$).

The correctness of (2.17) follows from the fact that for transformations (2.15)

$$\det \frac{\partial x}{\partial \tilde{x}} = \mathrm{Ber}\left(\frac{\partial(x,\theta)}{\partial(\tilde{x},\tilde{\theta})}\right)^{1/2}, \qquad \left(\mathrm{Ber}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\det(A - BD^{-1}C)}{\det D}\right). \tag{2.18}$$

The action of operator $\Delta^{\#}$ corresponds to the action of exterior differential:

$$\Delta^{\#} \circ \tau^{\#}_{M} = \tau^{\#}_{M} \circ d\,, \qquad \text{and} \qquad \tau^{\#}_{M}(\mathbf{T}\rfloor w) = \tau_{M}(\mathbf{T}) \cdot \tau^{\#}_{M}(w)\,, \tag{2.19}$$

where $\mathbf{T}\rfloor w$ is the inner product of polyvectorial field $\mathbf{T}$ with differential form w.

If the semidenstity $\mathbf{s}$ in ΠT^*M corresponds to the volume form (differential n-form) w on M and the structure of special odd sympelctic superspace on ΠT^*M is defined by the square of this semidenstiy ($d\mathbf{v} = \mathbf{s}^2$) then the action of $\Delta_{d\mathbf{v}}$ corresponds to the divergence w.r.t. to the volume form w on M: $\Delta_{d\mathbf{v}} \circ \tau_{M} = \tau_{M} \circ \mathrm{div}_w$. (See also [6,7].)

Of course the essential difference of the odd symplectic superspace ΠT^*M from the cotangent bundle T^*M is that canonical transformations in ΠT^*M are not exhausted by (2.15). In a general case canonical transformations in ΠT^*M mix forms of different degrees. For example if we consider the action of Hamiltonian $Q = L\theta_1 \ldots \theta_n$ on the differential form $w = dx^1 \wedge \ldots \wedge dx^n$ then we obtain using (2.11) that $\delta w = (\tau^{\#}_{M})^{-1}\delta_Q \tau^{\#}_{M} w = dL$. Only in the special case if an odd Hamiltonian Q corresponds to the vector field $T^i\frac{\partial}{\partial x^i}$ ($Q = T^i\theta_i$), then this Hamiltonian induces infinitezimal canonical transformation, which corresponds to the point transformation on M: $(\tau^{\#}_{M})^{-1}\delta_Q \tau^{\#}_{M} w$ is equal to Lie derivative of w along the vector field $T^i\frac{\partial}{\partial x^i}$.

The initial space M is the Lagrangian $(n.0)$-dimensional surface in ΠT^*M. If $L^{n.0}$ is an arbitrary $(n.0)$-dimensional Lagrangian surface in ΠT^*M then there exist Darboux coordinates such that $L^{n.0}$ is given in these coordinates locally by the equations $\theta_1 = \ldots = \theta_n = 0$. A general canonical transformation transforms the initial space M to some $(n.0)$-dimensional Lagrangian surface in ΠT^*M. An arbitrary semidensity $\mathbf{s}$ on ΠT^*M corresponds by (2.17) to the linear combination of differential forms on $L^{n.0}$, via the map

* The map $\tau^{\#}$: $w \to (\tau^{\#}(w))^2$ transforms differential forms on M to density (volume form) on ST^*M. This map was constructed in [7] via the superspace STM associated to tangent bundle TM and additional arbitrary volume form on ST^*M.

$\tau^{\#}_L$. (In the case if we consider the points of superspace as Λ-points where Λ is an arbitrary Grassmann algebra, (see the footnote in the beginning of the section) then one have to consider differential forms with coefficients in this algebra Λ.)

This example is basic one, because locally every odd symplectic superspace can be considered as superspace ΠT^*L associated to the cotangent bundle T^*L of its $(n.0)$-dimensional Lagrangian surface L. The semidensity can be integrated over arbitrary $(n-k.k)$-dimensional Lagrangian surface in an odd symplectic superspace (See [7] for the corresponding construction and [6] for explicit formula.) In the case if the Lagrangian surface is $(n.0)$-dimensional, the integral of semidensity $\mathbf{s}$ over this surface L is nothing but the integral of corresponding differential form $(\tau^{\#}_L)^{-1}\mathbf{s}$ by this surface. In general case if C^k is surface embedded in M^n and L is $(k.n-k)$-dimensional Lagrangian surface in ΠT^*M associated to the conormal bundle of C^k in T^*M, then the integral of a differential form w over the surface C^k is equal to the integral of the semidensity $\tau^{\#}_M w$ over Lagrangian surface $L^{k.n-k}$ [7].

