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BIG IMAGES OF TWO-DIMENSIONAL PSEUDOREPRESENTATIONS

ANDREA CONTI, JACLYN LANG, AND ANNA MEDVEDOVSKY

Abstract. For an odd prime p, we study the image of a continuous 2-dimensional (pseudo)representation ρ of a profinite group with coefficients in a local pro-p domain A. Under mild conditions, Bellaïche has proved that the image of ρ contains a nontrivial congruence subgroup of $\mathrm{SL}_2(B)$ for a certain subring B of A. We prove that the ring B can be slightly enlarged and then described in terms of the *conjugate self-twists* of ρ, symmetries that naturally constrain its image; hence this new B is optimal. We use this result to recover, and in some cases improve, the known large-image results for Galois representations arising from elliptic and Hilbert modular forms due to Serre, Ribet and Momose, and Nekovář, and p-adic Hida or Coleman families of elliptic modular forms due to Hida, Lang, and Conti–Iovita–Tilouine.

1. Introduction

Let p be an odd prime, A a local pro-p domain with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F} := A/\mathfrak{m}$, and Π a profinite group. Let $\rho\colon \Pi \to \mathrm{GL}_2(A)$ be a continuous representation[(i)] with the property that the residual representation $\overline{\rho} := \rho \bmod \mathfrak{m}$ is *residually multiplicity-free*: either absolutely irreducible, or a sum of two distinct characters to $\mathbb{F}^\times$. Our goal is to study the image of ρ from an algebraic perspective, with an eye towards applications to modular forms and their p-adic families. In those settings Π is a Galois group and A is a finite extension of $\mathbb{Z}_p$ or of $\mathbb{Z}_p[\![X_1, \dots, X_n]\!]$ for some $n \geq 1$. Roughly, the objective is to show that the image of ρ is as big as possible.

Note that if ρ, or its restriction to an index-2 subgroup of Π, is reducible, then the image of ρ is both well understood and not big. Similarly, one cannot expect a big-image result when the image of ρ is isomorphic to that of $\overline{\rho}$, as happens when ρ arises from a modular form of weight one. Let us call these three kinds of representations *a priori small*.

Suppose now that ρ is not a priori small. How big can we expect its image to be? We cannot expect ρ to be surjective as its determinant need not be surjective. Nor can we expect the image of ρ to contain $\mathrm{SL}_2(A)$ unless the image of $\overline{\rho}$ contains $\mathrm{SL}_2(\mathbb{F})$. We settle on the idea of *fullness*. If B is any ring and $\mathfrak{b}$ is any nonzero ideal of B, the subgroup of $\mathrm{SL}_2(B)$ given by the kernel of reduction modulo $\mathfrak{b}$ is a *congruence subgroup* of $\mathrm{SL}_2(B)$ (of *level* $\mathfrak{b}$):

$$\Gamma_B(\mathfrak{b}) := \ker\big(\mathrm{SL}_2(B) \to \mathrm{SL}_2(B/\mathfrak{b})\big).$$

If the image of ρ, up to conjugation, contains a congruence subgroup for some subring B of A, we say that ρ is *B-full*. In the special case where A is a finite extension of $\mathbb{Z}_p$, the representation ρ is A-full if and only if it contains an open subgroup of $\mathrm{SL}_2(A)$.

It turns out that ρ may have symmetries that prevent it from being A-full. If σ is an automorphism of the Galois closure of A and χ is a character of Π, the pair (σ, χ) is a *conjugate self-twist* of ρ if applying the automorphism gives the same representation as twisting by the character: ${}^\sigma\rho \cong \chi \otimes \rho$. If (σ, χ) is a conjugate self-twist of ρ, then ρ cannot be A-full: indeed, since $\sigma\big(\operatorname{tr}\rho(g)\big) = \operatorname{tr}{}^\sigma\rho(g) = \chi(g)\operatorname{tr}\rho(g)$, the trace of $\rho(g)$ is an eigenvector for σ viewed as a linear map over the scalars fixed by σ. But the trace of a congruence subgroup of A is not so constrained.

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(i) In fact we consider 2-dimensional pseudorepresentations, but we content ourselves with true representations for the purposes of this introductory section.

Therefore, the best we can hope for is fullness with respect to A^{Σ_ρ}, the subring of A fixed by the conjugate self-twists of ρ.

If ρ comes from a classical or Hilbert modular form or a Hida family, we know that ρ is A^{Σ_ρ}-full from work of Ribet, Momose, Nekovář, and the second author of the present work (see Theorems 1.1 and 1.2 below). For technical reasons, in the general case it is easier to work with a different subring of A with the same field of fractions as A^{Σ_ρ}: let $A_0 = A_0(\rho)$ be the closed subring of A topologically generated by the set

$$\left\{ \frac{(\operatorname{tr}\rho(g))^2}{\det \rho(g)} : g \in \Pi \right\}.$$

It is clear that A_0 depends on ρ only up to twist. In fact, A_0 is the closed algebra generated by the trace of $\operatorname{ad}\rho$. The main theorem of this paper (Theorem 1.4 below) states that under mild conditions, if ρ is not a priori small, then ρ is A_0-full. The result is optimal in the sense that, if ρ is B-full for some subring B of A, then the field of fractions of B must be contained in the field of fractions of A_0.[(ii)]

1.1. **History.** We survey the known big-image results, using the terminology introduced above.

1.1.1. *Classical modular forms.* The big-image line of inquiry began in the late 60s, when Serre showed that if ρ comes from the p-adic Tate module (including for $p = 2$) of a non-CM elliptic curve over a number field F, so that $\Pi = \operatorname{Gal}(\overline{F}/F)$ and $A = A_0 = \mathbb{Z}_p$, then ρ is $\mathbb{Z}_p$-full [Ser68, Theorem IV.2.2].[(iii)]

In the 80s, Ribet and Momose generalized Serre's theorem to elliptic modular forms. Let f be a cuspidal non-CM eigenform of weight at least 2. Given a prime p and an embedding $\iota_p \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, one can associate to f a 2-dimensional Galois representation $\rho = \rho_{\iota_p}$ of $\Pi = \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over a finite extension A of $\mathbb{Z}_p$.

Theorem 1.1 ([Mom81, Theorem 4.1], [Rib85, Theorem 3.1])**.** *For all but finitely many primes p, the representation ρ is A^{Σ_ρ}-full. Hence it is also A_0-full.*[(iv)]

More recently, Nekovář generalized Theorem 1.1 to representations coming from Hilbert modular forms, in which case Π is the absolute Galois group of a totally real number field and A is still a finite extension of $\mathbb{Z}_p$ [Nek12, Appendices B.3–B.6].

1.1.2. *Families of p-adic modular forms.* Although we have stated the work of Serre, Ribet, Momose, and Nekovář for a fixed prime p to better fit our p-adic framework, all of these theorems are actually adelic open-image results proved using geometric methods. Much work has been done to generalize such theorems to groups other than GL_2, but that is not the direction that interests us. Rather, we are interested in fixing p and deforming representations p-adically, which necessitates a completely different approach. There has been some progress in this direction in special cases. Recall that we are assuming throughout that $p \neq 2$.

(ii) We do not know whether ρ is A^{Σ_ρ}-full in general. If A^{Σ_ρ} is finitely generated as an A_0-module, then ρ is A^{Σ_ρ}-full by Lemma 2.16. This condition can be verified when ρ comes from a classical modular form or a Hida family. On the other hand, A^{Σ_ρ}-fullness always implies A_0-fullness.

(iii) Serre's result is better known as an open-image theorem; and in fact he shows much more: the image of all the p-adic Tate modules for all p, including $p = 2$, at once is open adelically.

(iv) Like Serre, Ribet and Momose prove stronger adelic big-image results, including for $p = 2$. In particular, for the finitely many primes p where the statement of Theorem 1.1 fails, $\operatorname{Im}\rho \cap \mathrm{SL}_2(A)$ contains, with finite index and up to conjugation, an open subgroup of the norm-one elements of the maximal order in the nonsplit quaternion algebra over the fraction field of A_0.

First we suppose that ρ arises from a non-CM cuspidal Hida family. In this case $\Pi = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and A is a domain that is finite over $\Lambda := \mathbb{Z}_p[\![X]\!]$.[(v)] When A is a constant extension of Λ and the image of $\overline{\rho}$ contains $\mathrm{SL}_2(\mathbb{F}_p)$, Boston [MW86, Proposition 3] and Fischman [Fis02, Theorem 4.8] show that the image of ρ contains $\mathrm{SL}_2(A^{\Sigma_\rho})$. In particular, ρ is A_0-full. More recently, Hida proved that if $\overline{\rho}$ is locally-at-p multiplicity-free then ρ is Λ-full [Hid15, Theorem I], but his work did not relate Λ to A_0 or conjugate self-twists of ρ. The second author of the present article then improved Hida's result from Λ-fullness to A_0-fullness under the assumption that $\overline{\rho}$ is absolutely irreducible, proving the following result.

Theorem 1.2. [Lan16, Theorem 2.4] *Assume that $\mathbb{F} \neq \mathbb{F}_3$. If ρ arises from a non-CM cuspidal Hida family, and $\overline{\rho}$ is absolutely irreducible and multiplicity free when restricted to a certain finite-index subgroup of the decomposition group at p, then ρ is A^{Σ_ρ}-full. Therefore it is A_0-full.*

The case when ρ arises from a Coleman family was studied by the first author of the present work with Iovita and Tilouine in [CIT16]. In this case we again have $\Pi = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and A is a domain over Λ. In [CIT16, Theorem 6.2] it is proved that, under hypotheses similar to those in Theorem 1.2, a certain Lie algebra attached to $\mathrm{Im}\,\rho$ contains the Lie algebra of a congruence subgroup of $A^{\Sigma'_\rho}$, where Σ'_ρ denotes the conjugate self-twists of ρ that fix Λ. This strongly suggests that ρ should be $A^{\Sigma'_\rho}$-full, though this statement does not follow from [CIT16]. We do not know whether $\Sigma'_\rho = \Sigma_\rho$; see Question 6.3 and the surrounding discussion.

Both Hida and the first author of the present work with Iovita and Tilouine consider questions related to the level of a representation – that is, the largest congruence subgroup contained in the image. Although our techniques are amenable to such questions, we do not discuss them in this paper.

1.1.3. *General p-adic families.* Both [Hid15] and [Lan16] rely in a key way on results of Pink [Pin93] classifying, for odd p, pro-p subgroups of $\mathrm{SL}_2(A)$ in terms of a correspondence with purely algebraically defined "Pink-Lie" algebras. The analogous role in [CIT16] is played by traditional rigid-analytic Lie theory (whence also the different form of the conclusion in that case). Although the big-image theorems in all three of [Hid15, Lan16, CIT16] are stated in terms of pure algebra — a feature that is most clear in the fullness results of [Hid15] and [Lan16] — nonetheless all of these results are fundamentally arithmetic in nature: they rely on special information about the restriction of ρ to the local Galois group at p, and they all use the classical results of Ribet and Momose as input.

In contrast, Bellaïche in [Bel18] studies the image of $\rho\colon \Pi \to \mathrm{GL}_2(A)$ in a purely algebraic way. More precisely, he systematically applies Pink's theory from [Pin93] to images of 2-dimensional (pseudo)representations. His main application is to density results for mod-p modular forms, but along the way he also proves the following theorem.

Theorem 1.3. [Bel18, Theorem 7.2.3] *Assume that Π satisfies Mazur's p-finiteness condition*[(vi)]. *Assume further the following* regularity condition: $\mathrm{Im}\,\overline{\rho}$ *contains an element with eigenvalues in $\mathbb{F}_p^\times$ whose ratio is not ± 1. If ρ has* constant determinant *(i.e., $\det\rho \cong \det\overline{\rho}$) and is not a priori small, then there is a subring A_B of A such that ρ is A_B-full.*

See Theorem 2.31 and the discussion following it for the definition of the ring A_B appearing in Theorem 1.3.

Unfortunately, it is not straightforward to relate Bellaïche's ring A_B to the rings A_0 or A^{Σ_ρ} from previous results. Indeed, there appears to be no conceptual interpretation of Bellaïche's ring

(v) When $\overline{\rho}$ is reducible, ρ may be realizable only over the field of fractions of A. Hida in [Hid15] avoids this by assuming that ρ is realized over A; we work with generalized matrix algebras, following Bellaïche [Bel18], and hence do not need this assumption.

(vi) See Definition 2.7.

A_B. The goal of the present work is to refine the definition of A_B and then give it a conceptual interpretation. Under mild assumptions we thus recover, and in the case of p-adic families improve, the results mentioned above in a uniform and purely algebraic way. See Theorem 1.4 for the mild hypotheses we impose on $\overline{\rho}$. We point out that prior to Bellaïche's work, Hida's work was the only fullness result when $\overline{\rho}$ is reducible and ρ comes from a p-adic family of modular forms. In the case of Coleman families, a true fullness result was not previously known.

1.2. **The main result.** Recall that p is a fixed odd prime. From now on assume that Π (a profinite group) satisfies Mazur's p-finiteness condition. Also recall that A is a local pro-p domain with residue field $\mathbb{F}$, and that $\rho\colon \Pi \to \mathrm{GL}_2(A)$ is a continuous, residually multiplicity-free representation.

Let $\mathbb{E}$ denote the residue field of A_0. We say that $\overline{\rho}$ is *regular* if there is some $g_0 \in \Pi$ such that $\overline{\rho}(g_0)$ has eigenvalues $\lambda_0, \mu_0 \in \overline{\mathbb{F}}_p^\times$ with $\lambda_0\mu_0^{-1} \in \mathbb{E}^\times \setminus \{\pm 1\}$. (See Remark 2.28 for an analysis of this condition.)

Let ζ_n denote a primitive n-th root of unity. The main theorem of this paper is the following.

Theorem 1.4. *Assume that $\overline{\rho}$ is regular. If the projective image of $\overline{\rho}$ is isomorphic to S_4, assume further that $\overline{\rho}$ is good*[(vii)]*. If ρ is not a priori small, then ρ is A_0-full.*

In fact we prove something slightly more general in that we can replace ρ by a pseudodeformation $(t, d)\colon \Pi \to A$ of $\overline{\rho}$. See Theorem 5.17 for the most general statement of the main result.

Broadly speaking, there are two main steps in the proof of Theorem 1.4. First, we modify Bellaïche's ring A_B in a fairly minor way. His ring is defined as the subring of A given by some generators. We consider instead the $W(\mathbb{E})$-subalgebra of A generated by the same elements. We denote by A'_B our modification of A_B for the purposes of this introduction. We prove a theorem analogous to Theorem 1.3 with A'_B in place of A_B; see Corollary 3.8 for the precise statement. Although this is a small improvement, it is crucial for the second step. It also allows our regularity hypothesis to be weaker than that of Bellaïche.

The second step in the proof is to relate A'_B to the subring A^{Σ_ρ} of A fixed by conjugate self-twists of ρ (see Definition 2.32). These two rings coincide when $\overline{\rho}$ is regular, has constant determinant, and has no conjugate self-twists; see the first paragraph of Section 5. In general, we prove that they have the same field of fractions and that A^{Σ_ρ} is finite as a module over A'_B. (This is the "conceptual interpretation" of A_B to which we referred above.) As we show in Lemma 2.16, if $A_1 \subseteq A_2$ are domains with the same field of fractions such that A_2 is a finite type A_1-module, a representation ρ is A_2-full if and only if it is A_1-full. Thus we reach the conclusion of Theorem 1.4 from the modified fullness result of Bellaïche found in the first step.

Let us point out some features of the statement of Theorem 1.4. First, the group Π can be quite general. For that reason, representations coming from Hilbert modular forms and their p-adic families are no more difficult than representations coming from elliptic modular forms or their p-adic families. Similarly, since Π need not be the absolute Galois group of a number field, the notion of oddness does not play a role in the paper. In particular, Theorem 1.4 applies to deformations of $\overline{\rho}$ when $\overline{\rho}$ is even.

Finally, we note that Theorem 1.4 gives optimal fullness results for certain arithmetically interesting families. For instance, if ρ arises from a Coleman family, or more generally from a p-adic family of Hilbert modular eigenforms, and if ρ satisfies the hypothesis of Theorem 1.4, then our result is an optimal fullness result, which was not previously known in any of these cases. By optimal, we mean that if ρ is full with respect to any ring B, then the field of fractions of B must be contained in that of A_0.

1.3. **Structure of the paper.** The article is organized as follows. In Section 2 we give definitions and results about pseudorepresentations, generalized matrix algebras, Pink-Lie algebras, and

[(vii)] See Definition 4.10.

a summary of Bellaïche's results from [Bel18]. We also introduce the notions of regularity and conjugate self-twists, which play a central role in the paper.

In both this paper and that of Bellaïche [Bel18], the key object of study is a certain Pink-Lie algebra L attached to $\operatorname{Im}\rho$. The Lie algebra L is a priori only a module over $\mathbb{Z}_p$, even when the ring A has dimension greater than one. In Section 3 we show that in fact L is always a $W(\mathbb{E})$-module. Moreover, we give minor conditions on ρ that ensure that L is a module over a ring that is, in general, much bigger ring than $W(\mathbb{E})$. We end Section 3 with the first main step of the proof. That is, we obtain a fullness result, analogous to Bellaïche's Theorem 1.3 above, with respect to the ring A'_B under a regularity assumption.

Section 4 is systematic study of conjugate self-twists of pseudorepresentations, especially their lifting properties to universal pseudodeformation rings and how they interact with regularity. These results are crucial when we relate A'_B to A^{Σ_ρ}. Section 4 is almost entirely independent of Section 3; only Proposition 3.1 is used.

The second main step of the proof, and the technical heart of the paper, is Section 5. The main goal is to prove that A'_B has the same field of fractions as A^{Σ_ρ}, the ring fixed by conjugate self-twists. This turns out to be intimately related to the question of whether conjugate self-twists of $\overline{\rho}$ lift to conjugate self-twists of ρ, which explains the need for many of the results from Section 4. Although much of Sections 3 and 5 are written under the assumption that ρ has constant determinant, in Section 5.5 we explain how to deduce fullness for nonconstant-determinant representations; see Theorem 5.17.

Finally, in Section 6 we show how to deduce the results cited at the beginning of this introduction from Theorem 1.4. We also obtain an improvement on Theorem 1.4 when the image of $\overline{\rho}$ is very large in Section 6.3.

Since the paper is rather long, we relegate a variety of lemmas about representation theory and commutative algebra to the appendix. We hope this improves the flow of the arguments presented in this paper while keeping it fairly self-contained.

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Contents

2. Background

We begin by establishing some notation and conventions that will be in force throughout the paper. All rings are unital. Given any ring R (not necessarily commutative), we will let $R^\times$ denote the multiplicative group of invertible elements in R. If R is a domain, then $Q(R)$ denotes its field of fractions. For any positive integer n, let ζ_n denote a primitive n-th root of unity. Given a finite field $\mathbb{F}$, the ring of Witt vectors will be denote by $W(\mathbb{F})$, which is isomorphic to $\mathbb{Z}_p[\zeta_{q-1}]$, where q is the size of $\mathbb{F}$. Let $s\colon \mathbb{F}^\times \to W(\mathbb{F})^\times$ be the Teichmüller lift. We shall extend it to $s\colon \mathbb{F} \to W(\mathbb{F})$ by defining $s(0) := 0$. If A is a $W(\mathbb{F})$-algebra, we will often use $W(\mathbb{F})$ to denote the image of $W(\mathbb{F})$ in A under the structure map. In particular, s will often be viewed as having values in A by composing with the structure map.

Throughout the paper we fix a prime $p \neq 2$. Unless otherwise noted, A will always denote a local pro-p commutative ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F}$. Thus we can take square roots of elements $x \in 1 + \mathfrak{m}$ via the formula

$$\sqrt{x} := \sum_{n=0}^{\infty} \binom{1/2}{n} (x-1)^n.$$

In particular, when we write $\sqrt{x}$, we always choose the root congruent to 1 modulo $\mathfrak{m}$.

If M is a subset of a $W(\mathbb{F})$-module N, then we will write $W(\mathbb{F})N$ for the $W(\mathbb{F})$-linear span of M in N.

Whenever we conjugate an element x by g, we mean $g^{-1}xg$ (as opposed to gxg^{-1}).

Finally, if $\overline{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ is a representation over a finite field $\mathbb{F}$, we can compose $\overline{\rho}$ with the natural projection $\mathbb{P}\colon \mathrm{GL}_2(\mathbb{F}) \to \mathrm{PGL}_2(\mathbb{F})$. We shall refer to the image of Π under the composition $\mathbb{P} \circ \overline{\rho}$ as *the projective image of* $\overline{\rho}$. It is well known that the projective image of $\overline{\rho}$ is either cyclic, dihedral, or isomorphic to A_4, S_4, A_5 or one of $\mathrm{PSL}_2(\mathbb{F}')$ or $\mathrm{PGL}_2(\mathbb{F}')$ for some subfield $\mathbb{F}'$ of $\mathbb{F}$ [Dic58, Chpater XII]. If $\mathbb{P}\overline{\rho}(\Pi) \cong A_4$ (respectively, S_4, A_5), we say that $\overline{\rho}$ is *tetrahedral* (respectively, *octahedral*, *icosahedral*). If $\overline{\rho}$ is tetradhedral, octahedral, or icosahedral, then we say that $\overline{\rho}$ is *exceptional*. If $\mathbb{P}\overline{\rho}(\Pi)$ contains $\mathrm{PSL}_2(\mathbb{F}_p)$ and $\overline{\rho}$ is not exceptional, then we say that $\overline{\rho}$ is *large*. Be warned that there are exceptional isomorphisms $\mathrm{PSL}_2(\mathbb{F}_3) \cong A_4, \mathrm{PGL}_2(\mathbb{F}_3) \cong S_4, \mathrm{PSL}_2(\mathbb{F}_5) \cong A_5$.

2.1. Pseudorepresentations.

In this section we give the introductory definitions and notation related to pseudorepresentations. Although equivalent definitions go back to work of Taylor and Wiles, which was formalized by Rouquier, our notation follows most closely that of Chenevier since we consider a pseudorepresentation to have both a "trace" and a "determinant" (see for instance [Che14, Example 1.8]). Let Π be a group and A a local pro-p commutative ring.

Definition 2.1. A 2-dimensional *pseudorepresentation* of Π with values in A is a pair of functions $t\colon \Pi \to A$ and $d\colon \Pi \to A^\times$ such that

(1) $d(gh) = d(g)d(h)$ for all $g, h \in \Pi$;
(2) $t(gh) = t(hg)$ for all $g, h \in \Pi$;
(3) $t(1) = 2$;
(4) $t(gh) + d(h)t(gh^{-1}) = t(g)t(h)$ for all $g, h \in \Pi$.

If Π is a topological group, we say that a pseudorepresentation $(t, d)\colon \Pi \to A$ is *continuous* if t and d are continuous.

If $2 \in A^\times$ and $(t,d)\colon \Pi \to A$ is a pseudorepresentation, then $d(g) = \frac{t(g)^2 - t(g^2)}{2}$. Furthermore, if $\rho\colon \Pi \to \mathrm{GL}_2(A)$ is a (continuous) representation, then $(\operatorname{tr}\rho, \det\rho)$ is a (continuous) pseudorepresentation. It is straightforward to check that if $(t,d)\colon \Pi \to A$ is a pseudorepresentation and $\chi\colon \Pi \to A^\times$ is a character, then $(\chi t, \chi^2 d)$ is also a pseudorepresentation, called the *twist of* (t,d) *by* χ. The following definition takes familiar properties of representations and translates them into properties of pseudorepresentations.

Definition 2.2. Let $(t,d)\colon \Pi \to A$ be a pseudorepresentation.

(1) We say (t,d) is *reducible* if $t = \chi_1 + \chi_2$ with $\chi_i \colon \Pi \to A^\times$ characters. Otherwise we say that (t,d) is *irreducible*.
(2) We say (t,d) is *dihedral* if it is irreducible and there is a nontrivial character $\eta\colon \Pi \to A^\times$ such that $(\eta t, \eta^2 d) = (t,d)$.
(3) We say that (t,d) is *a priori small* if it is irreducible, not dihedral, and $s(t) \neq t$, where $s\colon \mathbb{F} \to A$ is the Teichmüller lift.

Occasionally it will be convenient to assume that d is a finite order character whose order is a power of 2. This can be achieved when $2 \in A^\times$ by multiplying the pseudorepresentation with a character, as the following lemma demonstrates.

Lemma 2.3. *Assume that $2 \in A^\times$. Let $d\colon \Pi \to A^\times$ be a character. Then there is a character $\chi\colon \Pi \to A^\times$ such that the order of $d\chi^2$ is a power of 2.*

Proof. Let $\overline{d}\colon \Pi \to \mathbb{F}^\times$ the reduction of d modulo $\mathfrak{m}$. Let $d_0 := ds(\overline{d})^{-1}\colon \Pi \to 1+\mathfrak{m}$. Write $s(\overline{d}) = d_1 d_2$ for characters $d_i\colon \Pi \to s(\mathbb{F}^\times)$ such that d_1 has odd order a and d_2 has order a power of 2. Since A is local and $2 \in A^\times$, it follows that the function $d_0^{1/2}\colon \Pi \to A^\times$ given by $g \mapsto \sqrt{d_0(g)}$ is a character such that $(d_0^{1/2})^2 = d_0$. Then the character $\chi := (d_1^{\frac{a+1}{2}} d_0^{1/2})^{-1}$ satisfies the lemma. □

Let R be a topological ring and S a subring of R. We will say that S is *topologically generated* by at set X if S is the smallest closed subring of R containing X. Similarly, we can talk about an additive subgroup or a $W(\mathbb{F})$-algebra topologically generated by a set. If $(t,d)\colon \Pi \to A$ is a pseudorepresentation, we call the subring of A topologically generated by $t(\Pi)$ the *trace algebra* of (t,d).

Definition 2.4. Let $(t,d)\colon \Pi \to A$ be a pseudorepresentation. We write $A_0(t)$, or just A_0 when (t,d) is clear, for the subring of A topologically generated by $\{t(g)^2/d(g)\colon g \in \Pi\}$. When $A = \mathbb{F}$, we write $\mathbb{E}$ for A_0.

Definition 2.5. If $\mathbb{F}'$ is a subfield of $\mathbb{F}$, then we say that a semisimple representation $\overline{\rho}$ is *multiplicity-free over* $\mathbb{F}'$ if either $\overline{\rho}$ is absolutely irreducible or $\overline{\rho} \cong \chi_1 \oplus \chi_2$ such that $\chi_1, \chi_2\colon \Pi \to \mathbb{F}'^\times$ are distinct characters.

Fix a continuous representation $\overline{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ that is multiplicity-free over $\mathbb{F}$.

Definition 2.6. A pseudorepresentation $(t,d)\colon \Pi \to A$ is a *pseudodeformation* of $\overline{\rho}$ if $(t,d) \equiv (\operatorname{tr}\overline{\rho}, \det\overline{\rho}) \bmod \mathfrak{m}$. A pseudodeformation satisfying $d = s(\det\overline{\rho})$ is said to have *constant determinant.*

Let $\mathcal{C}$ be the category of local pro-p commutative rings with residue field $\mathbb{F}$, which have a natural $W(\mathbb{F})$-algebra structure, and with morphisms being local continuous $W(\mathbb{F})$-algebra homomorphisms. We are interested in the deformation functors

$$\begin{aligned} F\colon \mathcal{C} &\to \text{ SETS} \\ A &\mapsto \{(t,d)\colon \Pi \to A \text{ pseudodeformation of } \overline{\rho}\}. \end{aligned}$$

and

$$\begin{aligned} G\colon \mathcal{C} &\to \text{ SETS} \\ A &\mapsto \{(t,d) \in F(A)\colon d = s(\det\overline{\rho})\}. \end{aligned}$$

These functors are always representable. In order for the representing ring to be Noetherian, we need to impose a finiteness condition on Π du to Mazur, which we now recall.

Definition 2.7. [Maz89] A profinite group Π *satisfies the p-finiteness condition* if, for every open subgroup Π_0 of Π, the set $\mathrm{Hom}(\Pi_0, \mathbb{F}_p)$ is finite.

It is well known that F is represented by a pro-p local *Noetherian* $W(\mathbb{F})$-algebra $\tilde{\mathcal{A}}$ whenever Π is a profinite group that satisfies Mazur's p-finiteness condition. See, for example, [Che14, Proposition 3.3] or [Böc13, Proposition 2.3.1]. Let $(t^{\mathrm{univ}}, d^{\mathrm{univ}})\colon \Pi \to \tilde{\mathcal{A}}$ be the universal pseudodeformation of $\overline{\rho}$. It is easy to see that the constant-determinant condition is a deformation condition. Indeed, let $\mathfrak{a}$ be the ideal of $\tilde{\mathcal{A}}$ topologically generated by $\{d^{\mathrm{univ}}(g) - s(\det \overline{\rho}(g))\colon g \in \Pi\}$. Then $\mathcal{A} \coloneqq \tilde{\mathcal{A}}/\mathfrak{a}$ represents G. In particular, $\mathcal{A}$ is also a pro-p local Noetherian $W(\mathbb{F})$-algebra with residue field $\mathbb{F}$. We shall often use $(T, d)\colon \Pi \to \mathcal{A}$ to denote the universal constant-determinant pseudodeformation.

The following notion of admissibility was introduced by Bellaïche in [Bel18] and plays a central role in his work.

Definition 2.8. [Bel18, Section 5.2] A tuple $(\Pi, \overline{\rho}, t, d)$ is an *admissible pseudodeformation over* A if the following conditions are satisfied:

(1) Π is a profinite group that satisfies the p-finiteness condition;
(2) $\overline{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ is a continuous representation that is multiplicity-free over $\mathbb{F}$;
(3) $(t, d)\colon \Pi \to A$ is a continuous pseudodeformation of $\overline{\rho}$;
(4) $d(g) \in s(\mathbb{F}^{\times})$ for all $g \in \Pi$;
(5) A is topologically generated by $t(\Pi)$ as a $W(\mathbb{F})$-algebra.

2.2. **GMAs and (t, d)-representations.** It is natural to ask when a given pseudodeformation $(t, d)\colon \Pi \to A$ arises as the trace and determinant of an actual representation $\rho\colon \Pi \to \mathrm{GL}_2(A)$. This has been studied in great generality; see the introduction of Chenevier's [Che14] for a thorough history. Bellaïche and Chenevier [BC09, Section 1.4] have shown that the answer is that (t, d) always comes from a representation if one allows something more general than matrix algebras for the target. In Section 2.2 we summarize Bellaïche's [Bel18, Section 2], where he specializes Chenevier's work specifically to the 2-dimensional setting. All proofs that can be found Bellaïche's work are omitted.

Definition 2.9. A *generalized matrix algebra (GMA)* over a commutative ring A is given by a tuple of data (A, B, C, m), where B and C are A-modules, $m\colon B \times C \to A$ is a morphism of A-modules satisfying

$$m(b, c)b' = m(b', c)b \text{ and } m(b, c')c = m(b, c)c' \text{ for all } b, b' \in B, c, c' \in C.$$

Given such data, define $R \coloneqq A \oplus B \oplus C \oplus A = \begin{pmatrix} A & B \\ C & A \end{pmatrix}$ and give R a ring structure via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \coloneqq \begin{pmatrix} aa'+m(b,c') & ab'+bd' \\ a'c+dc' & dd'+m(b',c) \end{pmatrix},$$

where $a, a', d, d' \in A, b, b' \in B, c, c' \in C$. We refer to the GMA given by (A, B, C, m) simply by R. If A is a topological ring and B, C are topological A-modules, then R inherits a natural topology, and we call R a *topological GMA* if m is continuous. We say that R is *faithful* if m is nondegenerate as a pairing of A-modules.

The following lemma shows that when A is a domain, faithful GMAs can be embedded into a matrix algebra over the field of fractions of A.

Lemma 2.10. [Bel18, Lemmas 2.2.2, 2.2.3] *Assume that A is a domain with field of fractions K and that $R = \begin{pmatrix} A & B \\ C & A \end{pmatrix}$ is a faithful GMA over A. Then there exist embeddings of A-modules $B, C \hookrightarrow K$ such that (identifying B, C with their images in K), $m\colon B \times C \to A$ is induced by multiplication in K. In particular, if $BC \neq 0$, then $R \otimes_A K$ is isomorphic over K as a GMA to $M_2(K)$.*

We recall the following result of Bellaïche, which explains that any pseudorepresentation can be realized as the trace of a GMA-valued representation.

Proposition 2.11. [Bel18, Proposition 2.4.2] *Let $\overline{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ be multiplicity-free over $\mathbb{F}$. Let $(t, d)\colon \Pi \to A$ be a pseudodeformation of $\overline{\rho}$.*

(1) *There exists a faithful GMA R over A and a morphism of groups $\rho\colon \Pi \to R^\times$ such that* $\operatorname{tr}\rho = t, \det\rho = d$*, and* $A\rho(\Pi) = R$*.*
(2) *If (ρ, R) and (ρ', R') are as in (1), then there is a unique isomorphism of A-algebras $\Psi\colon R \to R'$ such that $\Psi \circ \rho = \rho'$.*
(3) *If $g_0 \in \Pi$ such that $\overline{\rho}(g_0)$ has distinct eigenvalues $\lambda_0, \mu_0 \in \mathbb{F}^\times$, then there exists (ρ, R) as in (1) such that $\rho(g_0)$ is diagonal and $\rho(g_0) \equiv \left(\begin{smallmatrix} \lambda_0 & 0 \\ 0 & \mu_0 \end{smallmatrix}\right) \bmod \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of A.*
(4) *If $g_0 \in \Pi$ and $(\rho, R), (\rho', R')$ are as in (3), then the unique isomorphism of A-algebras $\Psi\colon R \to R'$ such that $\Psi \circ \rho = \rho'$ is an isomorphism of GMAs.*
(5) *If $\overline{\rho}$ is irreducible and (ρ, R) is as in (1), then $R = (A, B, C, m, R)$ is isomorphic to $M_2(A)$ as a GMA over A. If $\overline{\rho}$ is reducible, then $BC \subset \mathfrak{m}$.*
(6) *Assume that A is Noetherian and Π satisfies the p-finiteness condition. If (t, d) is continuous, then for (ρ, R) as in (1), R is of finite type as an A-module. If R is given its unique topology as an A-algebra, then ρ is continuous.*

Following Bellaïche, we make the following definitions.

Definition 2.12. [Bel18, Definition 2.4.3] A representation $\rho\colon \Pi \to R^\times$ satisfying condition (1) in Proposition 2.11 is called a *(t, d)-representation*. If in addition ρ satisfies condition (3), then we say that ρ is *adapted* to (g_0, λ_0, μ_0).

In fact, it is often useful to have the following strengthening of Proposition 2.11(3).

Lemma 2.13. *Let $\overline{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ be multiplicity-free over $\mathbb{F}$ and $\lambda_0 \neq \mu_0 \in \mathbb{F}^\times$ be the eigenvalues of an element in $\operatorname{Im}\overline{\rho}$. Let $(t, d)\colon \Pi \to A$ be a pseudodeformation of $\overline{\rho}$. Then there exists $g_0 \in \Pi$ and a (t, d)-representation ρ adapted to (g_0, λ_0, μ_0) such that $\rho(g_0) = \left(\begin{smallmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{smallmatrix}\right)$.*

Proof. Let $g_0' \in \Pi$ be any element such that $\overline{\rho}(g_0')$ has eigenvalues λ_0, μ_0. Then Proposition 2.11(3) guarantees the existence of a (t, d)-representation $\rho\colon \Pi \to R^\times$ adapted to (g_0', λ_0, μ_0). By [Bel18, Theorem 6.2.1], it follows that $\left(\begin{smallmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{smallmatrix}\right) \in \operatorname{Im}\rho$. Let g_0 be any element in $\rho^{-1}\left(\begin{smallmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{smallmatrix}\right)$. Then ρ is a (t, d)-representation adapted to (g_0, λ_0, μ_0) and $\rho(g_0) = \left(\begin{smallmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{smallmatrix}\right)$. ☐

2.3. **Fullness.** In Section 2.3 we define the notion of fullness for a pseudorepresentation, which will be our measure for the size of its image. Recall that A is a pro-p local ring. We assume throughout Section 2.3 that A is a domain with field of fractions K. For any nonzero A-ideal $\mathfrak{a}$, let

$$\Gamma_A(\mathfrak{a}) \coloneqq \ker(\mathrm{SL}_2(A) \to \mathrm{SL}_2(A/\mathfrak{a})) = \left\{ \left(\begin{smallmatrix} 1+a & b \\ c & 1+d \end{smallmatrix}\right) \in \mathrm{SL}_2(A) \colon a, b, c, d \in \mathfrak{a} \right\},$$

the congruence subgroup of $\mathrm{SL}_2(A)$.

