CONFORMAL QUATERNIONIC CONTACT CURVATURE AND THE LOCAL SPHERE THEOREM

STEFAN IVANOV AND DIMITER VASSILEV

Abstract. A curvature-type tensor invariant called quaternionic contact (qc) conformal curvature is defined on a qc manifolds in terms of the curvature and torsion of the Biquard connection. The discovered tensor is similar to the Weyl conformal curvature in Riemannian geometry and to the Chern-Moser invariant in CR geometry. It is shown that a qc manifold is locally qc conformal to the standard flat qc structure on the quaternionic Heisenberg group, or equivalently, to the standard 3-sasakian structure on the sphere if and only if the qc conformal curvature vanishes.

Contents

1. Introduction

A quaternionic contact (qc) structure introduced in [Biq1, Biq2] appears naturally as the conformal boundary at infinity of the quaternionic hyperbolic space. Such structures are also relevant for the quaternionic contact Yamabe problem which is naturally connected with the extremals and the best constant in an associated Sobolev-type embedding on the quaternionic Heisenberg group [IMV1, IMV2].

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A qc structure on a real (4n+3)-dimensional manifold M is a codimension three distribution H locally given as the kernel of 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ and the three 2-forms $d\eta_i|_H$ are the fundamental 2-forms of a quaternionic structure on H, i.e., there exists a Riemannian metric g on H and three local almost complex structures I_i on H satisfying the commutation relations of the imaginary quaternions, $I_1 I_2 I_3 = -1$, such that, $d\eta_i|_H = 2g(I_i.,.)$. The 1-form η is determined up to a conformal factor and the action of $SO(3)$ on $\mathbb{R}^3$, and therefore H is equipped with a conformal class $[g]$ of Riemannian metrics and a 2-sphere bundle of almost complex structures, the quaternionic bundle $\mathbb{Q}$. The 2-sphere bundle of one forms determines uniquely the associated metric and a conformal change of the metric is equivalent to a conformal change of the one forms. To every metric in the fixed conformal class one can associate a linear connection ∇ preserving the qc structure, see [Biq1], which we shall call the Biquard connection.

The transformations preserving a given quaternionic contact structure η, i.e. $\bar\eta = \mu\Psi\eta$ for a positive smooth function μ and a $SO(3)$ matrix Ψ with smooth functions as entries, are called *quaternionic contact conformal (qc conformal for short) transformations*.

Examples of QC manifolds are given in [Biq1, Biq2, IMV1, D1]. In particular, any totally umbilic hypersurface of a quaternionic Kähler or hyperKähler manifold carries such a structure [IMV1]. A basic example is provided by any 3-Sasakian manifold which can be defined as a $(4n+3)$-dimensional Riemannian manifold whose Riemannian cone is a hyperKähler manifold. It was shown in [IMV1] that the torsion endomorphism of the Biquard connection is the obstruction for a given qc-structure to be locally 3-Sasakian, up to a multiplication with a constant factor and a $SO(3)$-matrix.

The quaternionic Heisenberg group $\boldsymbol{G}(\mathbb{H})$ with its "standard" left-invariant qc structure is the unique (up to a $SO(3)$-action) example of a qc structure with flat Biquard connection [IMV1]. As a manifold $\boldsymbol{G}(\mathbb{H}) = \mathbb{H}^n \times \mathrm{Im}\,\mathbb{H}$, while the group multiplication is given by

$$(q', \omega') = (q_o, \omega_o) \circ (q, \omega) = (q_o + q, \omega + \omega_o + 2\,\mathrm{Im}\, q_o\, \bar q),$$

where $q,\ q_o \in \mathbb{H}^n$ and $\omega, \omega_o \in \mathrm{Im}\,\mathbb{H}$. The standard flat quaternionic contact structure is defined by the left-invariant quaternionic contact form $\tilde\Theta = (\tilde\Theta_1, \tilde\Theta_2, \tilde\Theta_3) = \frac{1}{2}\,(d\omega - q' \cdot d\bar q' + dq' \cdot \bar q')$, where . denotes the quaternion multiplication.

The aim of this paper is to find a tensor invariant characterizing locally the qc structures which are quaternionic contact conformally equivalent to the flat qc-structure. With this goal in mind, we describe a curvature-type tensor W^{qc} defined in terms of the curvature and torsion of the Biquard connection by (4.14), whose form is similar to the Weyl conformal curvature in Riemannian geometry (see e.g. [E]) and to the Chern-Moser invariant in CR geometry [ChM], see also [W]. We call W^{qc} the *quaternionic contact conformal curvature, qc conformal curvature for short*. The main purpose of this article is to prove the following two facts.

Theorem 1.1. *The qc conformal curvature W^{qc} is invariant under quaternionic contact conformal transformations.*

Theorem 1.2. *A qc structure on a (4n+3)-dimensional smooth manifold is locally quaternionic contact conformal to the standard flat qc structure on the quaternionic Heisenberg group $\boldsymbol{G}(\mathbb{H})$ if and only if the qc conformal curvature vanishes, $W^{qc} = 0$.*

The quaternionic Cayley transform establishes a conformal quaternionic contact automorphism between the standard 3-Sasaki structure on the quaternionic sphere S^{4n+3} and the standard qc structure on $\boldsymbol{G}(\mathbb{H})$ [IMV1]. As a consequence of Theorem 1.2 and the fact that quaternionic Cayley transform is a quaternionic contact conformal equivalence between the 3-sasakian structure on the sphere and the flat qc structure on $\boldsymbol{G}(\mathbb{H})$, we obtain

Corollary 1.3. *A qc manifold is locally quaternionic contact conformal to the quaternionic sphere S^{4n+3} if and only if the qc conformal curvature vanishes, $W^{qc} = 0$.*

We note that for locally 3-Sasakian manifolds a curvature invariant under quaternionic contact conformal transformations preserving the 3-Sasakian condition is defined in [AK] and it is shown that its vanishing is equivalent the structure to be locally isometric to the 3-Sasaki structure on the sphere.

Our investigations are close to the classical approach used by H.Weyl, see e.g. [E], while [ChM], [W] and [AK] followed the Cartan method of equivalence.

Organization of the paper. The paper relies heavily on the notion of Biquard connection introduced in [Biq1] and the properties of its torsion and curvature discovered in [IMV1]. In order to make the present paper self-contained, in Section 2 we give a review of the notion of a quaternionic contact structure and collect formulas and results from [Biq1] and [IMV1] that will be used in the subsequent sections.

Convention 1.4. *We use the following conventions:*

a) We shall use X, Y, Z, U to denote horizontal vector fields, i.e. $X, Y, Z, U \in H$;
b) $\{e_1, \dots, e_{4n}\}$ denotes an orthonormal basis of the horizontal space H;
c) The summation convention over repeated vectors from the basis $\{e_1, \dots, e_{4n}\}$ will be used. For example, for a (0,4)-tensor P, the formula $k = P(e_b, e_a, e_a, e_b)$ means

$$k = \sum_{a,b=1}^{4n} P(e_b, e_a, e_a, e_b);$$

d) The triple (i, j, k) denotes any cyclic permutation of $(1, 2, 3)$. In particular, any equation involving i, j, k holds for any such permutation.
e) s and t will be any numbers from the set $\{1, 2, 3\}$, $\quad s, t \in \{1, 2, 3\}$.
f) We shall denote with ∇h the horizontal gradient of the function h, while dh means as usual the differential of the function h, or the full gradient of h.

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2. Quaternionic contact manifolds

In this section we will briefly review the basic notions of quaternionic contact geometry and recall some results from [Biq1] and [IMV1].

For the purposes of this paper, a quaternionic contact (QC) manifold $(M, g, \mathbb{Q})$ is a $4n + 3$ dimensional manifold M with a codimension three distribution H equipped with a metric g and an Sp(n)Sp(1) structure, i.e., we have

i) a 2-sphere bundle $\mathbb{Q}$ over M of almost complex structures $I_s : H \to H, \quad I_s^2 = -1$, satisfying the commutation relations of the imaginary quaternions $I_1 I_2 = -I_2 I_1 = I_3$ and $\mathbb{Q} = \{aI_1 + bI_2 + cI_3 : \ a^2 + b^2 + c^2 = 1\}$;
ii) H is locally the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ satisfying the compatibility condition $\quad 2g(I_s X, Y) = d\eta_s(X, Y)$.

A QC manifold $(M, \bar{g}, \mathbb{Q})$ is called quaternionic contact conformal (qc-conformal for short) to $(M, g, \mathbb{Q})$ if $\bar{g} \in [g]$. In that case, if $\bar{\eta}$ is a corresponding associated one-form with complex structures $\bar{I}_s$, we have $\bar{\eta} = \mu \Psi \eta$ for some $\Psi \in SO(3)$ and a positive function μ. In particular, starting with a QC manifold (M, η) and defining $\bar{\eta} = \mu \eta$ we obtain a QC manifold $(M, \bar{\eta})$ qc-conformal to the original one.

On a quaternionic contact manifold there exists a canonical connection defined in [Biq1] when the dimension $(4n + 3) > 7$, and in [D] in the 7-dimensional case.

Theorem 2.1. [Biq1] *Let $(M, g, \mathbb{Q})$ be a quaternionic contact manifold of dimension $4n + 3 > 7$ and a fixed metric g on H in the conformal class $[g]$. Then there exists a unique connection ∇ with torsion T on M^{4n+3} and a unique supplementary subspace V to H in TM, such that:*

i) ∇ *preserves the decomposition* $H \oplus V$ *and the* $Sp(n)Sp(1)$*-structure on* H, $\nabla g = 0, \nabla \mathbb{Q} \subset \mathbb{Q}$;
ii) for $X, Y \in H$, *one has* $T(X,Y) = -[X,Y]_{|V}$;
iii) for $\xi \in V$, *the endomorphism* $T(\xi, .)_{|H}$ *of* H *lies in* $(sp(n) \oplus sp(1))^{\perp} \subset so(4n)$;

We shall call the above connection *the Biquard connection*. Biquard [Biq1] also described the supplementary subspace V, namely, locally V is generated by vector fields $\{\xi_1, \xi_2, \xi_3\}$, such that

$$\eta_s(\xi_k) = \delta_{sk}, \qquad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \quad (\xi_s \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_s)_{|H}. \tag{2.1}$$

The vector fields ξ_1, ξ_2, ξ_3 are called Reeb vector fields or fundamental vector fields.

If the dimension of M is seven, the conditions (2.1) do not always hold. Duchemin shows in [D] that if we assume, in addition, the existence of Reeb vector fields as in (2.1), then Theorem 2.1 holds. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2.1).

Notice that equations (2.1) are invariant under the natural $SO(3)$ action. Using the triple of Reeb vector fields we extend g to a metric on M by requiring $span\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and $g(\xi_s, \xi_k) = \delta_{sk}$. The extended metric does not depend on the action of $SO(3)$ on V, but it changes in an obvious manner if η is multiplied by a conformal factor. Clearly, the Biquard connection preserves the extended metric on $TM, \nabla g = 0$. We shall also extend the quternionic structure by setting $I_{s|V} = 0$. The fundamental 2-forms ω_s of the quaternionic structure Q are defined by

$$2\omega_{s|H} = d\eta_{s|H}, \qquad \xi \lrcorner \omega_s = 0, \quad \xi \in V. \tag{2.2}$$

Due to (2.2), the torsion restricted to H has the form

$$T(X,Y) = -[X,Y]_{|V} = 2\omega_1(X,Y)\xi_1 + 2\omega_2(X,Y)\xi_2 + 2\omega_3(X,Y)\xi_3. \tag{2.3}$$

The properties of the Biquard connection are encoded in the properties of the torsion endomorphism $T_\xi = T(\xi, .) : H \to H, \quad \xi \in V$. Recall that any endomorphism Ψ of H can be decomposed with respect to the quaternionic structure $(\mathbb{Q}, g)$ uniquely into $Sp(n)$-invariant parts as follows $\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}$, where Ψ^{+++} commutes with all three I_i, Ψ^{+--} commutes with I_1 and anti-commutes with the others two and etc. The two $Sp(n)Sp(1)$-invariant components are given by

$$\Psi_{[3]} = \Psi^{+++}, \qquad \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}. \tag{2.4}$$

Denoting the corresponding (0,2) tensor via g by the same letter one sees that the $Sp(n)Sp(1)$-invariant components are the projections on the eigenspaces of the Casimir operator

$$\dagger = I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3 \tag{2.5}$$

corresponding, respectively, to the eigenvalues 3 and -1, see [CSal]. If $n = 1$ then the space of symmetric endomorphisms commuting with all $I_i, i = 1, 2, 3$ is 1-dimensional, i.e. the [3]-component of any symmetric endomorphism Ψ on H is proportional to the identity, $\Psi_{[3]} = \frac{|\Psi|^2}{4} Id_{|H}$.

Decomposing the endomorphism $T_\xi \in (sp(n) + sp(1))^{\perp}$ into symmetric part T^0_ξ and skew-symmetric part b_ξ, $T_\xi = T^0_\xi + b_\xi$, we summarize the description of the torsion due to O. Biquard in the following Proposition.

Proposition 2.2. [Biq1] *The torsion* T_ξ *is completely trace-free,*

$$tr\, T_\xi = g\,(T_\xi(e_a), e_a) = 0, \quad tr\, T_\xi \circ I = g\,(T_\xi(e_a), Ie_a) = 0, \quad I \in Q.$$

The symmetric part of the torsion has the properties:

$$T^0_{\xi_s} I_s = -I_s T^0_{\xi_s}, \quad I_2 T^{0+--}_{\xi_2} = I_1 T^{0-+-}_{\xi_1}, \quad I_3 T^{0-+-}_{\xi_3} = I_2 T^{0--+}_{\xi_2}, \quad I_1 T^{0--+}_{\xi_1} = I_3 T^{0+--}_{\xi_3}.$$

The skew-symmetric part can be represented as follows $b_{\xi_s} = I_s u$, *where* u *is a traceless symmetric (1,1)-tensor on* H *which commutes with* I_1, I_2, I_3.