Now using these constructions we return to statements (1.2) of BV formalism geometry.

One have straightforwardly deal with an odd symplectic superspace provided with semidensity $\mathbf{s}=\sqrt{d\mathbf{v}}=(f|dxd\theta|^{1/2})$, which corresponds to the exponent of master-action $f=e^{2S}$.

The master-equation (1.2b) in terms of the operator $\Delta^{\#}$ can be rewritten as the condition of the semidensity (2.12b) vanishing: $\Delta^{\#}\mathbf{s}=0$. It means according to (2.19), that the differential form $(\tau^{\#})^{-1}\mathbf{s}$ corresponding to the semidensity $\sqrt{d\mathbf{v}}$ is closed.

Under the infinitezimal transformation to another Darboux coordinates, according (2.11) this semidenity transforms as

$$\delta_Q\mathbf{s}=\Delta^{\#}(Q\mathbf{s})\,, \tag{2.20}$$

and corresponding form $w=(\tau^{\#})^{-1}\mathbf{s}$ changes on the exact form $d({\tau^{\#}}^{-1}(Q\mathbf{s}))$.

If in some Darboux coordinates this semidensity is expressed locally by the formula

$$\mathbf{s}=s(x,\theta)|dxd\theta|^{1/2}=(a(x)+a^i(x)\theta_i+\ldots+c\theta_1\ldots\theta_n)|dxd\theta|^{1/2}\,, \tag{2.21}$$

then c is a constant, because the form $(\tau^{\#})^{-1}\mathbf{s}$ is closed. From (2.20) it follows that this constant does not depend on the choice of Darboux coordinates. In fact it characterizes cohomological class of the form $w=(\tau^{\#})^{-1}\mathbf{s}$ on the $(0.n)$-dimensional Lagrangian plane (see [7]):

$$c=\int_{L^{0.n}}\mathbf{s}=\int_{x^1=x^1_0,x^2=x^2_0,\ldots,x^n=x^n_0}s(x,\theta)d\theta_1\ldots d\theta_n\,. \tag{2.22}$$

One can say more: the condition of closedness of the form corresponding to the semidensity $\mathbf{s}$ means that by choosing appropriate Hamiltonian $Q(z,t)$, and integrating the relation (2.20) one comes to the canonical transformation to new Darboux coordinates in which the semidensity (2.21) is expressed locally by the following formula:

$$\mathbf{s}=s(x,\theta)|dxd\theta|^{1/2}=(1+c\theta_1\dots\theta_n)\,|dxd\theta|^{1/2}\,. \tag{2.23}$$

We find this canonical transformation in the way similar to [7] using the correspondence between semidensities and differential forms and the basic formula of the differential calculus of forms [20].

Let $\mathbf{s}_0$ and $\mathbf{b}$ be arbitrary "closed" semidensities, (i.e., the corresponding differential forms are closed). We consider the odd function Q which obeys to relation $Q\mathbf{s}_t=-\mathbf{b}$, where $\mathbf{s}_t=\mathbf{s}_0+t\Delta^{\#}\mathbf{b}$. (We suppose that the semidensity $\mathbf{s}_t$ is not degenerated, i.e. n-form corresponding to $\mathbf{s}_t$ is not equal to zero). One can see that canonical transformation $z=z(\tilde z,t)$ which is defined by the equation:

$$\begin{cases}\frac{dz(\tilde z,t)}{dt}=\{Q,z(\tilde z,t)\}\\ z(\tilde z,t)|_{t=0}=\tilde z\end{cases},$$

transforms the semidenstiy $\mathbf{s}_t$ to the semidensity $\mathbf{s}_0$ for an arbitrary t. This follows from initial conditions and from the fact that

$$\frac{d\mathbf{s}_t}{dt}=\Delta^{\#}\mathbf{b}+\delta_Q\mathbf{s}_t=\Delta^{\#}(\mathbf{b}+Q\mathbf{s}_t)=0\,.$$

We put $\mathbf{s}_0$ to be equal to the semidenstiy (2.23) and choose the semidensity $\mathbf{b}$ such that $\mathbf{s}_1$ is equal to the semidenstiy (2.21). (This is possible, because locally every closed k-form is exact, if $k\geq 1$.) *

We see that if the condition (1.2b) is obeyed, then the condition of $c\neq 0$ in (2.21—2.23) is the obstacle to the condition (1.2a).