Definition 2.14. Let A' is a subring of A. Let R be a faithful GMA over A, and consider it as a subalgebra of $M_2(K)$ via Lemma 2.10. Let G be a subgroup of $R^\times$. We say that G is *A'-full* if there exists a nonzero A'-ideal $\mathfrak{a}'$ and $x \in \mathrm{GL}_2(K)$ such that

$$xGx^{-1} \supseteq \Gamma_{A'}(\mathfrak{a}'),$$

where we also consider $\Gamma_{A'}(\mathfrak{a}')$ inside of $M_2(K)$ via the natural inclusion $\mathrm{SL}_2(A') \hookrightarrow M_2(K)$. If $\rho\colon \Pi \to R^\times$ is a representation, we say that ρ is *A'-full* if $\rho(\Pi)$ is A'-full. We say that a pseudorepresentation $(t, d)\colon \Pi \to A$ is *A'-full* if there exists a (t, d)-representation that is A'-full.

Remark 2.15. Fullness is a well-defined notion for pseudorepresentations in the sense that if there exists a (t, d)-representation $\rho\colon \Pi \to R^\times$ that is A'-full, then every (t, d)-representation $\rho'\colon \Pi \to R'^\times$ is A'-full. To see this, we just have to verify that the A-algebra isomorphism $\Psi\colon R \to R'$ such that $\rho' = \Psi \circ \rho$ from Proposition 2.11 is given by conjugation by an element of $\mathrm{GL}_2(K)$. Consider $\Psi \otimes 1\colon R \otimes_A K \to R' \otimes_A K$ and recall that $R \otimes_A K \cong M_2(K) \cong R' \otimes_A K$ by Lemma 2.10. By the Skolem-Noether theorem, it follows that $\Psi \otimes 1$ (and hence Ψ) is conjugation by an element of $\mathrm{GL}_2(K)$.

The notion of fullness is meant to replace Bellaïche's notion of congruence large-image [Bel18, Definition 7.2.1]. The advantage of our definition is that, given an admissible pseudodeformation $(\Pi, \overline{\rho}, t, d)$ over A whose image we wish to study, we do not have to create an admissible pseudodeformation over a subring A' of A in order to conclude that it is A'-full (cf. [Bel18, Theorem 7.2.3]). The disadvantage of Definition 2.14 is that it relies on the fact that A is a domain in order to embed R into $M_2(K)$ and thus to be able to compare subgroups of $R^\times$ with congruence subgroups of subrings of A. We remark however that Bellaïche's Theorem 1.3 is also only valid for domains [Bel18, Theorem 7.2.3].

In general a pseudorepresentation is A'-full for more than one choice of A'. The next lemma shows that there is not a single optimal ring for fullness results. It plays a key role in our arguments in Section 5.

Lemma 2.16. *Let $A_1 \subseteq A_2$ be domains. The following conditions are equivalent:*
(1) A_1 contains a nonzero ideal of A_2;
(2) there exists $y \in A_1 \setminus \{0\}$ such that $yA_2 \subseteq A_1$;
(3) every nonzero ideal of A_1 contains a nonzero ideal of A_2.
These equivalent conditions imply that A_2 and A_1 have the same field of fractions. If moreover A_1 is Noetherian, then conditions (1,2,3) are equivalent to:
(4) $Q(A_2) = Q(A_1)$ and A_2 is a finitely generated A_1-module.

Proof. For (1) implies (2), take y to be any nonzero element of the nonzero ideal of A_2 contained in A_1. If (2) holds, then an arbitrary nonzero ideal $\mathfrak{a}$ of A_1 contains $(yA_2)\mathfrak{a}$, which is a nonzero ideal of A_2, implying (3). Clearly (3) implies (1).

To see that $Q(A_2) = Q(A_1)$ under either of (1,2,3), note that any $x \in A_2$ can be written as $(yx)/y \in Q(A_1)$ with y as in (2).

For the rest of the proof, assume that A_1 is Noetherian. Suppose first that either of (1,2,3) holds. If I is a non-zero ideal of A_2 contained in A_1, then it has a finite set of generators as an ideal of A_1, since A_1 is Noetherian. In particular, I is a finitely generated A_1-module. By replacing I with a smaller A_2-ideal, we can assume that I is principal in A_2, that is, $I = bA_2$ for some $b \in A_1$. Now choose a finite set of generators $\{bx_1, \dots, bx_n\}$ of bA_2 as an A_1-module, with $x_1, \dots, x_n$ in A_2. Then, for every y in A_1, by is a linear combination $\sum_i a_i b x_i$ for some $a_i \in A_1$, which means that $y = \sum_i a_i x_i$, so the set $\{x_1, \dots, x_n\}$ generates A_2 as an A_1-module.

Conversely, suppose that (4) is satisfied. Let $x_1, \dots, x_n$ be generators for A_2 as an A_1-module. Write $x_i = b_{i1}/b_{i2}$ with $b_{ij} \in A_1 \setminus \{0\}$. Set $b = \prod_{i=1}^n b_{i2} \in A_1 \setminus \{0\}$. Then $bx_i \in A_1$ for all i, and it follows that $bA_2 \subseteq A_1$, proving (2). ☐

Lemma 2.16 shows that if A_1, A_2 are subrings of A with the same field of fractions such that A_2 is a finitely generated A_1-module, then a pseudorepresentation is A_1-full if and only if it is A_2-full: indeed, if I_1 is an arbitrary nonzero ideal of A_1, Lemma 2.16 provides us with an ideal I_2 of A_2 contained in I_1, so that $\Gamma_{A_2}(I_2) \subset \Gamma_{A_1}(I_1)$. It is even easier to pass fullness from a ring to a subring so long as they have the same field of fractions, as the following lemma shows.

Lemma 2.17. *Assume that $A_1 \subseteq A_2$ are domains with the same field of fractions. If $\mathfrak{a}$ is a nonzero A_2-ideal, then $\mathfrak{a} \cap A_1$ is a nonzero A_1-ideal.*

Proof. It is clear that $\mathfrak{a} \cap A_1$ is an A_1-ideal. To see that it is nonzero, fix $e \in \mathfrak{a} \setminus \{0\}$. Then any element $a \in Q(A_1) = Q(A_2)$ can be written in the form αe with $\alpha \in Q(A_2)$. Assume that $a \in A_1 \setminus \{0\}$, and write $\alpha = \alpha_1/\alpha_2$ with $\alpha_i \in A_2 \setminus \{0\}$. Then we have

$$\alpha_1 e = \alpha_2 a.$$

But $\alpha_1 e \in \mathfrak{a}$ and $\alpha_2 a \in A_1 \setminus \{0\}$. □

Remark 2.18. When A is a Noetherian domain and A' is a closed (in particular local and complete) subring of A, an A-valued pseudorepresentation (t,d) is A'-full if and only if it is $(A')^{\mathrm{nm}} \cap A$-full, where $(A')^{\mathrm{nm}}$ denotes the normalization of A'. This follows from Lemmas 2.16 and 2.17, and the fact that the normalization of a complete local Noetherian domain A' is of finite type as an A'-module.

2.4. **Pink-Lie algebras.** In Section 2.4 we recall Pink's theory relating pro-p subgroups of $\mathrm{SL}_2(A)$ to closed Lie subalgebras of $\mathfrak{sl}_2(A)$ [Pin93]. In fact, we use Bellaïche's generalization to GMAs [Bel18, Section 4].

Recall that A is a local pro-p ring with $p \neq 2$. The assumption that $p \neq 2$ is critical for Pink's theory. Fix a compact topological GMA $R = \left(\begin{smallmatrix} A & B \\ C & A \end{smallmatrix}\right)$ over A. (The compactness condition is satisfied, for instance, when R is of finite type as an A-module.) Write

$$SR^\times := \{r \in R^\times \colon \det r = 1\}.$$

Let $\operatorname{rad} R$ be the Jacobson radical of R, and $R^1 := 1 + \operatorname{rad} R$. We let $SR^1 := SR^\times \cap R^1$, which is a closed normal pro-p subgroup of $R^\times$. See [Bel18, Remark 4.2.1] for an explicit description of these objects. Given any subset S of R, we write

$$S^0 := \{s \in S \colon \operatorname{tr} s = 0\}.$$

Then $(\operatorname{rad} R)^0$ has a Lie algebra structure with bracket given by $[r_1, r_2] := r_1 r_2 - r_2 r_1$.

For any topological group G and closed subgroup H of G, write (G,H) for the smallest closed subgroup of G containing $\{g^{-1}h^{-1}gh \colon g \in G, h \in H\}$. Fix a closed subgroup $\Gamma \subseteq SR^1$. Recall that the lower central series of Γ is defined by $\Gamma_1 := \Gamma$ and define $\Gamma_{n+1} := (\Gamma, \Gamma_n)$. We describe how Pink associates a filtration of Lie algebras to Γ [Pin93, Section 2].

Define a function

$$\begin{aligned} \Theta \colon R^\times &\to R^0 \\ r &\mapsto r - \frac{\operatorname{tr} r}{2}, \end{aligned}$$

where $(\operatorname{tr} r)/2$ is regarded as a scalar via the structure morphism $A \to R$. Let $L(\Gamma) = L_1(\Gamma)$ be the (additive) subgroup of $(\operatorname{rad} R)^0$ topologically generated by $\Theta(\Gamma)$. For $n \geq 2$, define $L_n(\Gamma)$ recursively as the subgroup of $(\operatorname{rad} R)^0$ topologically generated by the set

$$\{xy - yx \colon x \in L_1(\Gamma), y \in L_{n-1}(\Gamma)\}.$$

Although the $L_n(\Gamma)$ are a priori only subgroups of $(\operatorname{rad} R)^0$, Pink shows that they are closed under Lie brackets and form a descending filtration, as summarized in the following proposition, which is due to Pink when $R = M_2(A)$ [Pin93, Proposition 3.1, Proposition 2.3] and to Bellaïche in the GMA case [Bel18, Proposition 4.7.1].

Proposition 2.19. *For all $n \geq 1$, we have $L_{n+1}(\Gamma) \subseteq L_n(\Gamma)$. In particular, each $L_n(\Gamma)$ is a Lie subalgebra of $(\operatorname{rad} R)^0$.*

We emphasize that, a priori, each $L_n(\Gamma)$ is just a $\mathbb{Z}_p$-module, even if the ring A is very large. The point of Section 3 is to prove that, under mild conditions, $L_n(\Gamma)$ is in fact an algebra over an (in general) much larger ring.

Conversely, given a closed Lie subalgebra L of $(\operatorname{rad} R)^0$, define $H(L) := \Theta^{-1}(L) \cap SR^1$. Let $H_n := H(L_n(\Gamma))$. A priori, $H(L)$ is only a subset of SR^1. However, we have the following theorem of Pink [Pin93, Proposition 2.4, Theorem 2.7], which was generalized to GMAs by Bellaïche [Bel18, Theorem 4.7.3].

Theorem 2.20. *We have that H_n is a pro-p subgroup of SR^1. Furthermore, Γ is a normal subgroup of H_1, and H_1/Γ is abelian. For $n \geq 2$, we have $H_n = \Gamma_n$.*

Remark 2.21. Pink's construction satisfies the following two important properties.

(1) It is functorial with respect to surjective ring homomorphisms. Namely, let $\mathfrak{a}$ be a closed ideal of A and $\varphi\colon R \to R/\mathfrak{a}R$ the natural projection. Then for all $n \geq 1$ we have

$$\varphi(L_n(\Gamma)) = L_n(\varphi(\Gamma)).$$

(2) Pink's Lie algebra $L_n(\Gamma)$ is closed under conjugation by the normalizer of Γ in $R^\times$. This follows easily from the definitions since Θ is invariant under conjugation.

Let us give an example in the case when $R = M_2(A)$. For a nonzero A-ideal $\mathfrak{a}$, define

$$\mathfrak{sl}_2(\mathfrak{a}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) : a, b, c \in \mathfrak{a} \right\}.$$

Lemma 2.22. *Let $\mathfrak{a}$ be a closed ideal of A. Then $\Gamma_A(\mathfrak{a})$ is a closed pro-p subgroup of $\mathrm{GL}_2(A)$ and $L_n(\Gamma_A(\mathfrak{a})) = \mathfrak{sl}_2(\mathfrak{a}^n)$.*

Proof. For $x = \left(\begin{smallmatrix} 1+a & b \\ c & 1+d \end{smallmatrix}\right) \in \Gamma_A(\mathfrak{a})$ one has $\Theta(x) = \begin{pmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{pmatrix} \in \mathfrak{sl}_2(\mathfrak{a})$, so $L_1(\Gamma_A(\mathfrak{a})) \subseteq \mathfrak{sl}_2(\mathfrak{a})$. In particular, for any $b, c \in \mathfrak{a}$, we have $\Theta\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)$ and $\Theta\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right)$. For $a \in \mathfrak{a}$ we have $\left(\begin{smallmatrix} 1+2a & -2a \\ 2a & 1-2a \end{smallmatrix}\right) \in \Gamma_A(\mathfrak{a})$, and so

$$\Theta\left(\begin{smallmatrix} 1+2a & -2a \\ 2a & 1-2a \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & -2a \\ 2a & -a \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) + \Theta\left(\begin{smallmatrix} 1 & -2a \\ 0 & 1 \end{smallmatrix}\right) + \Theta\left(\begin{smallmatrix} 1 & 0 \\ 2a & 1 \end{smallmatrix}\right).$$

It follows that $\mathfrak{sl}_2(\mathfrak{a})$ is contained in the additive subgroup generated by $\Theta(\Gamma_A(\mathfrak{a}))$. Since $\mathfrak{sl}_2(\mathfrak{a})$ is closed in $\mathfrak{sl}_2(A)$, it follows that $\mathfrak{sl}_2(\mathfrak{a}) = L_1(\Gamma_A(\mathfrak{a}))$.

It is straightforward to calculate by induction on n that the subgroup topologically generated by

$$\{xy - yx \colon x \in \mathfrak{sl}_2(\mathfrak{a}), y \in \mathfrak{sl}_2(\mathfrak{a}^n)\}$$

is $\mathfrak{sl}_2(\mathfrak{a}^{n+1})$. That is, $L_n(\Gamma_A(\mathfrak{a})) = \mathfrak{sl}_2(\mathfrak{a}^n)$ for all $n \geq 1$. ☐

Corollary 2.23. *Let $\mathfrak{a}$ be a closed A-ideal different from A. Then $(\Gamma_A(\mathfrak{a}), \Gamma_A(\mathfrak{a}))' = \Gamma_A(\mathfrak{a}^2)$. This holds even for $\mathfrak{a} = A$ so long as $\mathbb{F}$ has more than three elements.*

Proof. First assume that $\mathfrak{a} \neq A$. By Theorem 2.20,

$$(\Gamma_A(\mathfrak{a}), \Gamma_A(\mathfrak{a})) = \Theta^{-1}(L_2(\Gamma_A(\mathfrak{a}))) \cap \Gamma_A(\mathfrak{m}).$$

By Lemma 2.22, $L_2(\Gamma_A(\mathfrak{a})) = \mathfrak{sl}_2(\mathfrak{a}^2)$.

Clearly $\Gamma_A(\mathfrak{a}^2) \subseteq \Theta^{-1}(\mathfrak{sl}_2(\mathfrak{a}^2)) \cap \Gamma_A(\mathfrak{m})$. We compute $\Theta^{-1}\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \cap \Gamma_A(\mathfrak{m})$ for $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathfrak{sl}_2(\mathfrak{a}^2)$. If $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Theta^{-1}\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \cap \Gamma_A(\mathfrak{m})$ then we must have $\beta = b, \gamma = c, \alpha - \delta = 2a$, and $1 = \alpha\delta - \beta\gamma$. From this one calculates that $\alpha = a \pm \sqrt{1 + a^2 + bc}$ and $\delta = -a \pm \sqrt{1 + a^2 + bc}$. But only one of these possibilities has $\alpha \equiv 1 \equiv \delta \bmod \mathfrak{m}$ and thus is in $\Gamma_A(\mathfrak{m})$. That is, there is a unique element in $\Theta^{-1}\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \cap \Gamma_A(\mathfrak{m})$. It follows that $\Theta^{-1}(\mathfrak{sl}_2(\mathfrak{a}^2)) \cap \Gamma_A(\mathfrak{m}) = \Gamma_A(\mathfrak{a}^2)$, as desired.

We now prove that $(\mathrm{SL}_2(A), \mathrm{SL}_2(A)) = \mathrm{SL}_2(A)$ when $\#\mathbb{F} > 3$. By the first statement in the corollary, we know that $\Gamma_A(\mathfrak{m}^2) \subseteq (\mathrm{SL}_2(A), \mathrm{SL}_2(A))$, so we may assume that $\mathfrak{m}^2 = 0$. Furthermore, the residual image of $(\mathrm{SL}_2(A), \mathrm{SL}_2(A))$ is $(\mathrm{SL}_2(\mathbb{F}), \mathrm{SL}_2(\mathbb{F}))$, which is equal to $\mathrm{SL}_2(\mathbb{F})$. Therefore, it suffices to show that $\left(\begin{smallmatrix} 1+a & b \\ c & 1-a \end{smallmatrix}\right) \in (\mathrm{SL}_2(A), \mathrm{SL}_2(A))$ for any with $a, b, c \in \mathfrak{m}$. Since $\mathfrak{m}^2 = 0$, we can decompose

$$\left(\begin{smallmatrix} 1+a & b \\ c & 1-a \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1+a & 0 \\ 0 & 1-a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ c & 1 \end{smallmatrix}\right).$$

Let $x \in A^\times$ such that $x^2 \not\equiv 1 \bmod \mathfrak{m}$, which exists since $\#\mathbb{F} > 3$. Note that for any $\beta, \gamma \in \mathfrak{m}$ we have

$$\left(\begin{smallmatrix} 1 & b \\ c & 1 \end{smallmatrix}\right) = \begin{pmatrix} 1 & b(1-x^2)^{-1} \\ c(1-x^{-2})^{-1} & 1 \end{pmatrix}\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right)\begin{pmatrix} 1 & -b(1-x^2)^{-1} \\ -c(1-x^{-2})^{-1} & 1 \end{pmatrix}\left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & x \end{smallmatrix}\right) \in (\mathrm{SL}_2(A), \mathrm{SL}_2(A))$$

and

$$\left(\begin{smallmatrix} 1+a & 0 \\ 0 & 1-a \end{smallmatrix}\right) = \begin{pmatrix} 1+\frac{a}{2} & 0 \\ 0 & 1-\frac{a}{2} \end{pmatrix}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\begin{pmatrix} 1-\frac{a}{2} & 0 \\ 0 & 1+\frac{a}{2} \end{pmatrix}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in (\mathrm{SL}_2(A), \mathrm{SL}_2(A)).$$

It follows that $\Gamma_A(\mathfrak{m}) \subseteq (\mathrm{SL}_2(A), \mathrm{SL}_2(A))$ and hence that $\mathrm{SL}_2(A)$ is its own topological derived subgroup. $\square$

Remark 2.24. In practice we will apply Lemma 2.22 and Corollary 2.23 not to A itself, but to a closed subring A' of A. That is, if $\mathfrak{a}'$ is a closed ideal of A', then $\Gamma_{A'}(\mathfrak{a}')$ is a pro-p subgroup of $\mathrm{GL}_2(A)$, and $L_n(\Gamma_{A'}(\mathfrak{a}')) = \mathfrak{sl}_2((\mathfrak{a}')^n)$. This follows easily from Lemma 2.22 since A' and $\mathfrak{a}'$ are closed in A.

Corollary 2.23 can be used to show that fullness does not change if we twist a pseudorepresentation by a character. Indeed, we have the following lemma.

Lemma 2.25. *Let Π be a profinite group and A a domain. Let $(t,d)\colon \Pi \to A$ be a continuous pseudorepresentation and $\chi\colon \Pi \to A^\times$ a continuous character. Let $A' \subseteq A$ be a closed subring. Then (t,d) is A'-full if and only if $(\chi t, \chi^2 d)$ is A'-full.*

Proof. Let $\Pi_0 \coloneqq \ker \chi$. It suffices to show that $(t|_{\Pi_0}, d|_{\Pi_0})$ is A'-full if (t,d) is A'-full. Let $\rho\colon \Pi \to R^\times$ be a (t,d)-representation such that $\operatorname{Im}\rho$ contains $\Gamma_{A'}(\mathfrak{a}')$ for some nonzero A'-ideal $\mathfrak{a}'$. Write $G \coloneqq \rho(\Pi)$.

Note that Π_0 is a closed subgroup of a compact group and hence compact. Since $\Pi/\Pi_0 \cong \operatorname{Im}\chi$, and hence $G/\rho(\Pi_0)$, is abelian, it follows that $\rho(\Pi_0)$ contains G', the topological derived subgroup of G. But $G \supseteq \Gamma_{A'}(\mathfrak{a}')$, and so G' contains the derived subgroup of $\Gamma_{A'}(\mathfrak{a}')$, which is $\Gamma_{A'}((\mathfrak{a}')^2)$ by Corollary 2.23. Since A is a domain and $\mathfrak{a}' \neq 0$, it follows that $(\mathfrak{a}')^2 \neq 0$ and hence $(t|_{\Pi_0}, d|_{\Pi_0})$ is A'-full. $\square$

2.5. **Decomposability and regularity.** In order to prove fullness theorems, it is useful to be able to decompose Pink's Lie algebra according to its entries. In Section 2.5 we define this precisely and then define regularity, which will turn out to be ensure that the Lie algebras of the representations we work with are decomposable.

Definition 2.26. [Bel18, Section 4.9] Let R be a GMA over A and L a closed subspace of $(\operatorname{rad} R)^0$. We say that L is *decomposable* if

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in L \text{ implies that } \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \in L \text{ and } \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in L.$$

We say that L is *strongly decomposable* if L is decomposable and

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in L \text{ implies that } \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in L \text{ and } \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \in L.$$

If $L_n(\Gamma) \subseteq R = \begin{pmatrix} A & B \\ C & A \end{pmatrix}$ is decomposable, we write

$$\begin{aligned} I_n(\Gamma) &\coloneqq \{a \in A\colon \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \in L_n(\Gamma)\}, \\ \nabla_n(\Gamma) &\coloneqq \{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in L_n(\Gamma)\}, \\ B_n(\Gamma) &\coloneqq \{b \in B\colon \exists c \in C \text{ such that } \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in L_n(\Gamma)\}, \\ C_n(\Gamma) &\coloneqq \{c \in C\colon \exists b \in B \text{ such that } \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in L_n(\Gamma)\}. \end{aligned}$$

Eventually, L will be a Pink-Lie algebra associated to some admissible pseudodeformation of $\overline{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$. Regularity is a condition on $\overline{\rho}$ that will allow us to decompose L, as we will see in Section 3.

Recall that $\mathbb{E}$ is the subfield of $\mathbb{F}$ generated by $\{(\operatorname{tr}\overline{\rho}(g))^2/\det\overline{\rho}(g)\colon g \in \Pi\}$. If λ_g, μ_g are the eigenvalues of $\overline{\rho}(g)$, then we see that $(\operatorname{tr}\overline{\rho}(g))^2/\det\overline{\rho}(g) = \lambda_g\mu_g^{-1} + \lambda_g^{-1}\mu_g + 2$. Hence $\mathbb{E}$ is generated over $\mathbb{F}_p$ by the set

$$(1) \qquad \{\lambda\mu^{-1} + \lambda^{-1}\mu\colon \lambda, \mu \text{ are the eigenvalues of } \overline{\rho}(g) \text{ for some } g \in \Pi\}.$$

In particular, g will not contribute to $\mathbb{E}$ if the multiplicative order of $\lambda_g\mu_g^{-1}$ is strictly less than 5. Using this reasoning, it is straightforward to calculate $\mathbb{E}$ when $\overline{\rho}$ exceptional. Namely, $\overline{\rho}$ is tetrahedral or octahedral, then $\mathbb{E} = \mathbb{F}_p$. If $\overline{\rho}$ is icosahedral, then $\mathbb{E} = \mathbb{F}_p(\zeta_5 + \zeta_5^{-1}) = \mathbb{F}_p(\sqrt{5})$.

Definition 2.27. Let $\overline{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ be a semisimple representation. We say that $\overline{\rho}$ is *regular* if there exists $g_0 \in \Pi$ such that $\overline{\rho}(g_0)$ has eigenvalues $\lambda_0, \mu_0 \in \overline{\mathbb{F}}_p^\times$ satisfying $\lambda_0\mu_0^{-1} \in \mathbb{E}^\times \setminus \{\pm 1\}$. We call g_0 a *regular element* for $\overline{\rho}$. If in addition $\lambda_0, \mu_0 \in \mathbb{E}^\times$, then we say that $\overline{\rho}$ is *strongly regular*.

Definition 2.27 is weaker than Bellaïche's definition of regularity [Bel18, Definition 7.2.1]. Given $g \in \Pi$, write λ, μ for the eigenvalues of $\overline{\rho}(g)$. By writing $\operatorname{tr} \overline{g} = \lambda + \mu$ and $\det \overline{\rho}(g) = \lambda\mu$, we see that $\lambda\mu^{-1} + \lambda^{-1}\mu \in \mathbb{E}$. Thus the only way $\overline{\rho}$ can fail to be regular is if, for every matrix in $\operatorname{Im}\overline{\rho}$ with eigenvalues λ, μ, either $\lambda\mu^{-1} = \pm 1$ or the unique quadratic extension of $\mathbb{E}$ is $\mathbb{E}(\lambda\mu^{-1})$.

Remark 2.28. Let us analyze regularity depending on the projective image of $\overline{\rho}$. With notation as in Definition 2.27, note that the order of $\lambda_0\mu_0^{-1}$ in $\mathbb{E}^\times$ corresponds to the order of $\overline{\rho}(g_0)$ in the projective image of $\overline{\rho}$.

(1) If $\overline{\rho}$ is large, then $\overline{\rho}$ is regular. Indeed, $\mathbb{P}\overline{\rho}(\Pi)$ contains $\mathrm{PSL}_2(\mathbb{E})$ up to conjugation. Since $\overline{\rho}$ is not exceptional, $\mathbb{E}^\times$ contains an element x such that $x^2 \neq \pm 1$. Then the image of $\overline{\rho}$ contains, up to conjugation, a scalar multiple of $\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1}\end{smallmatrix}\right)$, which satisfies the regularity property.

(2) If $\overline{\rho}$ is tetrahedral and $p > 3$, then a regular element must map to a 3-cycle in the projective image of $\overline{\rho}$, since the other elements of A_4 have order at most 2. Thus in this case regularity is equivalent to $\zeta_3 \in \mathbb{E} = \mathbb{F}_p$, which is equivalent to $p \not\equiv 2 \bmod 3$. By a similar argument we see that if $\overline{\rho}$ is octahedral and $p > 3$, then regularity is equivalent to one of ζ_3 or ζ_4 being in $\mathbb{E} = \mathbb{F}_p$, which is equivalent to $p \not\equiv 11 \bmod 12$. If $\overline{\rho}$ is icosahedral and $p \neq 5$, then regularity is equivalent to one of ζ_3 or ζ_5 being in $\mathbb{E} = \mathbb{F}_p(\sqrt{5})$, which is equivalent to $p \not\equiv 14 \bmod 15$.

(3) If $\overline{\rho} \cong \varepsilon \oplus \delta$, then we will see in Lemma 4.3 that $\overline{\rho}$ is regular if and only if $\varepsilon\delta^{-1}$ takes values in $\mathbb{E}^\times$.

(4) If $\overline{\rho} = \operatorname{Ind}_{\Pi_0}^{\Pi} \chi$ is dihedral, then elements in $\Pi \setminus \Pi_0$ have projective order 2, and so any regular element must lie in Π_0. Furthermore, elements in $\Pi \setminus \Pi_0$ have trace 0, and so the field $\mathbb{E}$ associated to $\overline{\rho}$ is the same as the field $\mathbb{E}$ associated to $\overline{\rho}|_{\Pi_0}$. Hence we are reduced to the previous case when $\overline{\rho}$ is reducible.

(5) If the projective image of $\overline{\rho}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$, or if $\mathbb{E} = \mathbb{F}_3$, then $\overline{\rho}$ is never regular. In particular, if $p = 3$ and $\overline{\rho}$ is tetrahedral or octahedral, then $\overline{\rho}$ is not regular. If $p = 5$ and $\overline{\rho}$ is icosahedral, then $\mathbb{P}\overline{\rho}(\Pi)$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_5)$. Thus $\mathbb{E} = \mathbb{F}_5$ and any potential regular element has eigenvalues in $(\mathbb{F}_5^\times)^2 = \{\pm 1\}$, so $\overline{\rho}$ is not regular in this case.

2.6. **Bellaïche's results.** The purpose of this section is to state Bellaïche's main results that form the basis for our work in this paper. We state them in slightly less generality than [Bel18, Section 6]. As before, A denotes a pro-p local ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F}$. In particular, A is naturally a topological $W(\mathbb{F})$-algebra.

If R is a faithful GMA over A, we define $s\colon R/\operatorname{rad} R \to R$ by

$$s\begin{pmatrix} a & b \\ c & d\end{pmatrix} := \begin{cases} \begin{pmatrix} s(a) & s(b) \\ s(c) & s(d)\end{pmatrix} & \text{if } R = M_2(A) \\ \begin{pmatrix} s(a) & 0 \\ 0 & s(s)\end{pmatrix} & \text{else.}\end{cases}$$

Note that in the latter case, we have a priori that $b = c = 0$.

Let us fix an admissible pseudodeformation $(\Pi, \overline{\rho}, t, d)$ over A. If $p = 3$, let us assume that $\overline{\rho}$ is not tetrahedral. By Proposition 2.11, there exists a (t,d)-representation $\rho\colon \Pi \to R^\times$. Given such a (t,d)-representation, write $G = G_\rho := \rho(\Pi)$ and $\Gamma = \Gamma_\rho := G \cap SR^1$. Furthermore, let $\overline{G}$ denote the image of G modulo $\operatorname{rad} R$. (Note that the image of $\overline{G}$ under an embedding $R/\operatorname{rad} R \to \mathrm{GL}_2(\mathbb{F})$ is a conjugate of $\overline{\rho}(\Pi)$.) We will write $L_n(\rho) := L_n(\Gamma_\rho)$ and analogously for $I_n(\rho), \nabla_n(\rho), B_n(\rho), C_n(\rho)$.

Bellaïche chooses his (t,d)-representations very carefully in order to give a nice description of their Pink-Lie algebras. How this is done depends upon the projective image of $\overline{\rho}$. Since $\overline{\rho}$ is multiplicity-free over $\mathbb{F}$, we can let $\lambda_0 \neq \mu_0 \in \overline{\mathbb{F}}_p^\times$ be the eigenvalues of a matrix $x_0 \in \operatorname{Im}\overline{\rho}$ chosen such that the following conditions are satisfied:

- if $\overline{\rho}$ is large, then $(\lambda_0\mu_0^{-1})^2 \neq 1$ and $\lambda_0, \mu_0 \in \mathbb{F}_p^\times$;
- if $p = 3$ and $\overline{\rho}$ is octahedral, then $\lambda_0\mu_0^{-1}$ is a primitive fourth root of unity;
- if $p = 5$ and $\overline{\rho}$ is icosahedral, then $\lambda_0\mu_0^{-1}$ is a primitive third root of unity;
- if $\overline{\rho}$ is exceptional and does not belong to one of the previous to scenarios, then $\lambda_0\mu_0^{-1}$ is a primitive third, fourth, or fifth root of unity;
- otherwise, the multiplicative order of $\lambda_0\mu_0^{-1}$ is equal to the maximal order of an element in the projective image of $\overline{\rho}$.

Definition 2.29. Suppose $(\Pi, \overline{\rho}, t, d)$ is an admissible pseudodeformation. We say that a (t, d)-representation ρ is *well adapted* if

(1) ρ is adapted to an element g_0 such that $\rho(g_0) = \begin{pmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{pmatrix}$, where λ_0, μ_0 satisfy the relevant property listed above (cf. Lemma 2.13);
(2) if the projective image of $\overline{\rho}$ is dihedral and nonabelian, then $\overline{G}$ contains a matrix of the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ with $bc^{-1} \in \mathbb{F}_p^\times$ and $s(\overline{G}) \subseteq \operatorname{Im}\rho$.

Bellaïche shows that well-adapted (t, d)-representations always exist, provided that one is willing to replace $\mathbb{F}$ by a quadratic extension in the dihedral case [Bel18, Proposition 6.3.2, Lemma 6.8.2].

Define $\mathbb{F}_q$ as in Table 1. We will see in Lemma 4.3 that if $\overline{\rho}$ is regular and reducible or dihedral, then $\mathbb{F}_q$ can be taken to be $\mathbb{E}$. If $\overline{\rho}$ is not projectively cyclic or dihedral, then $\mathbb{F}_q \subseteq \mathbb{E}$ by definition. (In the A_5 case, this follows from the calculation that $\mathbb{E} = \mathbb{F}_p(\sqrt{5})$ prior to Definition 2.27.)

Remark 2.30. Our definition of $\mathbb{F}_q$ differs from that of Bellaïche when $\overline{\rho}$ is exceptional. If $\overline{\rho}$ is tetrahedral, then Bellaïche defines $\mathbb{F}_q = \mathbb{F}_p(\zeta_3)$. If $\overline{\rho}$ is octahedral, he defines $\mathbb{F}_q$ to be $\mathbb{F}_p(\zeta_3)$ if the ratio $\lambda_0\mu_0^{-1}$ chosen prior to Definition 2.29 has order 3 and $\mathbb{F}_p(\zeta_4)$ if that ratio has order 4. If $\overline{\rho}$ is icosahedral, then he defines $\mathbb{F}_q = \mathbb{F}_p(\zeta_5)$. The key property that Bellaïche needs is that $\overline{\rho}$ can be conjugate so that its image lies in $Z\,\mathrm{GL}_2(\mathbb{F}_q)$ and $\begin{pmatrix} \lambda_0 & 0 \\ 0 & \mu_0 \end{pmatrix} \in \operatorname{Im}\overline{\rho}$, where Z is the group of scalar matrices in $\mathbb{F}$ (cf. [Bel18, Lemma 6.8.5]). This change of definition will be justified in Lemma 4.12.

Table 1. Definition of $\mathbb{F}_q$

the projective image of $\overline{\rho}$ is	$\mathbb{F}_q$
cyclic of order m or dihedral of order $2m$	any subfield of $\mathbb{F}$ such that $(m, q-1) > 2$
exceptional	$\mathbb{E}(\lambda_0\mu_0^{-1})$
otherwise	$\mathbb{F}_p$

The following theorem summarizes Bellaïche's results describing the structure of $W(\mathbb{F}_q)L_1(\rho)$ from [Bel18, Section 6].

Theorem 2.31 (Bellaïche)**.** *Let $(\Pi, \overline{\rho}, t, d)$ be an admissible pseudodeformation such that the projective image of $\overline{\rho}$ is not isomorphic to $\mathbb{Z}/2\mathbb{Z}$ nor $(\mathbb{Z}/2\mathbb{Z})^2$. Then every well-adapted (t, d)-representation $\rho\colon \Pi \to R^\times$ with $R = \begin{pmatrix} A & B \\ C & A \end{pmatrix}$ has the following properties:*

(1) $L_1(\rho)$ is decomposable;
(2) the ring A is equal to

$$\begin{cases} W(\mathbb{F}) + W(\mathbb{F})I_1(\rho) + W(\mathbb{F})I_1(\rho)^2 + W(\mathbb{F})B_1(\rho) & \text{if } \overline{\rho} \text{ is projectively dihedral} \\ W(\mathbb{F}) + W(\mathbb{F})I_1(\rho) + W(\mathbb{F})I_1(\rho)^2 & \text{otherwise;} \end{cases}$$

(3) $W(\mathbb{F})C_1(\rho) = C$ and $W(\mathbb{F})B_1(\rho) = B$;

(4) up to possibly replacing ρ with its conjugate by a certain matrix $\left(\begin{smallmatrix}1&0\\0&a\end{smallmatrix}\right)$ with $a \in A^\times$ when $\overline{\rho}$ is exceptional or large, $W(\mathbb{F}_q)L_1(\rho)$ is equal to

$$\begin{pmatrix} W(\mathbb{F}_q)I_1(\rho) & W(\mathbb{F}_q)B_1(\rho) \\ W(\mathbb{F}_q)C_1(\rho) & W(\mathbb{F}_q)I_1(\rho) \end{pmatrix}^0.$$

Furthermore

(i) $(W(\mathbb{F}_q)I_1(\rho))^3 \subseteq W(\mathbb{F}_q)I_1(\rho)$;
(ii) if $\overline{\rho}$ is not reducible, then $W(\mathbb{F}_q)C_1(\rho) = W(\mathbb{F}_q)B_1(\rho)$;
(iii) if $\overline{\rho}$ is exceptional or large, then $W(\mathbb{F}_q)B_1(\rho) = W(\mathbb{F}_q)I_1(\rho)$ and $(W(\mathbb{F}_q)I_1(\rho))^2 \subset W(\mathbb{F}_q)I_1(\rho)$.