If $n = 1$ *then* u *vanishes identically,* $u = 0$ *and the torsion is a symmetric tensor,* $T_\xi = T^0_\xi$.

The $sp(1)$-connection 1-forms are defined by $\nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j$, or equivalently determined with $\nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j$. The vanishing of the $sp(1)$-connection 1-forms on H is equivalent to the vanishing of the torsion endomorphism of the Biquard connection [IMV1].

2.1. **The Ricci type tensors.** Let $R = [\nabla, \nabla] - \nabla_{[\ ,\ ]}$ be the curvature tensor of ∇. We shall denote the curvature tensor of type (0,4) by the same letter, $R(A, B, C, D) := g(R(A, B)C, D)$, $A, B, C, D \in \Gamma(TM)$. The Ricci tensor and the scalar curvature $Scal$ of the Biquard connection, called *qc-Ricci tensor* and *qc-sacalr curvature*, respectively, are defined by

$$Ric(X, Y) = R(e_a, X, Y, e_a), \qquad Scal = Ric(e_a, e_a) = R(e_b, e_a, e_a, e_b).$$

The curvature of the Biquard connection admits also several horizontal traces, defined in [IMV1] by

$$4n\rho_s(X, Y) = R(X, Y, e_a, I_s e_a),\ 4n\tau_s(X, Y) = R(e_a, I_s e_a, X, Y),\ 4n\zeta_s(X, Y) = R(e_a, X, Y, I_s e_a).$$

The $sp(1)$-part of R is determined by the Ricci 2-forms by

$$R(A, B, \xi_i, \xi_j) = 2\rho_k(A, B), \qquad A, B \in \Gamma(TM). \tag{2.6}$$

According to [Biq1] the Ricci tensor restricted to H is a symmetric tensor. If the trace-free part of the qc-Ricci tensor is zero we call the quaternionic structure *a qc-Einstein manifold* [IMV1]. It is shown in [IMV1] that the trace part of these Ricci type contractions is proportional to the qc-scalar curvature and the trace-free part of ρ_s, τ_s, ζ_s vanish for exactly when the manifold is qc-Einstein (see also Theorem 2.4 below).

With the help of the operator $\dagger$ we introduced in [IMV1] two $Sp(n)Sp(1)$-invariant trace-free symmetric 2-tensors T^0, U on H as follows

$$T^0(X, Y) \overset{def}{=} g((T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3)X, Y), \quad U(X, Y) \overset{def}{=} g(uX, Y). \tag{2.7}$$

The tensor T^0 belongs to the [-1]-eigenspace while U is in the [3]-eigenspace of the operator $\dagger$, i.e., they have the properties:

$$T^0(X, Y) + T^0(I_1X, I_1Y) + T^0(I_2X, I_2Y) + T^0(I_3X, I_3Y) = 0, \tag{2.8}$$

$$3U(X, Y) - U(I_1X, I_1Y) - U(I_2X, I_2Y) - U(I_3X, I_3Y) = 0. \tag{2.9}$$

Applying Proposition 2.2 and equation (2.8), we obtain the following

Proposition 2.3. *The torsion components of the symmetric part of torsion endomorphism of Biquard connection satisfy the relations*

$$4T^0(\xi_s, I_sX, Y) = T^0(X, Y) - T^0(I_sX, I_sY). \tag{2.10}$$

It is shown in [IMV1] that all horizontal Ricci type contractions of the curvature of the Biquard connection can be expressed in terms of the torsion of the Biquard connection. With slight modification based on Proposition 2.3 we collect some facts from [IMV1, Theorem 1.3, Theorem 3.12, Corollary 3.14 and Proposition 4.3] in the next

Theorem 2.4. [IMV1] *On a $(4n + 3)$-dimensional QC manifold, $n > 1$ mthe next formulas hold*

$$\begin{aligned}
Ric(X, Y) &= (2n + 2)T^0(X, Y) + (4n + 10)U(X, Y) + \frac{Scal}{4n} g(X, Y)\\
\rho_s(X, I_sY) &= -\frac{1}{2}\Big[T^0(X, Y) + T^0(I_sX, I_sY)\Big] - 2U(X, Y) - \frac{Scal}{8n(n + 2)} g(X, Y),\\
\tau_s(X, I_sY) &= -\frac{n + 2}{2n}\Big[T^0(X, Y) + T^0(I_sX, I_sY)\Big] - \frac{Scal}{8n(n + 2)} g(X, Y),\\
\zeta_s(X, I_sY) &= \frac{2n + 1}{4n} T^0(X, Y) + \frac{1}{4n} T^0(I_sX, I_sY) + \frac{2n + 1}{2n} U(X, Y) + \frac{Scal}{16n(n + 2)} g(X, Y)\\
T(\xi_i, \xi_j) &= -\frac{Scal}{8n(n + 2)} \xi_k - [\xi_i, \xi_j]_H \qquad Scal = -8n(n + 2) g(T(\xi_1, \xi_2), \xi_3)\\
T(\xi_i, \xi_j, X) &= -\rho_k(I_iX, \xi_i) = -\rho_k(I_jX, \xi_j)\\
\rho_i(X, \xi_i) &= -\frac{X(Scal)}{32n(n + 2))} + \frac{1}{2}\left(-\rho_i(\xi_j, I_kX) + \rho_j(\xi_k, I_iX) + \rho_k(\xi_i, I_jX)\right).
\end{aligned}$$

For $n = 1$ the above formulas hold with $U = 0$.

In particular, the qc-Einstein condition is equivalent to the vanishing of the torsion endomorphism of the Biquard connection. If $Scal \neq 0$ the latter holds exactly when the qc-structure is 3-sasakian up to a multiplication by a constant and an $SO(3)$-matrix with smooth entries.

We derive from [[IMV1], Theorem 4.8]

Proposition 2.5. [IMV1] *On a qc manifold the following formula holds*

$$(n-1)(\nabla_{e_a}T^0)(e_a,X)+2(n+2)\nabla_{e_a}U)(e_a,X)-\frac{(n-1)(2n+1)}{8n(n+2)}d(Scal)(X)=0. \tag{2.11}$$

2.2. **Quaternionic Heisenberg group and the quaternionic Cayley transform.** Since our goal is to classify quaternionic contact manifolds locally conformal to the quaternionic Heisenberg group we recall briefly its definition together with the definition of the quaternionic Cayley transform as described in [IMV1, Section 5.2]. As a manifold the quaternionic Heisenberg group of topological dimension $4n+3$ is $\boldsymbol{G}(\mathbb{H}) = \mathbb{H}^n \times \operatorname{Im}\mathbb{H}$. The group law is given by $(q',\omega') = (q_o,\omega_o)\circ(q,\omega) = (q_o + q, \omega + \omega_o + 2\operatorname{Im} q_o\,\bar q)$, where $q,\ q_o \in \mathbb{H}^n$ and $\omega,\omega_o \in \operatorname{Im}\mathbb{H}$. We can identify the group $\boldsymbol{G}(\mathbb{H})$ with the boundary Σ of a Siegel domain in $\mathbb{H}^n\times\mathbb{H}$, $\Sigma = \{(q',p')\in\mathbb{H}^n\times\mathbb{H} : \Re\, p' = |q'|^2\}$. The Siegel domain Σ carries a natural group structure and the map $(q,\omega)\mapsto(q,|q|^2-\omega)\in\Sigma$ is an isomorphism between $\boldsymbol{G}(\mathbb{H})$ and Σ.

On the group $\boldsymbol{G}(\mathbb{H})$, the standard contact form, written as a purely imaginary quaternion valued form, is given by $2\tilde\Theta = (d\omega - q\cdot d\bar q + dq\cdot\bar q)$, where $\cdot$ denotes the quaternion multiplication,

$$\begin{aligned}\tilde\Theta_1 &= \frac12\,dx - x^a dt^a + t^a dx^a - z^a dy^a + y^a dz^a\\ \tilde\Theta_2 &= \frac12\,dy - y^a dt^a + z^a dx^a + t^a dy^a - x^a dz^a\\ \tilde\Theta_2 &= \frac12\,dz - z^a dt^a - y^a dx^a + x^a dy^a + t^a dz^a.\end{aligned} \tag{2.12}$$

Since $dp = q\cdot d\bar q + dq\cdot\bar q - d\omega$, under the identification of $\boldsymbol{G}(\mathbb{H})$ with Σ we also have $2\tilde\Theta = -dp' + 2dq'\cdot\bar q'$.

The left invariant flat connection on $\boldsymbol{G}(\mathbb{H})$ coincidies with the Biquard connection of the qc manifold $(\boldsymbol{G}(\mathbb{H}),\tilde\Theta)$ and, conversely, any qc manifold with flat Biquard connection is locally isomorphic to $\boldsymbol{G}(\mathbb{H})$ [IMV1].

The Cayley transform is the map $\mathcal{C}: S\mapsto\Sigma$ from the sphere $S = \{|q|^2+|p|^2=1\}\subset\mathbb{H}^n\times\mathbb{H}$ minus a point to the Heisenberg group Σ, with $\mathcal{C}$ defined by

$$(q',p') = \mathcal{C}\Big((q,p)\Big),\qquad q' = (1+p)^{-1}\,q,\qquad p' = (1+p)^{-1}\,(1-p)$$

and with an inverse map $(q,p) = \mathcal{C}^{-1}\Big((q',p')\Big)$ given by

$$q = 2(1+p')^{-1}\,q',\qquad p = (1-p')\,(1+p')^{-1}.$$

The unit sphere S carries a natural qc structure $\tilde\eta = dq\cdot\bar q + dp\cdot\bar p - q\cdot d\bar q - p\cdot d\bar p$ which has zero torsion and is 3-Sasakian up to a constant factor. In [IMV1] it was noted that the Cayley transform is a quaternionic contact conformal diffeomorphism between the quaternionic Heisenberg group with its standard quaternionic contact structure $\tilde\Theta$ and $S\setminus\{(-1,0)\}$ with the structure $\tilde\eta$

$$\lambda\cdot(\mathcal{C}_*\,\tilde\eta)\cdot\bar\lambda = \frac{8}{|1+p'|^2}\,\tilde\Theta,$$

where $\lambda = \frac{1+p'}{|1+p'|}$ is a unit quaternion.

3. Curvature and the Bianchi identities

Recall that an orthonormal frame

$$\{e_1, e_2 = I_1e_1, e_3 = I_2e_1, e_4 = I_3e_1\ldots, e_{4n} = I_3e_{4n-3}, \xi_1,\xi_2,\xi_3\}$$

is a qc-normal frame (at a point) if the connection 1-forms of the Biquard connection vanish (at that point). Lemma 4.5 in [IMV1] asserts that a qc-normal frame exists at each point of a QC manifold.

In general, to verify any $Sp(n)Sp(1)$-invariant tensor identity at a point it is sufficient to check it in a qc-normal frame at that point. Further, we work in a qc-normal frame.

Let $b(A,B,C)$ denote the Bianchi projector,

$$b(A,B,C) := \sum_{(A,B,C)} \Big\{ (\nabla_A T)(B,C) + T(T(A,B),C) \Big\}, \quad A,B,C \in \Gamma(TM), \tag{3.1}$$

where $\sum_{(A,B,C)}$ denotes the cyclic sum over the three tangent vectors. With this notation the first Bianchi identity reads as follows

$$\sum_{(A,B,C)} \Big\{ R(A,B,C,D) \Big\} = g\Big(b(A,B,C),D\Big) \quad A,B,C,D \in \Gamma(TM). \tag{3.2}$$

Theorem 3.1. *On a QC manifold the curvature of the Biquard connection satisfies the equalities:*

$$R(X,Y,Z,V) - R(Z,V,X,Y) = 2\sum_{s=1}^{3} \Big[\omega_s(X,Y)U(I_sZ,V) - \omega_s(Z,V)U(I_sX,Y) \Big] \tag{3.3}$$
$$-2\sum_{s=1}^{3} \Big[\omega_s(X,Z)T^0(\xi_s,Y,V) + \omega_s(Y,V)T^0(\xi_s,Z,X) - \omega_s(Y,Z)T^0(\xi_sX,V) - \omega_s(X,V)T^0(\xi_s,Z,Y) \Big].$$

$$\begin{aligned} &3R(X,Y,Z,V) - R(I_1X,I_1Y,Z,V) - R(I_2X,I_2Y,Z,V) - R(I_3X,I_3Y,Z,V) \\ &\quad = 2\Big[g(Y,Z)T^0(X,V) + g(X,V)T^0(Z,Y) - g(Z,X)T^0(Y,V) - g(V,Y)T^0(Z,X) \Big] \\ &- 2\sum_{s=1}^{3} \Big[\omega_s(Y,Z)T^0(X,I_sV) + \omega_s(X,V)T^0(Z,I_sY) - \omega_s(Z,X)T^0(Y,I_sV) - \omega_s(V,Y)T^0(Z,I_sX) \Big] \\ &+ \sum_{s=1}^{3} \Big[2\omega_s(X,Y)\Big(T^0(Z,I_sV) - T^0(I_sZ,V)\Big) - 8\omega_s(Z,V)U(I_sX,Y) - \frac{Scal}{2n(n+2)}\omega_s(X,Y)\omega_s(Z,V) \Big]; \end{aligned} \tag{3.4}$$

$$\begin{aligned} R(\xi_i,X,Y,Z) &= -(\nabla_X U)(I_iY,Z) + \omega_j(X,Y)\rho_k(I_iZ,\xi_i) - \omega_k(X,Y)\rho_j(I_iZ,\xi_i) \\ &- \frac{1}{4}\Big[(\nabla_Y T^0)(I_iZ,X) + (\nabla_Y T^0)(Z,I_iX) \Big] + \frac{1}{4}\Big[(\nabla_Z T^0)(I_iY,X) + (\nabla_Z T^0)(Y,I_iX) \Big] \\ &- \omega_j(X,Z)\rho_k(I_iY,\xi_i) + \omega_k(X,Z)\rho_j(I_iY,\xi_i) - \omega_j(Y,Z)\rho_k(I_iX,\xi_i) + \omega_k(Y,Z)\rho_j(I_iX,\xi_i) \end{aligned} \tag{3.5}$$

$$\begin{aligned} R(\xi_i,\xi_j,X,Y) &= (\nabla_{\xi_i} U)(I_jX,Y) - (\nabla_{\xi_j} U)(I_iX,Y) \\ &- \frac{1}{4}\Big[(\nabla_{\xi_i} T^0)(I_jX,Y) + (\nabla_{\xi_i} T^0)(X,I_jY) \Big] + \frac{1}{4}\Big[(\nabla_{\xi_j} T^0)(I_iX,Y) + (\nabla_{\xi_j} T^0)(X,I_iY) \Big] \\ &- (\nabla_X \rho_k)(I_iY,\xi_i) - \frac{Scal}{8n(n+2)}T(\xi_k,X,Y) - T(\xi_j,X,e_a)T(\xi_i,e_a,Y) + T(\xi_j,e_a,Y)T(\xi_i,X,e_a), \end{aligned} \tag{3.6}$$

where the Ricci 2-forms are given by

$$\begin{aligned} 3(2n+1)\rho_i(\xi_i,X) &= \frac{1}{4}(\nabla_{e_a}T^0)\Big[(e_a,X) - 3(I_ie_a,I_iX)\Big] - (\nabla_{e_a}U)(X,e_a) \\ &\qquad + \frac{2n+1}{16n(n+2)}X(Scal) \\ 3(2n+1)\rho_i(I_kX,\xi_j) &= -3(2n+1)\rho_i(I_jX,\xi_k) - \frac{(2n+1)(2n-1)}{16n(n+2)}X(Scal) \\ &\qquad + \frac{1}{4}(\nabla_{e_a}T^0)\Big[(4n+1)(e_a,X) + 3(I_ie_a,I_iX)\Big] + 2(n+1)(\nabla_{e_a}U)(X,e_a). \end{aligned} \tag{3.7}$$