Now we analyze the condition (1.2c). From (2.14) it follows that the condition (1.2c) means that the function (2.12d) is equal to an odd constant ν, and $\Delta^{\#}\mathbf{s}=\nu\mathbf{s}$. One can see using correspondence between semidensities and differential forms that all the solution to this equation are $\mathbf{s}=\Delta^{\#}\mathbf{h}-\nu\mathbf{h}$, where $\mathbf{h}$ is an arbitrary semidensity. The odd constant $\nu\neq 0$ is the obstacle to the condition (1.2b), if the condition (1.2c) is obeyed.

We come to the

Proposition

In the case if the odd symplectic superspace is provided by the volume form $d\mathbf{v}$*, such that* $\Delta^2_{d\mathbf{v}}=0$*, then to the volume form* $d\mathbf{v}$ *corresponds the odd constant* ν*:* $\Delta^{\#}\sqrt{d\mathbf{v}}=\nu\sqrt{d\mathbf{v}}$ *and the semidensity* $\sqrt{d\mathbf{v}}$ *has the form* $\Delta^{\#}\mathbf{h}-\nu\mathbf{h}$*.*

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* If we consider the case where $\mathbf{s}_0$ and $\mathbf{s}_1$ correspond to differential forms of the *same degree*, then we come to the standard statement of differential geometry about existence of local diffeomorphism which transforms one given closed form to another, if they differ by an exact form [20].

If the odd constant ν is equal to zero, then the master-equation $\Delta^{\#}\sqrt{d\mathbf{v}}=0$ holds. In this case to the volume form $d\mathbf{v}$ corresponds the constant c which is equal to the integral of semidensity $\sqrt{d\mathbf{v}}$ over $(0.n)$-dimensional Lagrangian plane (2.22). *The volume form in this case can be reduced to the form* (2.23).

In the case if this constant $c=0$ then the odd symplectic superspace provided with the volume form $d\mathbf{v}$ is the normal special odd symplectic superspace.

This Proposition removes uncorrectness of the considerations about equivalence of conditions (1.2a), (1.2b) and (1.2c) in the Statement 1 of Introduction, which was done in [6] and [7]. On the other hand the statements of this Proposition in non explicit way was contained in the statements of Lemma 4 and Theorem [5] of the paper [7]. The analysis performed here in terms of $\Delta^{\#}$ operator is essentially founded on these results.

3. Invariant densities on (1.1)- codimensional surfaces

In this section first we recall explicit formulae for the odd invariant semidensity on non-degenerated surfaces of codimension (1.1) embedded in an special odd symplectic superspace ([1,8]). Then we rewrite this semidensity in terms of the semidensity $\sqrt{d\mathbf{v}}$ of the ambient superspace and suggest the construction of pull-back of arbitrary semidensity from the ambient odd symplectic superspace on embedded (1.1)-codimensional surfaces. Using this construction for the semidensity $\Delta^{\#}\sqrt{d\mathbf{v}}$ we will construct the new densities on embedded non-degenerated surfaces.

The surface is called non-degenerate if the sympelctic structure of the space generates the symplectic structure on the embedded surface also, i.e. if the pull-back of the symplectic 2-form on the surface is non-degenerate 2-form. As usual we call this symplectic structure on an embedded surface *induced symplectic structure*.

Let z^A be Darboux coordinates on an special odd symplectic superspace $E^{n.n}$ with volume form $d\mathbf{v}=\rho(z)|dz|$. It is convenient here to use for Darboux coordinates notations $z^A=(x^\mu,\theta_\mu)$, $(\mu=(0,i)=(0,1,\ldots,n-1),\ i=(1,\ldots,n-1))$. Let $z(\zeta)$ be an arbitrary parametrization of an arbitrary non-degenerate surface of codimension (1.1), embedded in this special odd symplectic superspace. $(\zeta=(\xi^i,\eta_j)$, ξ^i and η_j are even and odd parameters respectively, $(i,j=1,\ldots,n-1)$.