For a subfield $\mathbb{F}'$ of $\mathbb{F}$, we shall often refer to the $W(\mathbb{F}')$-subalgebra of A generated by $I_1(\rho)$. We will denote it by $W(\mathbb{F}')[I_1(\rho)]$, which is simply equal to $W(\mathbb{F}')+W(\mathbb{F}')I_1(\rho)+W(\mathbb{F}')I_1(\rho)^2$ whenever $\mathbb{F}_q \subseteq \mathbb{F}'$. When $\overline{\rho}$ is not reducible or dihedral, we have $W(\mathbb{F}')[I_1(\rho)] = W(\mathbb{F}') + W(\mathbb{F}')I_1(\rho)$ by Theorem 2.31(ii).

Bellaïche uses Theorem 2.31 to deduce that, under certain hypotheses, the representation ρ is $A_B := \mathbb{Z}_p[I_1(\rho)]$-full. See Theorem 1.3 or [Bel18, Theorem 7.2.3] for a precise statement of his result.

2.7. **Conjugate self-twists.** The goal of this paper is to understand Bellaïche's ring A_B in terms of conjugate self-twists. Here we give the definitions and notation related to conjugate self-twists of pseudorepresentations. As usual, A is a pro-p local ring.

If σ is a ring automorphism of A and $f\colon \Pi \to A$ is a function, we write ${}^\sigma f\colon \Pi \to A$ for the function ${}^\sigma f(g) := \sigma(f(g))$.

Definition 2.32. Let $(t,d)\colon \Pi \to A$ be a constant-determinant pseudodeformation such that A is the trace algebra of (t,d). A *conjugate self-twist (CST)* of (t,d) is a pair (σ,η), where σ is a ring automorphism of A and $\eta\colon \Pi \to A^\times$ is a continuous character such that $({}^\sigma t, {}^\sigma d) = (\eta t, \eta^2 d)$.

It is easy to see that if (σ,η) is a conjugate self-twist of (t,d), then σ is determined by η. Furthermore, the pairs (σ,η) of conjugate self-twists of a pseudodeformation (t,d) form a group with the group law given by

$$(\sigma,\eta)\cdot(\tau,\chi) := (\sigma\circ\tau, {}^\sigma\chi\cdot\eta). \tag{2}$$

Let Σ_t^{pairs} denote the group of all conjugate self-twists of (t,d).

There is a natural group homomorphism $\Sigma_t^{\text{pairs}} \to \operatorname{Aut} A$ given by $(\sigma,\eta)\mapsto\sigma$. Let Σ_t denote the image of this map and let Σ_t^{di} denote the kernel of this map. Thus we have an exact sequence

$$1 \to \Sigma_t^{\text{di}} \to \Sigma_t^{\text{pairs}} \to \Sigma_t \to 1.$$

Since (t,d) is has constant determinant, it is straightforward to check that Σ_t is an abelian group. We will write A^{Σ_t} for the subring of A that is fixed (pointwise) by every element in Σ_t. When $(t,d) = (\operatorname{tr}\rho, \det\rho)$ for a representation $\rho\colon \Pi \to \mathrm{GL}_2(A)$, we will write $\Sigma_\rho^{\text{pairs}}$ and Σ_ρ in place of Σ_t^{pairs} and Σ_t.

Throughout most of the paper, we work with admissible pseudodeformations, which are constant-determinant pseudorepresentations for which A is the trace algebra. However, in applications one often wants to remove the constant-determinant condition. We still need a notion of conjugate-self twist in this setting, but the naive generalization of Definition 2.32 has some problems, as Example 2.35 below illustrates. We therefore make the following definition specifically in the case when A is a domain.

Definition 2.33. Let A be a domain with field of fractions K and K^{sep} a separable closure of K. Let $(t,d)\colon \Pi \to A$ be a pseudorepresentation. A *generalized conjugate self-twist* of (t,d) is a pair (σ,η), where $\sigma\colon K^{\text{sep}} \to K^{\text{sep}}$ is a field automorphism (not necessarily fixing K) and $\eta\colon \Pi \to K^{\text{sep}\times}$ is a group homomorphism such that $({}^\sigma t, {}^\sigma d) = (\eta t, \eta^2 d)$.

Lemma 2.34. *Let A be a domain. If $(t,d)\colon \Pi \to A$ is a constant-determinant pseudorepresentation such that A is the trace algebra of (t,d), then every generalized conjugate self-twist is a conjugate self-twist.*

Proof. Note that for any generalized conjugate self-twist we have $\eta^2 = {}^{\sigma}dd^{-1}$. If $d = s(\det \overline{\rho})$, then d takes values in the roots of unity in the image of $W(\mathbb{F})$ inside A. Thus σ must act by some integral power on d, and we find that η^2 is a power of d. It follows that η must be equal to a quadratic character times a power of d by the same argument as in [Mom81, Lemma 1.5] or [Lan16, Lemma 3.10]. In particular, η takes values in $A^\times$. Since A is generated as a $W(\mathbb{F})$-module by the values of t, it follows that σ is an automorphism of A. □

The following example shows that when a pseudorepresentation does not have constant determinant, it is necessary to consider generalized conjugate self-twists to understand the image of a (t,d)-representation.

Example 2.35. Let $p \neq 2$ be a prime. Let Π be the subgroup of $\mathrm{GL}_2(\mathbb{Z}_p[\sqrt[p]{p}])$ generated by $\mathrm{GL}_2(\mathbb{Z}_p)$ and the scalar matrix $\sqrt[p]{p}$. Then $\mathrm{GL}_2(\mathbb{Z}_p)$ is a normal subgroup of Π. Let $\rho\colon \Pi \to \mathrm{GL}_2(\mathbb{Z}_p[\sqrt[p]{p}])$ be the natural inclusion. There are no nontrivial $\mathbb{Z}_p$-algebra automorphisms of $\mathbb{Z}_p[\sqrt[p]{p}]$, but the pseudorepresentation (t,d) attached to ρ admits nontrivial generalized conjugate self-twists. Indeed, consider $\sigma\colon \mathbb{Z}[\sqrt[p]{p}] \to \mathbb{Z}[\sqrt[p]{p}]$ sending $\sqrt[p]{p}$ to $\zeta_p\sqrt[p]{p}$ and the character $\eta\colon \Pi \to \mathbb{Z}_p[\sqrt[p]{p}, \zeta_p]^\times$ with kernel $\mathrm{GL}_2(\mathbb{Z}_p)$ such that $\eta(\sqrt[p]{p}) = \zeta_p$. Then (σ,η) is a generalized conjugate self-twist of (t,d) that is not a conjugate self-twist. Furthermore, it is clear by inspection that $\mathbb{Z}_p$ is the largest ring with respect to which ρ is full.

If A is a domain and $(t,d)\colon \Pi \to A$ is a pseudorepresentation, let $\Sigma_t^{\mathrm{gen,pairs}}$ denote the group of all generalized conjugate-self twists of (t,d), with group law as in (2), and Σ_t^{gen} the image of the natural map $\Sigma_t^{\mathrm{gen,pairs}} \to \mathrm{Aut}(K^{\mathrm{sep}})$. Slightly abusing notation, we will write $K^{\Sigma_t^{\mathrm{gen}}}$ for the largest subfield of K pointwise fixed by all the elements of Σ_t^{gen}. That is, $K^{\Sigma_t^{\mathrm{gen}}} \coloneqq (K^{\mathrm{sep}})^{\Sigma_t^{\mathrm{gen}}}$.

3. Rings acting on Pink-Lie algebras

Throughout this section we fix a local pro-p ring A with residue field $\mathbb{F}$. Let $\overline{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation that is multiplicity-free over $\mathbb{F}$. Fix a pseudodeformation $(t,d)\colon \Pi \to A$ of $\overline{\rho}$, not necessarily admissible. Let A_0 be the subring of A topologically generated by $\{t(g)^2/d(g)\colon g \in \Pi\}$ and $\mathbb{E}$ the residue field of A_0. We let $\mathrm{ad}^0\colon \mathrm{GL}_2(K^{\mathrm{sep}}) \to \mathrm{GL}_3(K^{\mathrm{sep}})$ denote the representation of $\mathrm{GL}_2(K^{\mathrm{sep}})$ on $\mathfrak{sl}_2(K^{\mathrm{sep}}) \coloneqq \{x \in M_2(K^{\mathrm{sep}})\colon \operatorname{tr} x = 0\}$ by conjugation.

3.1. Conjugate self-twists and the adjoint representation.

In Section 3.1, we assume that A is a domain with field of fractions K. Fix a separable closure K^{sep} of K. The goal of Section 3.1 is to relate A_0 to the subfield of K^{sep} fixed by (generalized) conjugate self-twists. This is done in Proposition 3.1, but the main technical result that we need is Proposition 7.10, which can be found in the Appendix. In particular, we deduce a that $\mathbb{E} = \mathbb{F}^{\Sigma_{\overline{\rho}}}$.

Proposition 3.1. *The field of fractions of A_0 is equal to $K^{\Sigma_t^{\mathrm{gen}}}$. In particular, if $\chi\colon \Pi \to A^\times$ is a character, then $K^{\Sigma_t^{\mathrm{gen}}} = K^{\Sigma_{\chi t}^{\mathrm{gen}}}$.*

Proof. First we show that for all $g \in \Pi$, the element $t(g)^2/d(g)$ is fixed by every $\sigma \in \Sigma_t^{\mathrm{gen}}$. Indeed, for $(\sigma,\eta) \in \Sigma_t^{\mathrm{gen,pairs}}$, we have

$$\frac{{}^{\sigma}t(g)^2}{{}^{\sigma}d(g)} = \frac{\eta(g)^2 t(g)^2}{\eta(g)^2 d(g)} = \frac{t(g)^2}{d(g)}.$$

Thus if K_0 is the field of fractions of A_0, then $K_0 \subseteq (K^{\mathrm{sep}})^{\Sigma_t^{\mathrm{gen}}}$.

Choose a σ in $\mathrm{Aut}(K^{\mathrm{sep}})$ that fixes K_0 pointwise. We will show that $\sigma \in \Sigma_t^{\mathrm{gen}}$, and thus $K_0 = (K^{\mathrm{sep}})^{\Sigma_t^{\mathrm{gen}}}$. Since A is a domain, by Lemma 2.10 we can choose a (t,d)-representation

$\rho\colon \Pi \to \mathrm{GL}_2(K)$. Since σ fixes $t(g)^2/d(g)$ for all $g \in \Pi$, it follows from Lemma 7.3 that $\operatorname{tr}\operatorname{ad}^0 {}^\sigma\rho = \operatorname{tr}\operatorname{ad}^0 \rho$.

Let $g \in \Pi$. We claim that if $\operatorname{tr}\operatorname{ad}^0 {}^\sigma\rho(g) = \operatorname{tr}\operatorname{ad}^0 \rho(g)$, then $\operatorname{ad}^0 {}^\sigma\rho(g)$ and $\operatorname{ad}^0 \rho(g)$ have the same characteristic polynomials. (This is clear from $\operatorname{tr}\operatorname{ad}^0 {}^\sigma\rho = \operatorname{tr}\operatorname{ad}^0 \rho$ if the characteristic of K is not 3.) Write $\lambda(g), \mu(g) \in (K^{\mathrm{sep}})^\times$ for the eigenvalues of $\rho(g)$, and let $\alpha_g := \lambda(g)/\mu(g)$. Then the eigenvalues of $\operatorname{ad}^0 \rho(g)$ are $1, \alpha_g, \alpha_g^{-1}$. It follows that the eigenvalues of ${}^\sigma\rho(g)$ are $1, {}^\sigma\alpha_g, {}^\sigma\alpha_g^{-1}$. Hence the characteristic polynomial of $\operatorname{ad}^0 \rho(g)$ is

$$X^3 - (1 + \alpha_g + \alpha_g^{-1})X^2 + (1 + \alpha_g + \alpha_g^{-1})X - 1,$$

and the characteristic polynomial of $\operatorname{ad}^0 {}^\sigma\rho(g)$ is

$$X^3 - (1 + {}^\sigma\alpha_g + {}^\sigma\alpha_g^{-1})X^2 + (1 + {}^\sigma\alpha_g + {}^\sigma\alpha_g^{-1})X - 1.$$

We have already seen that $\operatorname{tr}\operatorname{ad}^0 {}^\sigma\rho(g) = \operatorname{tr}\operatorname{ad}^0 \rho(g)$, so ${}^\sigma\alpha_g + {}^\sigma\alpha_g^{-1} = \alpha_g + \alpha_g^{-1}$. Thus $\operatorname{ad}^0 {}^\sigma\rho(g)$ and $\operatorname{ad}^0 \rho(g)$ have the same characteristic polynomials.

Note that $\operatorname{ad}^0 \rho$ is semisimple since ρ is. By the Brauer-Nesbitt theorem, it follows that $\operatorname{ad}^0 {}^\sigma\rho \cong \operatorname{ad}^0 \rho$ over K^{sep}. By Proposition 7.10 there is a continuous character $\eta\colon \Pi \to (K^{\mathrm{sep}})^\times$ such that ${}^\sigma\rho \cong \eta \otimes \rho$. □

In particular, Proposition 3.1 implies that $\mathbb{E} = \mathbb{F}^{\Sigma_{\overline{\rho}}}$. We shall often make use of this characterization of $\mathbb{E}$ in the rest of the paper. Furthermore, we can use Proposition 3.1 to show that A_0-fullness can be deduced from fullness with respect to a certain ring fixed by conjugate self-twists.

Corollary 3.2. *Let $\chi\colon \Pi \to A^\times$ be a continuous character such that $(t', d') := (\chi t, \chi^2 d)$ has constant determinant. Let A' be the trace algebra of (t', d'). If (t', d') is $(A')^{\Sigma_{t'}}$-full, then (t, d) is A_0-full.*

Proof. Note that $A_0 \subseteq (A')^{\Sigma_{t'}}$ and that the two rings have the same field of fractions by Proposition 3.1. By Lemma 2.17, it follows that (t', d') is A_0-full. The corollary then follows from Lemma 2.25. □

3.2. **$L_2(\rho)$ is a $W(\mathbb{E})$-module.** For the rest of Section 3 we fix an admissible pseudodeformation $(\Pi, \overline{\rho}, t, d)$ over A. Recall that, a priori, Pink's construction only gives Lie algebras that are $\mathbb{Z}_p$-modules. The goal of Section 3.2 is to show that if ρ is a (t, d)-representation, then in fact its associated Lie algebras are modules over $W(\mathbb{E})$ (Proposition 3.4). Although this is a minor improvement on $\mathbb{Z}_p$ (indeed, it may be no improvement at all if $W(\mathbb{E}) = \mathbb{Z}_p$), it is an essential input for proving the results of Section 5.

We assume throughout Section 3.2 that the eigenvalues of $\overline{\rho}(g)$ are in $\mathbb{F}^\times$ for all $g \in \Pi$. This requires at most replacing $\mathbb{F}$ by its unique quadratic extension.

Let $\lambda \neq \mu \in \mathbb{F}^\times$ be the eigenvalues of a matrix in $\operatorname{Im}\overline{\rho}$. By Lemma 2.13, there is a (t, d)-representation $\rho_{\lambda,\mu}\colon \Pi \to R_{\lambda,\mu}^\times$ and $g_{\lambda,\mu} \in \Pi$ such that

$$\rho_{\lambda,\mu}(g_{\lambda,\mu}) = \begin{pmatrix} s(\lambda) & 0 \\ 0 & s(\mu) \end{pmatrix}.$$

Recall that $G_{\rho_{\lambda,\mu}} := \operatorname{Im}\rho_{\lambda,\mu}, \Gamma_{\rho_{\lambda,\mu}} := G_{\rho_{\lambda,\mu}} \cap SR^1_{\lambda,\mu}$, and $L_n(\rho_{\lambda,\mu}) := L_n(\Gamma_{\rho_{\lambda,\mu}})$. Since $\lambda \neq \mu$, the Lie algebra $L_1(\rho_{\lambda,\mu})$ is decomposable [Bel18, Corollary 6.2.2].

Lemma 3.3. *With the notation introduced at the beginning of Section 2.6, we have*
(1) $\nabla_1(\rho_{\lambda,\mu})$ and $L_2(\rho_{\lambda,\mu})$ are $W(\mathbb{F}_p(\lambda\mu^{-1} + \lambda^{-1}\mu))$-modules;
(2) if $L_1(\rho_{\lambda,\mu})$ is strongly decomposable, then $B_1(\rho_{\lambda,\mu}), C_1(\rho_{\lambda,\mu})$ are $W(\mathbb{F}_p(\lambda\mu^{-1} + \lambda^{-1}\mu))$-modules;
(3) if the projective image of $\overline{\rho}$ contains $\mathrm{PSL}_2(\mathbb{F}_p)$ and $p \geq 7$, then $I_1(\rho_{\lambda,\mu})$ is a $W(\mathbb{F}_p(\lambda\mu^{-1} + \lambda^{-1}\mu))$-module; after possibly replacing $\rho_{\lambda,\mu}$ with its conjugate by a certain $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ with $a \in A^\times$, one has that $L_1(\rho_{\lambda,\mu})$ is a $W(\mathbb{F}_p(\lambda\mu^{-1} + \lambda^{-1}\mu))$-module.

Proof. Note that

$$L_2(\rho_{\lambda,\mu}) = \left[I_1(\rho_{\lambda,\mu})\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right), \nabla_1(\rho_{\lambda,\mu})\right] + [\nabla_1(\rho_{\lambda,\mu}), \nabla_1(\rho_{\lambda,\mu})].$$

Furthermore, if $L_1(\rho_{\lambda,\mu})$ is strongly decomposable, then $\nabla_1(\rho_{\lambda,\mu}) \cong B_1(\rho_{\lambda,\mu}) \oplus C_1(\rho_{\lambda,\mu})$. Therefore the first statement implies the second, and it suffices to show that $\nabla(\rho_{\lambda,\mu})$ is a $W(\mathbb{F}_p(\lambda\mu^{-1}+\lambda^{-1}\mu))$-module.

To prove that

$$(s(\lambda)s(\mu)^{-1} + s(\lambda)^{-1}s(\mu))\nabla_1(\rho_{\lambda,\mu}) \subseteq \nabla_1(\rho_{\lambda,\mu}),$$

recall that $L_1(\rho_{\lambda,\mu})$ is closed under conjugation by $G_{\rho_{\lambda,\mu}}$ (in fact, by any element in the normalizer of $\Gamma_{\rho_{\lambda,\mu}}$). In particular, it is closed under conjugation by $\left(\begin{smallmatrix}s(\lambda)&0\\0&s(\mu)\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}s(\lambda)^{-1}&0\\0&s(\mu)^{-1}\end{smallmatrix}\right)$. Using this, a short matrix calculation shows that if $\left(\begin{smallmatrix}0&b\\c&0\end{smallmatrix}\right) \in \nabla_1(\rho_{\lambda,\mu})$ then

$$\left(\begin{smallmatrix}0&s(\lambda)s(\mu)^{-1}b\\s(\lambda)^{-1}s(\mu)c&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&s(\lambda)^{-1}s(\mu)b\\s(\lambda)s(\mu)^{-1}c&0\end{smallmatrix}\right) \in \nabla_1(\rho_{\lambda,\mu}).$$

Therefore $(s(\lambda)s(\mu)^{-1} + s(\lambda)^{-1}s(\mu))\nabla_1(\rho_{\lambda,\mu}) \subseteq \nabla_1(\rho_{\lambda,\mu})$.

Finally, if the projective image of $\overline{\rho}$ contains $\mathrm{PSL}_2(\mathbb{F}_p)$ and $p \geq 7$, then by Theorem 2.31, up to replacing $\rho_{\lambda,\mu}$ with its conjugate by a certain $\left(\begin{smallmatrix}1&0\\0&a\end{smallmatrix}\right)$ with $a \in A^\times$, we have

$$L_1(\rho_{\lambda,\mu}) = \left(\begin{smallmatrix}I_1(\rho_{\lambda,\mu})&I_1(\rho_{\lambda,\mu})\\I_1(\rho_{\lambda,\mu})&I_1(\rho_{\lambda,\mu})\end{smallmatrix}\right)^0,$$

and thus $B_1(\rho_{\lambda,\mu}) = I_1(\rho_{\lambda,\mu}) = C_1(\rho_{\lambda,\mu})$. (Note that conjugation by $\left(\begin{smallmatrix}1&0\\0&a\end{smallmatrix}\right)$ with $a \in A^\times$ does not change $I_1(\rho_{\lambda,\mu})$.) In particular, $L_1(\rho_{\lambda,\mu})$ is strongly decomposable. By the second statement of the lemma, we see that $I_1(\rho_{\lambda,\mu})$ is a $W(\mathbb{F}_p(\lambda\mu^{-1} + \lambda^{-1}\mu))$-module. The above description of $L_1(\rho_{\lambda,\mu})$ shows that it is also a $W(\mathbb{F}_p(\lambda\mu^{-1} + \lambda^{-1}\mu))$-module. ☐

Proposition 3.4. *Let $\rho\colon \Pi \to R^\times$ be a (t,d)-representation. Then $L_n(\rho)$ is a $W(\mathbb{E})$-module for all $n \geq 2$. If the projective image of $\overline{\rho}$ contains $\mathrm{PSL}_2(\mathbb{F}_p)$ and $p \geq 7$, then $L_1(\rho)$ is a $W(\mathbb{E})$-module.*

Proof. By (1) in Section 2.5, it suffices to show that $L_n(\rho)$ is closed under multiplication by $s(\lambda)s(\mu)^{-1} + s(\lambda)^{-1}s(\mu)$ for all $\lambda,\mu \in \overline{\mathbb{F}}_p^\times$ that are distinct eigenvalues of an element in $\mathrm{Im}\,\overline{\rho}$. Fix such λ,μ. Let $\rho_{\lambda,\mu}\colon \Pi \to R_{\lambda,\mu}^\times$ be the (t,d)-representation over A described prior to Lemma 3.3. Let us assume furthermore that, in the case when $\overline{\rho}$ is not projectively cyclic or dihedral, that we have already replaced $\rho_{\lambda,\mu}$ by its relevant diagonal conjugate so that the description of $W(\mathbb{F}_q)L_1(\rho_{\lambda,\mu})$ from Theorem 2.31 applies to $\rho_{\lambda,\mu}$.

Since $\rho\colon \Pi \to R^\times$ and $\rho_{\lambda,\mu}\colon \Pi \to R_{\lambda,\mu}^\times$ are both (t,d)-representations over A, it follows from Proposition 2.11 that there is a unique A-algebra isomorphism $\Psi\colon R_{\lambda,\mu} \to R$ such that $\rho = \Psi \circ \rho_{\lambda,\mu}$. We claim that this implies that $L_n(\rho) = \Psi(L_n(\lambda,\mu))$ for all $n \geq 1$. If this is true, then $L_2(\rho)$ is closed under multiplication by $s(\lambda)s(\mu)^{-1} + s(\lambda)^{-1}s(\mu) \in A$ since $L_2(\rho_{\lambda,\mu})$ is by Lemma 3.3 and Ψ is an A-algebra homomorphism. Since $L_2(\rho)$ is a $W(\mathbb{E})$-module, it follows immediately from the definition that $L_n(\rho)$ is a $W(\mathbb{E})$-module for all $n \geq 2$. Furthermore, the argument in this paragraph applies to $L_1(\rho)$ under the assumption that the projective image of $\overline{\rho}$ contains $\mathrm{PSL}_2(\mathbb{F}_p)$ for $p \geq 7$.

To see that $L_n(\rho) = \Psi(L_n(\rho_{\lambda,\mu}))$, note that $G_\rho = G_{\Psi\circ\rho_{\lambda,\mu}} = \Psi(G_{\rho_{\lambda,\mu}})$. Since Ψ is an algebra morphism, it follows that $\Psi(\mathrm{rad}\,R_{\lambda,\mu}) = \mathrm{rad}\,R$. Furthermore, since ρ and $\rho_{\lambda,\mu}$ are both (t,d)-representations, it follows that Ψ preserves determinants. Therefore $\Psi(SR^1_{\lambda,\mu}) \supset SR^1$. Since Ψ is a continuous algebra homomorphism, it follows directly from the definition of Θ that $\Psi(L_1(\rho_{\lambda,\mu})) = L_1(\rho)$ and hence $\Psi(L_n(\rho_{\lambda,\mu})) = L_n(\rho)$ for all $n \geq 1$. ☐

3.3. **$L_2(\rho)$ is a $W(\mathbb{E})[I_1(\rho)]$-module.** In Section 3.3 we use Bellaïche's work to show that, for any well-adapted (t,d)-representation ρ, $L_n(\rho)$ is a module over a ring comparable to A. This is the key input into Corollary 3.8, which is our improvement on Bellaïche's fullness theorem.

Proposition 3.5. *Let ρ be a (t,d)-representation adapted to (g_0,λ_0,μ_0). Then*

(1) $L_2(\rho)$ is a module over $W(\mathbb{E})[I_1(\rho)^2] := W(\mathbb{E}) + W(\mathbb{E})I_1(\rho)^2$;
(2) if $n \geq 1$ and $L_n(\rho)$ is strongly decomposable, then $L_{n+1}(\rho)$ is a module over $W(\mathbb{E})[I_1(\rho)]$;
(3) if the projective image of $\overline{\rho}$ contains $\mathrm{PSL}_2(\mathbb{F}_p)$ for $p \geq 7$, then up to replacing ρ with its conjugate by some $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$ with $a \in A^\times$, $L_1(\rho)$ is a module over $W(\mathbb{E})[I_1(\rho)]$.

Proof. Since ρ is adapted to (g_0, λ_0, μ_0), it follows that $L_1(\rho)$ is decomposable [Bel18, Corollary 6.2.2]. Note that $[I_1(\rho)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \nabla_1(\rho)] \subset \nabla_1(\rho)$ since $L_1(\rho)$ is a Lie algebra. That is, for all $a \in I_1(\rho)$ and $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \in \nabla_1(\rho)$, we have $2a\left(\begin{smallmatrix} 0 & b \\ -c & 0 \end{smallmatrix}\right) \in \nabla_1(\rho)$. To prove the first statement, we can apply this fact a second time to $\alpha \in I_1(\rho)$ and $2a\left(\begin{smallmatrix} 0 & b \\ -c & 0 \end{smallmatrix}\right)$ to see that $4a\alpha\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \in \nabla_1(\rho)$. Therefore $\nabla_1(\rho)$ is closed under multiplication by $I_1(\rho)^2$. Since

$$L_2(\rho) = \left[I_1(\rho)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \nabla_1(\rho)\right] + [\nabla_1(\rho), \nabla_1(\rho)],$$

we see that $L_2(\rho)$ is closed under multiplication by $I_1(\rho)^2$.

For the second statement, if $L_n(\rho)$ is strongly decomposable, then we can write

$$L_n(\rho) = I_n(\rho)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \oplus B_n(\rho)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \oplus C_n(\rho)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right).$$

By calculating $\left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right]$ and $\left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\right]$, we find that $I_1(\rho)B_n(\rho) = B_{n+1}(\rho) \subset B_n(\rho)$ and $I_1(\rho)C_n(\rho) = C_{n+1}(\rho) \subset C_n(\rho)$. Therefore $B_n(\rho), C_n(\rho)$ are closed under multiplication by $I_1(\rho)$. Since

$$L_{n+1}(\rho) = [B_n(\rho)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), C_n(\rho)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)] + [I_1(\rho)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), B_n(\rho)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)] + [I_1(\rho)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), C_n(\rho)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)],$$

it follows that $L_{n+1}(\rho)$ is closed under multiplication by $I_1(\rho)$.

The first two results now follow from Proposition 3.4. The last statement follows from Theorem 2.31 and Proposition 3.4. □

Remark 3.6. It would be nice to remove the assumption that $L_1(\rho)$ is strongly decomposable and still conclude that $L_2(\rho)$ is a $W(\mathbb{E})[I_1(\rho)]$-module, but we do not see a way to do this.

3.4. **Regularity implies $W(\mathbb{E})[I_1(\rho)]$-fullness.** The goal of Section 3.4 is to establish a slightly stronger version of [Bel18, Theorem 7.2.3], which is Bellaïche's Theorem 1.3 of the introduction. We do so in Corollary 3.8 below. Our result is different from that of Bellaïche mainly in that we can weaken his definition of regularity and enlarge his ring $\mathbb{Z}_p[I_1(\rho)]$ to $W(\mathbb{E})[I_1(\rho)]$.

Throughout Section 3.4 the ring A will be a local pro-p domain with residue field $\mathbb{F}$ and field of fractions K. We fix an admissible pseudodeformation $(\Pi, \overline{\rho}, t, d)$ over A throughout this section. If ρ is a (t, d)-representation that is adapted to some (g_0, λ_0, μ_0), then $L_1(\rho)$ is decomposable by [Bel18, Corollary 6.2.2]. Thus $I_1(\rho)$ is defined. We write K_1 for the field of fractions of $W(\mathbb{E})[I_1(\rho)]$.

Proposition 3.7. *Assume that $\overline{\rho}$ is regular. Let $\rho\colon \Pi \to R^\times$ be a (t, d)-representation adapted to (g_0, λ_0, μ_0) for a regular element g_0 such that $\rho(g_0) = \begin{pmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{pmatrix}$. If $B_1(\rho), C_1(\rho) \neq 0$, then ρ is $W(\mathbb{E})[I_1(\rho)]$-full.*

Proof. It is easy to see that the eigenvalues of $\rho(g_0) = \begin{pmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{pmatrix}$ acting on $L_n(\rho)$ by conjugation are $1, s(\lambda_0)s(\mu_0^{-1}), s(\lambda_0^{-1})s(\mu_0)$, which are distinct elements of $W(\mathbb{E})^\times$ since g_0 is a regular element. Since $L_n(\rho)$ is a $W(\mathbb{E})$-module for $n \geq 2$ by Proposition 3.4, it follows that $L_n(\rho)$ is the direct sum of the eigenspaces for the conjugation action of $\rho(g_0)$. Thus, $L_n(\rho)$ is strongly decomposable for $n \geq 2$. By Proposition 3.5, it follows that $L_n(\rho)$ is an $W(\mathbb{E})[I_1(\rho)]$-module for $n \geq 3$.

Since A is a domain, we may view R inside of $M_2(K)$ by Lemma 2.10. Note that if $B_1(\rho), C_1(\rho) \neq 0$, then since $I_n(\rho), B_n(\rho), C_n(\rho) \subset K$, it follows that $I_n(\rho), B_n(\rho)$, and $C_n(\rho)$ are nonzero for all $n \geq 1$. In particular, $I_3(\rho), B_3(\rho), C_3(\rho)$ are nonzero $W(\mathbb{E})[I_1(\rho)]$-modules.

Define

$$R_1 := \begin{pmatrix} W(\mathbb{E})[I_1(\rho)] & B_3(\rho) \\ C_3(\rho) & W(\mathbb{E})[I_1(\rho)] \end{pmatrix}.$$

Then R_1 is a faithful GMA over $W(\mathbb{E})[I_1(\rho)]$. By the proof of [Bel18, Lemma 2.2.2], if $0 \neq b_0 \in B_3(\rho)$ and $x = \left(\begin{smallmatrix} 1 & 0 \\ 0 & b_0 \end{smallmatrix}\right)$, it follows that $xR_1x^{-1} \subseteq \mathrm{GL}_2(K_1)$. Thus, by replacing ρ with $x\rho x^{-1}$, which is still a (t,d)-representation adapted to (g_0, λ_0, μ_0) that sends g_0 to $\begin{pmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{pmatrix}$, we may assume that $B_3(\rho), C_3(\rho) \subseteq K_1$. (Note that $I_1(\rho) = I_1(x\rho x^{-1})$.)

Note that any nonzero $W(\mathbb{E})[I_1(\rho)]$-submodule of K_1 contains a nonzero element of $W(\mathbb{E})[I_1(\rho)]$ and thus contains a non-zero $W(\mathbb{E})[I_1(\rho)]$-ideal. Therefore there exists a nonzero $W(\mathbb{E})[I_1(\rho)]$-ideal $\mathfrak{a}$ contained in $I_3(\rho) \cap B_3(\rho) \cap C_3(\rho)$. Hence $\mathfrak{sl}_2(\mathfrak{a}) \subseteq L_3(\rho)$. Using Theorem 2.20 we deduce that $\Gamma_{W(\mathbb{E})[I_1(\rho)]}(\mathfrak{a}) \subset \operatorname{Im}\rho$ and ρ is $W(\mathbb{E})[I_1(\rho)]$-full. □

Corollary 3.8. *Assume that $\overline{\rho}$ is regular. Let ρ be a well-adapted (t,d)-representation adapted to (g_0, λ_0, μ_0) for a regular element g_0 such that $\rho(g_0) = \left(\begin{smallmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{smallmatrix}\right)$. If (t,d) is not a priori small, then (t,d) is $W(\mathbb{E})[I_1(\rho)]$-full.*

Proof. By Proposition 3.7 it suffices to show that $B_1(\rho), C_1(\rho) \neq 0$. We do this by analyzing the different possibilities for $\overline{\rho}$. By Lemma 7.5, we see that either $\overline{\rho}$ is reducible, dihedral, or $\mathrm{ad}^0\,\overline{\rho}$ is irreducible. The proof is easiest in the last case. Indeed, the assumption that $t \neq s(t)$ implies that $\Gamma \coloneqq \Gamma_\rho$ is not trivial. Recall that Γ is equipped with a filtration

$$\Gamma_n \coloneqq \Gamma \cap \Gamma_A(\mathfrak{m}^n)$$

such that $\Gamma_n/\Gamma_{n+1} \hookrightarrow \mathfrak{sl}_2(\mathbb{F})$. This embedding is equivariant with respect to the adjoint action of $\overline{G}$. Since Γ is nontrivial, there is some n such that Γ_n/Γ_{n+1} contains a nontrivial element. Since $\mathrm{ad}^0\,\overline{\rho}$ is irreducible, it follows that we must have $\Gamma_n/\Gamma_{n+1} \cong \mathfrak{sl}_2(\mathbb{F})$. In particular, $B_1(\rho), C_1(\rho) \neq 0$.

Now suppose that $\overline{\rho}$ is reducible. Since ρ is well adapted by assumption, it follows that ρ is adapted to (g_0, λ_0, μ_0), where $\overline{\rho}(g_0)$ generates the projective image of $\overline{\rho}$. In particular, ρ is automatically adapted to a regular element. Suppose for contradiction that $C_1(\rho) = 0$ (respectively, $B_1(\rho) = 0$). Then Γ_ρ is contained in the upper (respectively, lower) triangular matrices. By [Bel18, Theorem 6.2.1], we know that $s(\overline{G}) \subset G_\rho$ since ρ is well adapted. Thus $G_\rho = s(\overline{G})\Gamma_\rho$. But then G_ρ is contained in the upper (respectively, lower) triangular matrices, and hence ρ is reducible. Therefore t is the sum of two continuous characters $\Pi \to A^\times$, a contradiction. Thus $B_1(\rho), C_1(\rho) \neq 0$ if $\overline{\rho}$ is reducible.

Finally suppose that $\overline{\rho}$ is dihedral. By Lemma 7.7 there is a unique subgroup Π_0 of index 2 in Π such that $\overline{\rho} \cong \mathrm{Ind}_{\Pi_0}^{\Pi}\chi$ for some character $\chi\colon \Pi_0 \to \mathbb{F}^\times$. Applying the reducible case to $\overline{\rho}|_{\Pi_0}$, we see that either $\rho|_{\Pi_0}$ is reducible or $B_1(\rho|_{\Pi_0}), C_1(\rho|_{\Pi_0}) \neq 0$. The first possibility is not allowed by hypothesis, so we must have $B_1(\rho|_{\Pi_0}), C_1(\rho|_{\Pi_0}) \neq 0$. But $B_1(\rho|_{\Pi_0}) \subseteq B_1(\rho)$ and $C_1(\rho|_{\Pi_0}) \subseteq C_1(\rho)$, which proves the desired result when $\overline{\rho}$ is dihedral. □

Remark 3.9. Although fullness with respect to a particular ring is a property of the pseudorepresentation (t,d) (as discussed in Remark 2.15), Corollary 3.8 does not imply that any (t,d)-representation ρ is $W(\mathbb{E})[I_1(\rho)]$-full. We just know that there exists one (call it ρ), and any other (t,d)-representation ρ' is $W(\mathbb{E})[I_1(\rho)]$-full. The point is that the ring $W(\mathbb{E})[I_1(\rho)]$ depends on the choice of ρ. Part of the advantage of relating $W(\mathbb{E})[I_1(\rho)]$ to A^{Σ_t} and A_0 (which will be done in Section 5) is that the latter rings can be defined in terms of (t,d), independent of a choice of ρ.

4. Conjugate self-twists

In Section we analyze conjugate self-twists of admissible pseudodeformations. Section 4.1 is devoted to understanding conjugate self-twists of $\overline{\rho}$. In Section 4.2 we introduce the notion of a good octahedral representation and explain how to choose a good basis for $\overline{\rho}$ that will be used in Section 5. In Section 4.3 we show that any conjugate self-twist of an admissible pseudodeformation lifts to a conjugate self-twist of the universal constant-determinant pseudorepresentation $(T,d)\colon \Pi \to \mathcal{A}$

defined in Section 2.1. Finally, Section 4.4 contains a lemma concerning residually trivial conjugate self-twists.