Proof. The first Bianchi identity (3.2) yields [Biq1]

$$
\begin{aligned}
(3.8)\quad R(A,B,C,D)-R(C,D,A,B)&=\frac{1}{2}\,g\big(b(A,B,C),D\big)+\frac{1}{2}\,g\big(b(B,C,D),A\big)\\
&-\frac{1}{2}\,g(b(A,C,D),B)-\frac{1}{2}\,\big(b(A,B,D),C\big),\quad A,B,C,D\in\Gamma(TM).
\end{aligned}
$$

With the help of Proposition 2.2 and equation (2.3) the identity (3.3) follows.
Recall the following equality [IMV1, Lemma 3.8]

$$
(3.9)\qquad R(X,Y,I_iZ,I_iV)=R(X,Y,Z,V)-2\rho_j(X,Y)\omega_j(Z,V)-2\rho_k(X,Y)\omega_k(Z,V).
$$

Taking into account (3.3) and (3.9), the properties of the torsion listed in Propositions 2.2 and 2.3, together with equations (2.8), (2.9) and (2.10) we find

$$
\begin{aligned}
(3.10)\quad & R(X,Y,Z,V)-R(I_iX,I_iY,Z,V)=2\omega_j(X,Y)\rho_j(Z,V)+2\omega_k(X,Y)\rho_k(Z,V)\\
& 4\omega_j(X,Y)U(I_jZ,V)+4\omega_k(X,Y)U(I_kZ,V)-4\omega_j(Z,V)U(I_jX,Y)-4\omega_k(Z,V)U(I_kX,Y)\\
&+2\Big[g(Y,Z)T^0(\xi_i,I_iX,V)+g(X,V)T^0(\xi_i,I_iZ,Y)-g(Z,X)T^0(\xi_i,I_iY,V)-g(V,Y)T^0(\xi_i,I_iZ,X)\Big]\\
&+2\Big[\omega_i(Y,Z)T^0(\xi_i,X,V)+\omega_i(X,V)T^0(\xi_i,Z,Y)-\omega_i(X,Z)T^0(\xi_i,Y,V)-\omega_i(Y,V)T^0(\xi_i,Z,X)\Big]\\
&-\frac{1}{2}\Big[\omega_j(Y,Z)\Big(T^0(X,I_jV)-T^0(I_iX,I_kV)\Big)+\omega_k(Y,Z)\Big(T^0(X,I_kV)+T^0(I_iX,I_jV)\Big)\Big]\\
&-\frac{1}{2}\Big[\omega_j(X,V)\Big(T^0(Y,I_jZ)-T^0(I_iY,I_kZ)\Big)+\omega_k(X,V)\Big(T^0(Y,I_kZ)+T^0(I_iY,I_jZ)\Big)\Big]\\
&+\frac{1}{2}\Big[\omega_j(X,Z)\Big(T^0(Y,I_jV)-T^0(I_iY,I_kV)\Big)+\omega_k(X,Z)\Big(T^0(Y,I_kV)+T^0(I_iY,I_jV)\Big)\Big]\\
&+\frac{1}{2}\Big[\omega_j(Y,V)\Big(T^0(X,I_jZ)-T^0(I_iX,I_kZ)\Big)+\omega_k(Y,V)\Big(T^0(X,I_kZ)+T^0(I_iX,I_jZ)\Big)\Big].
\end{aligned}
$$

Now, equality (3.4) follows from (3.10) and Theorem 2.4.
Invoking (3.1) and applying (2.3) and Theorem 2.4 we have

$$
\begin{aligned}
(3.11)\quad b(\xi_i,X,Y,Z)&=-(\nabla_XT)(\xi_i,Y,Z)+(\nabla_YT)(\xi_i,X,Z)\\
&+2\omega_j(X,Y)\rho_k(I_iZ,\xi_i)-2\omega_k(X,Y)\rho_j(I_iZ,\xi_i).
\end{aligned}
$$

A substitution of (3.11) in (3.8) implies (3.5).
If we take the trace in (3.5) and apply the sixth formula in Theorem 2.4 we come to

$$
(3.12)\qquad n\rho_i(\xi_i,X)=\frac{1}{8}(\nabla_{e_a}T^0)\Big[(e_a,X)-(I_ie_a,I_iX)\Big]-\frac{1}{2}\rho_k(I_jX,\xi_i)-\frac{1}{2}\rho_j(I_iX,\xi_k).
$$

Summing (3.12) and the last formula in Theorem 2.4, we obtain

$$
(3.13)\qquad (n+1)\rho_i(\xi_i,X)+\frac{1}{2}\rho_i(I_kX,\xi_j)=\frac{1}{8}(\nabla_{e_a}T^0)\Big[(e_a,X)-(I_ie_a,I_iX)\Big]+\frac{X(Scal)}{32n(n+2)}.
$$

The second Bianchi identity

$$
(3.14)\qquad \sum_{(A,B,C)}\Big\{(\nabla_AR)(B,C,D,E)+R(T(A,B),C,D,E)\Big\}=0
$$

and (2.3) give

$$
(3.15)\qquad \sum_{(X,Y,Z)}\Big[(\nabla_XR)(Y,Z,V,W)+2\sum_{s=1}^{3}\omega_s(X,Y)R(\xi_s,Z,V,W)\Big]=0.
$$

We obtain (3.15) and (2.3) that

$$(3.16)\quad (\nabla_{e_a}R)(X,Y,Z,e_a)-(\nabla_X Ric)(Y,Z)+(\nabla_Y Ric)(X,Z) \\ -2\sum_{s=1}^{3}\Big[R(\xi_s,Y,Z,I_sX)-R(\xi_s,X,Z,I_sY)+\omega_s(X,Y)Ric(\xi_s,Z)=0.$$

Letting $X=e_a, Y=I_ie_a$ in (3.15) we find

$$(3.17)\quad (\nabla_{e_a}R)(I_ie_a,Z,V,W)+2n(\nabla_Z\tau_i)(V,W) \\ +2(2n-1)R(\xi_i,Z,V,W)+2R(\xi_j,I_kZ,V,W)-2R(\xi_k,I_jZ,V,W)=0.$$

After taking the trace in (3.17) and applying the formulas in Theorem 2.4 we come to

$$(3.18)\quad (2n-1)\rho_i(\xi_i,X)-2\rho_i(I_kX,\xi_j)= \\ -\frac{1}{4}\Big[(\nabla_{e_a}T^0)(e_a,X)+(\nabla_{e_a}T^0)(I_ie_a,I_iX)\Big]-(\nabla_{e_a}U)(X,e_a)+\frac{2n-1}{16n(n+2)}X(Scal).$$

Now, (3.13) and (3.18) yield (3.7).

Finally, from (3.1) and an application of (2.3) and Proposition 2.4 we verify that (3.6) holds. □

As consequence of Theorem 3.1 we obtain the next important Proposition.

Proposition 3.2. *A QC manifold is locally isomorphic to the quaternionic Heisenberg group exactly when the curvature of the Biquard connection restricted to H vanishes, $R_{|_H}=0$.*

Proof. Taking into account [IMV1, Proposition 4.11], in order to see the claim it is sufficient to show that the (full) curvature tensor vanishes. From $R_{|_H}=0$ we can conclude, cf. [IMV1, Proposition 4.2, Proposition 4.3, Theorem 4.9], that the vertical distribution V is involutive and

$$(3.19)\qquad \rho_{t_{|_H}}=\tau_{t_{|_H}}=\zeta_{t_{|_H}}=\rho_t(\xi,.)_{|_H}=\zeta_t(\xi,.)_{|_H}=\tau_t(\xi,.)_{|_H}=Ric(\xi,.)_{|_H}=T(\xi_s,.)=0.$$

Applying (3.19) to (3.5) and (3.6) allows us to conclude $R(\xi,X,Y,Z)=R(\xi_i,\xi_j,X,Y)=0$. Furthermore, (2.6) yields $R(X,Y,\xi_i,\xi_j)=2\rho_k(X,Y)=0$, $R(X,\xi,\xi_i,\xi_j)=2\rho_k(X,\xi)=0$, and $4nR(\xi_s,\xi_t,\xi_i,\xi_j)=8n\rho_k(\xi_s,\xi_t)=2R(\xi_s,\xi_t,e_a,I_ke_a)=0$, which ends the proof. □

4. Quaternionic contact conformal curvature. Proof of Theorem 1.1

In this section we define the quaternionic contact conformal curvature and prove Theorem 1.1.

4.1. **Conformal transformations.** A conformal quaternionic contact transformation between two quaternionic contact manifold is a diffeomorphism Φ which satisfies $\Phi^*\eta=\mu\,\Psi\cdot\eta$ for some positive smooth function μ and some matrix $\Psi\in SO(3)$ with smooth functions as entries, where $\eta=(\eta_1,\eta_2,\eta_3)^t$ is considered as an element of $\mathbb{R}^3$. The Biquard connection does not change under rotations, i.e., the Biquard connection of $\Psi\cdot\eta$ and η coincides. Hence, studying qc conformal transformations we may consider only transformations $\Phi^*\eta=\mu\,\eta$.

We recall the formulas for the conformal change of the corresponding Biquard connections from [IMV1]. Let h be a positive smooth function on a QC manifold (M,η). Let $\bar\eta=\frac{1}{2h}\eta$ be a conformal deformation of the QC structure η. We will denote the objects related to $\bar\eta$ by over-lining the same object corresponding to η. Thus, $d\bar\eta=-\frac{1}{2h^2}\,dh\wedge\eta+\frac{1}{2h}d\eta$, $\bar g=\frac{1}{2h}g$.

The new triple $\{\bar\xi_1,\bar\xi_2,\bar\xi_3\}$, determined by the conditions (2.1) defining the Reeb vector fields, is $\bar\xi_s=2h\,\xi_s+I_s\nabla h$. The horizontal sub-Laplacian and the norm of the horizontal gradient are defined respectively by $\triangle h=tr^g_H(\nabla dh)=\nabla dh(e_\alpha,e_\alpha)$, $|\nabla h|^2=dh(e_\alpha)\,dh(e_\alpha)$. The Biquard connections ∇ and $\bar\nabla$ are connected by a (1,2) tensor S,

$$(4.1)\qquad \bar\nabla_AB=\nabla_AB+S_AB,\qquad A,B\in\Gamma(TM).$$

Condition (2.3) yields $g(S_XY,Z)-g(S_YX,Z)=-h^{-1}\sum_{s=1}^3\omega_s(X,Y)dh(I_sZ)$, while $\bar\nabla\bar g=0$ implies $g(S_XY,Z)+g(S_XZ,Y)=-h^{-1}dh(X)g(Y,Z)$. The last two equations determine $g(S_XY,Z)$,

$$
\begin{aligned}
(4.2)\quad g(S_XY,Z)=-(2h)^{-1}\{dh(X)g(Y,Z)-\sum_{s=1}^3 dh(I_sX)\omega_s(Y,Z)\\
+dh(Y)g(Z,X)+\sum_{s=1}^3 dh(I_sY)\omega_s(Z,X)-dh(Z)g(X,Y)+\sum_{s=1}^3 dh(I_sZ)\omega_s(X,Y)\}.
\end{aligned}
$$

Using Theorem 2.1 we obtain after some calculations

$$
\begin{aligned}
(4.3)\quad g(\bar T_{\bar\xi_1}X,Y)-2hg(T_{\xi_1}X,Y)-g(S_{\bar\xi_1}X,Y)\\
=-\nabla dh(X,I_1Y)+h^{-1}(dh(I_3X)dh(I_2Y)-dh(I_2X)dh(I_3Y)).
\end{aligned}
$$

The identity $d^2=0$ yields $\nabla dh(X,Y)-\nabla dh(Y,X)=-dh(T(X,Y))$. Applying (2.3), we have

$$
(4.4)\qquad \nabla dh(X,Y)=[\nabla dh]_{[sym]}(X,Y)-\sum_{s=1}^3 dh(\xi_s)\omega_s(X,Y),
$$

where $[.]_{[sym]}$ denotes the symmetric part of the corresponding (0,2)-tensor. Decomposing (4.3) into [3] and [-1] parts according to (2.4), using the properties of the torsion tensor T_{ξ_i} and (2.7) we come to the next transformation formula [IMV1]

$$
\begin{aligned}
(4.5)\quad g(S_{\bar\xi_i}X,Y) &= -\frac14\Big[-\nabla dh(X,I_iY)+\nabla dh(I_iX,Y)-\nabla dh(I_jX,I_kY)+\nabla dh(I_kX,I_jY)\Big]\\
&-(2h)^{-1}\Big[dh(I_kX)dh(I_jY)-dh(I_jX)dh(I_kY)+dh(I_iX)dh(Y)-dh(X)dh(I_iY)\Big]\\
&+\frac{1}{4n}\left(-\triangle h+2h^{-1}|\nabla h|^2\right)\omega_i(X,Y)-dh(\xi_k)\omega_j(X,Y)+dh(\xi_j)\omega_k(X,Y).
\end{aligned}
$$

4.2. **Quaternionic contact conformal curvature.** Let $(M,g,\mathbb{Q})$ be a (4n+3)-dimensional QC manifold. We consider the symmetric (0,2) tensor L defined on H by the equality

$$
\begin{aligned}
(4.6)\quad L(X,Y)=\Big(\frac{1}{4(n+1)}Ric_{[-1]}+\frac{1}{2(2n+5)}Ric_{[3][0]}+\frac{1}{32n(n+2)}Scal\,g\Big)(X,Y)\\
=\frac12T^0(X,Y)+U(X,Y)+\frac{Scal}{32n(n+2)}g(X,Y),
\end{aligned}
$$

where $Ric_{[-1]}$ is the [-1]-part of the Ricci tensor and $Ric_{[3][0]}$ is the trace-free [3]-part of Ric and we use the identities in Theorem 2.4 to obtain the second equality.