As it was mentioned in Introduction there is no non-trivial invariant density of the rank $k=1$ on non-degenerate surfaces [19]. The invariant semidensity of the rank $k=2$ (depending on first and second derivatives of $z(\zeta)$) is given by the following formula:

$$A\left(z(\zeta),\frac{\partial z}{\partial\zeta},\frac{\partial^2 z}{\partial\zeta\partial\zeta}\right)|d\zeta|^{1/2}=$$

$$\left(\Psi^A\frac{\partial\log\rho(z)}{\partial z^A}-\Psi^A\Omega_{AB}\frac{\partial^2 z^B}{\partial\zeta^\alpha\partial\zeta^\beta}\Omega^{\alpha\beta}(z(\zeta))(-1)^{B(\alpha+\beta)+\alpha}\right)|d\zeta|^{1/2}, \tag{3.1}$$

where $\Omega_{AB}dz^A dz^B$ is the two-form defining the odd sympelctic structure on $E^{n.n}$ and $\Omega^{\alpha\beta}$ is the tensor inversed to the two-form which defines induced symplectic structure on the surface. The vector field $\mathbf{\Psi}$ is defined as follows: one have to consider the pair of vectors $(\mathbf{H}, \mathbf{\Psi})$, $\mathbf{H}$-even and $\mathbf{\Psi}$-odd, which are symplectoorthogonal to the surface and obey to the following conditions:

$$\Omega(\mathbf{H}, \mathbf{\Psi}) = 1,\ \Omega(\mathbf{\Psi}, \mathbf{\Psi}) = 0 \quad (\text{symplectoorthonormality conditions})\,, \tag{3.2}$$

$$d\mathbf{v}\left(\left\{\frac{\partial z}{\partial \zeta}\right\}, \mathbf{H}, \mathbf{\Psi}\right) = 1 \quad (\text{volume form normalization conditions})\,. \tag{3.3}$$

These conditions fix uniquely the vector field $\mathbf{\Psi}$. (See for details [1]).

The explicit expression for this semidensity was calculated in [8] in terms of dual densities: If (1.1)-codimensional supersurface M is given not by parametrization, but by the equations $f = 0, \varphi = 0$, where f is an even function and φ is an odd function then to the semidensity (3.1) there corresponds the dual semidensity:

$$\tilde{A}\Big|_{f=\varphi=0} = \frac{1}{\sqrt{\{f,\varphi\}}}\left(\Delta_{d\mathbf{v}} f - \frac{\{f,f\}}{2\{f,\varphi\}}\Delta_{d\mathbf{v}}\varphi - \frac{\{f,\{f,\varphi\}\}}{\{f,\varphi\}} - \frac{\{f,f\}}{2\{f,\varphi\}^2}\{\varphi,\{f,\varphi\}\}\right). \tag{3.4}$$

This function is multiplied by the square root of the corresponding Berezinian (superdeterminant) under the transformation $f \to af + \alpha\varphi$, $\varphi \to \beta f + b\varphi$, which does not change the surface M.

Now we rewrite the semidensity (3.1) straightforwardly via the semidensity $\sqrt{d\mathbf{v}}$ on the ambient special odd symplectic superspace $E^{n.n}$.

For this purpose we consider for every given non-degenerate surface M of codimension (1.1) embedded in $E^{n.n}$ Darboux coordinates (in a vicinity of an arbitrary point) such that in these Darboux coordinates this surface M locally is given by equations

$$x^0 = \theta_0 = 0\,. \tag{3.5}$$

We call these Darboux coordinates adjusted to the surface M. We note that if $(x^0, x^i, \theta_0, \theta_j)$ are Darboux coordinates in $E^{n.n}$ adjusted to the surface M, then (x^i, θ_j) are Darboux coordinates on the surface $M^{n-1.n-1}$ w.r.t. the induced symplectic structure. We consider the following parametrization of the surface M in adjusted Darboux coordinates:

$$\begin{cases} x^0 = 0, \theta_0 = 0, \\ x^i = \xi^i, \theta_i = \eta_i\,, \text{ for } 1 \le i \le n-1 \end{cases} \tag{3.6}$$