4.1. **Residual conjugate self-twists.** Fix a representation $\overline{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ that is multiplicity free over $\mathbb{F}$. It will be important to have a good understanding of the group $\Sigma_{\overline{\rho}}^{\mathrm{pairs}}$ of conjugate self-twists of $\overline{\rho}$. Indeed, we will see in Section 4.3 that $\Sigma_{\overline{\rho}}^{\mathrm{pairs}}$ controls the conjugate self-twists of any constant-determinant pseudodeformation of $\overline{\rho}$.

We begin by studying $\Sigma_{\overline{\rho}}^{\mathrm{di}}$ and use that to show that $\Sigma_{\overline{\rho}}^{\mathrm{pairs}}$ is finite.

Lemma 4.1.

(1) If the projective image of $\overline{\rho}$ is not dihedral or cyclic of order 2, then $\Sigma_{\overline{\rho}}^{\mathrm{di}}$ is trivial.

(2) If the projective image of $\overline{\rho}$ is either a nonabelian dihedral group or has order 2, then $\Sigma_{\overline{\rho}}^{\mathrm{di}}$ has order 2.

(3) If the projective image of $\overline{\rho}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, then $\Sigma_{\overline{\rho}}^{\mathrm{di}}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

Proof. Recall that $\mathbb{P}\colon \mathrm{GL}_2(\mathbb{F}) \to \mathrm{PGL}_2(\mathbb{F})$ denotes the natural projection. We claim that if $\overline{\rho}$ is irreducible, then the following sets are in bijection:

(a) $\Sigma_{\overline{\rho}}^{\mathrm{di}} \setminus \{(1,1)\}$;
(b) subgroups Π_0 of Π such that $[\Pi\colon \Pi_0] = 2$ and $\overline{\rho}(\Pi_0)$ is abelian;
(c) subgroups H of $\mathbb{P}\overline{\rho}(\Pi)$ such that $[\mathbb{P}\overline{\rho}(\Pi)\colon H] = 2$ and H is abelian.

Indeed, the maps between them can be described as follows. Given $(1,\eta) \in \Sigma_{\overline{\rho}}^{\mathrm{di}} \setminus \{(1,1)\}$, let $\Pi_0 \coloneqq \ker \eta$. The fact that $[\Pi\colon \Pi_0] = 2$ follows from Lemma 7.7, and Lemma 7.6 shows that $\overline{\rho}(\Pi_0)$ is abelian. Conversely, given Π_0 as in (b), let $\eta_{\Pi_0}\colon \Pi \to \Pi/\Pi_0 \cong \{\pm 1\}$ be the natural projection. Note that $\overline{\rho}|_{\Pi_0}$ is reducible since $\overline{\rho}(\Pi_0)$ is abelian. Let $\chi\colon \Pi_0 \to \mathbb{F}^\times$ be a constituent of $\overline{\rho}|_{\Pi_0}$. Then $\overline{\rho} \cong \mathrm{Ind}_{\Pi_0}^{\Pi} \chi$ by Frobenius reciprocity since $\overline{\rho}$ is irreducible. Thus $(1, \eta_{\Pi_0}) \in \Sigma_{\overline{\rho}}^{\mathrm{di}} \setminus \{(1,1)\}$ by Lemma 7.7.

Given Π_0 as in (b), let $H \coloneqq \mathbb{P}\overline{\rho}(\Pi_0)$. Given H as in (c), let $\Pi_0 \coloneqq \mathbb{P}\overline{\rho}^{-1}(H)$. It is clear that $[\Pi\colon \Pi_0] = 2$. That $\overline{\rho}(\Pi_0)$ is abelian follows from the fact that H is abelian and scalar matrices commute with everything.

When $\overline{\rho}$ is irreducible, the lemma now follows from counting subgroups as in (c) in each of the possible projective images of $\overline{\rho}$. (The fact that elements in $\Sigma_{\overline{\rho}}^{\mathrm{di}}$ have order at most 2 follows from the fact that $\det \overline{\rho} = \eta^2 \det \overline{\rho}$ and so $\eta^2 = 1$.)

Finally, suppose that $\overline{\rho} = \varepsilon \oplus \delta$. If $(1,\eta) \in \Sigma_{\overline{\rho}}^{\mathrm{di}}$ and η is nontrivial, then we must have $\eta\varepsilon = \delta$ and $\eta\delta = \varepsilon$. Thus

$$\varepsilon\delta^{-1} = \eta = \delta\varepsilon^{-1},$$

which implies that $\varepsilon\delta^{-1}$ has order 2. But the projective image of $\overline{\rho}$ is isomorphic to the image of $\varepsilon\delta^{-1}$. Thus $\Sigma_{\overline{\rho}}^{\mathrm{di}}$ is trivial unless the projective image of $\overline{\rho}$ has order 2, in which case there is one nontrivial element. ☐

Corollary 4.2. *If $\overline{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ is a multiplicity-free representation, then $\Sigma_{\overline{\rho}}^{\mathrm{pairs}}$ is finite.*

Proof. Since $\mathbb{F}$ is a finite field, there are only finitely many automorphisms of $\mathbb{F}$. Fix $\sigma \in \Sigma_{\overline{\rho}}$. We need to show that there are only finitely many characters $\eta\colon \Pi \to \mathbb{F}^\times$ such that $(\sigma,\eta) \in \Sigma_{\overline{\rho}}^{\mathrm{pairs}}$. But if $(\sigma,\eta_1), (\sigma,\eta_2) \in \Sigma_{\overline{\rho}}^{\mathrm{pairs}}$, then $(\mathrm{id}, \eta_1\eta_2^{-1}) \in \Sigma_{\overline{\rho}}^{\mathrm{di}}$, and $\Sigma_{\overline{\rho}}^{\mathrm{di}}$ is finite by Lemma 4.1. ☐

Recall that $\mathbb{E}$ is the subfield of $\mathbb{F}$ generated by $\{(\mathrm{tr}\,\overline{\rho}(g))^2/\det \overline{\rho}(g)\colon g \in \Pi\}$, and $\overline{\rho}$ is *regular* if $\mathrm{Im}\,\overline{\rho}$ contains an element with eigenvalues $\lambda_0, \mu_0 \in \overline{\mathbb{F}}^\times$ such that $\lambda_0\mu_0^{-1} \in \mathbb{E}^\times \setminus \{\pm 1\}$ (Definition 2.27). We now show that if $\overline{\rho}$ is reducible and regular, then one can eliminate the conjugate self-twists of $\overline{\rho}$ by twisting $\overline{\rho}$ by a character.

Lemma 4.3. *Suppose that $\overline{\rho} = \varepsilon \oplus \delta$ and $\overline{\rho}$ is regular. If $(\sigma, \eta) \in \Sigma_{\overline{\rho}}^{\mathrm{pairs}}$, then ${}^{\sigma}\varepsilon = \eta\varepsilon$ and ${}^{\sigma}\delta = \eta\delta$. In particular, $\varepsilon\delta^{-1}$ takes values in $\mathbb{E}$.*

Proof. It suffices to show that if $\overline{\rho}$ is regular then we cannot have ${}^{\sigma}\varepsilon = \eta\delta$ and ${}^{\sigma}\delta = \eta\varepsilon$. If this were true, then we would have ${}^{\sigma}\varepsilon\delta^{-1} = \eta = {}^{\sigma}\delta\varepsilon^{-1}$, which implies that

$$(3) \qquad {}^{\sigma}(\varepsilon\delta^{-1}) = \delta\varepsilon^{-1}.$$

Since $\overline{\rho}$ is regular, there is some $g \in \Pi$ such that $\varepsilon(g)\delta(g)^{-1} \in \mathbb{E} \setminus \{\pm 1\}$. As $\mathbb{E}$ is fixed by σ by Proposition 3.1, it follows from (3) that $\varepsilon(g)\delta(g)^{-1} = \pm 1$, a contradiction.

The last sentence in the statement of the lemma follows from the fact that, for any $\sigma \in \Sigma_{\overline{\rho}} = \mathrm{Gal}(\mathbb{F}/\mathbb{E})$, we have ${}^{\sigma}\varepsilon\varepsilon^{-1} = \eta = {}^{\sigma}\delta\delta^{-1}$ and hence $\varepsilon\delta^{-1}$ is fixed by $\mathrm{Gal}(\mathbb{F}/\mathbb{E})$. ☐

Corollary 4.4. *Suppose $\overline{\rho} = \varepsilon \oplus \delta$ and $\overline{\rho}$ is regular. Then $\overline{\rho}' := \overline{\rho} \otimes \delta^{-1}$ has no conjugate self-twists.*

Proof. Since $\overline{\rho}$ is regular, its projective image cannot have order 2. Therefore it suffices to show that $\Sigma_{\overline{\rho}'}$ is trivial by Lemma 4.1. Let $\mathbb{F}'$ be the extension of $\mathbb{F}_p$ generated by the trace of $\overline{\rho}'$. Then $\Sigma_{\overline{\rho}'} = \mathrm{Gal}(\mathbb{F}'/\mathbb{E})$, so it suffices to show that $\mathbb{F}' \subseteq \mathbb{E}$. But $\mathbb{F}'$ is generated by the values of $\varepsilon\delta^{-1}$, which takes values in $\mathbb{E}$ by Lemma 4.3. ☐

Define

$$(4) \qquad \Pi_0(\overline{\rho}) := \bigcap_{(\sigma,\eta)\in\Sigma_{\overline{\rho}}^{\mathrm{pairs}}} \ker \eta.$$

Note that $\Pi_0(\overline{\rho})$ is sensitive to twisting $\overline{\rho}$ by a character since this operation can change $\Sigma_{\overline{\rho}}^{\mathrm{pairs}}$. In Section 5 it will be important to understand what happens to $\overline{\rho}$ when it is restricted to $\Pi_0(\overline{\rho})$, especially when $\overline{\rho}$ is absolutely irreducible and regular. We investigate these properties now.

Lemma 4.5. *Assume that $\overline{\rho}$ is exceptional or large. If the order of $\det \overline{\rho}$ is a power of 2, then $\overline{\rho}|_{\Pi_0(\overline{\rho})}$ is absolutely irreducible.*

Proof. If $(\sigma, \eta) \in \Sigma_{\overline{\rho}}^{\mathrm{pairs}}$, then η^2 is equal to a power of $\det \overline{\rho}$. Thus it follows from the hypothesis in the lemma that the order of η is a power of 2. Hence $[\Pi\colon \Pi_0(\overline{\rho})]$ is a power of 2 and $\Pi/\Pi_0(\overline{\rho})$ is abelian.

By hypothesis, the projective image of $\overline{\rho}$ is isomorphic to one of $A_4, S_4, A_5, \mathrm{PSL}_2(\mathbb{E}), \mathrm{PGL}_2(\mathbb{E})$. None of $A_4, A_5, \mathrm{PSL}_2(\mathbb{E})$ has a subgroup of 2-power index with abelian quotient. Both S_4 and $\mathrm{PGL}_2(\mathbb{E})$ have a unique proper 2-power index subgroup with abelian quotient, namely A_4 and $\mathrm{PSL}_2(\mathbb{E})$, respectively. Therefore the projective image of $\overline{\rho}|_{\Pi_0(\overline{\rho})}$ is one of $A_4, S_4, A_5, \mathrm{PSL}_2(\mathbb{E}), \mathrm{PGL}_2(\mathbb{E})$. It follows that $\overline{\rho}|_{\Pi_0(\overline{\rho})}$ is absolutely irreducible. ☐

Note that by Lemma 2.3 we may always twist $\overline{\rho}$ by a character to assume that the order of $\det \overline{\rho}$ is a power of 2.

It remains to treat the case when $\overline{\rho}$ is dihedral.

Proposition 4.6. *Assume that $\overline{\rho}$ is regular dihedral, say $\overline{\rho} = \mathrm{Ind}_{\Pi_0}^{\Pi} \chi$. Then $\overline{\rho}|_{\Pi_0(\overline{\rho})}$ is multiplicity free over $\mathbb{E}$. Furthermore, given $g \in \Pi_0$, we have $g \in \Pi_0(\overline{\rho})$ if and only if $\chi(g) \in \mathbb{E}^{\times}$.*

Proof. Since $\overline{\rho}$ is regular, it follows from Lemma 7.7 that there is a unique subgroup Π_0 of Π of index 2 such that $\overline{\rho} \cong \mathrm{Ind}_{\Pi_0}^{\Pi} \chi$ for some character $\chi\colon \Pi_0 \to \mathbb{F}^{\times}$. For any $h \in \Pi$, define $\chi^h\colon \Pi_0 \to \mathbb{F}^{\times}$ by $\chi^h(g) := \chi(h^{-1}gh)$. The character χ^h only depends on the class of h in Π/Π_0. Fix an element $c \in \Pi \setminus \Pi_0$. Fix a generator $\sigma \in \Sigma_{\overline{\rho}} = \mathrm{Gal}(\mathbb{F}/\mathbb{E})$, and choose η such that $(\sigma, \eta) \in \Sigma_{\overline{\rho}}^{\mathrm{pairs}}$. (Note that there are two choices for η, and they differ by the character $\eta_0\colon \Pi \twoheadrightarrow \Pi/\Pi_0 \cong \{\pm 1\}$.) Then $\Pi_0(\overline{\rho}) = \ker \eta_0 \cap \ker \eta$ since σ generates $\Sigma_{\overline{\rho}}$. Therefore $\Pi_0(\overline{\rho}) = \ker \eta|_{\Pi_0}$.

Note that any regular element for $\overline{\rho}$ must be in Π_0 since elements in $\Pi \setminus \Pi_0$ have projective order 2. By applying Lemma 4.3 to $\overline{\rho}|_{\Pi_0}$, we find that ${}^{\sigma}\chi = \eta\chi$ and ${}^{\sigma}\chi^c = \eta\chi^c$. Hence $\eta|_{\Pi_0} = {}^{\sigma}\chi\chi^{-1}$, so

$g \in \Pi_0(\overline{\rho})$ if and only if $\chi(g) \in \mathbb{E}^\times$. In particular, $\ker\chi \subseteq \Pi_0(\overline{\rho})$. Furthermore, using the fact that ${}^\sigma\chi\chi^{-1} = \eta|_{\Pi_0} = {}^\sigma\chi^c(\chi^c)^{-1}$, we find that the character χ/χ^c takes values in $\mathbb{E}^\times$.

We know that $\overline{\rho}|_{\Pi_0(\overline{\rho})}$ is multiplicity free over $\mathbb{E}$ if and only if there is some $g \in \Pi_0(\overline{\rho})$ such that $\chi(g) \neq \chi^c(g)$. If $\ker\chi \neq \ker\chi^c$, then we can choose $g \in \ker\chi \setminus \ker\chi^c$. Then $\chi(g) = 1 \neq \chi^c(g)$ and $g \in \Pi_0(\overline{\rho})$ by the previous paragraph. Therefore we may assume that $\ker\chi = \ker\chi^c$.

Let n denote the order of χ. Since $\ker\chi = \ker\chi^c$, we have that $\chi^c = \chi^a$ for some $a \in (\mathbb{Z}/n\mathbb{Z})^\times$. Note that $\chi^{c^2} = \chi$ since $c^2 \in \Pi_0$. Therefore

$$\chi = \chi^{c^2} = (\chi^c)^c = (\chi^a)^c = (\chi^c)^a = (\chi^a)^a = \chi^{a^2}.$$

Fix $g_0 \in \Pi_0$ such that $\overline{\rho}(g_0)$ generates the projective image of $\overline{\rho}(\Pi_0)$. We will show that $h \coloneqq g_0^{a-1} \in \Pi_0(\overline{\rho})$ and $\chi(h) \in \mathbb{E}^\times$ with $\chi(h) \neq \chi^c(h)$. First we calculate, using the fact that $\chi^{a^2} = \chi$,

$$\chi^c(h) = \chi^a(g_0^{a-1}) = \chi^{a^2}(g_0)\chi^{-1}(g_0) = 1.$$

Hence $h \in \ker\chi^c = \ker\chi \subseteq \Pi_0(\overline{\rho})$. On the other hand,

$$\chi(h) = \chi(g_0^{a-1}) = \chi^c(g_0)\chi^{-1}(g_0).$$

We saw in the second paragraph that χ/χ^c is an $\mathbb{E}$-valued character. Furthermore, $\chi^c(g_0)/\chi(g_0) \neq 1$ since g_0 was chosen has a generator of the projective image of $\overline{\rho}(\Pi_0)$, which is isomorphic to the image of χ/χ^c. ☐

Let us note a useful consequence of Proposition 4.6.

Corollary 4.7. *Assume that $\overline{\rho}$ is regular and dihedral and that the order of $\det\overline{\rho}$ is a power of 2. Let σ be a generator of $\Sigma_{\overline{\rho}}$ and $\eta\colon \Pi \to \mathbb{F}^\times$ a character such that $(\sigma,\eta) \in \Sigma_{\overline{\rho}}^{\text{pairs}}$. Then either $\Sigma_{\overline{\rho}}$ is trivial or $\overline{\rho}|_{\ker\eta}$ is absolutely irreducible.*

Proof. Write $\overline{\rho} = \operatorname{Ind}_{\Pi_0}^{\Pi}\chi$ and fix $c \in \Pi\setminus\Pi_0$. We shall make frequent use of Lemma 7.7 in this proof without referencing it every time. We saw in the proof of Proposition 4.6 that $\Pi_0(\overline{\rho}) = \Pi_0 \cap \ker\eta$. Thus Proposition 4.6 implies that $\chi|_{\Pi_0\cap\ker\eta} \neq \chi^c|_{\Pi_0\cap\ker\eta}$.

If $\ker\eta \neq \Pi_0\cap\ker\eta$, then $[\ker\eta\colon \Pi_0\cap\ker\eta] = 2$ since $[\Pi\colon \Pi_0] = 2$. Thus $\overline{\rho}|_{\ker\eta} \cong \operatorname{Ind}_{\Pi_0\cap\ker\eta}^{\ker\eta}\chi|_{\Pi_0\cap\ker\eta}$. Since $\chi|_{\Pi_0\cap\ker\eta} \neq \chi^c|_{\Pi_0\cap\ker\eta}$ it follows that $\overline{\rho}|_{\ker\eta}$ is irreducible.

If $\ker\eta = \Pi_0 \cap \ker\eta$ then $\Pi_0 \supseteq \ker\eta$ and $\Pi/\ker\eta$ is a cyclic group whose order is a power of 2 since η^2 is a power of $\det\overline{\rho}$. If $\Pi_0 \neq \ker\eta$, then there is a subgroup $\ker\eta \subseteq \Pi' \subset \Pi_0$ such that $[\Pi_0\colon \Pi'] = 2$. Note that $\chi|_{\Pi'} \neq \chi^c|_{\Pi'}$ since $\chi|_{\ker\eta} \neq \chi^c|_{\ker\eta}$. Then $\overline{\rho}|_{\Pi_0} \cong \operatorname{Ind}_{\Pi'}^{\Pi_0}\chi|_{\Pi'}$ is irreducible, a contradiction since $\overline{\rho} \cong \operatorname{Ind}_{\Pi_0}^{\Pi}\chi$. Thus we must have $\Pi_0 = \ker\eta$. Therefore $\overline{\rho} \cong \overline{\rho}\otimes\eta$ and so σ, and hence $\Sigma_{\overline{\rho}}$, is trivial. ☐

In Section 5 we will assume that the order of $\det\overline{\rho}$ is a power of 2, which is possible by twisting by Lemma 2.3. A large part of the reason for that assumption is that it guarantees that $[\mathbb{F}\colon \mathbb{E}]$ can be taken to be a power of 2 as well, as the next lemma shows. We need this in an induction argument in Section 5. Given any $\mathbb{F}$-valued function f and any subfield $\mathbb{F}'$ of $\mathbb{F}$, let us write $\mathbb{F}'(f)$ for the subfield of $\mathbb{F}$ generated over $\mathbb{F}'$ by the values of f.

Lemma 4.8. *Assume that the order of $\det\overline{\rho}$ is a power of 2. Then the degree of $\mathbb{F}_p(\operatorname{tr}\overline{\rho})$ over $\mathbb{E}$ is a power of 2.*

Proof. Let $d \coloneqq \det\overline{\rho}$. Since the order of d is a power of 2, the degree of $\mathbb{E}(d)$ over $\mathbb{E}$ is a power of 2. But, for an arbitrary $g' \in \Pi$, the extension $\mathbb{E}(\operatorname{tr}\overline{\rho}(g'))$ is at most quadratic over $\mathbb{E}(d)$ because $\operatorname{tr}\overline{\rho}(g')$ satisfies

$$d(g')x^2 - (\operatorname{tr}\overline{\rho}(g'))^2/d(g') \in \mathbb{E}(d)[x].$$

The field $\mathbb{F}_p(\operatorname{tr}\overline{\rho})$ is obtained from $\mathbb{E}(d)$ by adding the values of $\operatorname{tr}\overline{\rho}$ on finitely many elements of Π. ☐

In Section 5 we will be interested in gradings coming from conjugate self-twists. To be able to apply Lemma 7.19 in those situations, we now verify one of the hypotheses.

Lemma 4.9. *Assume that both the order of* $\det \overline{\rho}$ *and* $[\mathbb{F} \colon \mathbb{E}]$ *are powers of 2. If* $n = \#\Sigma_{\overline{\rho}}$*, then* $\mathbb{F}$ *contains a primitive* n*-th root of unity. In particular, condition* $(*)$ *from Section 7.3 is satisfied.*

Proof. Let $d := \det \overline{\rho}$, and write 2^s for the order of d. We have that $\mathbb{E}(d)$ contains a primitive 2^s-th root of unity. If $[\mathbb{F} \colon \mathbb{E}(d)] = 2^r$, then $\mathbb{F}$ contains a primitive 2^{r+s}-th root of unity. On the other hand,

$$n = \#\Sigma_{\overline{\rho}} = [\mathbb{F} \colon \mathbb{E}] = 2^r[\mathbb{E}(d) \colon \mathbb{E}].$$

Since d has order 2^s, it follows that $[\mathbb{E}(d) \colon \mathbb{E}]$ divides 2^{s-1}. Thus n divides 2^{r+s-1}, and so $\mathbb{F}$ contains a primitive n-th root of unity. □

4.2. **A good basis for $\overline{\rho}$.** In Section 5 we will need to carefully choose a basis for $\overline{\rho}$ that has many good properties. In this section we explain how to find this basis when $\overline{\rho}$ is exceptional or large. Let us first define an extra condition on octahedral representations.

Definition 4.10. We say a regular octahedral representation $\overline{\rho}$ is *good* if one of the following properties is satisfied:

(1) $p \equiv 1 \bmod 3$;
(2) $\overline{\rho}$ is strongly regular;
(3) there is a regular element $g_0 \in \Pi$ such that $g_0^2 \in \Pi_0(\overline{\rho})$.

We shall need to know that if $\overline{\rho}$ is good, then twisting away the odd part of the determinant of $\overline{\rho}$ gives a representation that is also good.

Lemma 4.11. *Let* $\overline{\rho} \colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ *be a good representation. Let* $\chi \colon \Pi \to \mathbb{F}^\times$ *be the unique odd-order character such that the order of* $\chi^2 \det \overline{\rho}$ *is a power of 2. Then* $\overline{\rho} \otimes \chi$ *is good.*

Proof. First note that twisting by any character does not change the projective image, so $\overline{\rho} \otimes \chi$ is octahedral. Regularity is also invariant under twisting. The claim is clear if $p \equiv 1 \bmod 3$, so we assume that $p \equiv 2 \bmod 3$. The regularity assumption then implies that $\zeta_4 \in \mathbb{F}_p$ by Remark 2.28. As in the proof of Lemma 2.3, decompose $\det \overline{\rho} = d_1 d_2$, where $d_i \colon \Pi \to \mathbb{F}^\times$ are characters such that the order of d_1 is odd and the order of d_2 is a power of 2.

First suppose that $\overline{\rho}$ is strongly regular. Then there is a matrix $g_0 \in \Pi$ such that $\overline{\rho}(g_0)$ has eigenvalues $\lambda_0, \mu_0 \in \mathbb{E}^\times$ such that $\lambda_0 \mu_0^{-1} = \zeta_4$. We have $\lambda_0 \mu_0 = \det \overline{\rho}(g_0) = d_1(g_0) d_2(g_0)$. Note that any $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{E})$ fixes $\lambda_0 \mu_0$ since $\lambda_0, \mu_0 \in \mathbb{E}$. Therefore $\sigma(d_1(g_0) d_2(g_0)) = d_1(g_0) d_2(g_0)$. But since $d_1(g_0)$ is an odd order root of unity and $d_2(g_0)$ is a 2-power order root of unity, it follows that σ must fix both $d_1(g_0)$ and $d_2(g_0)$. Write a for the order of d_1. Then $\chi = d_1^{-(a+1)/2}$ by the proof of Lemma 2.3. In particular, $\chi(g_0) \in \mathbb{E}^\times$. Thus the eigenvalues $\chi(g_0)\lambda_0$ and $\chi(g_0)\mu_0$ of $(\overline{\rho} \otimes \chi)(g_0)$ are in $\mathbb{E}$. Thus g_0 is a strongly regular element for $\overline{\rho} \otimes \chi$, as desired.

Finally, suppose that there is a regular element $g_0 \in \Pi$ such that $g_0^2 \in \Pi_0(\overline{\rho})$. Let σ be a generator for $\mathrm{Gal}(\mathbb{F}/\mathbb{E})$ and let $\eta \colon \Pi \to \mathbb{F}^\times$ such that $(\sigma, \eta) \in \Sigma_{\overline{\rho}}^{\mathrm{pairs}}$. Then $\Pi_0(\overline{\rho}) = \ker \eta$ and $\Pi_0(\overline{\rho} \otimes \chi) = \ker {}^\sigma\chi\chi^{-1}\eta$. Since $g_0^2 \in \Pi_0(\overline{\rho})$ and ${}^\sigma\det \overline{\rho} = \eta^2 \det \overline{\rho}$, it follows that $\det \overline{\rho}(g_0) \in \mathbb{E}$. But $\det \overline{\rho}(g_0) = d_1(g_0) d_2(g_0)$, and since $d_1(g_0)$ is an odd order root of unity and $d_2(g_0)$ has 2-power order, it follows that both $d_1(g_0)$ and $d_2(g_0)$ are in $\mathbb{E}$. Therefore $\chi(g_0) = d_1^{-(a+1)/2}(g_0) \in \mathbb{E}$. Thus $g_0^2 \in \ker {}^\sigma\chi\chi^{-1}\eta = \Pi_0(\overline{\rho} \otimes \chi)$. □

Finally we describe the basis of $\overline{\rho}$ that we shall want to work with in Section 5. Let Z denote the group of scalar matrices in $\mathrm{GL}_2(\mathbb{F})$. The following lemma justifies our definition of $\mathbb{F}_q$ in Section 2.6 for exceptional representations.

Lemma 4.12. *Up to conjugation, the image of $\overline{\rho}$ is contained in $Z\,\mathrm{GL}_2(\mathbb{E})$. If $\mathbb{F}_q$ is an extension of $\mathbb{E}$ and $\lambda_0, \mu_0 \in \overline{\mathbb{F}}_p^\times$ are eigenvalues of a matrix in the image of $\overline{\rho}$ such that $\lambda_0\mu_0^{-1} \in \mathbb{F}_q$, then we may further conjugate $\overline{\rho}$ to assume that $\left(\begin{smallmatrix}\lambda_0 & 0\\ 0 & \mu_0\end{smallmatrix}\right) \in \mathrm{Im}\,\overline{\rho}$ and $\mathrm{Im}\,\overline{\rho} \subseteq Z\,\mathrm{GL}_2(\mathbb{F}_q)$.*

Proof. By Proposition 3.1, $\mathbb{E} = \mathbb{F}^{\Sigma_{\overline{\rho}}}$. First we show that $\overline{\rho}$ can be conjugated to land in $Z\,\mathrm{GL}_2(\mathbb{E})$. Let $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{E})$ be a generator and η a character such that $(\sigma, \eta) \in \Sigma_{\overline{\rho}}$. Then there is some $x \in \mathrm{GL}_2(\mathbb{F})$ such that for all $g \in \Pi$, we have ${}^\sigma\overline{\rho}(g) = x^{-1}\eta(g)\overline{\rho}(g)x$. By a theorem of Serge Lang [Lan56, Corollary to Theorem 1], it follows that there is some $y \in \mathrm{GL}_2(\mathbb{F})$ such that $x = {}^\sigma y y^{-1}$. Thus ${}^\sigma(y^{-1}\overline{\rho}(g)y) = \eta(g)(y^{-1}\overline{\rho}(g)y)$. Replacing $\overline{\rho}$ by its conjugate by y, we have that the projective image of $\overline{\rho}$ is fixed by $\mathrm{Gal}(\mathbb{F}/\mathbb{E})$, and hence the image of $\overline{\rho}$ lands in $Z\,\mathrm{GL}_2(\mathbb{E})$, as desired.

If $\mathbb{F}_q, \lambda_0, \mu_0$ are as in the statement of the lemma, then $\overline{\rho}$ can be further conjugated such that $\left(\begin{smallmatrix}\lambda_0 & 0\\ 0 & \mu_0\end{smallmatrix}\right) \in \mathrm{Im}\,\overline{\rho}$ while preserving the property that the image of $\overline{\rho}$ is in $Z\,\mathrm{GL}_2(\mathbb{F}_q)$. ☐

Proposition 4.13. *Let $\overline{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ be exceptional or large and regular. If $\overline{\rho}$ is octahedral, assume further that $\overline{\rho}$ is good. Assume that the order of $\det\overline{\rho}$ is a power of 2. Then there is a regular element $g_0 \in \Pi$ and a basis for $\overline{\rho}$ such that the following are simultaneously true:*

(1) $\mathrm{Im}\,\overline{\rho} \subseteq Z\,\mathrm{GL}_2(\mathbb{E})$
(2) $\overline{\rho}(g_0) = \left(\begin{smallmatrix}\lambda_0 & 0\\ 0 & \mu_0\end{smallmatrix}\right)$*;*
(3) if $p \geq 7$ and $\overline{\rho}$ is large, then $\lambda_0, \mu_0 \in \mathbb{F}_p^\times$;
(4) there is a positive integer n such that $g_0 \in \Pi_0(\overline{\rho})$ and $\overline{\rho}(g_0^n)$ is not scalar.

Proof. By Lemma 4.12 we can always conjugate $\overline{\rho}$ so that $\mathrm{Im}\,\overline{\rho} \subseteq Z\,\mathrm{GL}_2(\mathbb{E})$. If $g_0 \in \Pi$ is a regular element and λ_0 and μ_0 are the eigenvalues of $\overline{\rho}(g_0)$, then we may assume further that $\overline{\rho}(g_0) = \left(\begin{smallmatrix}\lambda_0 & 0\\ 0 & \mu_0\end{smallmatrix}\right)$.

If $\overline{\rho}$ is large, then up to conjugation, $\mathrm{Im}\,\overline{\rho} \supseteq \mathrm{SL}_2(\mathbb{F}_p)$. Indeed, up to conjugation we may assume that the projective image of $\overline{\rho}$ contains $\mathrm{PSL}_2(\mathbb{E})$. Therefore there is some $\lambda \in \mathbb{F}^\times$ such that $\lambda\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) \in \mathrm{Im}\,\overline{\rho}$. Note that the n-th power of this matrix is $\lambda^n\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right)$. Since $\lambda \in \mathbb{F}^\times$, its order m is prime to p. Therefore we can write $1 = am + bp \equiv am \bmod p$ for some $a, b \in \mathbb{Z}$. Thus

$$\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) = \lambda^{am}\left(\begin{smallmatrix}1 & am\\ 0 & 1\end{smallmatrix}\right) = (\lambda\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right))^{am} \in \mathrm{Im}\,\overline{\rho}.$$

Similarly, $\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right) \in \mathrm{Im}\,\overline{\rho}$. Since $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$ generate $\mathrm{SL}_2(\mathbb{F}_p)$, it follows that $\mathrm{SL}_2(\mathbb{F}_p) \subseteq \mathrm{Im}\,\overline{\rho}$.

If $p \geq 7$, then we can choose $\alpha \in \mathbb{F}_p^\times$ such that $\alpha^2 \neq \pm 1$. Then any $g_0 \in \Pi$ such that $\overline{\rho}(g_0)$ has eigenvalues α, α^{-1} satisfies the first three conditions. Note that $\mathbb{P}\overline{\rho}(g_0) \in \mathrm{PSL}_2(\mathbb{E})$. Recall that $\Pi_0(\overline{\rho})$ is a normal subgroup of 2-power index in Π since the order of $\det\overline{\rho}$ is a power of 2. Furthermore, $\Pi/\Pi_0(\overline{\rho})$ is abelian. Therefore $\mathbb{P}\overline{\rho}(\Pi_0(\overline{\rho}))$ is either $\mathrm{PGL}_2(\mathbb{E})$ or $\mathrm{PSL}_2(\mathbb{E})$. In either case, we can find $g_0 \in \Pi_0(\overline{\rho})$ such that $\overline{\rho}(g_0)$ has eigenvalues α, α^{-1}. Thus all of the properties of the proposition are satisfied for this choice of g_0.

Next suppose that $\overline{\rho}$ is either tetrahedral or icosahedral. Once again, $\mathbb{P}\overline{\rho}(\Pi_0(\overline{\rho}))$ is a normal subgroup of $\mathbb{P}\overline{\rho}(\Pi)$ with 2-power index and abelian quotient. Since $\mathbb{P}\overline{\rho}(\Pi)$ is isomorphic to one of A_4 or A_5, it follows that $\mathbb{P}\overline{\rho}(\Pi_0(\overline{\rho})) = \mathbb{P}\overline{\rho}(\Pi)$. In particular, one can choose the regular element g_0 to be in $\Pi_0(\overline{\rho})$, and the resulting representation satisfies all of the desired conditions.

Finally, suppose that $\overline{\rho}$ is octahedral and good. If $p \equiv 1 \bmod 3$ then any $g_0 \in \Pi$ such that $\mathbb{P}\overline{\rho}(g_0)$ has order 3 is a regular element. Since $\mathbb{P}\overline{\rho}(\Pi_0(\overline{\rho}))$ is a normal subgroup of $\mathbb{P}\overline{\rho}(\Pi)$ with 2-power index and abelian quotient, it follows that $\Pi_0(\overline{\rho})$ contains an element g_0 such that $\mathbb{P}\overline{\rho}(g_0)$ has order 3. Such a g_0 satisfies all of the necessary conditions.

Next suppose that $p \equiv 2 \bmod 3$ and that $\overline{\rho}$ is strongly regular. Let $g_0 \in \Pi$ be a strongly regular element. Then $\overline{\rho}(g_0) = \left(\begin{smallmatrix}\lambda\zeta_4 & 0\\ 0 & \lambda\end{smallmatrix}\right)$ for some $\lambda \in \mathbb{E}^\times$. (Note that $\zeta_4 \in \mathbb{F}_p$ since $\overline{\rho}$ is regular and $p \equiv 2 \bmod 3$.) We claim that $g_0 \in \Pi_0(\overline{\rho})$. Indeed, let σ be a generator of $\mathrm{Gal}(\mathbb{F}/\mathbb{E})$ and η a character such that $(\sigma, \eta) \in \Sigma_{\overline{\rho}}^{\mathrm{pairs}}$. Then $\Pi_0(\overline{\rho}) = \ker\eta$. Since $\lambda, \zeta_4 \in \mathbb{E}^\times$ we have

$$\lambda(\zeta_4 + 1) = {}^\sigma(\lambda(\zeta_4 + 1)) = {}^\sigma\mathrm{tr}\,\overline{\rho}(g_0) = \eta(g_0)\,\mathrm{tr}\,\overline{\rho}(g_0) = \eta(g_0)\lambda(\zeta_4 + 1).$$

As $\zeta_4 + 1 \neq 0$ it follows that $\eta(g_0) = 1$, and so $g_0 \in \Pi_0(\overline{\rho})$, as claimed. Therefore g_0 satisfies all of the necessary conditions.

Finally suppose that $p \equiv 2 \bmod 3$ and there is a regular element $g_0 \in \Pi$ such that $g_0^2 \in \Pi_0(\overline{\rho})$. Note that $\mathbb{P}\overline{\rho}(g_0)$ has order 4 since $p \not\equiv 1 \bmod 3$. Therefore $\mathbb{P}\overline{\rho}(g_0^2)$ is nontrivial, so $\overline{\rho}(g_0^2)$ is not scalar. Therefore g_0 satisfies all of the conditions of the proposition. ☐

Note that the g_0 chosen in Proposition 4.13 satisfies all of the conditions prior to Definition 2.29. In particular, if $(t, d)\colon \Pi \to A$ is any admissible pseudodeformation of $\overline{\rho}$, then any (t, d)-representation that is adapted to the element g_0 from Proposition 4.13 is well adapted.