Let us denote the trace-free part of L with L_0, hence

$$
(4.7)\qquad L_0=\frac{1}{4(n+1)}Ric_{[-1]}+\frac{1}{2(2n+5)}Ric_{[3][0]}=\frac12T^0+U,
$$

We employ the notation for the Kulkarni-Nomizu product of two (not necessarily symmetric) tensors, for example,

$$
(\omega_s\owedge L)(X,Y,Z,V):=\omega_s(X,Z)L(Y,V)+\omega_s(Y,V)L(X,Z)-\omega_s(Y,Z)L(X,V)-\omega_s(X,V)L(Y,Z).
$$

We also note explicitly that following usual conventions we have

$$
I_sL\,(X,Y)=g(I_sLX,Y)=-L(X,I_sY).
$$

Now, define the (0,4) tensor WR on H as follows

$$
\begin{aligned}
(4.8)\quad WR(X,Y,Z,V) &= R(X,Y,Z,V) + (g \owedge L)(X,Y,Z,V) + \sum_{s=1}^{3}(\omega_s \owedge I_sL)(X,Y,Z,V)\\
&\quad -\frac{1}{2}\sum_{(i,j,k)}\omega_i(X,Y)\Big[L(Z,I_iV)-L(I_iZ,V)+L(I_jZ,I_kV)-L(I_kZ,I_jV)\Big]\\
&\quad -\sum_{s=1}^{3}\omega_s(Z,V)\Big[L(X,I_sY)-L(I_sX,Y)\Big]+\frac{1}{2n}(trL)\sum_{s=1}^{3}\omega_s(X,Y)\omega_s(Z,V),
\end{aligned}
$$

where $\sum_{(i,j,k)}$ denotes the cyclic sum.

A substitution of (4.6) and (4.7) in (4.8), invoking also (2.8) and (2.9), gives

$$
\begin{aligned}
(4.9)\quad WR(X,Y,Z,V) &= R(X,Y,Z,V) + (g \owedge L_0)(X,Y,Z,V) + \sum_{s=1}^{3}(\omega_s \owedge I_sL_0)(X,Y,Z,V)\\
&-\frac{1}{2}\sum_{s=1}^{3}\Big[\omega_s(X,Y)\Big\{T^0(Z,I_sV)-T^0(I_sZ,V)\Big\}+\omega_s(Z,V)\Big\{T^0(X,I_sY)-T^0(I_sX,Y)-4U(X,I_sY)\Big\}\Big]\\
&+\frac{Scal}{32n(n+2)}\Big[(g\owedge g)(X,Y,Z,V)+\sum_{s=1}^{3}\Big((\omega_s\owedge\omega_s)(X,Y,Z,V)+4\omega_s(X,Y)\omega_s(Z,V)\Big)\Big].
\end{aligned}
$$

Proposition 4.1. *The tensor WR is completely trace-free, i.e.*

$$Ric(WR)=\rho_s(WR)=\tau_s(WR)=\zeta_s(WR)=0.$$

Proof. Proposition 2.3, (2.8), (2.9) and (4.6) imply the following identities

$$(4.10)\quad T^0(\xi_s,I_sX,Y)=\frac{1}{2}\Big[L(X,Y)-L(I_sX,I_sY)\Big]$$

$$(4.11)\quad U(X,Y)=\frac{1}{4}\Big[L(X,Y)+L(I_1X,I_1Y)+L(I_2X,I_2Y)+L(I_3X,I_3Y)-\frac{1}{n}tr\,L\,g(X,Y)\Big]$$

$$
\begin{aligned}
(4.12)\quad T(\xi_i,X,Y)&=-\frac{1}{2}\Big[L(I_iX,Y)+L(X,I_iY)\Big]+U(I_iX,Y)\\
&=-\frac{1}{4}L(I_iX,Y)-\frac{3}{4}L(X,I_iY)-\frac{1}{4}L(I_kX,I_jY)+\frac{1}{4}L(I_jX,I_kY)-\frac{1}{4n}(tr\,L)\,g(I_iX,Y).
\end{aligned}
$$

After a substitution of (4.10) and (4.11) in the first four equations of Theorem 2.4 we derive

$$
\begin{aligned}
Ric(X,Y)&=\frac{2n+3}{2n}tr\,L\,g(X,Y)\\
&\quad+\frac{8n+11}{2}L(X,Y)+\frac{3}{2}\Big[L(I_iX,I_iY)+L(I_jX,I_jY)+L(I_kX,I_kY)\Big]\\
\rho_i(X,Y)&=L(X,I_iY)-L(I_iX,Y)-\frac{1}{2n}trL\,\omega_i(X,Y)\\
(4.13)\quad \tau_i(X,Y)&=-\frac{1}{n}tr\,L\,\omega_i(X,Y)\\
&\quad-\frac{n+2}{2n}\Big[L(I_iX,Y)-L(X,I_iY)+L(I_kX,I_jY)-L(I_jX,I_kY)\Big]\\
\zeta_i(X,Y)&=\frac{2n-1}{8n^2}tr\,L\,\omega_i(X,Y)\\
&\quad+\frac{3}{8n}L(I_iX,Y)-\frac{8n+3}{8n}L(X,I_iY)+\frac{1}{8n}\Big[L(I_kX,I_jY)-L(I_jX,I_kY)\Big].
\end{aligned}
$$

Taking the corresponding traces in (4.8), using also (4.13), we can verify the claim. $\square$

Comparing (4.9) with (3.4) we obtain the next Proposition.

Proposition 4.2. *On a QC manifold the [-1]-part with respect to the first two arguments of the tensor WR vanishes identically,*

$$WR_{[-1]}(X,Y,Z,V)=\frac{1}{4}\Big[3WR(X,Y,Z,V)-\sum_{s=1}^{3}WR(I_sX,I_sY,Z,V)\Big]=0.$$

The [3]-part with respect to the first two arguments of the tensor WR is determined completely by the torsion and the scalar curvature as follows

$$\begin{aligned}(4.14)\quad WR_{[3]}(X,Y,Z,V)&=\frac{1}{4}\Big[WR(X,Y,Z,V)+\sum_{s=1}^{3}WR(I_sX,I_sY,Z,V)\Big]\\ &=\frac{1}{4}\Big[R(X,Y,Z,V)+\sum_{s=1}^{3}R(I_sX,I_sY,Z,V)\Big]-\frac{1}{2}\sum_{s=1}^{3}\omega_s(Z,V)\Big[T^0(X,I_sY)-T^0(I_sX,Y)\Big]\\ &\quad+\frac{Scal}{32n(n+2)}\Big[(g\owedge g)(X,Y,Z,V)+\sum_{s=1}^{3}(\omega_s\owedge\omega_s)(X,Y,Z,V)\Big]\\ &\quad+(g\owedge U)(X,Y,Z,V)+\sum_{s=1}^{3}(\omega_s\owedge I_sU)(X,Y,Z,V).\end{aligned}$$

Definition 4.3. *We denote the [3]-part of the tensor WR described in (4.14) by W^{qc}, $W^{qc}:=WR_{[3]}$ and call it* the quaternionic contact conformal curvature.

4.3. **Proof of Theorem 1.1.** The relevance of WR is partially justified by the following Theorem.

Theorem 4.4. *The tensor WR is invariant under qc conformal transformations, i.e. if*

$$\bar\eta=(2h)^{-1}\Psi\eta \quad then \qquad WR_{\bar\eta}=WR_\eta,$$

for any smooth positive function h and any $SO(3)$-matrix Ψ.

Proof. After a long computation based on (4.1), (4.2), (4.5), and a suitable computer program the relation between the curvature tensors $\bar R$ and R was computed by I.Minchev [M] and presented to us in 10 pages. After a careful study of the structure of the equation we put the output in the following form

$$\begin{aligned}(4.15)\quad &2hg(\bar R(X,Y)Z,V)-g(R(X,Y)Z,V)\\ &=-g\owedge M(X,Y,Z,V)-\sum_{s=1}^{3}\omega_s\owedge(I_sM)(X,Y,Z,V)\\ &+\frac{1}{2}\sum_{(i,j,k)}\omega_i(X,Y)\Big[M(Z,I_iV)-M(I_iZ,V)+M(I_jZ,I_kV)-M(I_kZ,I_jV)\Big]\\ &-g(Z,V)\Big[M(X,Y)-M(Y,X)\Big]+\sum_{s=1}^{3}\omega_s(Z,V)\Big[M(X,I_sY)-M(Y,I_sX)\Big]\\ &-\frac{1}{2n}(trM)\sum_{s=1}^{3}\omega_s(X,Y)\omega_s(Z,V)+\frac{1}{2n}\sum_{(i,j,k)}M_i\Big[\omega_j(X,Y)\omega_k(Z,V)-\omega_k(X,Y)\omega_j(Z,V)\Big],\end{aligned}$$

where the (0,2) tensor M is given by

$$(4.16)\quad M(X,Y)=\frac{1}{2h}\Big(\nabla dh(X,Y)-\frac{1}{2h}\Big[dh(X)dh(Y)+\sum_{s=1}^{3}dh(I_sX)dh(I_sY)+\frac{1}{2}g(X,Y)|dh|^2\Big]\Big)$$

and $trM=M(e_a,e_a)$, $M_s=M(e_a,I_se_a)$ are its traces. Using (4.16) and (4.4), we obtain

$$(4.17)\qquad trM=(2h)^{-1}\Big(\triangle h-(n+2)h^{-1}|dh|^2\Big),\qquad M_s=-2n\,h^{-1}dh(\xi_s).$$

After taking the traces in (4.15), using (4.16) and the fact that the [3]-component $(\nabla dh)_{[3]}$ of ∇dh on H is symmetric, we obtain

$$\overline{Ric} - Ric = 4(n+1)M_{[sym]} + 6M_{[3]} + \frac{2n+3}{2n} trM\, g, \quad \frac{\overline{Scal}}{2h} - Scal = 8(n+2)trM. \tag{4.18}$$

The $Sp(n)Sp(1)$-invariant, [-1] and [3], parts of (4.18) are

$$(\overline{Ric} - Ric)_{[-1]} = (n+1)M_{[sym][-1]}, \quad (\overline{Ric} - Ric)_{[3]} = \frac{2n+5}{2}M_{[3]} + \frac{2n+3}{2n}(tr\, M)\, g. \tag{4.19}$$

The identities in Theorem 2.4, equations (4.18) and (4.19) yield

$$\begin{aligned} M_{[sym]} = &\Big(\frac{1}{4(n+1)}\overline{Ric}_{[-1]} + \frac{1}{2(2n+5)}\overline{Ric}_{[3]} - \frac{2n+3}{32n(n+2)(2n+5)}\overline{Scal}\,\overline{g}\Big)\\ &- \Big(\frac{1}{4(n+1)}Ric_{[-1]} + \frac{1}{2(2n+5)}Ric_{[3]} - \frac{2n+3}{32n(n+2)(2n+5)}Scal\, g\Big)\\ &= \Big[\frac{1}{2}\overline{T^0} + \overline{U} + \frac{\overline{Scal}}{32n(n+2)}\,\overline{g}\Big] - \Big[\frac{1}{2}T^0 + U + \frac{Scal}{32n(n+2)}\, g\Big]. \end{aligned} \tag{4.20}$$

Now, from (4.16) and (4.4) we obtain

$$M(X,Y) = M_{[sym]}(X,Y) - \frac{1}{2h}\sum_{s=1}^{3} dh(\xi_s)\omega_s(X,Y). \tag{4.21}$$

Substituting (4.20) in (4.21), inserting the obtained equality in (4.15), and using (4.17) completes the proof of Theorem 4.4. □

At this point, a combination of Theorem 4.4 and Proposition 4.2 ends the proof of Theorem 1.1 as well.

5. Converse problem. Proof of Theorem 1.2

Suppose $W^{qc} = 0$, hence $WR = 0$ by Proposition 4.2. In order to prove Theorem 1.2 we search for a conformal factor such that after a conformal transformation using this factor the new qc structure has Biquard connection which is flat when restricted to the common horizontal space H. After we achieve this task we can invoke Proposition 3.2 and conclude that the given structure is locally qc conformally flat. With this considerations in mind, it is then sufficient to find (locally) a solution h of equation (4.21) with $M_{[sym]} = -L$. In fact, a substitution of (4.21) in (4.15) and an application of the condition $W^{qc} = 0 = WR$ allows us to see that the qc structure $\bar\eta = \frac{1}{2h}\eta$ has flat Biquard connection.