The conditions of symplectoorthonormality in (3.2) give that $\mathbf{H} = a\frac{\partial}{\partial x^0} + \beta\frac{\partial}{\partial \theta_0}$ and $\mathbf{\Psi} = \frac{1}{a}\frac{\partial}{\partial \theta_0}$, where a and β are arbitrary numbers (a even and β-odd). The condition (3.3) of the volume form $d\mathbf{v} = \rho|dz|$ normalization:

$$\rho|dxd\theta|\left(\frac{\partial}{\partial x^1}, \dots \frac{\partial}{\partial x^{n-1}}, \mathbf{H}; \frac{\partial}{\partial \theta_1}, \dots \frac{\partial}{\partial \theta_{n-1}}, \mathbf{\Psi}\right) = \rho a^2 = 1\,,$$

gives that

$$a = \frac{1}{\sqrt{\rho}} \quad \text{and} \quad \mathbf{\Psi} = \sqrt{\rho}\frac{\partial}{\partial\theta_0} \,. \tag{3.7}$$

From (3.6) and (3.7) it follows that the semidensity (3.1) on M in adjusted coordinates is reduced to

$$A = \sqrt{\rho}\frac{\partial \log\rho}{\partial\theta_0}|\widetilde{dxd\theta}|^{1/2}\Big|_{x^0=\theta^0=0} = 2\frac{\partial\sqrt{\rho}}{\partial\theta_0}|\widetilde{dxd\theta}|^{1/2}\Big|_{x^0=\theta^0=0} =$$

$$2\frac{\partial\sqrt{\rho}}{\partial\theta_0}\mathrm{Ber}\left(\frac{\partial(x^i,\theta_j)}{\partial(\xi^i,\eta_j)}\right)^{1/2}|d\xi d\eta|^{1/2}\Big|_{x^0=\theta^0=0}, \tag{3.8}$$

where $\widetilde{dxd\theta} = dx^1\dots dx^{n-1}d\theta_1\dots d\theta_{n-1}$.

The infinitezimal transformation from adjusted Darboux coordinates to another adjusted Darboux coordinates is generated by an odd Hamiltonian Q, which according to (3.5) subject to the folowing conditions:

$$\frac{\partial Q(x,\theta)}{\partial x^0}\Big|_{x^0=\theta_0=0} = \frac{\partial Q(x,\theta)}{\partial\theta^0}\Big|_{x^0=\theta_0=0} = 0 \quad . \tag{3.9}$$

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The differential δ_Q (2.11) corresponding to the action of infinitezimal transformation generating by an arbitrary odd Hamiltonian Q subject to condition (3.9) on the semidenstiy $\sqrt{\rho}|dz|^{1/2}$ has to commute with the derivative along θ_0 on the surface:

$$\left(\frac{\partial}{\partial\theta_0}\delta_Q - \widetilde{\delta_Q}\frac{\partial}{\partial\theta_0}\right)\Big|_{x^0=\theta_0=0} = 0\,. \tag{3.10}$$

Here δ_Q acts on a semidensity in the symplectic space $E^{n,n}$ and $\widetilde{\delta_Q}$ acts on a semidensity on the surface M provided with induced symplectic structure. One can check the condition (3.10), using (2.11) and noting that (x^i,θ_i) are Darboux coordinates on the surface M w.r.t. induced symplectic structure on $M^{n-1.n-1}$.

The formula (3.8) defines correctly semidensity on the surface M not only for semidenstiy, related with the volume form, but for arbitrary semidenstiy, even in the case if it is an odd semidensity, and the corresponding volume form is equal to zero. We come to the following statement

Theorem *To every semidensity* $\mathbf{s}$ *in the odd symplectic superspace* $E^{n.n}$ *corresponds semidensity* $\mathcal{A}(\mathbf{s})$ *defined on non-degenerated* (1.1)*-codimensional surfaces embedded in this superspace.*

The value of this semidensity $\mathcal{A}(\mathbf{s})$ *on every* (1.1)*-codimensional surface* M *in Darboux coordinates adjusted to the surface* M *is given by the equation:*

$$\mathcal{A}(M,\,\mathbf{s}) = \frac{\partial s(x^\mu,\theta_\nu)}{\partial\theta_0}\Big|_{x^0=\theta_0=0}|\widetilde{dxd\theta}|^{1/2}, \quad (\mathbf{s} = s(x^\mu,\theta_\nu)|dxd\theta|^{1/2})\,. \tag{3.11}$$