4.3. **Lifting conjugate self-twists.** Let Π be a profinite group satisfying the p-finiteness condition. Fix a multiplicity-free representation $\overline{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$. Recall from Section 2.1 that there is a local pro-p Noetherian $W(\mathbb{F})$-algebra $\mathcal{A}$ with maximal ideal $\mathfrak{m}_{\mathcal{A}}$ and residue field $\mathbb{F}$ and a pseudodeformation $(T, d)\colon \Pi \to \mathcal{A}$ that is universal among all constant-determinant pseudodeformations of $\overline{\rho}$. The purpose of Section 4.3 is to show that every conjugate self-twist of $\overline{\rho}$, and in fact of every constant-determinant pseudodeformation of $\overline{\rho}$, can be lifted to a conjugate self-twist of (T, d) (see Proposition 4.14 and Corollary 4.15 below).

Since we are working only with constant-determinant pseudodeformations, we shall identify any $\mathbb{F}$-valued character η with the $W(\mathbb{F})$-valued character $s(\eta)$. Furthermore, we will consider η as being valued in any $W(\mathbb{F})$-algebra via the structure map. If σ is an automorphism of $\mathbb{F}$, we write $W(\sigma)$ the automorphism of $W(\mathbb{F})$ induced by σ.

We introduce some notation that will be used in the proof of Proposition 4.14. For any $W(\mathbb{F})$-algebra A, let $A^\sigma := A \otimes_{W(\mathbb{F}), W(\sigma)} W(\mathbb{F})$, where $W(\mathbb{F})$ is considered as a $W(\mathbb{F})$-algebra via $W(\sigma)$. We can equip A^σ with two different $W(\mathbb{F})$-algebra structures by letting $W(\mathbb{F})$ act either on the first or second factor of the tensor product. In what follows, we refer to these actions respectively as the *first* or *second $W(\mathbb{F})$-algebra structure* on A^σ. Let $\iota(\sigma, A)\colon A \to A^\sigma$ be the natural map given by $\iota(\sigma, A)(a) = a \otimes 1$. It is an isomorphism of rings with inverse given by $\iota(\sigma^{-1}, A^\sigma)$ since $(A^\sigma)^{\sigma^{-1}}$ can be naturally identified with A as a $W(\mathbb{F})$-algebra. Furthermore, $\iota(\sigma, A)$ is a morphism of $W(\mathbb{F})$-algebras with respect to the first structure on A^σ. Note that if we view $\mathcal{A}^\sigma$ with respect to its second $W(\mathbb{F})$-algebra structure, its residue field is $\mathbb{F} \otimes_{\mathbb{F},\sigma} \mathbb{F}$, which is identified with $\mathbb{F}$ via $x \otimes y \mapsto \sigma(x)y$. The proof of the following proposition is a more streamlined treatment of the arguments in [Lan16, Section 2].

Proposition 4.14. *Let $(\sigma, \eta) \in \Sigma_{\overline{\rho}}^{\mathrm{pairs}}$. Then there is an automorphism $\tilde{\sigma}$ of $\mathcal{A}$ such that $(\tilde{\sigma}, \eta) \in \Sigma_T^{\mathrm{pairs}}$ and $\tilde{\sigma}$ induces σ modulo $\mathfrak{m}_{\mathcal{A}}$. Furthermore, for any w in the image of $W(\mathbb{F})$ in $\mathcal{A}$, we have $\tilde{\sigma}(w) = W(\sigma)(w)$.*

Note that any such $\tilde{\sigma}$ is necessarily unique, because it is determined by the character η.

Proof. Note that $\eta T\colon \Pi \to \mathcal{A}$ is the universal constant-determinant pseudodeformation of $\eta \otimes \overline{\rho} \cong {}^\sigma\overline{\rho}$. We claim that, considering $\mathcal{A}^{\sigma^{-1}}$ with its second $W(\mathbb{F})$-algebra structure, $\iota(\sigma^{-1}, \mathcal{A}) \circ (\eta T)$ is a constant-determinant pseudodeformation of $\overline{\rho}$. Indeed, reducing $\iota(\sigma^{-1}, \mathcal{A}) \circ (\eta T)$ modulo the maximal of ideal of $\mathcal{A}^{\sigma^{-1}}$ gives

$$\iota(\sigma^{-1}, \mathbb{F}) \circ (\eta \operatorname{tr} \overline{\rho}) = {}^\sigma\operatorname{tr} \overline{\rho} \otimes_{\mathbb{F},\sigma^{-1}} 1 = 1 \otimes \operatorname{tr} \overline{\rho},$$

which is identified with $\operatorname{tr} \overline{\rho}$ under the identification of $\mathbb{F} \otimes_{\mathbb{F},\sigma^{-1}} \mathbb{F}$ with $\mathbb{F}$ discussed prior to the proposition.

By universality, there is a unique $W(\mathbb{F})$-algebra homomorphism $\alpha\colon \mathcal{A} \to \mathcal{A}^{\sigma^{-1}}$, where $\mathcal{A}^{\sigma^{-1}}$ is given its second $W(\mathbb{F})$-algebra structure, such that

$$\alpha \circ T = \iota(\sigma^{-1}, \mathcal{A}) \circ (\eta T).$$

Since $\iota(\sigma, \mathcal{A}^{\sigma^{-1}})$ is the inverse of $\iota(\sigma^{-1}, \mathcal{A})$, we have that

$$\iota(\sigma, \mathcal{A}^{\sigma^{-1}}) \circ \alpha \circ T = \eta T. \tag{5}$$

Define $\tilde{\sigma} := \iota(\sigma, \mathcal{A}^{\sigma^{-1}}) \circ \alpha$, which is a ring endomorphism of $\mathcal{A}$. The relation (5) implies that $\tilde{\sigma}$ is an automorphism of $\mathcal{A}$ since the image of T topologically generates $\mathcal{A}$ as a $W(\mathbb{F})$-module [Bel18, Proposition 5.3.3] and η takes values in $W(\mathbb{F})$. The relation (5) also shows that $(\tilde{\sigma}, \eta) \in \Sigma_T^{\text{pairs}}$.

Finally, let $w \in W(\mathbb{F})$. Since α is a $W(\mathbb{F})$-algebra homomorphism with respect to the second $W(\mathbb{F})$-algebra structure on $\mathcal{A}^{\sigma^{-1}}$, we have that

$$\tilde{\sigma}(w) = \iota(\sigma, \mathcal{A}^{\sigma^{-1}}) \circ \alpha(w) = \iota(\sigma, \mathcal{A}^{\sigma^{-1}})(1 \otimes w) = \iota(\sigma^{-1}, \mathcal{A})^{-1}(W(\sigma)(w) \otimes 1) = W(\sigma)(w).$$

□

For the rest of Section 4.3, fix a local profinite $W(\mathbb{F})$ algebra A with residue field $\mathbb{F}$ and a constant-determinant pseudodeformation $(t, d)\colon \Pi \to A$ of $\overline{\rho}$. Assume that A is the $W(\mathbb{F})$-algebra generated by $t(\Pi)$. Let $\alpha_t\colon \mathcal{A} \to A$ be the unique $W(\mathbb{F})$-algebra homomorphism such that $\alpha \circ T = t$ given by universality. The following corollary shows that conjugate self-twists of (t, d) also lift to conjugate self twists of (T, d).

Corollary 4.15. *Given $(\sigma, \eta) \in \Sigma_t^{\text{pairs}}$, there is a unique $(\tilde{\sigma}, \eta) \in \Sigma_T^{\text{pairs}}$ such that $\alpha_t \circ \tilde{\sigma} = \sigma \circ \alpha_t$.*

Proof. Let $\overline{\sigma}$ denote the automorphism of $\mathbb{F}$ induced by σ. Let $\tilde{\sigma}$ be the automorphism of $\mathcal{A}$ given by Proposition 4.14 lifting $\overline{\sigma}$ to $\mathcal{A}$. Then we just have to show that $\alpha_t \circ \tilde{\sigma} = \sigma \circ \alpha_t$. Note that σ acts by $W(\overline{\sigma})$ on the image of $W(\mathbb{F})$ in A, so $\sigma^{-1} \circ \alpha \circ \tilde{\sigma}$ is a $W(\mathbb{F})$-algebra homomorphism. Thus by universality, it suffices to show that $t = \sigma^{-1} \circ \alpha_t \circ \tilde{\sigma} \circ T$. Since η takes values in $W(\mathbb{F})$ and α_t is a $W(\mathbb{F})$-algebra homomorphism, we have that

$$\sigma^{-1} \circ \alpha_t \circ \tilde{\sigma} \circ T = \sigma^{-1} \circ \alpha_t(\eta T) = \sigma^{-1}(\eta t) = \sigma^{-1}(\sigma(t)) = t.$$

□

We end Section 4.3 with some observations about the consequences of Proposition 4.14 and Corollary 4.15. They give the following commutative diagram with exact lines.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Sigma_t^{\text{di}} & \longrightarrow & \Sigma_t^{\text{pairs}} & \longrightarrow & \Sigma_t & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Sigma_T^{\text{di}} & \longrightarrow & \Sigma_T^{\text{pairs}} & \longrightarrow & \Sigma_T & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \cong & & \downarrow & & \\
1 & \longrightarrow & \Sigma_{\overline{\rho}}^{\text{di}} & \longrightarrow & \Sigma_{\overline{\rho}}^{\text{pairs}} & \longrightarrow & \Sigma_{\overline{\rho}} & \longrightarrow & 1
\end{array}$$

We write

$$\beta_t\colon \Sigma_t \to \Sigma_{\overline{\rho}}$$

for the composition of the vertical maps on the right in the above diagram. It is induced by the composition $\tilde{\beta}_t\colon \Sigma_t^{\text{pairs}} \to \Sigma_{\overline{\rho}}^{\text{pairs}}$ of the middle maps, which reflects the fact that every conjugate self-twist of (t, d) induces a conjugate self-twist of $\overline{\rho}$. Combining Corollary 4.15 with Corollary 4.2, we see that Σ_t^{pairs} is a finite group for any constant-determinant pseudodeformation (t, d) of $\overline{\rho}$.

In this paper, we will only be concerned with pseudodeformations (t, d) of $\overline{\rho}$ that are not a priori small. Under this assumption, if $t \neq \operatorname{tr} \overline{\rho}$ then $\Sigma_t^{\text{di}} = 1$ and $\Sigma_T^{\text{di}} = 1$ by Lemma 4.1(1). In particular, $\Sigma_t^{\text{pairs}} = \Sigma_t$ and $\Sigma_T^{\text{pairs}} = \Sigma_T$, so (except for $\overline{\rho}$) a conjugate self-twist (σ, η) is determined uniquely by the automorphism σ.

4.4. **The action of $\ker\beta_t$ on $I_1(\rho)$ and $B_1(\rho)$.** Nontrivial elements in $\ker\beta_t$ will complicate matters in Section 5. Lemma 4.16 explains how a nontrivial element $\tau\in\ker\beta_t$ interacts with $I_1(\rho)$ and $B_1(\rho)$ for a well-adapted (t,d)-representation ρ when $\overline{\rho}$ is projectively dihedral and nonabelian. For $\varepsilon\in\{+,-\}$, let

$$A^{\varepsilon}:=\{a\in A\colon {}^{\tau}a=\varepsilon a\}.$$

Lemma 4.16. *Assume that $\overline{\rho}$ is projectively dihedral and nonabelian. Suppose there exists $1\neq\tau\in\ker\beta_t$. If $\rho\colon\Pi\to\mathrm{GL}_2(A)$ is a well-adapted (t,d)-representation, then A^+ is generated by $\mathbb{Z}_p+I_1(\rho)+I_1(\rho)^2$ as a $W(\mathbb{F})$-module and A^- is generated by $B_1(\rho)$ as a $W(\mathbb{F})$-module.*

Proof. Since $A=A^+\oplus A^-$ and A is generated by $\mathbb{Z}_p+I_1(\rho)+I_1(\rho)^2+B_1(\rho)$ as a $W(\mathbb{F})$-module by Bellaïche's Theorem 2.31, it suffices to show that τ acts trivially on $I_1(\rho)$ and by -1 on $B_1(\rho)$. Let $\eta\colon\Pi\to\{\pm1\}$ be the unique quadratic character such that $\overline{\rho}\cong\overline{\rho}\otimes\eta$. (It is unique by Lemma 4.1 since the projective image of $\overline{\rho}$ is not isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$). Since $\tau\in\ker\beta_t$, it follows that η must be the character such that $(\tau,\eta)\in\Sigma_t^{\text{pairs}}$.

We first prove that $I_1(\rho)$ is fixed by τ. As usual, let $\Gamma:=\operatorname{Im}\rho\cap\Gamma_A(\mathfrak{m})$. Recall that by definition $I_1(\rho)$ is the $\mathbb{Z}_p$-module topologically generated by $\{\alpha-\delta\colon \left(\begin{smallmatrix}1+\alpha & b\\ c & 1+\delta\end{smallmatrix}\right)\in\Gamma\}$. Let $g\in\ker\eta$. Since ρ is well adapted, we can write

$$\rho(g)=\gamma\cdot\begin{pmatrix}s(\lambda_0) & 0\\ 0 & s(\mu_0)\end{pmatrix}$$

for some $\gamma\in\Gamma$ and $\lambda_0,\mu_0\in\mathbb{F}^\times$. Write $\gamma=\left(\begin{smallmatrix}1+\alpha & b\\ c & 1+\delta\end{smallmatrix}\right)$ with $\alpha-\delta=2a$, $\left(\begin{smallmatrix}a & b\\ c & -a\end{smallmatrix}\right)\in L_1(\rho)$ and $0=\alpha+\delta+\alpha\delta-bc$. Then we have

$$\rho(g)=\begin{pmatrix}s(\lambda_0)(1+\alpha) & s(\mu_0)b\\ s(\lambda_0)c & s(\mu_0)(1+\delta)\end{pmatrix}.$$

Since $g\in\ker\eta$, it follows that

$$s(\lambda_0)(1+\alpha)+s(\mu_0)(1+\delta)=\operatorname{tr}\rho(g)={}^{\tau}(\operatorname{tr}\rho(g))=s(\lambda_0)(1+{}^{\tau}\alpha)+s(\mu_0)(1+{}^{\tau}\delta),$$

since τ acts trivially on $W(\mathbb{F})$. Thus, we obtain

$$s(\lambda_0\mu_0^{-1})(\alpha-{}^{\tau}\alpha)={}^{\tau}\delta-\delta$$

for all λ_0,μ_0 such that $\left(\begin{smallmatrix}\lambda_0 & 0\\ 0 & \mu_0\end{smallmatrix}\right)\in\operatorname{Im}\overline{\rho}$. As the projective image of $\overline{\rho}$ is not isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$, it follows that $\lambda_0\mu_0^{-1}$ takes at least two distinct values in $\mathbb{F}^\times$. Thus it follows that $\alpha-{}^{\tau}\alpha=0={}^{\tau}\delta-\delta$. Since $I_1(\rho)$ is generated by $\alpha-\delta$ with α,δ as above, it follows that $I_1(\rho)$ is fixed by τ.

The proof that τ acts by -1 on $B_1(\rho)$ is similar. Namely, recall that $B_1(\rho)$ is topologically generated by $\{b,c\in A\colon \left(\begin{smallmatrix}1+\alpha & b\\ c & 1+\delta\end{smallmatrix}\right)\in\Gamma\}$. Let $g\in\Pi\setminus\ker\eta$. Again since ρ is well adapted, we can write

$$\rho(g)=\gamma\cdot\begin{pmatrix}0 & s(\lambda_0)\\ s(\mu_0) & 0\end{pmatrix}$$

for some $\gamma\in\Gamma,\lambda_0,\mu_0\in\mathbb{F}^\times$. As above, write $\gamma=\left(\begin{smallmatrix}1+\alpha & b\\ c & 1+\delta\end{smallmatrix}\right)$. Then we have

$$\rho(g)=\begin{pmatrix}s(\mu_0)b & s(\lambda_0)(1+\alpha)\\ s(\mu_0)(1+\delta) & s(\lambda_0)c\end{pmatrix}.$$

Since $g\notin\ker\eta$, it follows that

$$s(\mu_0)b+s(\lambda_0)c=\operatorname{tr}\rho(g)=-{}^{\tau}(\operatorname{tr}\rho(g))=-s(\mu_0){}^{\tau}b-s(\lambda_0){}^{\tau}c.$$

Thus

$$s(\mu_0\lambda_0^{-1})(b+{}^{\tau}b)=-(c+{}^{\tau}c)$$

for all $\left(\begin{smallmatrix} 0 & \lambda_0 \\ \mu_0 & 0 \end{smallmatrix}\right) \in \operatorname{Im}\overline{\rho}$. Once again, since the projective image of $\overline{\rho}$ is not isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$, it follows that $\lambda_0\mu_0^{-1}$ takes at least two distinct values in $\mathbb{F}^\times$. Therefore $b + {}^\tau b = 0 = c + {}^\tau c$. Since $B_1(\rho)$ is generated by such b and c, it follows that τ acts on $B_1(\rho)$ by -1. □

5. Relating $W(\mathbb{E})[I_1(\rho)]$ and A^{Σ_t}

With the exception of Section 5.5, throughout Section 5 we fix a local pro-p domain A and an admissible pseudodeformation $(\Pi, \overline{\rho}, t, d)$ over A. In view of Corollary 3.2 and Corollary 3.8, we want to relate A^{Σ_t} to $W(\mathbb{E})[I_1(\rho)]$ for some well chosen (t,d)-representation ρ. Let us point out an easy case when this is possible. If $\overline{\rho}$ has no conjugate self-twists, then $\mathbb{E} = \mathbb{F}$ by Proposition 3.1 and $\Sigma_t = 1$ by the diagram following Corollary 4.15. Furthermore, the assumption that $\Sigma_{\overline{\rho}}^{\text{pairs}} = 1$ implies that $\overline{\rho}$ is not dihedral and so $A = W(\mathbb{F})[I_1(\rho)]$ by Theorem 2.31. Therefore we have

$$W(\mathbb{E})[I_1(\rho)] = W(\mathbb{F})[I_1(\rho)] = A = A^{\Sigma_t}.$$

The goal of Section 5 is to prove that (t,d) is A^{Σ_t}-full. The case when $\overline{\rho}$ is reducible is easily done in Proposition 5.1, so from Section 5.2 onwards we always assume that $\overline{\rho}$ is irreducible. In light of Corollary 3.8, the strategy is to prove that, under certain conditions on $\overline{\rho}$ and a good choice of a (t,d)-representation ρ, the two rings $W(\mathbb{E})[I_1(\rho)]$ and A^{Σ_t} have the same fields of fractions and A^{Σ_t} is finitely generated as a $W(\mathbb{E})[I_1(\rho)]$-module. This is done in Corollary 5.15, although key parts of it are proved in Corollary 5.9 and Proposition 5.14. Lemma 2.16 then implies that ρ is A^{Σ_t}-full whenever ρ is $W(\mathbb{E})[I_1(\rho)]$-full. In Corollary 5.16, we combine Corollary 3.8, which established $W(\mathbb{E})[I_1(\rho)]$-fullness, with Corollary 5.15 to show that (t,d) is A^{Σ_t}-full under mild assumptions on $\overline{\rho}$.

Let us now establish some assumptions on our fixed residual representation $\overline{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$. Assume that $\overline{\rho}$ is regular and, after Section 5.1, absolutely irreducible. Whenever $\overline{\rho}$ is absolutely irreducible, assume further that $\det\overline{\rho}$ is a power of 2, which can always be achieved by twisting $\overline{\rho}$ by a character by Lemma 2.3. Furthermore, the twisting operation does not change the field $\mathbb{E}$ by Proposition 3.1. Assume that $[\mathbb{F}\colon \mathbb{E}]$ is a power of 2, which is possible by Lemma 4.8. Note that we do not require $\mathbb{F}$ to be the trace algebra of $\overline{\rho}$ since one may need to make a quadratic extension of the trace algebra in order to make representations well-adapted in the dihedral case.

5.1. The reducible case.

When $\overline{\rho}$ is reducible, we can use Corollary 3.8 to show that (t,d) is A^{Σ_t}-full.

Proposition 5.1. *Suppose that $\overline{\rho} = \varepsilon \oplus \delta$ and that $\overline{\rho}$ is regular. Let (t,d) be a pseudodeformation of $\overline{\rho}$ that is not a priori small. Then (t,d) is A^{Σ_t}-full.*

Proof. Let $(t', d') = (s(\delta^{-1})t, s(\delta^{-1})^2 d)$, which is a pseudodeformation of $\overline{\rho}' := \overline{\rho} \otimes \delta^{-1}$. Let A' be the subring of A topologically generated by $t'(\Pi)$. Note that the residue field of A' is $\mathbb{E}$ since $\overline{\rho}'$ has no conjugate self-twists by Corollary 4.4. Then (Π, ρ', t', d') is an admissible pseudorepresentation over A'. Note that (t', d') is not a priori small since (t,d) is not. By Corollary 3.8, there is a well-adapted (t', d')-representation ρ' such that (t', d') is $W(\mathbb{E})[I_1(\rho')]$-full. But $A' = W(\mathbb{E})[I_1(\rho')]$ by Theorem 2.31. Since $\overline{\rho}'$ has no conjugate self-twists by Corollary 4.4, it follows that $\Sigma_{t'}$ is trivial. Thus $(A')^{\Sigma_{t'}} = A' = W(\mathbb{E})[I_1(\rho')]$.

By Lemma 2.25 it follows that (t,d) is $(A')^{\Sigma_{t'}}$-full. We know that A^{Σ_t} and $(A')^{\Sigma_{t'}}$ have the same fields of fractions by Proposition 3.1. Furthermore A is obtained by adjoining the values of $s(\delta)$ to A'. Therefore A, and hence A^{Σ_t}, is finitely generated over A'. Therefore (t,d) is A^{Σ_t}-full by Lemma 2.16. □

5.2. **Choosing a good (t,d)-representation.** Throughout Sections 5.2–5.4 we fix an absolutely irreducible regular representation $\overline{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ such that the order of $\det \overline{\rho}$ is a power of 2. We assume that $[\mathbb{F}\colon \mathbb{E}]$ is a power of 2 by Lemma 4.8. If $\overline{\rho}$ is octahedral, we assume further that $\overline{\rho}$ is good. Furthermore, we fix a good basis for $\overline{\rho}$ as follows. If $\overline{\rho}$ is exceptional or large, choose a basis and a regular element $g_0 \in \Pi$ such that Proposition 4.13 holds. If $\overline{\rho} = \mathrm{Ind}_{\Pi_0}^{\Pi} \chi$ is dihedral, assume that $\overline{\rho}(\Pi_0)$ is diagonal and $\mathrm{Im}\,\overline{\rho}$ contains a matrix $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ such that $bc^{-1} \in \mathbb{F}_p$, which is possible by [Bel18, Proposition 6.3.2].

Recall that $(T,d)\colon \Pi \to \mathcal{A}$ is the universal constant-determinant pseudorepresentation. Part of our arguments will require appealing to a universal (T,d)-representation. This requires choosing a good (T,d)-representation ρ^{univ} and also choosing our (t,d)-representation to be compatible with ρ^{univ}. In particular, we want $I_1(\rho^{\mathrm{univ}})$ to be fixed by all conjugate self-twists of (T,d). In Section 5.2 we make these choices and compatibilities precise.

Fix a generator σ_1 of $\Sigma_{\overline{\rho}} = \mathrm{Gal}(\mathbb{F}/\mathbb{E})$. We want to choose a character $\eta_1\colon \Pi \to \mathbb{F}^\times$ such that $(\sigma_1, \eta_1) \in \Sigma_{\overline{\rho}}^{\mathrm{pairs}}$. There is a unique choice for η_1 when $\overline{\rho}$ is not dihedral. If $\overline{\rho}$ is dihedral and $\Sigma_t = 1$, choose η_1 to be the trivial character. Recall from the end of Section 4.3 that $\beta_t\colon \Sigma_t \to \Sigma_{\overline{\rho}}$ is given by reducing automorphisms of A modulo $\mathfrak{m}$. If $\overline{\rho}$ is dihedral and $\ker \beta_t = 1$ but $\Sigma_t \neq 1$, then there is a unique complement to $\Sigma_{\overline{\rho}}^{\mathrm{di}}$ in $\Sigma_{\overline{\rho}}^{\mathrm{pairs}}$ that contains $\tilde{\beta}_t(\Sigma_t^{\mathrm{pairs}})$. Choose η_1 such that (σ_1, η_1) generates that complement. Otherwise, when $\overline{\rho}$ is dihedral, we may take η_1 to be either of the two characters such that $(\sigma_1, \eta_1) \in \Sigma_{\overline{\rho}}^{\mathrm{pairs}}$. Recall from (4) in Section 4.1 that $\Pi_0(\overline{\rho})$ is the intersection of the kernels of all characters that occur in conjugate self-twists of $\overline{\rho}$. Define

$$\Pi_1 \coloneqq \begin{cases} \ker \eta_1 & \text{if } \overline{\rho} \text{ is dihedral} \\ \Pi_0(\overline{\rho}) & \text{else.} \end{cases}$$

Let A_1 be the subring of A topologically generated by $t(\Pi_1)$. Note that $\overline{\rho}|_{\Pi_1}$ is absolutely irreducible by Lemma 4.5 and Corollary 4.7.

Proposition 5.2. *There exists a well-adapted (t,d)-representation $\rho\colon \Pi \to \mathrm{GL}_2(A)$ such that $\rho|_{\Pi_1}$ takes values in $\mathrm{GL}_2(A_1)$ and such that ρ is adapted to a regular element.*

Proof. With the exception of the well-adaptedness statement, the proof is well known since $\overline{\rho}|_{\Pi_1}$ is absolutely irreducible. Indeed, a theorem of Rouquier [Rou96, Theorem 5.1] and Nyssen [Nys96] tells us that there are representations $\rho\colon \Pi \to \mathrm{GL}_2(A)$ and $\rho_1\colon \Pi_1 \to \mathrm{GL}_2(A_1)$ such that $\mathrm{tr}\,\rho = t$ and $\mathrm{tr}\,\rho_1 = t|_{\Pi_1}$. By a theorem of Carayol and Serre, $\rho|_{\Pi_1}$ and ρ_1 are conjugate by a matrix in $\mathrm{GL}_2(A)$ [Car94, Théorème 1].

For the well-adapteness statement, let us first assume that $\overline{\rho}$ is not dihedral. Choose ρ adapted to g_0 and ρ_1 adapted to g_0^n with g_0 and n as in Proposition 4.13. Then the matrix $M \in \mathrm{GL}_2(A)$ such that $M^{-1}\rho|_{\Pi_1} M = \rho_1$ commutes with $\rho(g_0^n) = \left(\begin{smallmatrix} s(\lambda_0^n) & 0 \\ 0 & s(\mu_0^n) \end{smallmatrix}\right) = \rho_1(g_0^n)$. Since $\lambda_0^n \neq \mu_0^n$ by Proposition 4.13, it follows that M must be diagonal. In particular, M commutes with $\rho(g_0)$. Hence $M^{-1}\rho M$ is still adapted to g_0 and satisfies the properties in the statement of the proposition.

The idea is similar when $\overline{\rho}$ is dihedral, except we can no longer assume that ρ_1 is adapted to the $g_0 \in \Pi_0$ such that $\overline{\rho}(g_0)$ generates the unique index-2 subgroup of the projective image of $\overline{\rho}$, because g_0 may not be in Π_1. Let ρ be a well-adapted (t,d)-representation, say adapted to g_0 with $\rho(g_0) = \left(\begin{smallmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{smallmatrix}\right)$.

Since $\overline{\rho}$ is regular, it follows that $\overline{\rho}|_{\Pi_0(\overline{\rho})}$ is multiplicity free over $\mathbb{E}$ by Proposition 4.6. Therefore, since ρ is well adapted, the image of ρ contains a matrix of the form $\left(\begin{smallmatrix} s(\lambda) & 0 \\ 0 & s(\mu) \end{smallmatrix}\right)$ with $\lambda \neq \mu$ and $\lambda, \mu \in \mathbb{E}^\times$. Let $h \in \Pi$ such that $\rho(h) = \left(\begin{smallmatrix} s(\lambda) & 0 \\ 0 & s(\mu) \end{smallmatrix}\right)$.

We claim that $h \in \Pi_1$. It suffices to prove that $h \in \Pi_0(\overline{\rho})$ since $\Pi_0(\overline{\rho}) = \Pi_0 \cap \ker \eta_1 \subset \ker \eta_1 = \Pi_1$. Note that $h \in \Pi_0$ since $\overline{\rho}(h)$ is diagonal. By Proposition 4.6, $h \in \Pi_0(\overline{\rho})$ if and only if the eigenvalues of $\overline{\rho}(h)$ are in $\mathbb{E}^\times$. But the eigenvalues of $\overline{\rho}(h)$ are λ, μ, which were chosen to be in $\mathbb{E}^\times$. Therefore $h \in \Pi_1$.

By [Bel18, Proposition 2.4.2] we may assume that ρ_1 in the first paragraph of this proof is adapted to h. Therefore the matrix $M \in \mathrm{GL}_2(A)$ such that $M^{-1}\rho|_{\Pi_1}M = \rho_1$ commutes with $\rho(h) = \left(\begin{smallmatrix} s(\lambda) & 0 \\ 0 & s(\mu) \end{smallmatrix}\right) = \rho_1(h)$. Since $\lambda \neq \mu$, it follows that M is a diagonal matrix. Note that the second property in Definition 2.29 is unchanged by conjugation by a diagonal matrix. Therefore $M^{-1}\rho M$ is still well adapted and satisfies the statement of the proposition. □

Corollary 5.3. *There exists a well-adapted (t,d)-representation $\rho\colon \Pi \to R^\times$ such that $I_1(\rho) \subseteq A^{\Sigma_t}$. If $\overline{\rho}$ is dihedral and $\sigma \in \Sigma_t$ such that σ and $\ker \beta_t$ generate Σ_t, then we may assume furthermore that $B_1(\rho)$ is pointwise fixed by σ.*

Proof. Let ρ be the (t,d)-representation from Proposition 5.2. Since the order of $\det \overline{\rho}$ is a power of 2, it follows that $[\Pi\colon \Pi_0(\overline{\rho})]$ is a power of 2. Since Γ is pro-p and $p \neq 2$, it follows that $\Gamma \subseteq \rho(\Pi_0(\overline{\rho})) \subseteq \mathrm{GL}_2(A_1)$. Therefore $L_1(\rho) \subseteq \mathfrak{sl}_2(A_1)$, and so $I_1(\rho), B_1(\rho) \subseteq A_1$.

Let $(\sigma, \eta) \in \Sigma_t^{\mathrm{pairs}}$ such that $\Pi_1 \subseteq \ker \eta$. Then for all $g \in \Pi_1$ we have

$$^{\sigma}t(g) = \eta(g)t(g) = t(g),$$

and thus A_1 is contained in the subring of A fixed by σ.

If $\ker \beta_t = 1$, then every $(\sigma, \eta) \in \Sigma_t^{\mathrm{pairs}}$ satisfies $\Pi_1 \subseteq \ker \eta$ by definition of Π_1. Thus if $\ker \beta_t = 1$, then $A_1 \subseteq A^{\Sigma_t}$, and hence $I_1(\rho), B_1(\rho) \subseteq A^{\Sigma_t}$.

Now suppose that $\overline{\rho}$ is dihedral and $\ker \beta_t \neq 1$. Then half of the elements $(\sigma, \eta) \in \Sigma_t^{\mathrm{pairs}}$ satisfy $\ker \eta \subseteq \ker \eta_1 = \Pi_1$, namely all those in the preimage under β_t of the subgroup generated by (σ_1, η_1) in $\Sigma_{\overline{\rho}}^{\mathrm{pairs}}$. This proves the statement about $B_1(\rho)$ in the dihedral case. To see that $I_1(\rho)$ is fixed by all conjugate self-twists, it remains to show that $I_1(\rho)$ is fixed by the nontrivial element in $\ker \beta_t$. This follows from Lemma 4.16. □

In light of Corollary 5.3, let us fix a well-adapted (T,d)-representation $\rho^{\mathrm{univ}}\colon \Pi \to \mathrm{GL}_2(\mathcal{A})$ such that $I_1(\rho^{\mathrm{univ}}) \subseteq \mathcal{A}^{\Sigma_T}$. Assume furthermore in the case when the projective image of $\overline{\rho}$ is not dihedral that we have conjugated ρ^{univ} by the relevant diagonal element so that Theorem 2.31 applies to ρ^{univ}, and thus to any quotient of ρ^{univ}. In the case when $\overline{\rho}$ is dihedral, we need to choose a complement to $\ker \beta_T$ in Σ_T, whose generator we will denote by ν. We choose ν such that $(\nu, \eta_1) \in \Sigma_T^{\mathrm{pairs}}$, where η_1 is the character fixed prior to Proposition 5.2. By Corollary 5.3, we may and do assume that $B_1(\rho^{\mathrm{univ}})$ is fixed by ν.

The universal property of $(\mathcal{A}, (T,d))$ gives a surjective $W(\mathbb{F})$-algebra homomorphism $\alpha_t\colon \mathcal{A} \to A$. Let $\rho_t := \alpha_t \circ \rho^{\mathrm{univ}}\colon \Pi \to \mathrm{GL}_2(A)$. It is a (t,d)-representation such that $I_1(\rho_t) \subseteq A^{\Sigma_t}$ by the diagram following Corollary 4.15. Furthermore, if $\overline{\rho}$ is dihedral and $\ker \beta_t = 1$, then $B_1(\rho_t) \subseteq A^{\Sigma_t}$ as well. By the functoriality of Pink-Lie algebras with respect to quotient maps, we have that $\alpha_t(I_1(\rho^{\mathrm{univ}})) = I_1(\rho_t)$ and $\alpha_t(B_1(\rho^{\mathrm{univ}})) = B_1(\rho_t)$. All of our theorems below will be specifically for this well-chosen representation ρ_t.

Recall that by Theorem 2.31

$$A = \begin{cases} W(\mathbb{F})[I_1(\rho_t)] + W(\mathbb{F})B_1(\rho_t) & \text{if } \overline{\rho} \text{ is dihedral} \\ W(\mathbb{F})[I_1(\rho_t)] & \text{else.} \end{cases}$$

By Lemma 4.16 and the fact that $B_1(\rho_t) \subseteq A^{\Sigma_t}$ if $\overline{\rho}$ is dihedral and $\ker \beta_t = 1$, it follows that

$$A^{\Sigma_t} = \begin{cases} W(\mathbb{F}^{\beta_t(\Sigma_t)})[I_1(\rho_t)] + W(\mathbb{F}^{\beta_t(\Sigma_t)})B_1(\rho_t) & \text{if } \overline{\rho} \text{ dihedral and} \ker \beta_t = 1 \\ W(\mathbb{F}^{\beta_t(\Sigma_t)})[I_1(\rho_t)] & \text{else.} \end{cases}$$

We therefore define

$$J = J(\rho_t) := \begin{cases} W(\mathbb{E})I_1(\rho_t) + W(\mathbb{E})I_1(\rho_t)^2 + W(\mathbb{E})B_1(\rho_t) & \text{if } \overline{\rho} \text{ is dihedral and} \ker\beta_t = 1 \\ W(\mathbb{E})I_1(\rho_t) + W(\mathbb{E})I_1(\rho_t)^2 & \text{else.} \end{cases}$$

We claim that $J \subset \mathfrak{m}$ is a multiplicatively closed $W(\mathbb{E})$-module by Theorem 2.31. The key is to note that, since $\overline{\rho}$ is regular and ρ_t is well adapted, it follows that Bellaïche's field $\mathbb{F}_q$ from Table 1 is contained in $\mathbb{E}$. Therefore it follows from Theorem 2.31 that $(W(\mathbb{E})I_1(\rho_t))^3 \subseteq W(\mathbb{E})I_1(\rho_t)$ and $W(\mathbb{E})I_1(\rho_t)B_1(\rho_t) \subseteq W(\mathbb{E})B_1(\rho_t)$ and $(W(\mathbb{E})B_1(\rho_t))^2 \subseteq W(\mathbb{E})I_1(\rho_t)$, which proves that J is multiplicatively closed. Define

$$\mathfrak{A} := W(\mathbb{F}) + W(\mathbb{F})J.$$

We have $\mathfrak{A} = A$ unless $1 \neq \ker\beta_t$, in which case $\mathfrak{A} = A^+$ by Lemma 4.16.