Let us consider the following overdetermined system of partial differential equations with respect to an unknown function u

$$\begin{aligned} \nabla du(X,Y) = -du(X)du(Y) + \sum_{s=1}^{3}\Big[du(I_sX)du(I_sY) - du(\xi_s)\omega_s(X,Y)\Big]&\\ + \frac{1}{2}g(X,Y)|\nabla u|^2 - L(X,Y)& \end{aligned} \tag{5.1}$$

$$\begin{aligned} \nabla du(X,\xi_i) = \mathbb{B}(X,\xi_i) - L(X,I_i du) + \frac{1}{2}du(I_iX)|\nabla u|^2&\\ - du(X)du(\xi_i) - du(I_jX)du(\xi_k) + du(I_kX)du(\xi_j)& \end{aligned} \tag{5.2}$$

$$\nabla du(\xi_i,\xi_i) = -\mathbb{B}(\xi_i,\xi_i) + \mathbb{B}(I_i du,\xi_i) + \frac{1}{4}|\nabla u|^4 - (du(\xi_i))^2 + (du(\xi_j))^2 + (du(\xi_k))^2, \tag{5.3}$$

$$\nabla du(\xi_j,\xi_i) = -\mathbb{B}(\xi_j,\xi_i) + \mathbb{B}(I_i du,\xi_j) - 2du(\xi_i)du(\xi_j) - \frac{Scal}{16n(n+2)}du(\xi_k) \tag{5.4}$$

$$\nabla du(\xi_k,\xi_i) = -\mathbb{B}(\xi_k,\xi_i) + \mathbb{B}(I_i du,\xi_k) - 2du(\xi_i)du(\xi_k) + \frac{Scal}{16n(n+2)}du(\xi_j). \tag{5.5}$$

Here the tensor L is given by (4.6), while the tensors $\mathbb{B}(X,\xi_i)$ and $\mathbb{B}(\xi_i,\xi_j)$ do not depend on the unknown function u and will be determined later in (5.10) and (5.24), respectively. If we make the substitution

$$2u = \ln h, \qquad 2h du = dh, \qquad \nabla dh = 2h\nabla du + 4h du\otimes du,$$

in (4.16) we recognize that (4.21) transforms into (5.1). Therefore, it is sufficient to show that the system (5.1)-(5.5) admits (locally) a smooth solution.

The integrability conditions for the above considered over-determined system are furnished by the Ricci identity

$$\nabla^2 du(A,B,C) - \nabla^2 du(B,A,C) = -R(A,B,C,du) - \nabla du((T(A,B),C), \quad A,B,C\in\Gamma(TM). \tag{5.6}$$

Since (5.6) is $Sp(n)Sp(1)$-invariant it is sufficient to check it in a qc-normal frame.

The proof of Theorem 1.2 will be achieved by considering all possible cases of (5.6). It will be presented as a sequel of subsections, which occupy the rest of this section. Let us remind that the goal is to show that the vanishing of the qc conformal tensor W^{qc} implies the validity of (5.6) which guaranties the existence of a local smooth solution to the system (5.1)-(5.5).

5.1. Case 1, $A,B,C\in H$. Integrability condition (5.9).

When we consider equation (5.6) on H it takes the form

$$\begin{aligned}\nabla^2 du(Z,X,Y) - \nabla^2 du(X,Z,Y) = {}& -R(Z,X,Y,du)\\ & - 2\omega_1(Z,X)\nabla du(\xi_1,Y) - 2\omega_2(Z,X)\nabla du(\xi_2,Y) - 2\omega_3(Z,X)\nabla du(\xi_3,Y),\end{aligned} \tag{5.7}$$

where we have used (2.3). The identity $d^2u=0$ gives

$$\nabla du(X,\xi_s) - \nabla du(\xi_s,X)) = du(T(\xi_s,X)) = T(\xi_s,X,du) \tag{5.8}$$

After we take a covariant derivative of (5.1) along $Z\in H$, substitute the derivatives from (5.1) and (5.2), then anti-commute the covariant derivatives, substitute the result in (5.7) and use (4.8) with $WR=0$ we obtain, after some standard calculations, that the integrability condition in this case is

$$\begin{aligned}(\nabla_Z L)(X,Y) - (\nabla_X L)(Z,Y)&\\ = \sum_{s=1}^{3}\Big[\omega_s(Z,Y)\mathbb{B}(X,\xi_s) &- \omega_s(X,Y)\mathbb{B}(Z,\xi_s) + 2\omega_s(Z,X)\mathbb{B}(Y,\xi_s)\Big].\end{aligned} \tag{5.9}$$

For example, we check below that the term involving $\omega_1(Z,X)$ is $2\mathbb{B}(Y,\xi_1)$. Indeed, the coefficient of $\omega_1(Z,X)$ in (5.7) is calculated to be

$$\begin{aligned}&-\frac{1}{2}\Big[L(Y,I_1\nabla u) - L(I_1Y,\nabla u) + L(I_2Y,I_3\nabla u) - L(I_3Y,I_2\nabla u)\Big] + \frac{Scal}{16n(n+2)}du(I_1Y)\\ &\quad - 2\nabla du(\xi_1,Y) + du(I_1Y)|\nabla u|^2 - 2du(\xi_1)du(Y) - 2du(\xi_3)du(I_2Y) + 2du(\xi_2)du(I_3Y)\\ &= -\frac{1}{2}\Big[T^0(Y,I_1\nabla u) - T^0(I_1Y,\nabla u)\Big] + \frac{Scal}{16n(n+2)}du(I_1Y) + 2L(Y,I_1\nabla u) + 2du(T(\xi_1,Y)) + 2\mathbb{B}(Y,\xi_1)\\ &= -\frac{1}{2}\Big[T^0(Y,I_1\nabla u) - T^0(I_1Y,\nabla u)\Big] + T^0(Y,I_1\nabla u) + 2U(Y,I_1\nabla u) + 2\Big[du(T^0(\xi_1,Y)) + U(I_1Y,\nabla u)\Big]\\ &\qquad + 2\mathbb{B}(Y,\xi_1) = 2\mathbb{B}(Y,\xi_1),\end{aligned}$$

where we used (5.8), (4.6) and the properties of the torsion described in (2.8),(2.9) and Proposition 2.3.

At this point we determine the tensors $\mathbb{B}(X,\xi_s)$. Thus, we take the traces in (5.9) which give the next sequence of equalities

$$(5.10)\qquad \begin{aligned} &(\nabla_{e_a}L)(I_ie_a,I_iX)=(4n+1)\mathbb{B}(I_iX,\xi_i)-\mathbb{B}(I_jX,\xi_j)-\mathbb{B}(I_kX,\xi_k)\\ &\sum_{s=1}^3\mathbb{B}(I_sX,\xi_s)=\frac13\Big[(\nabla_X tr\,L-(\nabla_{e_a}L)(e_a,X)\Big]=\frac{1}{4n-1}\sum_{s=1}^3(\nabla_{e_a}L)(I_se_a,I_sX)\\ &\mathbb{B}(X,\xi_i)=\frac{1}{2(2n+1)}\Big[(\nabla_{e_a}L)(I_ie_a,X)+\frac13\Big((\nabla_{e_a}L)(e_a,I_iX)-\nabla_{I_iX}tr\,L\Big)\Big],\end{aligned}$$

where the second equality in (5.10) is precisely equivalent to (2.11).

We turn to a useful technical

Lemma 5.1. *The condition (5.9) is equivalent to*

$$(\nabla_Z L)(X,Y)-(\nabla_X L)(Z,Y)=0\qquad mod\quad g,\omega_1,\omega_2,\omega_3.$$

Proof. The condition of the lemma implies

$$(5.11)\quad (\nabla_Z L)(X,Y)-(\nabla_X L)(Z,Y)=g(Z,Y)C(X)-g(X,Y)C(Z)+\sum_{s=1}^3\Big[\omega_s(Z,Y)\mathbb{B}(X,\xi_s)-\omega_s(X,Y)\mathbb{B}(Z,\xi_s)+2\omega_s(Z,X)B(Y,\xi_s)\Big],$$

for some tensors $C(X), B(X,\xi_s)$ due to the vanishing of the cyclic sum $\sum_{(Z,X,Y)}[(\nabla_Z L)(X,Y)-(\nabla_X L)(Z,Y)]=0$. Taking traces in (5.11) we obtain

$$\begin{aligned} &(\nabla_{e_a}L)(I_ie_a,I_iX)=(4n+1)\mathbb{B}(I_iX,\xi_i)-\mathbb{B}(I_jX,\xi_j)-\mathbb{B}(I_kX,\xi_k)+C(I_iX)\\ &(\nabla_{e_a}L)(e_a,X)-\nabla_X tr\,L=\sum_{s=1}^3(-3\mathbb{B}(I_sX,\xi_s)+(4n-1)C(X))\\ &\sum_{s=1}^3(\nabla_{e_a}L)(I_se_a,I_sX)=\sum_{s=1}^3(4n-1)\mathbb{B}.(I_sX,\xi_s)+C(I_sX)\end{aligned}$$

The last two equalities together with (2.11) and its consequences (5.10) yield

$$(5.12)\qquad (4n-1)^2C(X)+3\sum_{s=1}^3 C(I_sX)=0.$$

Solving the linear sistem (5.12), we see $((4n-1)^4+3^4)C(X)=0$. Hence, $C(X)=0$. □

Proposition 5.2. *If $W^{qc}=0$ then the condition (5.9) holds.*

Proof. Suppose $W^{qc}=0$, use (3.16) and apply (4.8) to calculate

$$(5.13)\quad (\nabla_{e_a}R)(X,Y,Z,e_a)=-(\nabla_Y L)(X,Z)+(\nabla_X L)(Y,Z)$$
$$+\sum_{s=1}^3\Big[(\nabla_{I_sY}L)(X,I_sZ)-(\nabla_{I_sX}L)(Y,I_sZ)+(\nabla_{I_sZ}L)(X,I_sY)-(\nabla_{I_sZ}L)(I_sX,Y)\Big]\quad \mathrm{mod}\quad g,\omega_s.$$

Substituting (4.10), (4.11) in (3.5) we come to

$$(5.14)\quad -2\sum_{s=1}^3\Big[R(\xi_s,Y,Z,I_sX)-R(\xi_s,X,Z,I_sY)\Big]$$
$$=\sum_{s=1}^3\Big[(\nabla_{I_sY}L)(X,I_sZ)-(\nabla_{I_sX}L)(Y,I_sZ)+(\nabla_{I_sY}L)(I_sX,Z)-(\nabla_{I_sX}L)(I_sY,Z)\Big]$$
$$+\frac32\sum_{s=1}^3\Big[(\nabla_Y L)(X,Z)-(\nabla_X L)(Y,Z)+(\nabla_Y L)(I_sX,I_sZ)-(\nabla_X L)(I_sY,I_sZ)\Big]\quad \mathrm{mod}\quad g,\omega_s.$$

The second Bianchi identity gives $\sum_{(X,Y,Z)} \nabla_X \rho_i(Y,Z) = 0 \mod g, \omega_s$. Use (4.13) to see

$$(5.15)\quad 3\Big((\nabla_Y L)(X,Z) - (\nabla_X L)(Y,Z)\Big) + \sum_{s=1}^{3}\Big((\nabla_Y L)(I_sX, I_sZ) - (\nabla_X L)(I_sY, I_sZ)\Big) + \sum_{s=1}^{3}\Big[(\nabla_{I_sZ} L)(X, I_sY) - (\nabla_{I_sZ} L)(I_sX, Y)\Big] = 0 \mod g, \omega_s.$$

A substitution of (5.13), (5.14), (5.15) and (4.13) in (3.16) shows, after some standard calculations, the following identity

$$(5.16)\quad (4n+3)\Big[(\nabla_Y L)(X,Z) - (\nabla_X L)(Y,Z)\Big] + \sum_{s=1}^{3}\Big[(\nabla_{I_sY} L)(I_sX, Z) - (\nabla_{I_sX} L)(I_sY, Z)\Big]$$
$$+2\sum_{s=1}^{3}\Big[(\nabla_Y L)(I_sX, I_sZ) - (\nabla_{I_sX} L)(Y, I_sZ) + \nabla_{I_sY} L)(X, I_sZ) - (\nabla_X L)(I_sY, I_sZ)\Big] = 0 \mod g, \omega_s.$$

Taking the [3]-component with respect to X, Y in (5.16) yields

$$(5.17)\quad (\nabla_Y L)(X,Z) - (\nabla_X L)(Y,Z) + \sum_{s=1}^{3}\Big[(\nabla_{I_sY} L)(I_sX, Z) - (\nabla_{I_sX} L)(I_sY, Z)\Big] = 0 \mod g, \omega_s.$$

A substitution of (5.17) in (5.16) gives

$$(5.18)\quad 2n\Big[(\nabla_Y L)(X,Z) - (\nabla_X L)(Y,Z)\Big] + \sum_{s=1}^{3}\Big[(\nabla_{I_sY} L)(X, I_sZ) - (\nabla_X L)(I_sY, I_sZ)\Big]$$
$$+ (\nabla_Y L)(X,Z) - (\nabla_X L)(Y,Z) + \sum_{s=1}^{3}\Big[(\nabla_Y L)(I_sX, I_sZ) - (\nabla_{I_sX} L)(Y, I_sZ)\Big] = 0 \mod g, \omega_s.$$

Taking the [-1]-component with respect to X, Z of (5.18), calculated with the help of (5.17), yields

$$(5.19)\quad (6n-1)\Big[(\nabla_Y L)(X,Z) - (\nabla_X L)(Y,Z)\Big] + 4\sum_{s=1}^{3}\Big[(\nabla_{I_sY} L)(X, I_sZ) - (\nabla_X L)(I_sY, I_sZ)\Big]$$
$$- (2n+1)\sum_{s=1}^{3}\Big[(\nabla_Y L)(I_sX, I_sZ) - (\nabla_{I_sX} L)(Y, I_sZ)\Big] = 0 \mod g, \omega_s.$$