The formula (3.11) has clear meaning in terms of differential forms. Consider the Lagrangian surface L in $E^{n.n}$, which is given in Darboux coordinates (x^μ, θ_ν) adjusted to the surface M by the equations $\theta_0 = \ldots = \theta_{n-1} = 0$. The intersection $\widetilde{L} = L \cap M$ of this Lagrangian surface with M will be Lagrangian subsurface in the symplectic manifold M, which is given by equations $\theta_1 = \ldots = \theta_{n-1} = 0$ on M. It is easy to see from (2.17) and (3.11) that if the semidensity $\mathbf{s}$ corresponds to the linear combination $\sum w_k$ of differential forms on L, $\mathbf{s} = \tau^{\#}_{L}(\sum w_k)$, then the semidensity (3.11) on M corresponds (up to the sign) to the pull-back on $\widetilde{L}$ of this differential form:

$$\mathcal{A}(M, \mathbf{s}) = -\widetilde{\tau^{\#}_{\widetilde{L}}} \circ \iota^* \circ (\tau^{\#}_{L})^{-1}(\mathbf{s})\,. \qquad (3.12)$$

ι: $\widetilde{L} \hookrightarrow L$ is the embedding map and $\widetilde{\tau^{\#}_{\widetilde{L}}}$ is the map (2.17) from differential forms on Lagrangian surface in M to semidensities on M. (See also the Example 2, below)

The statement of this Theorem gives us not only the alternative way to prove the correctness of the semidensity (3.1), but allows us to construct semidensity on embedded surfaces via odd semidensities on the ambient superspace, which cannot be yielded from volume forms.

The odd semidenstiy (3.1) is nothing but $\mathcal{A}(M, \sqrt{d\mathbf{v}})$. To the semidensity $\Delta^{\#}\sqrt{d\mathbf{v}}$ in the special odd symplectic superspace with volume form $d\mathbf{v}$ there corresponds the even semidensity $\mathcal{A}(M, \Delta^{\#}\sqrt{d\mathbf{v}})$ which cannot be represented (3.1)–like, because the square of the odd semidensity $\Delta^{\#}\sqrt{d\mathbf{v}}$ is equal to zero.

These semidensities correspond to differential forms on Lagrangian $(n-1.0)$-dimensional subsurfaces in M, hence they can be integrated over them. (Moreover they can be integrated over arbitrary $(n-1-k.k)$-dimensional Lagrangian subsurfaces in M also (see [7,6]).)

One can construct densities of weight $\sigma = 1$ via the semidensities $\mathcal{A}(M, \sqrt{d\mathbf{v}})$ and $\mathcal{A}(M, \Delta^{\#}\sqrt{d\mathbf{v}})$:

$$P_0 = \mathcal{A}^2(\Delta^{\#}\sqrt{d\mathbf{v}}) \quad \text{and} \quad P_1 = \mathcal{A}(\sqrt{d\mathbf{v}})\mathcal{A}(\Delta^{\#}\sqrt{d\mathbf{v}})\,. \qquad (3.13)$$

The density P_0 takes even values, the density P_1 takes odd values. In general case these densities give non-trivial integration objects over non-degenerated (1.1)-codimensional surfaces embedded in an special odd symplectic superspace with volume form $d\mathbf{v}$.

The densities P_0 and P_1 have rank $k = 4$ (i.e. depend on derivatives of the parametrization $z(\zeta)$ up fourth order). It follows from the fact that the semidensity $\mathcal{A}(\Delta^{\#}\sqrt{d\mathbf{v}})$ has the rank $k = 4$, because the semidensity $\mathcal{A}(\sqrt{d\mathbf{v}})$ which is equal to (3.1) has the rank $k = 2$. This is hidden in representation (3.8), where the function $\rho(z)$ corresponding to the volume form in adjusted coordinates depends non-explicitly on derivatives of surface parametrization $z(\zeta)$. For example consider the surface $x^0 = \Psi\xi^1\eta_1, \theta_0 = 0, x^1 = \xi^1, \theta_1 = \eta_1$ in the normal special odd symplectic superspace $E^{2.2}$ with volume form $d\mathbf{v} = |dx d\theta|$. ($\Psi$ is an odd constant.) In Darboux coordinates $\tilde{x}^0 = x^0 - \Psi x^1\theta_1$, $\tilde{\theta}_0 = \theta_0$, $\tilde{x}^1 = x^1 + \Psi\theta_0 x^1$, $\tilde{\theta}_1 = \theta_1 - \Psi\theta_0\theta_1$ adjusted to the surface M, the volume form is equal to $d\mathbf{v} = (1 + 2\Psi\tilde{\theta}_0)|d\tilde{x}d\tilde{\theta}|$, hence the semidensity (3.8) is equal to $\Psi|d\tilde{x}^1 d\tilde{\theta}_1|^{1/2}$.