Remark 5.4. The rings $W(\mathbb{E}) + J$ and A^{Σ_t} differ only in their constants, $W(\mathbb{E})$ versus $W(\mathbb{F}^{\beta_t(\Sigma_t)})$. Furthermore, $W(\mathbb{E}) + J$ is often equal to $W(\mathbb{E})[I_1(\rho_t)]$, and the goal of this section is to relate $W(\mathbb{E})[I_1(\rho_t)]$ with A^{Σ_t}. Assume for a moment that $W(\mathbb{E}) + J = W(\mathbb{E})[I_1(\rho_t)]$. Then the difference between $W(\mathbb{E})[I_1(\rho_t)]$ and A^{Σ_t} is entirely governed by understanding which elements of $\Sigma_{\overline{\rho}}$ lift to elements in Σ_t under β_t. In particular, when there are elements in $\Sigma_{\overline{\rho}}$ that do not lift to Σ_t, we will be interested in writing the extra elements in $W(\mathbb{F}^{\beta_t(\Sigma_t)})$ as quotients of elements in J to show that $Q(W(\mathbb{E})[I_1(\rho_t)]) = Q(A^{\Sigma_t})$.

5.3. **Lifting conjugate self-twists to $\mathfrak{A}$.** In Section 5.3 we study a condition on J, called *smallness* (Definition 5.6), that dictates which conjugate self-twists of $\overline{\rho}$ lift to conjugate self-twists of (t,d). This study culminates in Theorem 5.8. As a consequence, we prove in Corollary 5.9 that under such a smallness assumption, $A^{\Sigma_t} = W(\mathbb{E})[I_1(\rho_t)]$. The reader is advised that, with the exception of the motivational remark following Definition 5.6, the assumption that A is a domain is never used in Section 5.3.

Throughout Section 5.3, fix a subgroup $\Sigma \subseteq \Sigma_{\overline{\rho}}$, and let $\mathbb{F}' := \mathbb{F}^{\Sigma}$. Write $W := W(\mathbb{F})$ and $W' := W(\mathbb{F}')$. For an arbitrary ring $\mathcal{R}$ and a finite group X of ring automorphisms of $\mathcal{R}$, for any $\varphi \in \mathrm{Hom}(X, \mathcal{R}^\times)$, we write

$$\mathcal{R}^\varphi := \{s \in \mathcal{R} \colon {}^\sigma s = \varphi(\sigma)s, \forall \sigma \in X\}.$$

As explained at the beginning of Section 5, we assume that $[\mathbb{F} \colon \mathbb{E}]$ is a power of 2. By Lemma 4.9 we may apply Lemma 7.19 to conclude that $\mathbb{F} = \oplus_{\varphi\in\Sigma^*}\mathbb{F}^\varphi$, where $\Sigma^* := \mathrm{Hom}(\Sigma, \mathbb{F}^\times)$. Note that since $\Sigma = \mathrm{Gal}(W/W')$, it follows that this decomposition lifts to W. More precisely, viewing elements of Σ as automorphisms of W and elements of Σ^* as valued in $W^\times$ by composing with the Teichmüller map, we can define W^φ for each $\varphi \in \Sigma^*$. Then Lemma 7.19 gives $W = \oplus_{\varphi\in\Sigma^*}W^\varphi$.

For all $\varphi \in \Sigma$, define

$$\mathfrak{A}(\varphi) := W^\varphi + W^\varphi J,$$

where $W^\varphi J := \{\sum_i \alpha_i j_i | \alpha_i \in W^\varphi, j_i \in J\}$. Since $\mathfrak{A} = W + WJ$ it follows immediately that $\mathfrak{A} = \sum_{\varphi\in\Sigma^*}\mathfrak{A}(\varphi)$. We will be interested in understanding when this sum is direct, because in that case we will show that it is possible to find lifts of elements of Σ in Σ_t. If $\mathfrak{a}$ is an ideal of $\mathfrak{A}$ and $\varphi \in \Sigma^*$, let $\mathfrak{a}(\varphi) := \mathfrak{A}(\varphi) \cap \mathfrak{a}$ and let $(\mathfrak{A}/\mathfrak{a})(\varphi) \subset \mathfrak{A}/\mathfrak{a}$ be the image of $\mathfrak{A}(\varphi)$ under the natural projection $\mathfrak{A} \to \mathfrak{A}/\mathfrak{a}$.

Lemma 5.5. *The following are equivalent:*

(1) $\mathfrak{A} = \oplus_{\varphi\in\Sigma^*}\mathfrak{A}(\varphi)$.

(2) For every $\mathfrak{A}$-ideal $\mathfrak{a}$ such that $\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}\mathfrak{a}(\varphi)$, we have $\mathfrak{A}/\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}(\mathfrak{A}/\mathfrak{a})(\varphi)$. Furthermore, there exists at least one such ideal $\mathfrak{a}$.

(3) There exists an $\mathfrak{A}$-ideal $\mathfrak{a}$ such that $\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}\mathfrak{a}(\varphi)$ and $\mathfrak{A}/\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}(\mathfrak{A}/\mathfrak{a})(\varphi)$.

Proof. First we show that (1) implies (2). We can take $\mathfrak{a} = 0$ for the existence statement in (2). Now suppose that $\mathfrak{a}$ is an $\mathfrak{A}$-ideal such that $\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}\mathfrak{a}(\varphi)$. If $\sum_{\varphi\in\Sigma^*} \overline{a}_\varphi = 0 \in \mathfrak{A}/\mathfrak{a}$ with each $\overline{a}_\varphi \in (\mathfrak{A}/\mathfrak{a})(\varphi)$, then letting $a_\varphi \in \mathfrak{A}(\varphi)$ be a lift of $\overline{a}_\varphi$, we see that $\sum_{\varphi\in\Sigma^*} a_\varphi \in \mathfrak{a} = \oplus_{\varphi\in\Sigma^*}\mathfrak{a}(\varphi)$. Thus, there are $\alpha_\varphi \in \mathfrak{a}(\varphi)$ such that $\sum_{\varphi\in\Sigma^*} a_\varphi = \sum_{\varphi\in\Sigma^*} \alpha_\varphi$. Since $\mathfrak{A} = \oplus_{\varphi\in\Sigma^*}\mathfrak{A}(\varphi)$, it follows that $a_\varphi = \alpha_\varphi$ for all $\varphi \in \Sigma^*$. Thus $\overline{a}_\varphi = 0 \in \mathfrak{A}/\mathfrak{a}$ for all $\varphi \in \Sigma^*$ and hence $\mathfrak{A}/\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}(\mathfrak{A}/\mathfrak{a})(\varphi)$.

The fact that (2) implies (3) is trivial.

To see that (3) implies (1), suppose that $\mathfrak{a}$ is an $\mathfrak{A}$-ideal such that $\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}\mathfrak{a}(\varphi)$ and $\mathfrak{A}/\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}(\mathfrak{A}/\mathfrak{a})(\varphi)$. For each $\varphi \in \Sigma^*$ fix a set $S_\varphi \subset \mathfrak{A}(\varphi)$ of representatives of $(\mathfrak{A}/\mathfrak{a})(\varphi)$ such that $0 \in S_\varphi$. Suppose that $\sum_{\varphi\in\Sigma^*} a_\varphi = 0$ with $a_\varphi \in \mathfrak{A}(\varphi)$. Then there is a unique way to write each a_φ as

$$a_\varphi = s_\varphi + \alpha_\varphi$$

with $s_\varphi \in S_\varphi$ and $\alpha_\varphi \in \mathfrak{a}(\varphi)$. Modulo $\mathfrak{a}$, we see that

$$\sum_{\varphi\in\Sigma^*} \overline{s}_\varphi = 0.$$

Since $\mathfrak{A}/\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}(\mathfrak{A}/\mathfrak{a})(\varphi)$, it follows that $\overline{s}_\varphi = 0$ for all φ. As $0 \in S_\varphi$, it follows that $s_\varphi = 0$ for all $\varphi \in \Sigma^*$. Therefore $a_\varphi = \alpha_\varphi \in \mathfrak{a}(\varphi)$. As $\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}\mathfrak{a}(\varphi)$, it follows that each $a_\varphi = 0$. Thus $\mathfrak{A} = \oplus_{\varphi\in\Sigma^*}\mathfrak{A}(\varphi)$. □

Definition 5.6. Let $\mathbb{L}_2 \subset \mathbb{L}_1$ be subfields of $\mathbb{F}$. We say that J is *small with respect to* $\mathbb{L}_1/\mathbb{L}_2$ if

$$\ker(W(\mathbb{L}_1) \otimes_{W(\mathbb{L}_2)} W(\mathbb{L}_2)J \to W(\mathbb{L}_1)J) = 0,$$

where the map is given by multiplication inside $\mathfrak{A}$. Otherwise, we say that J is *big with respect to* $\mathbb{L}_1/\mathbb{L}_2$.

To motivate Definition 5.6, recall from Remark 5.4 that we need to be able to write elements of $W(\mathbb{F}^{\beta_t(\Sigma_t)})$ as quotients of elements in J whenever $\mathbb{F}^{\beta_t(\Sigma_t)} \neq \mathbb{E}$. Suppose that $\mathbb{L}_2 = \mathbb{E}, \mathbb{L}_1 = \mathbb{F}^{\beta_t(\Sigma_t)}$, and $[\mathbb{L}_1 \colon \mathbb{L}_2] = 2$. Write $\mathbb{L}_1 = \mathbb{L}_2(\alpha)$. Then $W(\mathbb{L}_1) = W(\mathbb{L}_2) \oplus s(\alpha)W(\mathbb{L}_2)$ and so

$$W(\mathbb{L}_1) \otimes_{W(\mathbb{L}_2)} W(\mathbb{L}_2)J = W(\mathbb{L}_2)J \oplus (s(\alpha)W(\mathbb{L}_2) \otimes_{W(\mathbb{L}_2)} W(\mathbb{L}_2)J).$$

If J is big with respect to $\mathbb{L}_1/\mathbb{L}_2$, then we can find $x, y \in W(\mathbb{L}_2)J \setminus \{0\}$ such that $x + s(\alpha)y = 0$. Thus $s(\alpha) = x/y$, and hence $W(\mathbb{L}_1)$ is in the field of fractions of any domain containing $W(\mathbb{L}_2)J$. In contrast, the following proposition shows that when J is small with respect to $\mathbb{F}/\mathbb{F}'$, elements of Σ can be lifted to automorphisms of $\mathfrak{A}$.

Proposition 5.7. *If J is small with respect to $\mathbb{F}/\mathbb{F}'$, then $\mathfrak{A} = \oplus_{\varphi\in\Sigma^*}\mathfrak{A}(\varphi)$. In this case, every $\overline{\sigma} \in \Sigma$ can be lifted to an automorphism σ of $\mathfrak{A}$ such that σ acts trivially on J, and a lift with this property is unique.*

Proof. Note that $\mathfrak{a} \coloneqq WJ$ is an $\mathfrak{A}$-ideal since $\mathfrak{A} = W + WJ$ and J is multiplicatively closed as discussed prior to Remark 5.4. The assumption that J is small with respect to $\mathbb{F}/\mathbb{F}'$ implies that $WJ = \oplus_{\varphi\in\Sigma^*}W^\varphi J$. Indeed,

$$\bigoplus_{\varphi\in\Sigma^*}(W^\varphi \otimes_{W'} W'J) = \Big(\bigoplus_{\varphi\in\Sigma^*} W^\varphi\Big) \otimes_{W'} W'J = W \otimes_{W'} W'J \hookrightarrow WJ,$$

and the image of $W^\varphi \otimes_{W'} W'J$ is exactly $W^\varphi J$. Since $WJ = \sum_{\varphi\in\Sigma^*} W^\varphi J$, it follows that the multiplication map is an isomorphism and thus WJ is graded by Σ^*. Note that $\mathfrak{a}(\varphi) = W^\varphi J$, so $\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}\mathfrak{a}(\varphi)$.

By Lemma 5.5, for the first statement of the proposition it suffices to show that $\mathfrak{A}/WJ = \oplus_{\varphi\in\Sigma^*}(\mathfrak{A}/WJ)(\varphi)$. Note that

$$\mathfrak{A}/WJ = (W + WJ)/WJ \cong W/(W \cap WJ)$$

and $W \cap WJ$ is a closed W-submodule of pW since $J \subseteq \mathfrak{m}_{\mathfrak{A}}$. Thus we have $W \cap WJ = p^n W$ and $\mathfrak{A}/WJ \cong W/p^n W$ for some $1 \le n \le \infty$, where $p^\infty W \coloneqq \{\infty\}$. Since W is graded by Σ^*, it follows from Lemma 5.5 that $W/p^n W$ is graded by Σ^* as well. Therefore $\mathfrak{A} = \oplus_{\varphi \in \Sigma^*} \mathfrak{A}(\varphi)$.

For the second statement, we let σ act by $W(\overline{\sigma})$ on W and trivially on J. The only question is to verify that this is well defined. Since $\mathfrak{A} = \oplus_{\varphi \in \Sigma^*} \mathfrak{A}(\varphi)$, it suffices to show that σ is well defined on each $\mathfrak{A}(\varphi)$. That is, we must show

$$\sum_{i=1}^{n} \alpha_i j_i = 0 \implies \sum_{i=1}^{n} W(\overline{\sigma})(\alpha_i) j_i = 0,$$

where $\alpha_i \in W^\varphi, j_i \in J$. Since J is small with respect to $\mathbb{F}/\mathbb{F}'$, $\sum_{i=1}^n \alpha_i j_i = 0$ implies that $\sum_{i=1}^n \alpha_i \otimes j_i = 0 \in W \otimes_{W'} W'J$. Since $\alpha_i \in W^\varphi$, we know that $W(\overline{\sigma})(\alpha_i) = s(\varphi(\overline{\sigma}))\alpha_i$ for all i. Hence

$$0 = \sum_{i=1}^{n} \alpha_i \otimes j_i \implies 0 = s(\varphi(\overline{\sigma})) \sum_{i=1}^{n} \alpha_i \otimes j_i.$$

Therefore $\sum_{i=1}^n W(\overline{\sigma})(\alpha_i) j_i = 0$, since it is the image of $s(\varphi(\overline{\sigma})) \sum_{i=1}^n \alpha_i \otimes j_i$ under $W \otimes_{W'} W'J \to WJ$. □

Now that we have lifted elements of Σ to automorphisms of $\mathfrak{A}$ under the smallness assumption, we would like to verify that the lifts are conjugate self-twists of (t, d) when $\mathfrak{A} = A$ and that they come from conjugate self-twists when $\mathfrak{A} = A^+$. (Recall that A^+ is only defined when $\ker \beta_t$ is nontrivial; see Lemma 4.16.)

Theorem 5.8. *If J is small with respect to $\mathbb{F}/\mathbb{F}'$ then Σ is contained in the image of $\beta_t \colon \Sigma_t \to \Sigma_{\overline{\rho}}$. Furthermore, every lift of $\overline{\sigma} \in \Sigma$ to Σ_t acts trivially on J.*

Proof. Fix $\overline{\sigma} \in \Sigma$. By Proposition 5.7, there is a unique $\sigma \in \operatorname{Aut} \mathfrak{A}$ that acts as $W(\overline{\sigma})$ on W and fixes J. If $\ker \beta_t = 1$, then $\mathfrak{A} = A$. If $\ker \beta_t \neq 1$, then $\mathfrak{A} = A^+$ and we need to extend σ to $A = A^+ \oplus A^-$. We do this by declaring that σ fixes A^-; we will still denote the automorphism of A by σ. We already know that σ acts trivially on J, so it is enough to prove that $\sigma \in \Sigma_t$.

Our strategy is to show that σ comes from an appropriate element of Σ_T. More precisely, we claim that there is some $(\tilde{\sigma}, \eta) \in \Sigma_T^{\text{pairs}}$ such that $\sigma \circ \alpha_t = \alpha_t \circ \tilde{\sigma}$, where $\alpha_t \colon \mathcal{A} \to A$ is the W-algebra homomorphism given by universality. If this is true, then for all $g \in \Pi$ we have

$${}^{\sigma}t(g) = \sigma \circ \alpha_t(T(g)) = \alpha_t \circ \tilde{\sigma}(T(g)) = \alpha_t(\eta(g)T(g)) = \eta(g)\alpha_t(T(g)) = \eta(g)t(g)$$

since α_t is a W-algebra homomorphism and $\eta(g) \in W$. Thus $\sigma \in \Sigma_t$.

First suppose that $\overline{\rho}$ is not dihedral. Then there is a unique lift $\tilde{\sigma}$ of $\overline{\sigma}$ in Σ_T by Lemma 4.1 and Proposition 4.14. By Proposition 4.14, we know that $\tilde{\sigma}$ acts as $W(\overline{\sigma})$ on the image of W in $\mathcal{A}$. Furthermore, $\tilde{\sigma}$ acts trivially on $I_1(\rho^{\text{univ}})$ by our fixed choice of ρ^{univ} after Corollary 5.3. Since $\overline{\rho}$ is not dihedral, it follows that $\mathcal{A} = W + WJ(\rho^{\text{univ}})$. By the construction of ρ^{univ} and ρ_t, we have $\alpha_t(I_1(\rho^{\text{univ}})) = I_1(\rho_t)$ and thus $\alpha_t(J(\rho^{\text{univ}})) = J(\rho_t) = J$. Recall that σ acts trivially on J. Both $\tilde{\sigma}$ and σ act on W by $W(\overline{\sigma})$. Thus for any $\sum_{i=1}^n a_i x_i$ with $a_i \in W, x_i \in J(\rho^{\text{univ}}) \cup \{1\}$, we have

$$\sigma \circ \alpha_t \left(\sum_{i=1}^{n} a_i x_i \right) = \sum_{i=1}^{n} W(\overline{\sigma})(a_i)\alpha_t(x_i) = \alpha_t \circ \tilde{\sigma} \left(\sum_{i=1}^{n} a_i x_i \right).$$

If $\overline{\rho}$ is dihedral, then there are two lifts of $\overline{\sigma}$ in Σ_T by Lemma 4.1 and Proposition 4.14. One acts on $\mathcal{A}^-$ by $+1$ and the other acts by -1 by Lemma 4.16 and since we chose ρ^{univ} such that $B_1(\rho^{\text{univ}})$ is fixed by ν, which generates a complement of $\ker \beta_T$. Let $\tilde{\sigma} \in \Sigma_T$ be the lift of $\overline{\sigma}$ that is in $\langle \nu \rangle$. Thus $\tilde{\sigma}$ acts trivially on $J(\rho^{\text{univ}})$ and $B_1(\rho^{\text{univ}})$. Then an argument similar to that in the previous paragraph shows that $\sigma \circ \alpha_t = \alpha_t \circ \tilde{\sigma}$. □

Corollary 5.9. *If J is small with respect to $\mathbb{F}/\mathbb{E}$ then $A^{\Sigma_t} = W(\mathbb{E}) + J$. Suppose furthermore that either $\overline{\rho}$ is not dihedral or $\ker \beta_t \neq 1$. Then $A^{\Sigma_t} = W(\mathbb{E})[I_1(\rho_t)]$.*

Proof. By Theorem 5.8 applied to $\Sigma = \Sigma_t$, the map β_t is a surjection and Σ_t acts trivially on J. If $\ker \beta_t = 1$, then $A = \mathfrak{A} = W + WJ$, so $A^{\Sigma_t} = W(\mathbb{E}) + J$.

If $\ker \beta_t \neq 1$, then $A = W + WI_1(\rho_t) + WI_1(\rho_t)^2 + WB_1(\rho_t)$ and $J = W(\mathbb{E})I_1(\rho_t) + W(\mathbb{E})I_1(\rho_t)^2$. Note that $A^{\Sigma_t} \subseteq A^+ = W + WJ$ since the nontrivial element in $\ker \beta_t$ acts by -1 on $B_1(\rho_t)$ by Lemma 4.16. As above, we have that

$$A^{\Sigma_t} = (W + WJ)^{\Sigma_t} = W(\mathbb{E}) + J.$$

The last sentence in the statement of the corollary follows from the definition of J. □

Remark 5.10. Note that none of the arguments in Section 5.3 require that A is a domain. In particular, when J is small with respect to $\mathbb{F}/\mathbb{E}$ and either $\overline{\rho}$ is not dihedral or $\ker \beta_t \neq 1$, Corollary 5.9 gives a conceptual interpretation of the ring $W(\mathbb{E})[I_1(\rho_t)]$.

5.4. **When J is big with respect to $\mathbb{F}/\mathbb{E}$.** Corollary 5.9 requires the assumption that J is small with respect to $\mathbb{F}/\mathbb{E}$. We do not always expect this to be true. The purpose of Section 5.4 is to show that A^{Σ_t} and $W(\mathbb{E})[I_1(\rho_t)]$ have the same field of fractions and A^{Σ_t} is a finite type $W(\mathbb{E})[I_1(\rho_t)]$-module even without the smallness assumption. This is done in Corollary 5.15, although the two key inputs to that theorem are Propositions 5.13 and 5.14. Then we can apply Lemma 2.16 and Corollary 3.8 to conclude that ρ_t, and thus (t, d), is A^{Σ_t}-full in Corollary 5.16.

The discussion following Definition 5.6 shows why one may expect to get $Q(A^{\Sigma_t}) = Q(W(\mathbb{E})[I_1(\rho_t)])$ when smallness fails and $[\mathbb{F}^{\beta_t(\Sigma_t)} \colon \mathbb{E}] = 2$. Unfortunately, the assumption that $[\mathbb{F}^{\beta_t(\Sigma_t)} \colon \mathbb{E}] = 2$ is rather critical to that argument. This is the primary reason we insist that $[\mathbb{F} \colon \mathbb{E}]$ be a power of 2 throughout this section. It allows us to split up the extension $\mathbb{F}^{\beta_t(\Sigma_t)}/\mathbb{E}$ into a series of quadratic extensions, and thus we can apply the argument following Definition 5.6 inductively. This is the essential idea of the argument; we now prepare some notation to formalize it.

Write $[\mathbb{F}^{\beta_t(\Sigma_t)} \colon \mathbb{E}] = 2^n$ for some $n \geq 0$. Let us define some notation related to the intermediate fields between $\mathbb{F}^{\beta_t(\Sigma_t)}$ and $\mathbb{E}$. For integers $0 \leq i \leq n$, let $\mathbb{E}_i$ be the unique extension of $\mathbb{E}$ of degree 2^i. In particular, $\mathbb{E}_0 = \mathbb{E}$ and $\mathbb{E}_n = \mathbb{F}^{\beta_t(\Sigma_t)}$, and $[\mathbb{E}_i \colon \mathbb{E}_{i-1}] = 2$ for all $1 \leq i \leq n$. For $0 \leq i \leq n$, let W_i denote the image of $W(\mathbb{E}_i)$ in $\mathfrak{A}$. Define

$$\mathfrak{A}_i := W_i + W_i J \subseteq \mathfrak{A}.$$

In particular, $\mathfrak{A}_0 = W(\mathbb{E}) + J$ and $\mathfrak{A}_n = W(\mathbb{F}^{\beta_t(\Sigma_t)}) + W(\mathbb{F}^{\beta_t(\Sigma_t)})J$. Since A is a domain, so are all of the $\mathfrak{A}_i$, and we write $Q(\mathfrak{A}_i)$ for the field of fractions of $\mathfrak{A}_i$.

In the case when $\mathfrak{A} = A^+$, there is a 2-to-1 group homomorphism $\Sigma_t \to \operatorname{Aut} A^+$ given by restricting elements of Σ_t to A^+. Let $\Sigma_t(\mathfrak{A})$ denote the image of this map when $\mathfrak{A} = A^+$, and otherwise (that is, whenever $\ker \beta_t = 1$) let $\Sigma_t(\mathfrak{A}) = \Sigma_t$. In either case we can, and do, identify $\Sigma_t(\mathfrak{A})$ with a subgroup of $\Sigma_{\overline{\rho}}$ via β_t and we have $\mathbb{E}_n = \mathbb{F}^{\beta_t(\Sigma_t(\mathfrak{A}))}$. We write $\Sigma_t(\mathfrak{A})^* := \operatorname{Hom}(\Sigma_t(\mathfrak{A}), \mathfrak{A}^\times)$.

We begin with two preliminary lemmas about the relationship between smallness and the $\mathbb{E}_i$.

Lemma 5.11. *We have that J is small with respect to $\mathbb{F}/\mathbb{E}_n$; that is,* $\ker(W \otimes_{W_n} W_n J \to WJ) = 0$.

Proof. Recall that WJ is an $\mathfrak{A}$-ideal that is stable under the action of $\Sigma_t(\mathfrak{A})$ since Σ_t fixes J by the construction of ρ_t. By Lemma 4.9, we can apply Lemma 7.19 with $X = \Sigma_t(\mathfrak{A})$. Therefore

$$WJ = \bigoplus_{\varphi \in \Sigma_t(\mathfrak{A})^*} (WJ)^\varphi,$$

where $(WJ)^\varphi := \{x \in WJ \colon {}^\sigma x = \varphi(\sigma)x, \forall \sigma \in \Sigma_t(\mathfrak{A})\}$. Recall that $W^\varphi J$ was defined prior to Lemma 5.5. We claim that

$$(WJ)^\varphi = W^\varphi J.$$

Clearly $(WJ)^\varphi \supseteq W^\varphi J$ since $\Sigma_t(\mathfrak{A})$ acts trivially on J. On the other hand,

$$WJ = \bigoplus_{\varphi \in \Sigma_t(\mathfrak{A})^*} (WJ)^\varphi = \sum_{\varphi \in \Sigma_t(\mathfrak{A})^*} (WJ)^\varphi \supseteq \sum_{\varphi \in \Sigma_t(\mathfrak{A})^*} W^\varphi J = WJ,$$

so we must have equality.

For each $\varphi \in \Sigma_t(\mathfrak{A})^*$, choose $x_\varphi \in \mathbb{F}^\varphi \setminus \{0\}$. Then $\{s(x_\varphi)\colon \varphi \in \Sigma_t(\mathfrak{A})^*\}$ is a W_n-basis for W. Thus we have

$$W \otimes_{W_n} W_n J = \bigoplus_{\varphi \in \Sigma_t(\mathfrak{A})^*} W_n s(x_\varphi) \otimes_{W_n} W_n J.$$

If $x \in \ker(W \otimes_{W_n} W_n J \to WJ)$, then we can write

$$x = \sum_{\varphi \in \Sigma_t(\mathfrak{A})^*} s(x_\varphi) \otimes y_\varphi$$

for some $y_\varphi \in W_n J$. Then we have

$$0 = \sum_{\varphi \in \Sigma_t(\mathfrak{A})^*} s(x_\varphi) y_\varphi$$

and $s(x_\varphi) y_\varphi \in W^\varphi J$. Since $WJ = \oplus_{\varphi \in \Sigma_t(\mathfrak{A})^*} W^\varphi J$, it follows that each $s(x_\varphi) y_\varphi = 0$. As $\mathfrak{A}$ is a domain and $s(x_\varphi) \neq 0$, it follows that $y_\varphi = 0$ for all $\varphi \in \Sigma_t(\mathfrak{A})^*$. □

Lemma 5.12. *We have*

$$\ker(W \otimes_{W_{n-1}} W_{n-1} J \to WJ) = W \otimes_{W_n} \ker(W_n \otimes_{W_{n-1}} W_{n-1} J \to W_n J).$$

Proof. Let $K \coloneqq \ker(W_n \otimes_{W_{n-1}} W_{n-1} J \to W_n J)$. We have an exact sequence of W_n-modules

$$0 \to K \to W_n \otimes_{W_{n-1}} W_{n-1} J \to W_n J \to 0.$$

Since W is free over W_n, tensoring with W over W_n gives an exact sequence

$$0 \to W \otimes_{W_n} K \to W \otimes_{W_{n-1}} W_{n-1} J \to W \otimes_{W_n} W_n J \to 0.$$

We can identify the last nonzero term in this sequence with WJ by Lemma 5.11. Thus $W \otimes_{W_n} K = \ker(W \otimes_{W_{n-1}} W_{n-1} J \to WJ)$. □

Proposition 5.13. *If J is big with respect to $\mathbb{F}/\mathbb{E}$ then $Q(\mathfrak{A}_n) = Q(\mathfrak{A}_{n-1})$.*

Proof. We claim that J is big with respect to $\mathbb{E}_n/\mathbb{E}_{n-1}$. Indeed, if J were small with respect to $\mathbb{E}_n/\mathbb{E}_{n-1}$, then J would be small with respect to $\mathbb{F}/\mathbb{E}_{n-1}$ by Lemma 5.12. Therefore we could apply Theorem 5.8 with $\Sigma = \mathrm{Gal}(\mathbb{F}/\mathbb{E}_{n-1})$, which implies that $\mathbb{E}_n \subseteq \mathbb{E}_{n-1}$, a contradiction.

Let $\{1, \alpha\}$ be an $\mathbb{E}_{n-1}$-basis for $\mathbb{E}_n$. Then $\{1, s(\alpha)\}$ is a W_{n-1}-basis for W_n and so

$$W_n \otimes_{W_{n-1}} W_{n-1} J = (W_{n-1} \otimes_{W_{n-1}} W_{n-1} J) \oplus (W_{n-1} s(\alpha) \otimes_{W_{n-1}} W_{n-1} J).$$

Since J is big with respect to $\mathbb{E}_n/\mathbb{E}_{n-1}$, there exist $x, y \in W_{n-1} J \setminus \{0\}$ such that

$$x + s(\alpha) y = 0.$$

Thus, $s(\alpha) = -x/y \in Q(\mathfrak{A}_{n-1})$. It follows that $W_n \subset Q(\mathfrak{A}_{n-1})$ and hence $Q(\mathfrak{A}_n) = Q(\mathfrak{A}_{n-1})$. □

Finally, we descend from $Q(\mathfrak{A}_n)$ to $Q(\mathfrak{A}_0)$ by induction on n.

Proposition 5.14. *For all $2 \leq k \leq n$, if $Q(\mathfrak{A}_k) = Q(\mathfrak{A}_{k-1})$ then $Q(\mathfrak{A}_{k-1}) = Q(\mathfrak{A}_{k-2})$. In particular, if J is big with respect to $\mathbb{F}/\mathbb{E}$, then $Q(A^{\Sigma_t}) = Q(W(\mathbb{E}) + J)$.*

Proof. Note that for any $k \geq 1$ we have $Q(\mathfrak{A}_k) = Q(\mathfrak{A}_{k-1})$ if and only if $W_k \subseteq Q(\mathfrak{A}_{k-1})$. Assume that $Q(\mathfrak{A}_k) = Q(\mathfrak{A}_{k-1})$ for some k, $2 \leq k \leq n$. Choose $\overline{\alpha} \in \mathbb{E}_{k-2}, \overline{\beta} \in \mathbb{E}_{k-1}$ such that $\mathbb{E}_{k-1} = \mathbb{E}_{k-2}(\sqrt{\overline{\alpha}})$ and $\mathbb{E}_k = \mathbb{E}_{k-1}(\sqrt{\overline{\beta}})$. Define $\alpha \coloneqq s(\overline{\alpha})$ and $\beta \coloneqq s(\overline{\beta})$, so $W_{k-1} = W_{k-2}(\sqrt{\alpha})$ and $W_k = W_{k-1}(\sqrt{\beta})$. It suffices to show that $\sqrt{\alpha} \in Q(\mathfrak{A}_{k-2})$.

Since $Q(\mathfrak{A}_k) = Q(\mathfrak{A}_{k-1})$, we can write $\sqrt{\beta} = x/y$ with $x, y \in \mathfrak{A}_{k-1} \setminus \{0\}$. By multiplying x and y by any nonzero element of $W_{k-1} J$, we may assume that $x, y \in W_{k-1} J \setminus \{0\}$.

Note that we can write $y = i_1 + \sqrt{\alpha} i_2$ with $i_1, i_2 \in W_{k-2} J$. If $i_1 - \sqrt{\alpha} i_2 \neq 0$, then by multiplying x and y by $i_1 - \sqrt{\alpha} i_2$, we may assume that $y \in W_{k-2} J \setminus \{0\}$. If $i_1 - \sqrt{\alpha} i_2 = 0$ and $y \notin W_{k-2} J$, then we must have $i_2 \neq 0$ since $y \neq 0$ and $\sqrt{\alpha} = i_1/i_2 \in Q(\mathfrak{A}_{k-2})$, as desired. We assume henceforth that $y \in W_{k-2}J$.

Write $x = a + b\sqrt{\alpha}$ for some $a, b \in W_{k-2}J$. Then we have $y\sqrt{\beta} = a + b\sqrt{\alpha}$ and thus

$$y^2\beta = a^2 + \alpha b^2 + 2ab\sqrt{\alpha}. \tag{6}$$

Since $\beta \in W_{k-1}$, we may write $\beta = e + f\sqrt{\alpha}$ for some $e, f \in W_{k-2}$. Note that $f \not\equiv 0 \bmod p$ since $[\mathbb{E}_k \colon \mathbb{E}_{k-1}] = 2$ and $\mathbb{E}_k = \mathbb{E}_{k-1}(\sqrt{\bar{\beta}})$. Substituting this into equation (6), we see that

$$(y^2 f - 2ab)\sqrt{\alpha} = a^2 + \alpha b^2 - y^2 e \in W_{k-2}J.$$

Note that $y^2 f - 2ab \in W_{k-2}J$ since all of $y, f, a, b \in W_{k-2}J$. If $y^2 f - 2ab \neq 0$, then we can conclude that $\sqrt{\alpha} \in Q(\mathfrak{A}_{k-2})$ as desired.

Henceforth, assume that $y^2 f = 2ab$. Then we also have $y^2 e = a^2 + \alpha b^2$. Thus $2f^{-1}eab = a^2 + \alpha b^2$. Note that $a, b \neq 0$ since $2ab = y^2 f$ and we know $y, f \neq 0$. Then we have

$$2f^{-1}e = \frac{a}{b} + \alpha\frac{b}{a}.$$

Therefore $\frac{a}{b}$ is a root of $t^2 - 2f^{-1}et + \alpha \in W_{k-2}[t]$. The discriminant of this polynomial is $4(f^{-2}e^2 - \alpha)$. We claim that $W_{k-1} \supseteq W_{k-2}(\sqrt{f^{-2}e^2 - \alpha})$. Note that since $\beta = s(\bar{\beta})$ and s is multiplicative we have that $e^2 - f^2\alpha = s(N_{\mathbb{E}_{k-1}/\mathbb{E}_{k-2}}(\bar{\beta}))$, and therefore $\sqrt{e^2 - f^2\alpha} = s(\sqrt{N_{\mathbb{E}_{k-1}/\mathbb{E}_{k-2}}(\bar{\beta})}) \in s(\mathbb{E}_{k-1}^{\times})$. Therefore $\frac{a}{b} \in W_{k-1}$. Thus we can write $\sqrt{\beta} = \frac{x/b}{y/b} = \frac{\frac{a}{b} + \sqrt{\alpha}}{y/b}$, and so

$$\frac{y}{b} = \left(\frac{a}{b} + \sqrt{\alpha}\right)\sqrt{\beta}^{-1}.$$

Note that $\frac{a}{b} + \sqrt{\alpha} \neq 0$ since $y \neq 0$. It follows that $(\frac{a}{b} + \sqrt{\alpha})\sqrt{\beta}^{-1}$ generates W_k over W_{k-1} since $\frac{a}{b} + \sqrt{\alpha} \in W_{k-1}$ and $\sqrt{\beta}$ generates W_k over W_{k-1}. Thus $(\frac{a}{b} + \sqrt{\alpha})\sqrt{\beta}^{-1} = \frac{y}{b} \in Q(\mathfrak{A}_{k-2})$ and so $W_k \subset Q(\mathfrak{A}_{k-2})$. Therefore $Q(\mathfrak{A}_k) = Q(\mathfrak{A}_{k-2})$.

For the second statement of the proposition, note that by Proposition 5.13 we have $Q(\mathfrak{A}_n) = Q(\mathfrak{A}_0)$ for all $0 \leq k \leq n$. We have $\mathfrak{A}_0 = W(\mathbb{E}) + J$ by definition. Since Σ_t acts trivially on J, it follows that

$$A^{\Sigma_t} = \mathfrak{A}^{\Sigma_t(\mathfrak{A})} = W_n + W_n J = \mathfrak{A}_n.$$

□

Corollary 5.15. *We have*

(1) $A^{\Sigma_t} \supseteq W(\mathbb{E})[I_1(\rho_t)]$;
(2) A^{Σ_t} *is a finitely generated* $W(\mathbb{E})[I_1(\rho_t)]$*-module;*
(3) A^{Σ_t} *has the same field of fractions as* $W(\mathbb{E})[I_1(\rho_t)]$.

Proof. The fact that $A^{\Sigma_t} \supseteq W(\mathbb{E})[I_1(\rho_t)]$ follows from the definition of $\mathbb{E}$ and Corollary 5.3.