The equations (5.18) and (5.19) lead to

$$(\nabla_Y L)(X,Z) - (\nabla_X L)(Y,Z) + \sum_{s=1}^{3}\Big[(\nabla_Y L)(I_sX, I_sZ) - (\nabla_{I_sX} L)(Y, I_sZ)\Big] = 0 \mod g, \omega_s.$$

The latter and (5.18) imply

$$(5.20)\quad (2n-1)\Big[(\nabla_Y L)(X,Z) - (\nabla_X L)(Y,Z)\Big] = 0 \mod g, \omega_s$$

and Lemma 5.1 completes the proof of (5.9). □

5.2. **Case 2, $A, B \in H$, $C \in V$. Integrability condition (5.23).**

In this case (5.6) turns into

$$(5.21)\quad \nabla^2 du(Z,X,\xi_i) - \nabla^2 du(X,Z,\xi_i) = -R(Z,X,\xi_i,du) - \nabla du(T(Z,X),\xi_i) =$$
$$- 2du(\xi_j)\rho_k(Z,X) + 2du(\xi_k)\rho_j(Z,X)$$
$$- 2\omega_i(Z,X)\nabla du(\xi_i,\xi_i) - 2\omega_j(Z,X)\nabla du(\xi_j,\xi_i) - 2\omega_k(Z,X)\nabla du(\xi_k,\xi_i),$$

after using (2.3) and (2.6). Taking a covariant derivative of (5.2) along $Z \in H$, substituting in the obtained equality (5.1) and (5.2), and anti-commuting the covariant derivatives we see

$$\begin{aligned}(5.22)\quad \nabla^2 du(Z,X,\xi_i) - \nabla^2 du(X,Z,\xi_i) &= (\nabla_Z\mathbb{B})(X,\xi_i) - (\nabla_X\mathbb{B})(Z,\xi_i)\\ &- (\nabla_Z L)(X,I_i du) + (\nabla_X L)(Z,I_i du) - L(X,\nabla_Z I_i du) + L(Z,\nabla_X I_i du)\\ &+ other\ \ terms\ \ comming\ \ from\ \ the\ \ use\ \ of\ \ (5.1)\ \ and\ \ (5.2).\end{aligned}$$

Substitute (5.22) into (5.21) use (5.9) proved in Proposition 5.2, also (5.3), (5.4), (5.5) and the second equation in (4.13) to get after some calculations that (5.21) is equivalent to

$$(5.23)\qquad (\nabla_Z\mathbb{B})(X,\xi_t) - (\nabla_X\mathbb{B})(Z,\xi_t) - L(Z,I_tL(X)) + L(X,I_tL(Z)) = \sum_{s=1}^{3} 2\mathbb{B}(\xi_s,\xi_t)\omega_s(Z,X),$$

which is the integrability condition in this case. The functions $\mathbb{B}(\xi_s,\xi_t)$ are uniquely determined by

$$(5.24)\qquad \mathbb{B}(\xi_s,\xi_t) = \frac{1}{4n}\Big[(\nabla_{e_a}\mathbb{B})(I_se_a,\xi_t) + L(e_a,e_b)L(I_te_a,I_se_b)\Big].$$

Proposition 5.3. *If $W^{qc}=0$ then the condition (5.23) holds.*

Proof. To prove the assertion it is sufficient to show that the left hand side of (5.23) vanishes *mod* ω_s. Differentiating (5.9) and taking the corresponding traces yields

$$\begin{aligned}(5.25)\quad (\nabla^2_{e_a,I_ie_a}L)(X,Y) - (\nabla^2_{e_a,X}L)(I_ie_a,Y) &= -(\nabla_Y\mathbb{B})(X,\xi_i) - 2(\nabla_X\mathbb{B})(Y,\xi_i)\\ &+ (\nabla_{I_kY}\mathbb{B})(X,\xi_j) + 2(\nabla_{I_kX}\mathbb{B})(Y,\xi_j) - (\nabla_{I_jY}\mathbb{B})(X,\xi_k) - 2(\nabla_{I_jX}\mathbb{B})(Y,\xi_k)\end{aligned}$$

$$\begin{aligned}(5.26)\quad (\nabla^2_{e_a,X}L)(I_ie_a,Y) - (\nabla^2_{e_a,Y}L)(I_ie_a,X) &= (\nabla_X\mathbb{B})(Y,\xi_i) - (\nabla_Y\mathbb{B})(X,\xi_i)\\ &+ (\nabla_{I_kY}\mathbb{B})(X,\xi_j) - (\nabla_{I_kX}\mathbb{B})(Y,\xi_j) - (\nabla_{I_jY}\mathbb{B})(X,\xi_k) + (\nabla_{I_jX}\mathbb{B})(Y,\xi_k) \quad \mathrm{mod}\quad \omega_s\end{aligned}$$

$$(5.27)\quad (\nabla^2_{X,e_a}L)(I_ie_a,Y) = (4n+1)(\nabla_X\mathbb{B})(Y,\xi_i) - (\nabla_X\mathbb{B})(I_kY,\xi_j) + (\nabla_X\mathbb{B})(I_jY,\xi_k) \quad \mathrm{mod}\quad \omega_s$$

$$(5.28)\quad -\nabla^2_{X,I_iY}tr\,L + (\nabla^2_{X,e_a}L)(e_a,I_iY) = 3(\nabla_X\mathbb{B})(Y,\xi_i) - 3(\nabla_X\mathbb{B})(I_kY,\xi_j) + 3(\nabla_X\mathbb{B})(I_jY,\xi_k).$$

From equalities (5.26) and (5.27) we obtain

$$\begin{aligned}(5.29)\quad \Big[\nabla^2_{X,e_a} - \nabla^2_{e_a,X}\Big]L(I_ie_a,Y) + \Big[\nabla^2_{e_a,Y} - \nabla^2_{Y,e_a}\Big]L(I_ie_a,X)&\\ = 4n\Big[(\nabla_X\mathbb{B})(Y,\xi_i) - (\nabla_Y\mathbb{B})(X,\xi_i)\Big]&\\ - \Big[(\nabla_X\mathbb{B})(I_kY,\xi_j) + (\nabla_{I_kY}\mathbb{B})(X,\xi_j) - (\nabla_Y\mathbb{B})(I_kX,\xi_j) - (\nabla_{I_kX}\mathbb{B})(Y,\xi_j)\Big]&\\ + \Big[(\nabla_X\mathbb{B})(I_jY,\xi_k) + (\nabla_{I_jY}\mathbb{B})(X,\xi_k) - (\nabla_Y\mathbb{B})(I_jX,\xi_k) - (\nabla_{I_jX}\mathbb{B})(Y,\xi_k)\Big] &\quad mod\quad \omega_s.\end{aligned}$$

On the other hand, the Ricci identities

$$\begin{aligned}(5.30)\quad \Big[\nabla^2_{X,e_a} - \nabla^2_{e_a,X}\Big]L(I_ie_a,Y) &= -R(X,e_a,Y,e_b)L(e_b,I_ie_a) - 4n\zeta_i(X,e_a)L(Y,e_a)\\ &2(\nabla_{\xi_i}L)(X,Y) - 2(\nabla_{\xi_j}L)(I_kX,Y) + 2(\nabla_{\xi_k}L)(I_jX,Y)\end{aligned}$$

and the first Bianchi identity (3.2) imply

$$\begin{aligned}(5.31)\quad &\Big[\nabla^2_{X,e_a} - \nabla^2_{e_a,X}\Big]L(I_ie_a,Y) + \Big[\nabla^2_{e_a,Y} - \nabla^2_{Y,e_a}\Big]L(I_ie_a,X) =\\ &- 2\Big[(\nabla_{\xi_j}L)(I_kX,Y) - (\nabla_{\xi_j}L)(X,I_kY)\Big] + 2\Big[(\nabla_{\xi_k}L)(I_jX,Y) - (\nabla_{\xi_k}L)(X,I_jY)\Big]\\ &+ 2T(\xi_i,Y,e_a)L(X,e_a) - 2T(\xi_j,Y,e_a)L(I_kX,e_a) + 2T(\xi_k,Y,e_a)L(I_jX,e_a)\\ &- 2T(\xi_i,X,e_a)L(Y,e_a) + 2T(\xi_j,X,e_a)L(I_kY,e_a) - 2T(\xi_k,X,e_a)L(I_jY,e_a)\\ &- R(X,Y,e_a,e_b)L(e_b,I_ie_a) - 4n[\zeta_i(X,e_a)L(Y,e_a) - \zeta_i(Y,e_a)L(X,e_a)] \quad mod\quad \omega_s.\end{aligned}$$

The second equality in (4.13) and a suitable contraction in the second Bianchi identity give the next two equations valid mod ω_s

$$\begin{aligned}(\nabla_{\xi_j}L)(X,I_kY)-(\nabla_{\xi_j}L)(I_kX,Y)&=(\nabla_{\xi_j}\rho_k)(X,Y)\\ &=(\nabla_X\rho_k)(\xi_j,Y)-(\nabla_Y\rho_k)(\xi_j,X)-\rho_k(T(\xi_j,X),Y)+\rho_k(T(\xi_j,Y),X)\\ (\nabla_{\xi_k}L)(X,I_jY)-(\nabla_{\xi_k}L)(I_jX,Y)&=(\nabla_{\xi_k}\rho_j)(X,Y)\\ &=(\nabla_X\rho_j)(\xi_k,Y)-(\nabla_Y\rho_j)(\xi_k,X)-\rho_j(T(\xi_k,X),Y)+\rho_j(T(\xi_k,Y),X).\end{aligned} \tag{5.32}$$

A substitution of (4.10), (4.11) in equations (3.7), together with a use of (5.9) and an application of (5.10) give the next

Lemma 5.4. *We have the following formulas for the Ricci 2-forms*

$$\begin{aligned}&\rho_k(\xi_i,X)=\mathbb{B}(X,\xi_j)-\mathbb{B}(I_kX,\xi_i) \qquad\qquad \rho_i(\xi_k,X)=-\mathbb{B}(X,\xi_j)-\mathbb{B}(I_iX,\xi_k)\\ &\rho_i(X,\xi_i)=-\frac{1}{4n}d(trL)(X)+\mathbb{B}(I_iX,\xi_i)\end{aligned} \tag{5.33}$$

When we take the covariant derivative of (5.33), substitute the obtained equalities together with (5.31), (5.32), in (5.29) we derive the formula

$$\begin{aligned}(4n+2)\Big[(\nabla_X\mathbb{B})(Y,\xi_i)-(\nabla_Y\mathbb{B})(X,\xi_i)\Big]+\Big[(\nabla_{I_jX}\mathbb{B})(I_jY,\xi_i)-(\nabla_{I_jY}\mathbb{B})(I_jX,\xi_i)\Big]\\ +\Big[(\nabla_{I_kX}\mathbb{B})(I_kY,\xi_i)-(\nabla_{I_kY}\mathbb{B})(I_kX,\xi_i)\Big]=F(X,Y)\quad mod\quad \omega_s,\end{aligned} \tag{5.34}$$

where the (0,2)-tensor F is defined by

$$\begin{aligned}F(X,Y)=&-R(X,Y,e_a,e_b)L(e_b,I_ie_a)-4n\Big[\zeta_i(X,e_a)L(Y,e_a)-\zeta_i(Y,e_a)L(X,e_a)\Big]\\ &+2T(\xi_i,Y,e_a)L(X,e_a)-2T(\xi_j,Y,e_a)L(I_kX,e_a)+2T(\xi_k,Y,e_a)L(I_jX,e_a)\\ &-2T(\xi_i,X,e_a)L(Y,e_a)+2T(\xi_j,X,e_a)L(I_kY,e_a)-2T(\xi_k,X,e_a)L(I_jY,e_a)\\ &+\rho_j(T(\xi_k,X),Y)-\rho_j(T(\xi_k,Y),X)+\rho_j(T(\xi_k,I_jX),I_jY)-\rho_j(T(\xi_k,I_jY),I_jX)\\ &-\rho_k(T(\xi_j,X),Y)+\rho_k(T(\xi_j,Y),X)-\rho_k(T(\xi_j,I_kX),I_kY)+\rho_k(T(\xi_j,I_kY),I_kX).\end{aligned} \tag{5.35}$$

Solving for $(\nabla_X\mathbb{B})(Y,\xi_i)-(\nabla_Y\mathbb{B})(X,\xi_i)$ we obtain

$$\begin{aligned}&16n(n+1)(2n+1)\Big[\nabla_X\mathbb{B})(Y,\xi_i)-(\nabla_Y\mathbb{B})(X,\xi_i)\Big]\\ &=(8n^2+8n+1)F(X,Y)+F(I_iX,I_iY)-(2n+1)\Big[F(I_jX,I_jY)+F(I_kX,I_kY)\Big]\quad mod\quad \omega_s.\end{aligned} \tag{5.36}$$

The condition $W^{qc}=0$ and (4.8) give

$$\begin{aligned}-R(X,Y,e_a,e_b)L(I_ie_a,e_b)=&\,4L(X,e_a)L(Y,I_ie_a)-2L(X,e_a)L(I_iY,e_a)\\ &+2L(I_iX,e_a)L(Y,e_a)+2L(X,e_a)L(I_jY,I_ke_a)-2L(I_kX,e_a)L(Y,I_je_a)\\ &-2L(X,e_a)L(I_kY,I_je_a)+2L(I_jX,e_a)L(Y,I_ke_a)-tr\,L\Big[L(X,I_iY)-L(I_iX,Y)\Big].\end{aligned} \tag{5.37}$$