One can construct densities using $\Delta^\#$-operator on embedded surfaces provided with the induced symplectic structure. For example one can consider on every non-degenerated surface $M^{n-1.n-1}$, the semidensity $\widetilde{\Delta^\#_M}\mathcal{A}(M, \sqrt{d\mathbf{v}})$, where $\widetilde{\Delta^\#_M}$ is the $\Delta^\#$ operator on the surface M w.r.t. to the induced symplectic structure. But in this case

$$\mathcal{A}(M, \Delta^\#\sqrt{d\mathbf{v}}) = -\widetilde{\Delta^\#_M}\mathcal{A}(M, \sqrt{d\mathbf{v}})\,. \tag{3.14}$$

This can be immediately checked in Darboux coordinates (3.5) adjusted to the surface M.

Finally we consider a simple example of these constructions.

Example 2. Let $E^{3.3}$ be a superspace associated to the 3-dimensional space E^3, $E^{3.3} = \Pi T^* E^3$. We consider on E^3 the differential form

$$w = -dx^0 \wedge dx^1 \wedge dx^2 + b_0 dx^0 + b_1 dx^1 + b_2 dx^2\,.$$

To this differential form according to (2.17) there corresponds a semidensity

$$\mathbf{s} = \tau^\#(w) = (1 + b_0\theta_1\theta_2 + b_1\theta_2\theta_0 + b_2\theta_0\theta_1)|dx^0 dx^1 dx^2 d\theta_0 d\theta_1 d\theta_2|^{1/2}\,.$$

Let M be a surface in $E^{3.3}$ which is defined by equations $x^0 = \theta_0 = 0$. M is associated to the space E^2 with coordinates (x^1, x^2). The value of the odd semidensity $\mathcal{A}(\mathbf{s})$ on M is equal to $(b_2\theta_1 - b_1\theta_2)|dx^1 dx^2 d\theta_1 d\theta_2|^{1/2}$. This corresponds to differential form $b_1 dx^1 + b_2 dx^2$, the pull-back of w on E^2. The value of even semidensity $\mathcal{A}(\Delta^\#\mathbf{s})$ on M is equal to $(\partial_2 b_1 - \partial_1 b_2)|dx^1 dx^2 d\theta_1 d\theta_2|^{1/2}$. This corresponds to differential form $d(b_1 dx^1 + b_2 dx^2)$ $= (\partial_2 b_1 - \partial_1 b_2) dx^1 \wedge dx^2$, the pull-back of dw on E^2.

The even density (volume form) on M is equal to $P_0 = (\partial_2 b_1 - \partial_1 b_2)^2 |dx^1 dx^2 d\theta_1 d\theta_2|$ and the odd density $P_1 = (\partial_2 b_1 - \partial_1 b_2)(b_1\theta_2 - b_2\theta_1)|dx^1 dx^2 d\theta_1 d\theta_2|$.

Discussions

We hope that considerations presented in this paper can be generalized for constructing densities in an odd symplectic superspace depending on higher order derivatives for surfaces of arbitrary dimension and for finding the complete set of local invariants of this geometry. In particular considerations which lead to Theorem follows that one can try to find non-trivial invariant densities on non-degenerated surfaces of codimension $(p.p)$ only if their rank is greater than p. It is interesting to analyze from this point of view relations between geometrical interpretations presented in this paper and in [1] where some relations of semidensity (3.1) with mean curvature in Riemanian geometry were indicated.

The densities presented in formulae (3.13) are needed to be investigated more in details. Particularly one have to present explicit formulae for them and consider the corresponding functionals over surfaces. These functionals are equal to zero in the special case if the volume form in the ambient special odd symplectic superspace obeys to BV-master equation. Do Euler-Lagrange motion equations for these functionals in a general case are equal identically to zero, as for usual Poincare-Cartan integral invariants (1.3)?

It can be interesting to note also that symmetry transformations of these functionals are not exhausted only by transformations induced by diffeomorphisms (2.15) of underlying space. General canonical transformations of superspace induce mixing of corresponding differential forms with different degrees. (See considerations after formula (2.19).)

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