If either $\bar{\rho}$ is not dihedral or $\ker \beta_t \neq 1$, then $A^{\Sigma_t} = W(\mathbb{F})^{\Sigma_t}[I_1(\rho_t)]$, which is finitely generated over $W(\mathbb{E})[I_1(\rho_t)]$ since $W(\mathbb{F})^{\Sigma_t}$ is finitely generated over $W(\mathbb{E})$. In the case when $\bar{\rho}$ is dihedral and $\ker \beta_t = 1$ we have to appeal to the commutative algebra results in Section 7.2. Recall from Lemma 4.16 that the universal constant-determinant pseudodeformation ring $\mathcal{A}$ decomposes as $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$, where the grading is given by the nontrivial element in $\ker \beta_T$. By Lemma 4.16 we have that $\mathcal{A}^+ = W(\mathbb{F})[I_1(\rho^{\mathrm{univ}})]$ and $\mathcal{A}^- = W(\mathbb{F})B_1(\rho^{\mathrm{univ}})$. Recall from the discussion in Section 2.1 that $\mathcal{A}$ is Noetherian. Thus, by Lemma 7.13 and Proposition 7.15 it follows that $\mathcal{A}^+$ is Noetherian ring and $\mathcal{A}$ is a Noetherian $\mathcal{A}^+$-module. Therefore, $\mathcal{A}^-$ is a Noetherian $\mathcal{A}^+$-module. The natural map $\alpha_t \colon \mathcal{A} \to A$ sends $\mathcal{A}^+$ onto $W(\mathbb{F})[I_1(\rho_t)]$ and $\mathcal{A}^-$ onto $W(\mathbb{F})B_1(\rho_t)$ since $\alpha_t(I_1(\rho^{\mathrm{univ}})) = I_1(\rho_t)$ and $\alpha_t(B_1(\rho^{\mathrm{univ}})) = B_1(\rho_t)$. Thus it follows that $W(\mathbb{F})[I_1(\rho_t)]$ is a Noetherian ring and $W(\mathbb{F})B_1(\rho_t)$ is a Noetherian $W(\mathbb{F})[I_1(\rho_t)]$-module. Since $W(\mathbb{F})$ is a Noetherian $W(\mathbb{E})$-module, it follows that

$A = W(\mathbb{F})[I_1(\rho_t)] + W(\mathbb{F})B_1(\rho_t)$ is a Noetherian $W(\mathbb{E})[I_1(\rho_t)]$-module. Hence A^{Σ_t} is a Noetherian (and thus finitely generated) $W(\mathbb{E})[I_1(\rho_t)]$-module.

The third point has largely been established already. When J is small with respect to $\mathbb{F}/\mathbb{E}$, it follows from Corollary 5.9. When J is big with respect to $\mathbb{F}/\mathbb{E}$ and either $\overline{\rho}$ is not dihedral or $\ker \beta_t \neq 1$, this follows from Proposition 5.14 since in those cases $W(\mathbb{E}) + J = W(\mathbb{E})[I_1(\rho_t)]$. Thus we may assume that $\overline{\rho}$ is dihedral, $\ker \beta_t = 1$, and J is big with respect to $\mathbb{F}/\mathbb{E}$. Then we have

$$Q(A^{\Sigma_t}) = Q(W(\mathbb{E}) + J)$$

by Proposition 5.14. Consider the ring $\mathcal{A}' \coloneqq W(\mathbb{E}) + J(\rho^{\mathrm{univ}}) + W(\mathbb{E})B_1(\rho^{\mathrm{univ}})$, a local Noetherian subring of $\mathcal{A}$. By Lemma 4.16, the nontrivial element in $\ker \beta_T$ is an involution on $\mathcal{A}'$. The natural map $\alpha_t \colon \mathcal{A} \to A$ restricts to a surjection $\mathcal{A}' \to W(\mathbb{E}) + J$. The image of $(\mathcal{A}')^+$ in $W(\mathbb{E}) + J$ is $W(\mathbb{E})[I_1(\rho_t)]$. Applying Proposition 7.16 to the quotient $\mathcal{A}' \to W(\mathbb{E}) + J$, it follows that $Q(W(\mathbb{E}) + J) = Q(W(\mathbb{E})[I_1(\rho_t)])$. Thus

$$Q(A^{\Sigma_t}) = Q(W(\mathbb{E}) + J) = Q(W(\mathbb{E})[I_1(\rho_t)]).$$

□

We now have the following corollary, which summarizes the most general theorem we have for images of admissible pseudodeformations.

Corollary 5.16. *Let $\overline{\rho} \colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ be a regular representation such that the order of $\det \overline{\rho}$ is a power of 2. If $\overline{\rho}$ is octahedral, assume furthermore that $\overline{\rho}$ is good. Let A be a domain and $(t, d) \colon \Pi \to A$ an admissible pseudodeformation of $\overline{\rho}$. If (t, d) is not a priori small, then (t, d) is A^{Σ_t}-full.*

Proof. By Corollary 3.8, ρ_t is $W(\mathbb{E})[I_1(\rho_t)]$-full. By Corollary 5.15 and Lemma 2.16, it follows that ρ_t is A^{Σ_t}-full. □

5.5. **Nonadmissible pseudorepresentations.** The results of [Bel18] and of the previous sections depend on the assumption that the pseudodeformations in question have constant determinant whose order is a power of 2. Having in mind applications to Galois representations arising from geometry, we want to transfer the results of the previous sections, specifically Corollary 5.16, to representations that do not have constant determinant.

Theorem 5.17. *Let p be an odd prime and A a local pro-p domain with residue field $\mathbb{F}$. Let Π be a profinite group satisfying Mazur's p-finiteness condition. Let $\overline{\rho} \colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ be a regular semisimple representation. If $\overline{\rho}$ is octahedral, assume further that $\overline{\rho}$ is good. If $(t, d) \colon \Pi \to A$ is a pseudodeformation of $\overline{\rho}$ that is not a priori small, then (t, d) is A_0-full.*

Proof. Let $\chi \colon \Pi \to A^\times$ be a character such that $(t', d') \coloneqq (\chi t, \chi^2 d)$ is a constant-determinant pseudorepresentation, and write $\overline{\rho}' \coloneqq \overline{\chi} \otimes \overline{\rho}$, where $\overline{\chi} \colon \Pi \to \mathbb{F}^\times$ is the reduction of χ modulo $\mathfrak{m}$. Assume that χ is chosen such that $\overline{\rho}'$ has no conjugate self-twists if $\overline{\rho}$ is reducible and the order of $\det \overline{\rho}'$ is a power of 2 if $\overline{\rho}$ is absolutely irreducible. This is possible by Corollary 4.4 in the reducible case and Lemma 2.3 in the absolutely irreducible case. Furthermore, note that if $\overline{\rho}$ is octahedral and good, then so is $\overline{\rho}'$ by Lemma 4.11. Let A' be the subring of A topologically generated by $t'(\Pi)$. We have seen in Proposition 5.1 and Corollary 5.16 that if $\overline{\rho}'$ is regular (and under the further assumption that $\overline{\rho}$ is good when $\overline{\rho}$ is octahedral) and (t', d') is not a priori small, then (t', d') is $(A')^{\Sigma_{t'}}$-full. This is sufficient by Corollary 3.2. □

Remark 5.18. Recall that if $(t, d) \colon \Pi \to A$ is an admissible pseudorepresentations, then A is topologically generated by $t(\Pi)$ as a $W(\mathbb{F})$-algebra. We remark that this property is not stable under the twisting operation used in the proof of Theorem 5.17: given (t, d) and (t', d') as in the proof of Theorem 5.17, such that A is generated by $t(\Pi)$ as a $W(\mathbb{F})$-algebra, it is not true in general that A is also generated by $t'(\Pi)$ as a $W(\mathbb{F})$-algebra. Take for instance $\Pi = \mathbb{Z}_p$, $A = \mathbb{Z}_p[\![T]\!]$, $\bar{\rho}$ trivial and t the trace of the representation $\rho(g) := \begin{pmatrix} 1+T & 0 \\ 0 & 1+T \end{pmatrix}^g$ for all $g \in \mathbb{Z}_p$. Then A is topologically generated as a $\mathbb{Z}_p$-algebra by the image of t, but not by the image of t'.

6. Applications

In this section we specialize Theorem 5.17 to some arithmetic settings, more specifically to representations coming from elliptic or Hilbert cuspidal eigenforms (Section 6.1) and cuspidal p-adic families of elliptic eigenforms (Section 6.2). We show how to recover, and in some cases improve, the results already present in the literature. We also show that one can obtain exceptionally strong fullness results when the image of $\overline{\rho}$ contains $\mathrm{SL}_2(\mathbb{E})$ (Section 6.3).

6.1. **Galois representations attached to non-CM cuspidal elliptic or Hilbert modular eigenforms.** Let F be a totally real field (possibly equal to $\mathbb{Q}$) and f a non-CM cuspidal Hilbert modular eigenform over F all of whose weights are at least 2. Fix an algebraic closure $\overline{F}$ of F, and let $G_F := \mathrm{Gal}(\overline{F}/F)$. Fix a prime $p > 2$ and an embedding $\iota_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let $\rho_{f,\iota_p}\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be the Galois representation attached to f in the usual way. We may, up to conjugation, view ρ_{f,ι_p} as taking values in $\mathrm{GL}_2(\mathcal{O})$ for some finite extension $\mathcal{O}$ of $\mathbb{Z}_p$. Let $\mathcal{O}_0$ be the ring of integers of $\overline{\mathbb{Q}}_p^{\Sigma^{\mathrm{gen}}_{\rho_{f,\iota_p}}}$, which is contained in $\mathcal{O}$. It is known that for all but finitely many primes p, the representation ρ_{f,ι_p} is $\mathcal{O}_0$-full [Mom81, Rib85, Nek12]. The goal of this subsection is to show that our results recover these classical fullness results, at least when ρ_{f,ι_p} satisfies the hypotheses necessary for our methods. In particular, since our methods are entirely agnostic about the group Π, they show that the classical fullness results can be obtained purely algebraically, that is, without any arithmetic input such as local information at the places where ρ_{f,ι_p} is ramified. (Note, however, that our methods are purely p-adic and thus say nothing about adelic openness, which is covered by the theorems in [Mom81, Rib85, Nek12].)

Let $\mathbb{F}$ be the residue field of $\mathcal{O}$ and $\overline{\rho}\colon G_F \to \mathrm{GL}_2(\mathbb{F})$ the semisimplification of the reduction of ρ_{f,ι_p} modulo $\mathfrak{m}_{\mathcal{O}}$.

Theorem 6.1. *Assume that $\overline{\rho}$ is regular. If $\overline{\rho}$ is octahedral, assume further that $\overline{\rho}$ is good. Then ρ_{f,ι_p} is $\mathcal{O}_0$-full.*

Proof. Let $(t,d) := (\operatorname{tr}\rho_{f,\iota_p}, \det\rho_{f,\iota_p})$. Then (t,d) is not reducible since f is cuspidal, (t,d) is not dihedral since f is not CM, and $t \neq s(t)$ since the weights of f are greater than 2. By Theorem 5.17, there exists a (t,d)-representation ρ that is A_0-full, where A_0 is the subring of $\mathcal{O}$ topologically generated by $\left\{\frac{t(g)^2}{d(g)} : g \in G_F\right\}$. By Proposition 3.1, A_0 and $\mathcal{O}_0$ have the same field of fractions. Furthermore, $A_0 \subseteq \mathcal{O}_0$ and $\mathcal{O}_0$ is a finite A_0-module since both A_0 and $\mathcal{O}_0$ are finite $\mathbb{Z}_p$-modules. Therefore ρ is $\mathcal{O}_0$-full by Lemma 2.16. Hence ρ_{f,ι_p} is $\mathcal{O}_0$-full by Remark 2.15. ☐

Remark 6.2. Our regularity condition is designed in part to rule out the representations at the finitely many primes where Momose, Ribet and Nekovář do not prove the fullness of ρ_{f,ι_p}. In order to make this point clear, we recall the result of Momose and Ribet. (Nekovář's result in the Hilbert case is analogous.) Let f be a cuspidal elliptic eigenform. Write K for the finite extension of $\mathbb{Q}$ containing the eigenvalues of f under the Hecke operators, and let K_0 be the subfield of K fixed by the group of generalized conjugate self-twists of ρ_{f,ι_p}. Let

$$H := \bigcap_{(\sigma,\eta)\in\Sigma^{\mathrm{pairs}}_{\rho_{f,\iota_p}}} \ker\eta.$$

It is an open subgroup of $G_{\mathbb{Q}}$. Momose and Ribet define a quaternion algebra D over K_0 that splits over K and satisfies the following property: for the place $\mathfrak{p}$ of K above p defined by ι_p, the image of the representation $\rho_{f,\iota_p}\colon H \to \mathrm{GL}_2(K_{\mathfrak{p}})$, up to conjugation in $\mathrm{GL}_2(K_{\mathfrak{p}})$, takes values in the unit group $\mathcal{O}_D^\times$ of an order $\mathcal{O}_D$ of $D(K_\mathfrak{p})$ and contains an open subgroup of $\mathcal{O}_D^\times$ as a finite index subgroup. In particular, ρ_{f,ι_p} is not full if $D(K_\mathfrak{p})$ is not split. In this case the residual representation attached to ρ_{f,ι_p} does not satisfy the regularity assumption: for a uniformizer π of $\mathcal{O}_D$, a matrix in $\mathcal{O}_D^\times/(1+\pi\mathcal{O}_D)$ cannot have distinct eigenvalues with ratio in $\mathbb{E}$.

6.2. Galois representations attached to p-adic families of modular forms.

6.2.1. *A question about pseudorepresentations arising from p-adic families of modular forms.* There is a rather subtle question in the case when (t,d) arises from a p-adic family of elliptic or Hilbert modular forms. In that case, $\Pi = \mathrm{Gal}(\overline{F}/F)$ for a totally real number field F, and the ring A is naturally an algebra over a power series ring Λ. The number of variables in Λ depends on the arithmetic of F. When $F = \mathbb{Q}$, we have $\Lambda = \mathbb{Z}_p[\![X]\!]$.

Question 6.3. If $(t,d)\colon \mathrm{Gal}(\overline{F}/F) \to A$ arises from a p-adic family of modular forms, is Λ contained in the field of fractions of A_0? In other words, is Λ fixed by all generalized conjugate self-twists of ρ?

The two questions above are equivalent by Proposition 3.1. Note that the answer to Question 6.3 must use the meaning of Λ in the setting of p-adic families of modular forms. Indeed, it is not difficult to construct an abstract continuous representation $\rho\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{Z}_p[\![X]\!])$ such that the automorphism $X \mapsto -X$ is a conjugate self-twist of ρ.

We answer Question 6.3 positively when $F = \mathbb{Q}$ and ρ is ordinary (see Proposition 6.6), but we do not treat any other case (Coleman families when $F = \mathbb{Q}$ or families of Hilbert modular forms). We now make Question 6.3 more precise for Coleman families, but we leave it open.

The rigid analytic space attached to (the formal scheme obtained from) a Hida family carries a map to a weight space, given by a disjoint union of p-adic wide open unit discs. Such a map induces the usual Λ-algebra structure of a Hida family, because Λ can be identified with the ring of rigid analytic functions bounded in norm by 1 on a connected component of the weight space. Coleman families (i.e. p-adic families of modular eigenforms of finite slope) are not defined over all of weight space in general. One can take as a domain of definition for a Coleman family a wide open disc D in the weight space, that is, a disc given by an increasing union of closed affinoid discs. Let $\mathcal{O}(D)$ be the ring of rigid analytic functions on D bounded in norm by 1; it is a Noetherian $\mathbb{Z}_p$-algebra. Restricting functions from a connected component of the weight space to D gives an embedding $\Lambda \hookrightarrow \mathcal{O}(D)$. Given a Coleman family over D, one can attach to it a pseudorepresentation $(t,d)\colon G_\mathbb{Q} \to A$ for an $\mathcal{O}(D)$-algebra A that is finite as an $\mathcal{O}(D)$-module. The pseudorepresentation (t,d) encodes the systems of eigenvalues of an abstract Hecke algebra acting on a space of "p-adic overconvergent modular forms with weight in D", of some fixed tame level and of arbitrary level at p. We then ask Question 6.3 for (t,d). If the answer is positive, our big image result implies a significant improvement on [CIT16, Theorem 6.2], where only a big image result for a Lie algebra attached to (t,d) is given, and only in the case when the residual representation of (t,d) is absolutely irreducible.

We point out that regardless of what the answer to Question 6.3 ends up being in the case of Coleman families, the fullness results we obtain in this paper are optimal, both for representations attached to Coleman families as well as any other representation that fits into our framework. In other words, if Λ is not fixed by some generalized conjugate self-twist of (t,d), then (t,d) cannot be full with respect to the ring used in [CIT16]. This is not incompatible with [CIT16, Theorem 6.2] because of the way the Lie algebra there is defined.

In Section 6.2.2, we work with a pseudorepresentation $(t,d)\colon \Pi \to A$, where A is a local pro-p domain. We give a criterion that can help determine when certain elements of A are fixed by generalized conjugate self-twists. This criterion can be useful in answering Question 6.3, as demonstrated in Section 6.2.3, where we use it to answer the question positively in the case of Hida families. This allows us to recover and improve upon the main theorem of the second author in [Lan16, Theorem 2.4].

6.2.2. *A criterion for being fixed by generalized conjugate self-twists.* Let A be a local pro-p domain. Fix a semisimple representation $\overline{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ and $(t,d)\colon \Pi \to A$ a pseudodeformation of $\overline{\rho}$, not necessarily constant determinant.

Lemma 6.4. *Assume that $\overline{\rho}$ is regular. If $\overline{\rho}$ is octahedral, assume further that $\overline{\rho}$ is good. Then there exists a (t,d)-representation ρ such that $L_1(\rho) \subseteq M_2((K^{\mathrm{sep}})^{\Sigma_t^{\mathrm{gen}}})$.*

Proof. Let $\chi\colon \Pi \to A^\times$ be the character described in the proof of Theorem 5.17, and let $(t',d') := (\chi t, \chi^2 d)$ and $\overline{\rho}' := \overline{\chi} \otimes \overline{\rho}$, where $\overline{\chi} := \chi \bmod \mathfrak{m}$. In particular, $\overline{\rho}'$ has no conjugate self-twists if $\overline{\rho}$ is reducible, and the order of $\det \overline{\rho}'$ is a power of 2 if $\overline{\rho}$ is irreducible. Write A' for the subring of A topologically generated by $t'(\Pi)$ and K' for the field of fractions of A'. We claim that it suffices to show that there exists a (t',d')-representation $\rho'\colon \Pi \to (R')^\times$ such that $L_1(\rho') \subseteq M_2((K')^{\Sigma_{t'}})$, where we are viewing $R' \subseteq M_2(K')$ by Lemma 2.10. Indeed, letting R be the A-span of R' in $\mathrm{GL}_2(K)$, we see that twisting ρ' by χ^{-1} gives a (t,d)-representation $\rho := \rho' \otimes \chi^{-1}\colon \Pi \to R^\times$. Let $\Gamma := \Gamma(\rho)$ and $\Gamma' := \Gamma(\rho')$. Note that $\Gamma \subseteq \Gamma'$ since elements in Γ have determinant 1 and Γ is a pro-p group while the finite order part of χ has prime-to-p order. Therefore by Proposition 3.1 we have

$$L_1(\Gamma) \subseteq L_1(\Gamma') \subseteq M_2((K')^{\Sigma_{t'}}) = M_2((K^{\mathrm{sep}})^{\Sigma_t^{\mathrm{gen}}}).$$

If $\overline{\rho}$ is reducible, note that $\Sigma_{t'} = 1$ since $\overline{\rho}'$ has no conjugate self-twists. Thus in this case ρ' exists by [Bel18, Proposition 2.4.2, Lemma 2.2.2].

Next assume that $\overline{\rho}$ is absolutely irreducible. Let $\rho_{t'}\colon \Pi \to \mathrm{GL}_2(A')$ be the (t',d')-representation given by Corollary 5.3. If $\ker \beta_{t'} = 1$, then Corollary 5.3 guarantees that $L_1(\rho') \subseteq M_2((K')^{\Sigma_{t'}})$. Thus we may assume that $\overline{\rho}$ is dihedral and $\ker \beta_{t'} \neq 1$.

If $B_1(\rho_{t'}) = 0$, then $L_1(\rho_{t'}) = \begin{pmatrix} I_1(\rho_{t'}) & 0 \\ 0 & I_1(\rho_{t'}) \end{pmatrix}^0$ satisfies the desired conditions by Corollary 5.3. Thus we may assume that $B_1(\rho_{t'}) \neq 0$. The problem with $\rho_{t'}$ is that $B_1(\rho_{t'})$ is not fixed by the nonidentity element $\tau \in \ker \beta_{t'}$. By Lemma 4.16, we know that τ acts on $B_1(\rho_{t'})$ by -1. Let $0 \neq b \in B_1(\rho_{t'})$, and let ρ' be the conjugate of $\rho_{t'}$ by $\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$. Then $L_1(\rho') = \begin{pmatrix} I_1(\rho_{t'}) & b^{-1}B_1(\rho_{t'}) \\ bB_1(\rho_{t'}) & I_1(\rho_{t'}) \end{pmatrix}^0$. Since τ acts on $B_1(\rho_{t'})$ by -1, it follows that $L_1(\rho') \subseteq M_2((K')^{\Sigma_{t'}})$, as desired. ☐

Note that in the case when $\overline{\rho}$ is dihedral and $\ker \beta_{t'} \neq 1$, the representation ρ' found in the proof of Lemma 6.4 is not well adapted. Thus there is no contradiction with Lemma 4.16.

Corollary 6.5. *Let $a \in A$. If there exists a (t,d)-representation ρ and $n \geq 1$ such that $aL_n(\rho) \subseteq L_1(\rho)$ and $L_n(\rho) \neq 0$, then ${}^\sigma a = a$ for all $\sigma \in \Sigma_t^{\mathrm{gen}}$.*

Proof. We claim first that in fact $aL_n(\rho) \subseteq L_1(\rho)$ for any (t,d)-representation ρ. Indeed, by [Bel18, Proposition 2.4.2] any other (t,d)-representation ρ_1 differs from ρ by an A-algebra isomorphism Ψ. In particular, $\Psi(L_n(\rho)) = L_n(\rho_1)$ and thus

$$aL_n(\rho_1) = a\Psi(L_n(\rho)) = \Psi(aL_n(\rho)) \subseteq \Psi(L_1(\rho)) = L_1(\rho_1).$$

By Lemma 6.4, we can find a (t,d)-representation ρ_1 such that $L_1(\rho_1) \subseteq M_2((K^{\mathrm{sep}})^{\Sigma_t^{\mathrm{gen}}})$. If $0 \neq x \in L_n(\rho_1)$, then we have $ax \in L_1(\rho_1)$. In particular, both ax and x are fixed by every $\sigma \in \Sigma_t^{\mathrm{gen}}$. Letting x_{ij} be any nonzero entry in x, we find that $a = (ax_{ij})/x_{ij} \in (K^{\mathrm{sep}})^{\Sigma_t^{\mathrm{gen}}}$. ☐

6.2.3. *Application to Hida families.* Finally, let $(t,d)\colon G_{\mathbb{Q}} \to A$ be the pseudorepresentation attached to a non-CM cuspidal Hida family, and let $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be the semisimplification of the residual representation. Recall that (t,d) is unramified outside a finite set of primes S of $\mathbb{Q}$. Let Π be the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside the primes in S, which satisfies the p-finiteness condition by the Hermite-Minkowski theorem. We view (t,d)

as a pseudorepresentation on Π. Recall that A_0 is the subring of A topologically generated by $\left\{\frac{t(g)^2}{d(g)} : g \in \Pi\right\}$.

Proposition 6.6. *Assume that $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is regular. If $\overline{\rho}$ is octahedral, assume furthermore that $\overline{\rho}$ is good. Then (t, d) is A_0-full, and $Q(A_0)$ contains $\Lambda = \mathbb{Z}_p[\![X]\!]$.*

Proof. Note that (t, d) is not reducible since the Hida family is cuspidal, and (t, d) is not dihedral since the Hida family is not CM. The fact that $t \neq s(t)$ follows from the fact that a Hida family has classical specializations of weight at least 2. Therefore we know that (t, d) is A_0-full by Theorem 5.17. We just need to verify that $Q(A_0)$, which is equal to $K^{\Sigma_t^{\mathrm{gen}}}$ by Proposition 3.1, contains Λ. That is, we must show that X is fixed by Σ_t^{gen}.

Since $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is regular and (t, d) arises from a Hida family, there is a (t, d)-representation ρ such that $\rho|_{G_{\mathbb{Q}_p}} = \left(\begin{smallmatrix} \varepsilon & * \\ 0 & \delta \end{smallmatrix}\right)$ with δ unramified and $\varepsilon(g_0) = \left(\begin{smallmatrix} 1+X & * \\ 0 & 1 \end{smallmatrix}\right)$ for some g_0 in the inertia subgroup at p.

By Corollary 6.5 it suffices to show that $XL_3(\rho) \subseteq L_3(\rho)$. As in [Lan16, Lemma 7.6] it follows that $\operatorname{Im} \rho$ is normalized by $\left(\begin{smallmatrix} s(\overline{\varepsilon}(g)) & 0 \\ 0 & s(\overline{\delta}(g)) \end{smallmatrix}\right)$, where $g \in G_{\mathbb{Q}_p}$ is a regular element. In particular, $L_2(\rho)$ is strongly decomposable by Proposition 3.4 since $s(\overline{\varepsilon}(g))s(\overline{\delta}(g))^{-1} \in \mathbb{E}^\times \setminus \{\pm 1\}$.

Furthermore, we have $\rho(g_0) = \left(\begin{smallmatrix} 1+X & u \\ 0 & 1 \end{smallmatrix}\right)$. Conjugating by this matrix and its inverse shows that $B_2(\rho), C_2(\rho)$ are closed under multiplication by X. Since $L_3(\rho) = [I_2(\rho), B_2(\rho)] \oplus [B_2(\rho), C_2(\rho)] \oplus [I_2(\rho), C_2(\rho)]$, it follows that $L_3(\rho)$ is closed under multiplication by X. By Corollary 6.5, it suffices to show that $L_3(\rho) \neq 0$. But this is clearly true since ρ is A_0-full. Therefore $X \in (K^{\mathrm{sep}})^{\Sigma_t^{\mathrm{gen}}}$. ☐

6.3. **Determining the image in the residually full case.** In this section we briefly study the image of "residually full" representations. That is, we assume that $\rho \colon \Pi \to \mathrm{GL}_2(A)$ is a continuous representation such that $\operatorname{Im} \overline{\rho} \supseteq \mathrm{SL}_2(\mathbb{E})$. Under this assumption we have a more precise understanding of the image of ρ than simply fullness.

Let $\rho \colon \Pi \to \mathrm{GL}_2(A)$ be a continuous representation such that $\operatorname{Im} \overline{\rho} \supseteq \mathrm{SL}_2(\mathbb{E})$. Let $\chi \colon \Pi \to A^\times$ be the character described in the proof of Theorem 5.17, and let $\rho' \coloneqq \chi \otimes \rho$. Assume that ρ is conjugated in such a way that Theorem 2.31 applies to ρ'.

Proposition 6.7. *Assume $p \geq 7$. Then*

(1) $\operatorname{Im} \rho' \supseteq \mathrm{SL}_2(W(\mathbb{E})[I_1(\rho')])$ as a finite index subgroup;
(2) $\operatorname{Im} \rho \supseteq \mathrm{SL}_2(W(\mathbb{E})[I_1(\rho')])$;
(3) $W(\mathbb{E})[I_1(\rho)]$ is the largest subring $\tilde{A}$ of A for which $\operatorname{Im} \rho \supseteq \mathrm{SL}_2(\tilde{A})$.

Proof. For ease of notation, let us write $A_1 \coloneqq W(\mathbb{E})[I_1(\rho')]$ and $\mathfrak{m}_1 \coloneqq \mathfrak{m}_{A_1}$.

Since $\operatorname{Im} \overline{\rho}' \supseteq \mathrm{SL}_2(\mathbb{E})$, it follows that $\operatorname{Im} \rho' \supseteq \mathrm{SL}_2(W(\mathbb{E}))$ by [Man15, Main Theorem]. In particular, $p \in I_1(\rho')$ and so $\mathfrak{m}_1 = I_1(\rho')$ by Theorem 2.31. Let $\tilde{G}'$ be the subgroup of $G' \coloneqq \operatorname{Im} \rho'$ generated by $\Gamma = \Gamma(\rho')$ and $\mathrm{SL}_2(W(\mathbb{E}))$. Then $\tilde{G}'$ is a finite index subgroup of G' since Γ is.

We claim that $\tilde{G}' = \mathrm{SL}_2(A_1)$. Indeed, note that $\Gamma = \Gamma_{A_1}(\mathfrak{m}_1)$ by [Bel18, Corollary 6.8.3] and the fact that $\mathfrak{m}_1 = I_1(\rho')$. In particular, this shows that $\tilde{G}' \subseteq \mathrm{SL}_2(A_1)$. In fact, $\tilde{G}'$ is a subgroup of $\mathrm{SL}_2(A_1)$ such that $\tilde{G}'/\Gamma = \mathrm{SL}_2(\mathbb{E}) = \mathrm{SL}_2(A_1)/\Gamma$. Thus we must have equality.

Now suppose that $\operatorname{Im} \rho' \supseteq \mathrm{SL}_2(\tilde{A})$ for some subring $\tilde{A}$ of A. Then $\Gamma \supseteq \Gamma_{\tilde{A}}(\mathfrak{m}_{\tilde{A}})$, which implies that $I_1(\rho') \supseteq \mathfrak{m}_{\tilde{A}}$. On the other hand, if $\operatorname{Im} \rho' \supseteq \mathrm{SL}_2(\tilde{A})$ then $\operatorname{Im} \overline{\rho}' \supseteq \mathrm{SL}_2(\tilde{A}/\mathfrak{m}_{\tilde{A}})$. By definition of $\mathbb{E}$, we know that $\mathbb{E}$ is the largest subfield of $\mathbb{F}$ such that $\operatorname{Im} \overline{\rho}' \supseteq \mathrm{SL}_2(\mathbb{E})$. Thus we must have $\tilde{A}/\mathfrak{m}_{\tilde{A}} \subseteq \mathbb{E}$. It follows that $\tilde{A} \subseteq W(\mathbb{E})[I_1(\rho')]$.

To see the statements about ρ, note that there is a character $\tilde{\chi} \colon \operatorname{Im} \rho \to A^\times$ such that $\operatorname{Im} \rho' = \{x\tilde{\chi}(x) \colon x \in \operatorname{Im} \rho\}$. Now $\tilde{\chi}$ must be trivial on $\mathrm{SL}_2(\tilde{A})$ for any ring $\tilde{A}$ whose residue field has more than three elements by Corollary 2.23. Therefore $\operatorname{Im} \rho$ and $\operatorname{Im} \rho'$ contain the same copies of SL_2. ☐

Remark 6.8. In the forthcoming work [AB19], Aryas-de-Reina and Böckle prove a large image result for a residually full representation $\Pi \to G(A)$, where G is an adjoint group and A is the ring of definition of the representation. It does not seem hard to recover Proposition 6.7 by applying their result to the projective representation $\mathbb{P}\rho\colon \Pi \to \mathrm{PGL}_2(A)$ attached to ρ, and using the fact that the ring of definition of $\mathbb{P}\rho$ is the ring fixed by the conjugate self-twists of ρ.

7. Appendix

7.1. **Representations with isomorphic adjoint differ by a character.** Throughout Section 7.1, let G be a compact topological group and K an algebraically closed topological field. All representations are assumed to be continuous. Let $\mathfrak{sl}_n(K)$ denote the K-vector space of $n \times n$-matrices of trace 0 and $\mathrm{ad}^0\colon \mathrm{GL}_n(K) \to \mathrm{GL}_{n^2-1}(K)$ the representation obtained by letting $\mathrm{GL}_n(K)$ act on $\mathfrak{sl}_n(K)$ by conjugation. The primary goal of this section is to prove that if $\rho_1, \rho_2\colon G \to \mathrm{GL}_2(K)$ are semisimple representations such that $\mathrm{ad}^0 \rho_1 \cong \mathrm{ad}^0 \rho_2$, then $\rho_1 \cong \rho_2 \otimes \eta$ for some character $\eta\colon G \to K^\times$. This is done in Proposition 7.10. The proof is easier when either K has characteristic 2 or when the ρ_i are not dihedral. These cases are treated first in Section 7.1.1. Section 7.1.2 is an analysis of dihedral representations that allows us to conclude Proposition 7.10 in full generality. The results of this section are probably well known to experts, but we give proofs for lack of a reference in the generality we need. We were guided by the MathOverflow answer [Ven]. In the nondihedral case, this result can be found in [KMP00, Lemma 2.9]. When the representations ρ_1 and ρ_2 arise from classical modular forms, the result can be found in [DK00, Appendix].

7.1.1. *The nondihedral case.* Given a representation $\rho\colon G \to \mathrm{GL}_n(K)$, we write ρ^* for its dual representation. That is, if V is the representation space of ρ, then $V^* := \mathrm{Hom}(V, K)$ is the representation space of ρ^* with G-action given by $(g\varphi)(v) := g\varphi(g^{-1}v)$. In terms of matrices, if we fix a basis for V and take the dual basis for V^*, then $\rho^*(g)$ is the inverse transpose of $\rho(g)$.

If ρ is 2-dimensional, then an explicit calculation shows that $\rho^* \cong \rho \otimes \Lambda^2\rho^*$, where Λ^2 denotes the second exterior power of ρ. (The conjugating matrix can be taken to be $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.) We have that

$$1 \oplus \mathrm{ad}^0 \rho \cong \rho \otimes \rho^*.$$

In particular, $\mathrm{ad}^0 \rho$ is self dual. Furthermore,

$$1 \oplus \mathrm{ad}^0 \rho \cong \rho \otimes \rho^* \cong \rho \otimes \rho \otimes \Lambda^2\rho^* \cong 1 \oplus (\mathrm{Sym}^2 \rho \otimes \Lambda^2\rho^*),$$

and so $\mathrm{ad}^0 \rho \cong \mathrm{Sym}^2 \rho \otimes \Lambda^2\rho^* = \mathrm{Sym}^2 \rho \otimes \det \rho^{-1}$.

The following lemma is essentially a version of Schur's lemma that will be useful in what follows.

Lemma 7.1. *If $\rho\colon G \to \mathrm{GL}_n(K)$ is a semisimple representation such that $\mathrm{ad}^0 \rho$ does not contain a copy of the trivial representation, then ρ is reducible.*

Proof. Let V be the K-vector space on which G acts via ρ. Then $\mathrm{End}\, V$ is the representation space for $1 \oplus \mathrm{ad}^0 \rho$, where 1 is the trivial representation, which corresponds to scalar endomorphisms of V. If $\mathrm{ad}^0 \rho$ contains a copy of the trivial representation, then there is a nonscalar $\varphi \in \mathrm{End}\, V$ that commutes with the action of G. By Schur's lemma, ρ must be reducible. $\square$

Proposition 7.2. *Let $\rho_1, \rho_2\colon G \to \mathrm{GL}_2(K)$ be semisimple representations. Assume that either both ρ_i are reducible or both $\mathrm{ad}^0 \rho_i$ are irreducible. If $\mathrm{ad}^0 \rho_1 \cong \mathrm{ad}^0 \rho_2$, then there is a character $\eta\colon G \to K^\times$ such that $\rho_1 \cong \eta \otimes \rho_2$.*

Proof. First suppose that both ρ_i are reducible. Write $\rho_i \cong \lambda_i \oplus \mu_i$ for $i = 1, 2$ and $\lambda_i, \mu_i\colon G \to K^\times$ group homomorphisms. It is straightforward to calculate

$$\lambda_1\mu_1^{-1} \oplus 1 \oplus \lambda_1^{-1}\mu_1 \cong \mathrm{ad}^0 \rho_1 \cong \mathrm{ad}^0 \rho_2 \cong \lambda_2\mu_2^{-1} \oplus 1 \oplus \lambda_2^{-1}\mu_2.$$

Thus, up to switching λ_2 and μ_2, we must have $\lambda_1\mu_1^{-1} = \lambda_2\mu_2^{-1}$. Set $\eta = \mu_1\mu_2^{-1}$. Then

$$\rho_1 \cong \lambda_1 \oplus \mu_1 = \lambda_2\mu_1\mu_2^{-1} \oplus \mu_1 = (\mu_1\mu_2^{-1}) \otimes (\lambda_2 \oplus \mu_2) \cong \eta \otimes \rho_2.$$

Now assume that both $\mathrm{ad}^0 \rho_i$ are irreducible. We begin by showing that $\rho_1 \otimes \rho_2$ must be reducible (which does not make use of the assumption that $\mathrm{ad}^0 \rho_i$ is irreducible). Indeed, by Lemma 7.1 if $\rho_1 \otimes \rho_2$ were irreducible then its endomorphism ring would contain a single copy of the trivial representation. But

$$\begin{aligned}\mathrm{End}(\rho_1 \otimes \rho_2) &= (\rho_1 \otimes \rho_2) \otimes (\rho_1 \otimes \rho_2)^* \cong (\rho_1 \otimes \rho_1^*) \otimes (\rho_2 \otimes \rho_2^*)\\ &\cong (1 \oplus \mathrm{ad}^0 \rho_1) \otimes (1 \oplus \mathrm{ad}^0 \rho_2)\\ &\cong 1 \oplus \mathrm{ad}^0 \rho_1 \oplus \mathrm{ad}^0 \rho_1 \oplus (\mathrm{ad}^0 \rho_1 \otimes \mathrm{ad}^0 \rho_1)\\ &\cong 1 \oplus \mathrm{ad}^0 \rho_1 \oplus \mathrm{ad}^0 \rho_1 \oplus (\mathrm{ad}^0 \rho_1 \otimes (\mathrm{ad}^0 \rho_1)^*),\end{aligned}$$

and $\mathrm{ad}^0 \rho_1 \otimes (\mathrm{ad}^0 \rho_1)^* \cong \mathrm{End}(\mathrm{ad}^0 \rho_1)$ contains a copy of the trivial representation, a contradiction.