Using (4.13), we get

$$\begin{aligned}-4n\Big[\zeta_i(X,e_a)L(Y,e_a)-\zeta_i(Y,e_a)L(X,e_a)\Big]=-(8n+3)L(X,e_a)L(Y,I_ie_a)\\ +\frac{3}{2}L(X,e_a)L(I_iY,e_a)-\frac{3}{2}L(I_iX,e_a)L(Y,e_a)-\frac{1}{2}L(X,e_a)L(I_jY,I_ke_a)+\frac{1}{2}L(I_kX,e_a)L(Y,I_je_a)\\ +\frac{1}{2}L(X,e_a)L(I_kY,I_je_a)-\frac{1}{2}L(I_jX,e_a)L(Y,I_ke_a)+\frac{2n-1}{2n}tr\,L\Big[L(X,I_iY)-L(I_iX,Y)\Big].\end{aligned} \tag{5.38}$$

From (5.37) and (5.38) we have

$$
\begin{aligned}
(5.39)\quad & -R(X,Y,e_a,e_b)L(I_ie_a,e_b)-4n\Big[\zeta_i(X,e_a)L(Y,e_a)-\zeta_i(Y,e_a)L(X,e_a)\Big]\\
&=-(8n-1)L(X,e_a)L(Y,I_ie_a)-\frac{1}{2}L(X,e_a)L(I_iY,e_a)+\frac{1}{2}L(I_iX,e_a)L(Y,e_a)+\frac{3}{2}L(X,e_a)L(I_jY,I_ke_a)\\
&\quad-\frac{3}{2}L(I_kX,e_a)L(Y,I_je_a)-\frac{3}{2}L(X,e_a)L(I_kY,I_je_a)+\frac{3}{2}L(I_jX,e_a)L(Y,I_ke_a)\\
&\quad-\frac{1}{2n}(trL)\Big[L(X,I_iY)-L(I_iX,Y)\Big]\\
&=-(8n-1)L(X,e_a)L(Y,I_ie_a)-\frac{1}{2n}(trL)\Big[L(X,I_iY)-L(I_iX,Y)\Big]\\
&\quad+\frac{1}{2}\Big[L(Y,e_a)L(I_iX,e_a)-L(X,e_a)L(I_iY,e_a)\Big]-\frac{3}{2}\Big[L(Y,e_a)L(I_jX,I_ke_a)-L(X,e_a)L(I_jY,I_ke_a)\Big]\\
&\quad+\frac{3}{2}\Big[L(Y,e_a)L(I_kX,I_je_a)-L(X,e_a)L(I_kY,I_je_a)\Big].
\end{aligned}
$$

Since ρ_s is a (1,1)-form with respect to I_s, see Proposition 2.4, we have

$$\rho_j(T(\xi_k,I_jX),I_jY)=\rho_j(e_a,I_jY)T(\xi_k,I_jX,e_a)=\rho_j(e_a,Y)T(\xi_k,I_jX,I_je_a).$$

Thus, using (4.13) we obtain the next sequence of equalities

$$
\begin{aligned}
(5.40)\quad & \rho_j(T(\xi_k,X),Y)+\rho_j(T(\xi_k,I_jX),I_jY)-\rho_k(T(\xi_j,X),Y)-\rho_k(T(\xi_j,I_kX),I_kY)\\
&=\Big[L(e_a,I_jY)-L(I_je_a,Y)-\frac{1}{2n}tr\,L\,\omega_j(e_a,Y)\Big]\Big[T(\xi_k,X,e_a)+T(\xi_k,I_jX,I_je_a)\Big]\\
&\quad-\Big[L(e_a,I_kY)-L(I_ke_a,Y)-\frac{1}{2n}tr\,L\,\omega_k(e_a,Y)\Big]\Big[T(\xi_j,X,e_a)+T(\xi_j,I_kX,I_ke_a)\Big]
\end{aligned}
$$

$$
\begin{aligned}
(5.41)\quad & \rho_j(T(\xi_k,X),Y)+\rho_j(T(\xi_k,I_jX),I_jY)-\rho_k(T(\xi_j,X),Y)-\rho_k(T(\xi_j,I_kX),I_kY)\\
&\quad-2T(\xi_i,X,e_a)L(Y,e_a)+2T(\xi_j,X,e_a)L(I_kY,e_a)-2T(\xi_k,X,e_a)L(I_jY,e_a)\\
&=L(e_a,Y)\Big[T(\xi_k,X,I_je_a)-T(\xi_k,I_jX,e_a)-T(\xi_j,X,I_ke_a)+T(\xi_j,I_kX,e_a)-2T(\xi_i,X,e_a)\Big]\\
&\quad+L(e_a,I_jY)\Big[T(\xi_k,I_jX,I_je_a)-T(\xi_k,X,e_a)\Big]-L(e_a,I_kY)\Big[T(\xi_j,I_kX,I_ke_a)-T(\xi_j,X,e_a)\Big]\\
&\quad-\frac{1}{2n}tr\,L\Big[-T(\xi_k,X,I_jY)+T(\xi_k,I_jX,Y)-T(\xi_j,I_kX,Y)+T(\xi_j,X,I_kY)\Big].
\end{aligned}
$$

The first line in (5.41) is equal to

$$(5.42)\quad \frac{1}{2n}tr\,L.\,L(I_iX,Y)+\frac{1}{2}L(Y,e_a)\Big[5L(X,I_ie_a)-L(I_iX,e_a)+L(I_jX,I_ke_a)-L(I_kX,I_je_a)\Big].$$

The second line in (5.41) is equal to

$$
\begin{aligned}
(5.43)\quad & \frac{1}{2n}tr\,L\Big[L(I_kX,I_jY)-L(I_jX,I_kY)\Big]+L(X,e_a)\Big[L(I_kY,I_je_a)-L(I_jY,I_ke_a)\Big]\\
&\quad-L(I_jX,e_a)L(I_jY,I_ie_a)-L(I_kX,e_a)L(I_kY,I_ie_a).
\end{aligned}
$$

The third line in (5.41) is equal to

$$(5.44)\quad -\frac{1}{2n}tr\,L\Big[L(I_kX,I_jY)-L(I_jX,I_kY)-L(X,I_iY)+L(I_iX,Y)\Big].$$

A substitution of (5.42), (5.43) and (5.44) in (5.41) gives

$$\begin{aligned}(5.45)\quad &\rho_j(T(\xi_k,X),Y)+\rho_j(T(\xi_k,I_jX),I_jY)-\rho_k(T(\xi_j,X),Y)-\rho_k(T(\xi_j,I_kX),I_kY)\\ &\qquad -2T(\xi_i,X,e_a)L(Y,e_a)+2T(\xi_j,X,e_a)L(I_kY,e_a)-2T(\xi_k,X,e_a)L(I_jY,e_a)\\ &=\frac{1}{2n}tr\,L.\,L(X,I_iY)+\frac12 L(Y,e_a)\Big[5L(X,I_ie_a)-L(I_iX,e_a)+L(I_jX,I_ke_a)-L(I_kX,I_je_a)\Big]\\ &\quad +L(X,e_a)\Big[L(I_kY,I_je_a)-L(I_jY,I_ke_a)\Big]-L(I_jX,e_a)L(I_jY,I_ie_a)-L(I_kX,e_a)L(I_kY,I_ie_a).\end{aligned}$$

The last four lines in (5.35) equal the skew symmetric sum of (5.45), which is equal to

$$\begin{aligned}(5.46)\quad &-5L(X,e_a)L(Y,I_ie_a)-\frac12\Big[L(Y,e_a)L(I_iX,e_a)-L(X,e_a)L(I_iY,e_a)\Big]\\ &+\frac32\Big[L(Y,e_a)L(I_jX,I_ke_a)-L(X,e_a)L(I_jY,I_ke_a)\Big]-\frac32\Big[L(Y,e_a)L(I_kX,I_je_a)-L(X,e_a)L(I_kY,I_je_a)\Big]\\ &\qquad +\frac{1}{2n}tr\,L\Big[L(X,I_iY)-L(I_iX,Y)\Big]-2L(I_jX,e_a)L(I_jY,I_ie_a)-2L(I_kX,e_a)L(I_kY,I_ie_a).\end{aligned}$$

A substitution of (5.39) and (5.46) in (5.35) yields

$$(5.47)\quad F(X,Y)=-4(2n+1)L(X,e_a)L(Y,I_ie_a)-2L(I_jX,e_a)L(I_jY,I_ie_a)-2L(I_kX,e_a)L(I_kY,I_ie_a).$$

Inserting (5.47) in (5.36) completes the proof of (5.23). □

5.3. **Case 3, $A\in V,\ B,C\in H$. Integrability condition(5.49).**

In this case (5.6) reads

$$(5.48)\quad \nabla^2du(\xi_i,X,Y)-\nabla^2du(X,\xi_i,Y)+\nabla du(T(\xi_i,X),Y)=-R(\xi_i,X,Y,du).$$

The identities below can be used to see that the integrability condition (5.48) reduces to

$$\begin{aligned}(5.49)\quad &(\nabla_{\xi_t}L)(X,Y)+(\nabla_X\mathbb{B})(Y,\xi_t)+L(Y,I_tL(X))+L(T(\xi_t,X),Y)+g(T(\xi_t,Y),L(X))\\ &\qquad\qquad =\sum_{s=1}^3\mathbb{B}(\xi_s,\xi_t)\omega_s(X,Y),\qquad t=1,2,3.\end{aligned}$$

Notice that (5.23) is the skew-symmetric part of (5.49).

We turn to the proof of the fact the vanishing of W^{qc} implies the validity of (5.49). When we take a covariant derivative along a Reeb vector field of (5.1) and a covariant derivative along a horizontal direction of (5.2), use (5.2), (5.1), (5.3), (5.4), (5.5), (5.8), we see that the left hand-side of (5.48) equals

$$\begin{aligned}(5.50)\quad &\nabla^2du(\xi_1,X,Y)-\nabla^2du(X,\xi_i,Y)+\nabla du(T(\xi_i,X),Y)\\ &=du(I_iY)\left[\mathbb{B}(I_iX,\xi_i)-\frac{1}{4n}d(tr\,L)(X)\right]+du(I_jY)\left[\mathbb{B}(I_jX,\xi_i)+\mathbb{B}(X,\xi_k)\right]\\ &\quad+du(I_kY)\left[\mathbb{B}(I_kX,\xi_i)-\mathbb{B}(X,\xi_j)\right]+g(X,Y)\mathbb{B}(du,\xi_i)\\ &\quad-\omega_i(X,Y)\mathbb{B}(I_idu,\xi_i)-\omega_j(X,Y)\mathbb{B}(I_jdu,\xi_i)-\omega_k(X,Y)\mathbb{B}(I_kdu,\xi_i)\\ &\quad-du(X)\mathbb{B}(Y,\xi_i)+du(I_iX)\mathbb{B}(I_iY,\xi_i)+du(I_jX)\mathbb{B}(I_jY,\xi_i)+du(I_kX)\mathbb{B}(I_kY,\xi_i)\\ &\quad+\frac14(\nabla_XL)(Y,I_idu)-\frac14\nabla_XL)(I_iY,du)-\frac14(\nabla_XL)(I_kY,I_jdu)+\frac14(\nabla_XL)(I_jY,I_kdu)\\ &\quad-(\nabla_{\xi_i}L)(X,Y)\quad-(\nabla_X\mathbb{B})(Y,\xi_i)+L(X,I_iLY)-T(\xi_i,X,LY)-T(\xi_i,Y,LX)\\ &\qquad\qquad+\omega_i(X,Y)\mathbb{B}(\xi_i,\xi_i)+\omega_j(X,Y)\mathbb{B}(\xi_i,\xi_j)+\omega_k(X,Y)\mathbb{B}(\xi_i,\xi_k).\end{aligned}$$

On the other hand, a substitution of (4.10) and (4.11) in (3.5), and an application of (5.33) together with the already proven (5.9) and (5.23), shows after a long standard calculations the following equality

$$
\begin{aligned}
(5.51)\quad R(\xi_i,X,Y,Z) &= \mathbb{B}(I_jZ,\xi_i)\omega_j(X,Y)+\mathbb{B}(I_kZ,\xi_i)\omega_k(X,Y)\\
&-\omega_i(Y,Z)\Big[\mathbb{B}(I_iX,\xi_i)-\frac{1}{4n}d(trL)(X)\Big]-\omega_j(Y,Z)\Big[\mathbb{B}(X,\xi_k)+\mathbb{B}(I_jX,\xi_i)\Big]\\
&+\omega_k(Y,Z)\Big[\mathbb{B}(X,\xi_j)-\mathbb{B}(I_kX,\xi_i)\Big]+g(X,Z)\mathbb{B}(Y,\xi_i)-\omega_i(X,Z)\mathbb{B}(I_iY,\xi_i)\\
&-\omega_j(X,Z)\mathbb{B}(I_jY,\xi_i)-\omega_k(X,Z)\mathbb{B}(I_kY,\xi_i)-g(X,Y)\mathbb{B}(Z,\xi_i)+\mathbb{B}(I_iZ,\xi_i)\omega_i(X,Y)\\
&+\frac14\Big[(\nabla_XL)(I_iY,Z)-(\nabla_XL)(Y,I_iZ)+(\nabla_XL)(I_kY,I_jZ)-(\nabla_XL)(I_jY,I_kZ)\Big].
\end{aligned}
$$

In the derivation of the above equation we used the next formulas comming from (5.10)

$$(5.52)\qquad (\nabla_{e_a,}L)(I_ie_a,X)=(4n+1)\mathbb{B}(X,\xi_i)-\mathbb{B}(I_kX,\xi_j)+\mathbb{B}(I_jX,\xi_k)$$

$$(5.53)\qquad (\nabla_{e_a}L)(e_a,X)=-3\mathbb{B}(I_iX,\xi_i)-3\mathbb{B}(I_jX,\xi_j)-3\mathbb{B}(I_kX,\xi_k)+d(trL)(X).$$

Substituting equations (5.51), with $Z=du$, and (5.50) in (5.48), we obtain (5.49).