Next we show that $\rho_1 \otimes \rho_2$ cannot be the sum of two 2-dimensional representations. Indeed, suppose that $\rho_1 \otimes \rho_2 \cong r_1 \oplus r_2$, where $r_1, r_2 \colon G \to \mathrm{GL}_2(K)$ are representations. Take the second exterior product on both sides. We have

$$\Lambda^2(\rho_1 \otimes \rho_2) \cong (\Lambda^2\rho_1 \otimes \mathrm{Sym}^2 \rho_2) \oplus (\mathrm{Sym}^2 \rho_1 \otimes \Lambda^2\rho_2)$$

and

$$\Lambda^2(r_1 \oplus r_2) \cong \Lambda^2 r_1 \oplus \Lambda^2 r_2 \oplus (r_1 \otimes r_2).$$

Since $\mathrm{ad}^0 \rho_i \cong \mathrm{Sym}^2 \rho_i \otimes \Lambda^2\rho_i^*$, we have $\mathrm{Sym}^2 \rho_1 \otimes \Lambda^2\rho_2 \cong \mathrm{Sym}^2 \rho_2 \otimes \Lambda^2\rho_1$. But if

$$(\Lambda^2\rho_1 \otimes \mathrm{Sym}^2 \rho_2)^{\oplus 2} \cong \Lambda^2 r_1 \oplus \Lambda^2 r_2 \oplus (r_1 \otimes r_2),$$

then this contradicts irreducibility of $\mathrm{ad}^0 \rho_i$. Thus $\rho_1 \otimes \rho_2$ must contain a 1-dimensional representation; call it χ. Then we claim that $\rho_2 \cong \rho_1^* \otimes \chi \cong \rho_1 \otimes \det \rho_1^{-1} \otimes \chi$, and so ρ_1 and ρ_2 differ by a twist.

To see that $\rho_2 \cong \rho_1^* \otimes \chi$, recall that $\rho_1 \otimes \rho_2 \cong \mathrm{Hom}(\rho_1^*, \rho_2)$. Thus having a 1-dimensional G-stable subspace corresponds to a nonzero linear map $\varphi \colon \rho_1^* \to \rho_2$ such that $g\varphi = \lambda(g)\varphi$ for some $\lambda(g) \in K^\times$ for all $g \in G$. Define $f \colon \rho_1^* \to \rho_2 \otimes \chi^{-1}$ by $v \mapsto \varphi(v) \otimes e$, where e is a basis for the 1-dimensional vector space on which G acts by χ. Note that $f \neq 0$ since $\varphi \neq 0$. It is straightforward to check that $f(gv) = gf(v)$ for all $g \in G$. Therefore $\mathrm{Hom}(\rho_1^*, \rho_2 \otimes \chi^{-1}) \neq 0$. Since ρ_1^* and $\rho_2 \otimes \chi^{-1}$ are irreducible, it follows that they must be isomorphic. ∎

The following observation can be checked easily via a direct calculation on 2×2-matrices.

Lemma 7.3. *For any $g \in \mathrm{GL}_2(K)$ with (not necessarily distinct) eigenvalues λ, μ, the eigenvalues of $\mathrm{ad}^0 g$ are $1, \lambda\mu^{-1}, \lambda^{-1}\mu$. In particular, we have*

$$\mathrm{tr}\,\mathrm{ad}^0 g = \frac{\mathrm{tr}(g)^2}{\det(g)} - 1.$$

We now give a different proof of Proposition 7.2 that works without any assumptions on ρ_i or $\mathrm{ad}^0 \rho_i$ in the case when K has characteristic 2. (This is not needed anywhere in the paper since we must avoid characteristic 2 for other reasons.)

Proposition 7.4. *Assume that the characteristic of K is 2. Let $\rho_1, \rho_2 \colon G \to \mathrm{GL}_2(K)$ be semisimple representations. If $\mathrm{ad}^0 \rho_1 \cong \mathrm{ad}^0 \rho_2$, then there is a character $\eta \colon G \to K^\times$ such that $\rho_1 \cong \rho_2 \otimes \eta$.*

Proof. Let $\eta^2 \colon G \to K^\times$ be given by $\eta^2(g) \coloneqq \det \rho_1(g) \det \rho_2(g)^{-1}$. Note that since K has characteristic 2, there is a unique square root of $\eta^2(g)$ in K; call it $\eta(g)$. The uniqueness of square roots in characteristic 2 implies that the function $\eta \colon G \to K^\times$ is multiplicative.

To see that $\rho_1 \cong \rho_2 \otimes \eta$, it suffices to prove that $\mathrm{tr}\,\rho_1 = \eta\,\mathrm{tr}\,\rho_2$ by the Brauer-Nesbitt theorem since ρ_1 and ρ_2 are semisimple. For this, we need another description of η.

Fix $g \in G$, and let a_i, b_i be the eigenvalues of $\rho_i(g)$. By Lemma 7.3, it follows that $\{a_1 b_1^{-1}, a_1^{-1} b_1\} = \{a_2 b_2^{-1}, a_2^{-1} b_2\}$. Note that if $a_i b_i^{-1} \neq a_i^{-1} b_i$, then up to switching the names of a_2 and b_2, we may assume that $a_1 b_1^{-1} = a_2 b_2^{-1}$. In fact, we may always assume this since K has characteristic 2. Indeed, if $a_i b_i^{-1} = a_i^{-1} b_i$, then $(a_i b_i^{-1})^2 = 1$, which has a unique solution in K. In this case $a_i = b_i$. Therefore in any case, given a_1, b_1, we may choose a_2 and b_2 such that $a_1 a_2^{-1} = b_1 b_2^{-1}$. Therefore

$$\eta(g) = \sqrt{\det \rho_1(g) \det \rho_2(g)^{-1}} = \sqrt{a_1 b_1 a_2^{-1} b_2^{-1}} = a_1 a_2^{-1} = b_1 b_2^{-1}.$$

Now we can verify that

$$\eta(g) \operatorname{tr} \rho_2(g) = \eta(g) a_2 + \eta(g) b_2 = (a_1 a_2^{-1}) a_2 + (b_1 b_2^{-1}) b_2 = a_1 + b_2 = \operatorname{tr} \rho_1(g).$$

□

7.1.2. *The dihedral case.* In Section 7.1.2 we assume for simplicity that the characteristic of K is not equal to 2. The goal of Section 7.1.2 is to remove the assumption that either both ρ_i are reducible or both $\operatorname{ad}^0 \rho_i$ are irreducible from Proposition 7.2. We begin with a lemma that shows that, in light of Proposition 7.2, we only need to consider the case when both ρ_1 and ρ_2 are dihedral representations.

Lemma 7.5. *If $\rho\colon G \to \mathrm{GL}_2(K)$ is irreducible but $\operatorname{ad}^0 \rho$ is reducible, then ρ is dihedral.*

Proof. If $\operatorname{ad}^0 \rho$ is reducible, then so is $\operatorname{Sym}^2 \rho$ and $\operatorname{Sym}^2 \rho^*$ since $\operatorname{ad}^0 \rho \cong \operatorname{Sym}^2 \rho \otimes \det \rho^{-1}$. But $\operatorname{Sym}^2 \rho^*$ can be identified with the action of G on the K-vector space of quadratic forms on K^2. Thus, there is a quadratic form Q on which G acts by a scalar. Since K is algebraically closed and $\operatorname{char} K \neq 2$, all quadratic forms are equivalent. In particular, we may assume that $Q(x, y) = xy$. But one checks immediately that the only matrices that preserve Q up to scalars are diagonal and antidiagonal. Thus ρ must be dihedral. □

The rest of this section is devoted to an analysis of dihedral representations.

Lemma 7.6. *Assume that $\rho\colon G \to \mathrm{GL}_2(K)$ is a nonscalar and semisimple representation. If $\rho \cong \eta \otimes \rho$ for some nontrivial character $\eta\colon G \to K^\times$, then the image of $\rho|_{\ker \eta}$ is abelian.*

Proof. This argument essentially comes from [Rib77, Proposition 4.4].

Note that $\det \rho = \eta^2 \det \rho$ and so $\eta^2 = 1$. Set $H \coloneqq \ker \eta$. Thus $[G\colon H] = 2$ since η is nontrivial. By assumption, there is a matrix $M \in \mathrm{GL}_2(K)$ such that $M\rho(g)M^{-1} = \eta(g)\rho(g)$ for all $g \in G$. In particular, $\rho(H)$ is contained in the commutant of M.

We claim that M is semisimple. It suffices to show that M has distinct eigenvalues. Up to a change of basis for ρ, we may assume that M is upper triangular, say $M = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$. The eigenvalues of M acting on $M_2(K)$ by conjugation are $1, 1, ac^{-1}, a^{-1}c$ by Lemma 7.3. Note that for any $g \in G \setminus H$, we have

$$M\rho(g)M^{-1} = -\rho(g).$$

Thus $-1 = ac^{-1}$, which implies that $a \neq c$ and thus M has distinct eigenvalues, as claimed. Therefore M is semisimple and so its commutant, and hence $\rho(H)$, is abelian. □

If H is a subgroup of G of index 2, then we shall always use c to denote a fixed element in $G \setminus H$. For a character $\chi\colon H \to K^\times$ and $g \in G$, we write $\chi^g\colon H \to K^\times$ for the character defined by $\chi^g(h) \coloneqq \chi(g^{-1}hg)$. It is not difficult to check that χ^g depends only on the coset of g in G/H. Set $\chi^- \coloneqq \chi/\chi^c$. We will write $\eta_H\colon G \to G/H \cong \{\pm 1\}$ for the canonical projection map. With this notation, we recall an explicit description of $\operatorname{Ind}_H^G \chi$. Namely, $\operatorname{Ind}_H^G \chi$ is isomorphic to the representation

$$g \mapsto \begin{cases} \begin{pmatrix} \chi(g) & 0 \\ 0 & \chi^c(g) \end{pmatrix} & \text{if } g \in H \\ \begin{pmatrix} 0 & \chi(gc) \\ \chi^c(gc^{-1}) & 0 \end{pmatrix} & \text{otherwise.} \end{cases} \tag{7}$$

Using Frobenius reciprocity it is easy to see that $\operatorname{Ind}_H^G \chi$ is irreducible if and only if $\chi \neq \chi^c$.

Lemma 7.7.
(1) If $\rho = \operatorname{Ind}_H^G \chi$ for a character $\chi\colon H \to K^\times$ and $[G\colon H] = 2$, then $\rho \cong \rho \otimes \eta_H$.
(2) Conversely, if $\rho\colon G \to \mathrm{GL}_2(K)$ is a dihedral representation, then there is a subgroup H of G of index 2 and a character $\chi\colon H \to K^\times$ such that $\rho \cong \operatorname{Ind}_H^G \chi$ and $\chi \neq \chi^c$.
(3) Furthermore, H as in (2) is unique unless $\chi^2 = (\chi^c)^2$.
(4) If $\chi^2 = (\chi^c)^2$ then there are exactly three index 2 subgroups H_i of G for $i = 1, 2, 3$ for which there exist characters $\chi_i\colon H_i \to K^\times$ such that $\rho \cong \operatorname{Ind}_{H_i}^G \chi_i$.

Proof. For the first point, note that χ is a constituent of $(\rho\otimes\eta_H)|_H = \rho|_H$. By Frobenius reciprocity and dimension counting, it follows that $\operatorname{Ind}_H^G \chi \cong \rho \otimes \eta_H$.

If ρ is dihedral, then there is a nontrivial character $\eta\colon G \to K^\times$ such that $\operatorname{tr}\rho = \eta \operatorname{tr}\rho$ and $\det\rho = \eta^2 \det\rho$. In particular, $\eta^2 = 1$ and so η is a quadratic character. Let $H := \ker\eta$. Then H is a subgroup of G of index 2 and $\rho|_H$ is reducible by Lemma 7.6. Let $\chi\colon H \to K^\times$ be one of the constituents of $\rho|_H$. By Frobenius reciprocity, $\operatorname{Ind}_H^G \chi$ is a constituent of ρ and we deduce equality for dimension reasons. Thus we have $\rho|_H = \chi \oplus \chi^c$. Since ρ is irreducible by the definition of being dihedral, it follows by Frobenius reciprocity that $\chi \neq \chi^c$. This finishes the proof of the second point.

For the third point, suppose that $\rho = \operatorname{Ind}_{H'}^G \chi'$ for some character $\chi'\colon H' \to K^\times$ and $[G\colon H'] = 2$. Let $c' \in G \setminus H'$. Then by restricting to H we have $\chi \oplus \chi^c = (\eta_{H'})|_H \cdot \chi \oplus (\eta_{H'})|_H \cdot \chi^c$. Thus we either have $\chi = (\eta_{H'})|_H \cdot \chi$ or $\chi = (\eta_{H'})|_H \cdot \chi^c$. In the first case, we see that $H = \ker \eta_{H'} = H'$. In the second case we conclude that $\chi^2 = (\chi^c)^2$ since $\eta_{H'}$ is quadratic.

Finally, suppose that $\chi^2 = (\chi^c)^2$. Then $H_0 := \ker(\chi/\chi^c)$ is a subgroup of index 2 in H. We claim that H_0 is normal in G. Recall that χ^c is independent of the choice of $c \in G \setminus H$. If $h \in H_0$ and $g \in G \setminus H$ then

$$\chi(g^{-1}hg)/\chi^c(g^{-1}hg) = \chi(g^{-1}hg)/\chi^g(g^{-1}hg) = \chi^g(h)/\chi(h) = (\chi/\chi^g)(h)^{-1} = (\chi/\chi^c)(h)^{-1} = 1.$$

Furthermore, the above calculation shows that the class of c generates a subgroup of G/H_0 of order 2 distinct from H. Thus G/H_0 is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. We claim that if H' is any of the three subgroups of G of index 2 containing H_0, then there is a character $\chi'\colon H' \to K^\times$ such that $\rho \cong \operatorname{Ind}_{H'}^G \chi'$. By Frobenius reciprocity, it suffices to show that $\rho|_{H'}$ is reducible. Since $\rho|_{H_0} = \chi|_{H_0} \oplus \chi^c|_{H_0}$, it follows from Frobenius reciprocity that $\rho|_{H'} = \operatorname{Ind}_{H_0}^{H'} \chi|_{H_0}$. But $\chi|_{H_0} = \chi^c|_{H_0}$ and so it follows (again by Frobenius reciprocity) that $\rho|_{H'}$ is reducible. ☐

Combining the following lemma with Frobenius reciprocity, we see that the irreducibility of $\operatorname{Ind}_H^G \chi$ is related to the question of whether the character $\chi\colon H \to K^\times$ extends to a character of G.

Lemma 7.8. *Let H be a subgroup of G of index 2 and $\chi\colon H \to K^\times$ a character. Then χ extends to a character of G if and only if $\chi = \chi^c$. If χ extends to a character of G, then there are exactly two different extensions, and they differ by η_H.*

Proof. If such an an extension exists, then certainly $\chi = \chi^c$. On the other hand, since $c^2 \in H$, we know that $\chi(c^2)$ is well defined. Since K is algebraically closed, we may choose a square root r of $\chi(c^2)$. Define a new character $\tilde{\chi}\colon G \to K^\times$ by

$$\tilde{\chi}(g) := \begin{cases} \chi(g) & \text{if } g \in H \\ r\chi(c^{-1}g) & \text{if } g \notin H. \end{cases}$$

To see that $\tilde{\chi}$ is a character, it suffices to verify that it is multiplicative. That is, one must check that $\tilde{\chi}(h)\tilde{\chi}(ch') = \tilde{\chi}(hch')$ and $\tilde{\chi}(ch)\tilde{\chi}(ch') = \tilde{\chi}(chch')$ for $h, h' \in H$. It is easy to see by direct computation that these are satisfied if $\chi = \chi^c$. If the characteristic of K is not 2, there are exactly two choices of square root of $\chi(c^2)$, which give two different characters $\tilde{\chi}$ differing exactly by η_H; in characteristic 2 there are no choices and only one extension. ☐

Lemma 7.9. *Let $\rho = \mathrm{Ind}_H^G \chi$ be a dihedral representation. Then $\mathrm{ad}^0 \rho \cong \eta_H \oplus \mathrm{Ind}_H^G \chi^-$. If $\mathrm{ad}^0 \rho$ is the sum of three characters, then $\chi^2 = (\chi^c)^2$ and $\mathrm{ad}^0 \rho \cong \eta_{H_1} \oplus \eta_{H_2} \oplus \eta_{H_3}$, where the H_i are the index 2 subgroups of G given in Lemma 7.7.*

Proof. The first claim is an explicit calculation. Let $e_1 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), e_2 := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), e_3 := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Assume that ρ is given by (7). Then with respect to the basis e_1, e_2, e_3 we see that

$$\mathrm{ad}^0(g) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \chi^-(g) & 0 \\ 0 & 0 & \chi^-(g)^{-1} \end{pmatrix} & \text{if } g \in H \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \chi^-(gc) \\ 0 & \chi^-(gc^{-1})^{-1} & 0 \end{pmatrix} & \text{otherwise.} \end{cases}$$

We observe that η_H appears in the upper left corner. Furthermore, $(\chi^-)^c = (\chi^-)^{-1}$. Therefore the lower right 2×2-matrix in $\mathrm{ad}^0 \rho$ is isomorphic to $\mathrm{Ind}_H^G \chi^-$ by (7). Thus $\mathrm{ad}^0 \rho \cong \eta_H \oplus \mathrm{Ind}_H^G \chi^-$.

If $\mathrm{ad}^0 \rho$ is the sum of three characters, then $\mathrm{Ind}_H^G \chi^-$ is reducible and thus $\chi^- = (\chi^-)^c$. That is, $\chi^2 = (\chi^c)^2$. By Lemma 7.7, it follows that there are exactly three subgroups H_i of G of index 2 for which $\rho \cong \mathrm{Ind}_{H_i}^G \chi_i$. By the above calculation, each η_{H_i} must be a constituent of $\mathrm{ad}^0 \rho$. By counting dimensions, we find that $\mathrm{ad}^0 \rho \cong \eta_{H_1} \oplus \eta_{H_2} \oplus \eta_{H_3}$. ☐

Proposition 7.10. *Let $\rho_1, \rho_2 \colon G \to \mathrm{GL}_2(K)$ be semisimple representations. If $\mathrm{ad}^0 \rho_1 \cong \mathrm{ad}^0 \rho_2$ then there is a character $\eta \colon G \to K^\times$ such that $\rho_1 \cong \eta \otimes \rho_2$.*

Proof. By Proposition 7.4, we may assume that the characteristic of K is not equal to 2. By Proposition 7.2 and Lemma 7.5 we may assume that both ρ_1 and ρ_2 are dihedral. By Lemma 7.9 there are index-2 subgroups H_i of G and characters $\chi_i \colon H_i \to K^\times$ such that $\rho_i \cong \mathrm{Ind}_{H_i}^G \chi_i$. Note that H_i can be read off from $\mathrm{ad}^0 \rho_i$ since η_{H_i} is a constituent of $\mathrm{ad}^0 \rho_i$ by Lemma 7.9 and $H_i = \ker \eta_{H_i}$. In particular, since $\mathrm{ad}^0 \rho_1 \cong \mathrm{ad}^0 \rho_2$, we may assume that $H := H_1 = H_2$. By Lemma 7.9 we have $\mathrm{Ind}_H^G \chi_1^- \cong \mathrm{Ind}_H^G \chi_2^-$. By restricting to H it follows that $\chi_1^- \oplus (\chi_1^-)^c \cong \chi_2^- \oplus (\chi_2^-)^c$, and so up to replacing χ_2 with χ_2^c (which is okay since $\mathrm{Ind}_H^G \chi_2 \cong \mathrm{Ind}_H^G \chi_2^c$), it follows that $\chi_1^- = \chi_2^-$. That is, $\chi_1 \chi_2^{-1} = (\chi_1 \chi_2^{-1})^c$. By Lemma 7.8 there is a character $\eta \colon G \to K^\times$ such that $\eta|_H = \chi_1 \chi_2^{-1}$. We claim that $\rho_1 \cong \eta \otimes \rho_2$. Indeed, this is true upon restriction to H since

$$\rho_1|_H = \chi_1 \oplus \chi_1^c = \eta|_H \otimes (\chi_2 \oplus \chi_2^c) = (\eta \otimes \rho_2)|_H.$$

Therefore $\rho_1 \cong \eta \otimes \rho_2$ by Frobenius reciprocity since ρ_1 is irreducible and thus $\chi_1 \neq \chi_1^c$. ☐

7.2. **Rings with involution.** Throughout Section 7.2, let A be a commutative Noetherian ring equipped with an involution $*$. Note that we will need to apply the results in this section to the universal constant-determinant pseudodeformation ring $\mathcal{A}$, so we cannot assume that A is a domain. Let $A^\varepsilon = \{a \in A \colon a^* = \varepsilon a\}$ for $\varepsilon \in \{+, -\}$. We will assume throughout that $*$ is not the identity on A so that $A^- \neq 0$. It is easy to see that A^+ is a subring of A and A^- is an A^+-module. The following results have been adapted from [Lan75] and [CL77], where they are presented in the context when A may be noncommutative.

Definition 7.11. We say that an A-ideal $\mathfrak{a}$ is a $*$-*ideal* if $\mathfrak{a}^* = \mathfrak{a}$. We say that A is $*$-*prime* if whenever $\mathfrak{a}$ and $\mathfrak{b}$ are $*$-ideals such that $\mathfrak{a}\mathfrak{b} = 0$ then either $\mathfrak{a} = 0$ or $\mathfrak{b} = 0$.

Lemma 7.12. *If A is $*$-prime then A is reduced.*

Proof. Let $0 \neq a \in A$ be nilpotent. Then there is a smallest integer $n > 1$ such that $a^n = 0$. Let $\mathfrak{a} = aA$ and $\mathfrak{b} = a^{n-1}A$. Note that $\mathfrak{a} \neq 0$ and $\mathfrak{b} \neq 0$ by the minimality of n. If $\mathfrak{a}$ and $\mathfrak{b}$ are $*$-ideals then we have reached a contradiction since $\mathfrak{a}\mathfrak{b} = a^n A = 0$. In particular, if $a + a^* = 0$ then $a^* = -a \in aA$ and so $\mathfrak{a}, \mathfrak{b}$ are $*$-ideals.

If $a + a^* \neq 0$, then $a + a^*$ is still nilpotent since A is commutative. By replacing a with $a + a^*$ in the above argument, we find that $\mathfrak{a}$ and $\mathfrak{b}$ are $*$-ideals and thus we reach a contradiction. □

Lemma 7.13. *If A is a Noetherian commutative ring such that $2 \in A^\times$, then A^+ is a Noetherian ring.*

Proof. The following argument comes from [CL77, Lemma]. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of ideals in A^+. Then $I_1 A \subseteq I_2 A \subseteq \cdots$ is an ascending chain of ideals in A. Since A is Noetherian, there is some n such that $I_n A = I_m A$ for all $m \geq n$.

Fix $m \geq n$ and $a \in I_m \subseteq A^+$. Since $a \in I_m A = I_n A$ we may write

$$a = \sum_i b_i x_i$$

with $b_i \in I_n$ and $x_i \in A$. Applying the involution $*$ yields

$$a = a^* = \sum_i b_i x_i^*.$$

Thus

$$2a = \sum_i b_i (x_i + x_i^*).$$

Since $x_i + x_i^* \in A^+$ and $2 \in A^\times$ it follows that $a = \frac{1}{2} \sum_i b_i (x_i + x_i^*) \in I_n$. In particular, $I_m = I_n$. □

We would like to show that A is finitely generated as an A^+-module, which is equivalent to A being a Noetherian A^+-module since A^+ is a Noetherian ring by Lemma 7.13. The following lemma follows the proof of [Lan75, Lemma 6].

Lemma 7.14. *If there is an element $d \in A^-$ that is not a zero divisor in A, then A is Noetherian as an A^+-module.*

Proof. Since d is not a zero divisor, it follows that A is isomorphic to dA as an A^+-module. On the other hand, for any $a \in A$ we can write

$$da = \frac{1}{2}(d(a - a^*)) + \frac{1}{2}(d(a + a^*)) \in A^+ + dA^+.$$

Thus dA is a submodule of the finitely generated A^+-module $A^+ + dA^+$. Since A^+ is Noetherian by Lemma 7.13, it follows that dA, and hence A, is a finitely generated (and hence Noetherian) A^+-module. □

Proposition 7.15. *If A is a commutative Noetherian ring with $2 \in A^\times$, then A is a Noetherian A^+-module.*

Proof. This proof combines elements of the proofs of [CL77, Theorem] and [Lan75, Theorem 7].

Suppose not. Let $\mathfrak{a}_0$ be the largest $*$-ideal of A such that $A/\mathfrak{a}_0$ is not a Noetherian A^+-module, which exists since A is a Noetherian ring and is not Noetherian as an A^+-module. Thus, by replacing A with $A/\mathfrak{a}_0$, we may assume that $A/\mathfrak{a}$ is a Noetherian A^+-module for any $*$-ideal $\mathfrak{a} \neq 0$.

We claim that, under this assumption, A is reduced. It suffices to show that A is $*$-prime by Lemma 7.12. Suppose that $\mathfrak{a}$ and $\mathfrak{b}$ are nonzero $*$-ideals of A such that $\mathfrak{a}\mathfrak{b} = 0$. Note that we can view $\mathfrak{a}$ as an $A/\mathfrak{b}$-module since $\mathfrak{a}\mathfrak{b} = 0$. We know that $\mathfrak{a}$ is Noetherian as an $A/\mathfrak{b}$-module since $\mathfrak{a}$ is Noetherian as an A-module. Furthermore, $A/\mathfrak{b}$ is a Noetherian A^+-module since $\mathfrak{b} \neq 0$. Thus $\mathfrak{a}$ is Noetherian as an A^+-module. We also know that $A/\mathfrak{a}$ is a Noetherian A^+-module since $\mathfrak{a} \neq 0$. Therefore A is a Noetherian A^+-module, a contradiction. Therefore A is $*$-prime and hence reduced.

Since A is a Noetherian ring, it has only finitely many minimal prime ideals; call them $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. Since A is reduced, we have that

$$\bigcap_{i=1}^{n} \mathfrak{p}_i = 0.$$

Note that $n = 1$ corresponds to the case when A is a domain, and in that case we have already seen that A is a Noetherian A^+-module by Lemma 7.14. Thus we assume henceforth that $n > 1$ and thus each $\mathfrak{p}_i \neq 0$.

If $\mathfrak{p}_i^* \cap \mathfrak{p}_i \neq 0$, then $\mathfrak{p}_i \cap \mathfrak{p}_i^*$ is a $*$-ideal and so $A/(\mathfrak{p}_i \cap \mathfrak{p}_i^*)$ is a Noetherian A^+-module. If every $\mathfrak{p}_i$ satisfies $\mathfrak{p}_i \cap \mathfrak{p}_i^* \neq 0$ then we can view A as a subring of

$$\oplus_{i=1}^{n} A/(\mathfrak{p}_i \cap \mathfrak{p}_i^*),$$

which is Noetherian as an A^+-module. In particular, A is a Noetherian A^+-module, a contradiction, which proves the proposition.

Suppose there is some k such that $\mathfrak{p}_k \cap \mathfrak{p}_k^* = 0$. It is easy to check that $\mathfrak{p}_k^*$ is another minimal prime ideal of A. We claim that $n = 2$ in this case. Indeed, if $\mathfrak{p}$ is any minimal prime ideal of A, then we have $\mathfrak{p}_k \mathfrak{p}_k^* \subseteq \mathfrak{p}_k \cap \mathfrak{p}_k^* = 0$ and thus

$$\mathfrak{p}_k \mathfrak{p}_k^* = 0 \in \mathfrak{p}.$$

Thus $\mathfrak{p} = \mathfrak{p}_k$ or $\mathfrak{p} = \mathfrak{p}_k^*$.

Let us write $\mathfrak{p} = \mathfrak{p}_k$ henceforth. We can embed A into $A/\mathfrak{p} \times A/\mathfrak{p}^*$ by identifying $a \in A$ with $(a + \mathfrak{p}, a + \mathfrak{p}^*)$. Note that $A^+ \cap \mathfrak{p} = 0$ since if $a \in A^+ \cap \mathfrak{p}$ then $a = a^* \in \mathfrak{p}^* \cap \mathfrak{p} = 0$. Similarly, $A^+ \cap \mathfrak{p}^* = 0$. In particular, A^+ injects into $A/\mathfrak{p}$ and is therefore a domain.

Note that by Lemma 7.14, we may assume that every element of A^- is a zero divisor in A. However, both $A/\mathfrak{p}$ and $A/\mathfrak{p}^*$ are domains, so the only zero divisors in $A/\mathfrak{p} \times A/\mathfrak{p}^*$ are elements of the form $(a + \mathfrak{p}, \mathfrak{p}^*)$ or $(\mathfrak{p}, a + \mathfrak{p}^*)$. Recall that $(A^-)^2 \subseteq A^+$. In particular, if $(a + \mathfrak{p}, \mathfrak{p}^*) \in A^-$, then $(a^2 + \mathfrak{p}, \mathfrak{p}^*) \in A^+$. That is, there is some $a_+ \in A^+$ such that $a_+ - a^2 \in \mathfrak{p}$ and $a_+ \in \mathfrak{p}^*$. But we have already seen that $A^+ \cap \mathfrak{p}^* = 0$. Similarly, any $(\mathfrak{p}, a + \mathfrak{p}^*) \in A^-$ must be trivial. In other words, $A^- = 0$, a contradiction. Therefore A must be Noetherian as an A^+-module. □

Given any ideal $\mathfrak{a}$ of A, we define $\mathfrak{a}^\varepsilon := \mathfrak{a} \cap A^\varepsilon$. We call $\mathfrak{a}$ a *graded* ideal if $\mathfrak{a} = \mathfrak{a}^+ \oplus \mathfrak{a}^-$.

Proposition 7.16. *Let A be a commutative local Noetherian ring such that A and A^+ have the same residue field. Assume that $2 \in A^\times$. If A' is the quotient of A by a nongraded prime ideal, then A' has the same field of fractions as the image of A^+ in A'.*

Proof. Write $f\colon A \to A'$ for the quotient map. It suffices to show that every element of $f(A^-)$ can be written as a quotient of elements in $f(A^+)$. Since the prime ideal $\mathfrak{p} = \ker f$ is assumed to be nongraded, it follows that there is some $a \in \mathfrak{p}$ such that, if we decompose $a = a^+ + a^-$ with $a^+ \in A^+$ and $a^- \in A^-$, then neither a^+ nor a^- is in $\mathfrak{p}$. It follows that $f(a^-) = -f(a^+)$, and so $f(a^-) \in f(A^+)$. Note that $f(a^-) \neq 0$ since $a^- \notin \mathfrak{p}$. For any $x \in A^-$ we have that $xa^- \in A^+$ since $(A^-)^2 \subseteq A^+$. Thus $f(x) = f(xa^-)/f(a^-) \in Q(f(A^+))$, as desired. □

7.3. **Automorphisms and gradings.** We recall how ring automorphisms give rise to gradings. The following lemma is standard, but we include a brief proof for the convenience of the reader.

Let A be a complete local ring and X a finite abelian subgroup of the group of ring automorphisms of A. Given a character $\varphi\colon X \to A^\times$, we define

$$A^\varphi := \{a \in A \colon {}^\sigma a = \varphi(\sigma)a, \forall \sigma \in X\}.$$

We write $\mu_n(A) := \{a \in A^\times \colon a^n = 1\}$.

Lemma 7.17. *Let $\mathbb{F}$ be a finite field of characteristic p and A a pro-p local ring with residue field $\mathbb{F}$. If $p \nmid n$, then $\mu_n(A) = s(\mu_n(\mathbb{F}))$.*

Proof. Suppose $p \nmid n$ and $a \in \mu_n(A)$. Then $\overline{a} \in \mu_n(\mathbb{F})$. Write $a = s(\overline{a}) + m$ for some $m \in \mathfrak{m}$. Then $1 = (s(\overline{a}) + m)^n$ implies that

$$0 = \sum_{i=1}^{n} \binom{n}{i} s(\overline{a})^{n-i} m^i = m \left(ns(\overline{a})^{n-1} + \sum_{i=2}^{n} \binom{n}{i} s(\overline{a})^{n-i} m^i \right).$$

Note that $ns(\overline{a})^{n-1} + \sum_{i=2}^{n} \binom{n}{i} s(\overline{a})^{n-i} m^i \equiv ns(\overline{a})^{n-1} \neq 0 \bmod \mathfrak{m}$ since $p \nmid n$. Thus $ns(\overline{a})^{n-1} + \sum_{i=2}^{n} \binom{n}{i} s(\overline{a})^{n-i} m^i \in A^\times$ and so $m = 0$. This shows that $\mu_n(A) \subseteq s(\mu_n(\mathbb{F}))$. The other containment is clear. □

Assume the following:

$(*)$ for every positive integer n, if X contains an element of order n, then $\#\mu_n(A) = n$.

Then one has $\#X = \#\operatorname{Hom}(X, A^\times)$. (It is easily checked when X is cyclic, and then for general X one applies the structure theorem of finite abelian groups.)

Corollary 7.18. *Assume $(*)$. If $p \nmid \#X$, then for any $1 \neq \sigma \in X$ we have*

$$\sum_{\varphi \in \operatorname{Hom}(X, A^\times)} \varphi(\sigma) = 0.$$

Proof. First suppose that X is cyclic of order n and σ is a generator for X. Then $\operatorname{Hom}(X, A^\times)$ is cyclic, generated by any φ_0 such that $\varphi_0(\sigma)$ is a primitive n-th root of unity. Let $H := \langle \varphi_0^k \rangle$ be a nontrivial subgroup of $\operatorname{Hom}(X, A^\times)$. Then by Lemma 7.17 we have

$$\sum_{\varphi \in H} \varphi(\sigma) = \sum_{i=0}^{n/k} \varphi_0^{ki}(\sigma) = \sum_{\omega \in \mu_{n/k}(A)} \omega = \sum_{\omega \in \mu_{n/k}(\mathbb{F})} s(\omega) = 0.$$

Now we allow X to be any finite abelian group such that $p \nmid \#X$ and σ any nontrivial element of X. Then we have an exact sequence

$$0 \to \operatorname{Hom}(X/\langle \sigma \rangle, A^\times) \to \operatorname{Hom}(X, A^\times) \to \operatorname{Hom}(\langle \sigma \rangle, A^\times).$$

Thus $\sum_{\varphi \in \operatorname{Hom}(X, A^\times)} \varphi(\sigma)$ is an integral multiple of

$$\sum_{\varphi \in H} \varphi(\sigma),$$

where H is the image of $\operatorname{Hom}(X, A^\times)$ in $\operatorname{Hom}(\langle \sigma \rangle, A^\times)$. This sum is 0 by the first paragraph of the proof. □

Lemma 7.19. *Let A and X be as above. Assume that $\#X \in A^\times$ and that condition $(*)$ holds. Then A admits a grading given by $A = \oplus_{\varphi \in \operatorname{Hom}(X, A^\times)} A^\varphi$. Furthermore, any $\mathbb{Z}[1/\#X][X]$-submodule $M \subseteq A$ decomposes as*

$$M = \oplus_{\varphi \in \operatorname{Hom}(X, A^\times)} M^\varphi,$$

where $M^\varphi := M \cap A^\varphi$.

Proof. For $\varphi \in \operatorname{Hom}(X, A^\times)$, define

$$e_\varphi := \frac{1}{\#X} \sum_{\sigma \in X} \varphi(\sigma) \sigma^{-1} \in \mathbb{Z}\left[\frac{1}{\#X}\right][X].$$

A straightforward computation shows that $\{e_\varphi \colon \varphi \in \operatorname{Hom}(X, A^\times)\}$ is an orthogonal system of idempotents in $\mathbb{Z}[1/\#X][X]$. (Note that Corollary 7.18 is needed to show that $\sum_\varphi e_\varphi = 1$.) There is a natural ring homomorphism

$$\mathbb{Z}\left[\frac{1}{\#X}\right][X] \to \operatorname{End} A,$$

and pushing forward the e_φ's to $\operatorname{End} A$ gives the result. □

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