In the proof of the integrability condition we shall use the following

Lemma 5.5. *For the vertical part of the Ricci 2-forms we have the equalities*

$$
(5.54)\qquad
\begin{aligned}
&\rho_i(\xi_j,\xi_k)=\frac{1}{8n^2}(tr\,L)^2-\mathbb{B}(\xi_j,\xi_j)-\mathbb{B}(\xi_k,\xi_k)\\
&\rho_i(\xi_i,\xi_j)=\frac{1}{4n}d(tr\,L)(\xi_j)+\mathbb{B}(\xi_i,\xi_k),\qquad \rho_i(\xi_i,\xi_k)\ =\frac{1}{4n}d(tr\,L)(\xi_k)-\mathbb{B}(\xi_i,\xi_j)
\end{aligned}
$$

Proof. From the formula for the curvature (3.6) and Proposition 2.2 it follows

$$
\begin{aligned}
4n\rho_i(\xi_i,\xi_k)&=(\nabla_{e_a}\rho_j)(I_je_a,\xi_k)+T(\xi_i,e_a,e_b)T(\xi_k,e_b,I_ie_a)-T(\xi_i,e_b,I_ie_a)T(\xi_k,e_a,e_b)\\
4n\rho_j(\xi_i,\xi_k)&=-(\nabla_{e_a}\rho_j)(I_ie_a,\xi_k)+T(\xi_i,e_a,e_b)T(\xi_k,e_b,I_je_a)-T(\xi_i,e_a,I_je_b)T(\xi_k,e_b,e_a).
\end{aligned}
$$

Lemma 5.4 allows us to compute the divergences

$$
\begin{aligned}
(\nabla e_a\rho_i)(I_ke_a,\xi_j)&=-(\nabla_{a_a}\mathbb{B})(I_ke_a,\xi_k)-(\nabla_{e_a}\mathbb{B})(I_je_a,\xi_j)\\
(\nabla_{e_a}\rho_i)(I_ie_a,\xi_j)&=-(\nabla_{e_a}\mathbb{B})(e_a,\xi_j)-(\nabla_{e_a}\mathbb{B})(I_ie_a,\xi_k).
\end{aligned}
$$

After a calculation in which we use the integrability condition (5.23), the preceding paragraphs imply the first equation of (5.54). For the calculation of $\rho_i(\xi_i,\xi_k)$ we use again (5.23) to obtain

$$(\nabla_{e_a}\mathbb{B})(I_ie_a,\xi_k)=-L(I_ie_a,I_ke_b)L(e_a,e_b)+4n\mathbb{B}(\xi_i,\xi_k).$$

Setting $s=i, Y=I_iX$ in (5.57), using (4.13), (4.4) with respect to the function $tr\,L$, together with Lemma 5.4 we obtain

$$
\begin{aligned}
(5.55)\quad \Big[(\nabla_{\xi_i}L(X,X)+(\nabla_X\mathbb{B})(X,\xi_i)\Big]&+\Big[(\nabla_{\xi_i}L(I_iX,I_iX)+(\nabla_{I_iX}\mathbb{B})(I_iX,\xi_i)\Big]=\\
&-\rho_i(e_a,X)\Big[T(\xi_i,I_iX,e_a)-T(\xi_i,X,I_ie_a)\Big].
\end{aligned}
$$

Take the trace in (5.55) and use the properties of the torsion listed in Proposition 2.2 to conclude

$$(5.56)\qquad 2\Big[(\nabla_{e_b}\mathbb{B})(e_b,\xi_i)+d(trL)(\xi_i)\Big]=2\rho_i(e_a,e_b)U(e_a,e_b)=0,$$

which implies the formula for $\rho_i(\xi_i,\xi_k)$ after a short computation.

Finally, with the help of $\rho_i(\xi_i,\xi_k)+\rho_j(\xi_j,\xi_k)=\frac{1}{16n(n+2)}\xi_k(Scal)=\frac{1}{2n}\xi_k(trL)$, cf. [IMV1, Proposition 4.4], we also obtain the formula for $\rho_i(\xi_i,\xi_j)$. ∎

Proposition 5.6. *If $W^{qc}=0$ then the condition (5.49) holds.*

Proof. It is sufficient to consider only the symmetric part of (5.49) since its skew-symmetric part is the already established (5.23).

Letting $A = \xi_i, B = X, C = Y, D = e_a, E = I_s e_a$ in the second Bianchi identity (3.14) we obtain

$$(5.57)\quad (\nabla_{\xi_i}\rho_s)(X,Y) - (\nabla_X\rho_s)(\xi_i,Y) + (\nabla_Y\rho_s)(\xi_i,X) + \rho_s(T(\xi_i,X),Y) - \rho_s(T(\xi_i,Y),X) + 2\sum_{t=1}^{3}\omega_t(X,Y)\rho_s(\xi_t,\xi_i).$$

Setting $s = j, Y = I_jX$ in (5.57), using (4.13), Lemma 5.4, Lemma 5.5 and (5.23), we calculate

$$(5.58)\quad \Big[(\nabla_{\xi_i}L(X,X) + (\nabla_X\mathbb{B})(X,\xi_i)\Big] + \Big[(\nabla_{\xi_i}L(I_jX,I_jX) + (\nabla_{I_jX}\mathbb{B})(I_jX,\xi_i)\Big] = \rho_j(e_a,X)\Big[T(\xi_i,X,I_je_a) - T(\xi_i,I_jX,e_a)\Big] + \Big[(\nabla_X\mathbb{B})(I_jX,\xi_k) - (\nabla_{I_jX}\mathbb{B})(X,\xi_k)\Big] - 2|X|^2\mathbb{B}(\xi_j,\xi_k) = 2L(X,I_ke_a)L(I_jX,e_a) + \rho_j(e_a,X)\Big[T(\xi_i,X,I_je_a) - T(\xi_i,I_jX,e_a)\Big].$$

Similarly, when we take $s = k, Y = I_kX$ in (5.57), use (4.13), Lemma 5.4, Lemma 5.5 and (5.23) it follows

$$(5.59)\quad \Big[(\nabla_{\xi_i}L(X,X) + (\nabla_X\mathbb{B})(X,\xi_i)\Big] + \Big[(\nabla_{\xi_i}L(I_kX,I_kX) + (\nabla_{I_kX}\mathbb{B})(I_kX,\xi_i)\Big] = \rho_k(e_a,X)\Big[T(\xi_i,X,I_ke_a) - T(\xi_i,I_kX,e_a)\Big] - \Big[(\nabla_X\mathbb{B})(I_kX,\xi_j) - (\nabla_{I_kX}\mathbb{B})(X,\xi_j)\Big] + 2|X|^2\mathbb{B}(\xi_j,\xi_k) = 2L(I_kX,I_je_a)L(X,e_a) + \rho_k(e_a,X)\Big[T(\xi_i,X,I_ke_a) - T(\xi_i,I_kX,e_a)\Big].$$

Finally, replace X with I_iX in (5.59), subtract the obtained equality from (5.58) and add the result to (5.55) to obtain

$$(5.60)\quad 2\Big[(\nabla_{\xi_i}L(X,X) + (\nabla_X\mathbb{B})(X,\xi_i)\Big] = 2L(X,I_ke_a)L(I_jX,e_a) - 2L(I_jX,I_je_a)L(I_iX,e_a) - \rho_i(e_a,X)\Big[T(\xi_i,I_iX,e_a) - T(\xi_i,X,I_ie_a)\Big] + \rho_j(e_a,X)\Big[T(\xi_i,X,I_je_a) - T(\xi_i,I_jX,e_a)\Big] + \rho_k(e_a,I_iX)\Big[T(\xi_i,I_jX,e_a) - T(\xi_i,I_iX,I_ke_a)\Big].$$

Now, using (4.12) and the second equality in (4.13) applied to (5.60) concludes, after some standard calculations, the proof of (5.49). □

5.4. **Cases 4 and 5,** $A, B \in V, C \in H$. **Integrability conditions (5.66), (5.64) and (5.61).**

Case 4, $\xi_i, \xi_j \in V, \quad Y \in H$. In this case (5.6) reads

$$(5.61)\quad \nabla^2 du(\xi_i,\xi_j,Y) - \nabla^2 du(\xi_j,\xi_i,Y) = -R(\xi_i,\xi_j,Y,du) - \nabla du(T(\xi_i,\xi_j),Y).$$

Working as in the previous case, using (5.3),(5.4), (5.5), substituting (4.10), (4.11) (4.13) into (3.6), one gets, after long standard calculations applying the already proven (5.9), (5.23) and (5.49), that (5.61) is equivalent to

$$(5.62)\quad (\nabla_{\xi_i}\mathbb{B})(X,\xi_j) - (\nabla_{\xi_j}\mathbb{B})(X,\xi_i) = L(X,I_je_a)\mathbb{B}(e_a,\xi_i) - L(X,I_ie_a)\mathbb{B}(e_a,\xi_j) - L(e_a,X)\rho_k(I_ie_a,\xi_i) - T(\xi_i,X,e_a)\mathbb{B}(e_a,\xi_j) + T(\xi_j,X,e_a)\mathbb{B}(e_a,\xi_i) + \frac{1}{n}(trL)\,\mathbb{B}(X,\xi_k) = \Big[2L(X,I_je_a) + T(\xi_j,X,e_a)\Big]\mathbb{B}(e_a,\xi_i) - \Big[2L(X,I_ie_a) + T(\xi_i,X,e_a)\Big]\mathbb{B}(e_a,\xi_j) + \frac{1}{n}(trL)\,\mathbb{B}(X,\xi_k).$$

where we used Lemma 5.4 to derive the second equality.

Case 5_a**,** $X \in H, \quad \xi_i, \xi_j \in V$. In this case (5.6) becomes

$$(5.63)\quad \nabla^2 du(X,\xi_i,\xi_j) - \nabla^2 du(\xi_i,X,\xi_j) = -R(X,\xi_i,\xi_j,du) + \nabla du(T(\xi_i,X),\xi_j) = 2du(\xi_i)\rho_k(X,\xi_i) - 2du(\xi_k)\rho_i(X,\xi_i) + T(\xi_i,X,e_a)\nabla du(e_a,\xi_j).$$

With a similar calculations as in the previous cases, we see that (5.63) is equivalent to

$$(5.64)\quad (\nabla_{\xi_i}\mathbb{B})(X,\xi_j)+(\nabla_X\mathbb{B})(\xi_i,\xi_j) \\ -2L(X,I_je_a)\mathbb{B}(e_a,\xi_i)+T(\xi_i,X,e_a)\mathbb{B}(e_a,\xi_j)-\frac{1}{2n}trL\,\mathbb{B}(X,\xi_k)=0.$$

Case 5_b**,** $X\in H,\quad \xi_j,\xi_j\in V$. In this case (5.6) reads

$$(5.65)\quad \nabla^2du(X,\xi_j,\xi_j)-\nabla^2du(\xi_j,X,\xi_j)=-R(X,\xi_j,\xi_j,du)+\nabla du(T(\xi_j,X),\xi_j)= \\ 2du(\xi_i)\rho_k(X,\xi_j)-2du(\xi_k)\rho_i(X,\xi_j)+T(\xi_j,X,e_a)\nabla du(e_a,\xi_j).$$

and (5.65) is equivalent to

$$(5.66)\quad (\nabla_{\xi_j}\mathbb{B})(X,\xi_j)+(\nabla_XB)(\xi_j,\xi_j)-2\mathbb{B}(\mathbf{e}_a,\xi_j)L(X,I_je_a)+T(\xi_j,X,e_a)\mathbb{B}(e_a,\xi_j)=0.$$

Proposition 5.7. *If $W^{qc}=0$ then the conditions 5.66, 5.64 and 5.61 hold.*

Proof. Differentiating the already proven (5.23) and taking the corresponding traces we get

$$(5.67)\quad (\nabla^2_{X,e_a}\mathbb{B})(I_ie_a,\xi_t)+2(\nabla_XL)(e_a,e_b)L(I_ie_a,I_te_b)=4n(\nabla_X\mathbb{B})(\xi_i,\xi_t)$$

$$(5.68)\quad (\nabla^2_{e_a,X}\mathbb{B})(I_ie_a,\xi_t)-(\nabla^2_{e_a,I_ie_a}\mathbb{B})(X,\xi_t)-2(\nabla_{\mathbf{e}_b}L)(X,I_te_a)L(I_ie_b,e_a) \\ -2(\nabla_{\mathbf{e}_b}L)(I_ie_b,e_a)L(X,I_te_a)=2(\nabla_X\mathbb{B})(\xi_i,\xi_t)-2(\nabla_{I_kX}\mathbb{B})(\xi_j,\xi_t)+2(\nabla_{I_jX}\mathbb{B})(\xi_k,\xi_t).$$

Subtracting (5.68) from (5.67) we obtain

$$(5.69)\quad \Big[\nabla^2_{X,e_a}-\nabla^2_{e_a,X}\Big]\mathbb{B}(I_ie_a,\xi_t)+(\nabla^2_{e_a,I_ie_a}\mathbb{B})(X,\xi_t)+2(\nabla_{\mathbf{e}_b}L)(I_ie_b,e_a)L(X,I_te_a) \\ +2\Big[(\nabla_XL)(e_a,e_b)-(\nabla_{\mathbf{e}_b}L)(X,e_a)\Big]L(I_ie_b,I_te_a) \\ =2(2n-1)(\nabla_X\mathbb{B})(\xi_i,\xi_t)+2(\nabla_{I_kX}\mathbb{B})(\xi_j,\xi_t)-2(\nabla_{I_jX}\mathbb{B})(\xi_k,\xi_t).$$

A use of the Ricci identities and (2.6) shows

$$(5.70)\quad \Big[\nabla^2_{X,e_a}-\nabla^2_{e_a,X}\Big]\mathbb{B}(I_ie_a,\xi_i) \\ =-R(X,e_a,I_ie_a,e_b)\mathbb{B}(e_b,\xi_i)-R(X,e_a,\xi_i,\xi_s)\mathbb{B}(I_ie_a,\xi_s)-2\omega_s(X,e_a)(\nabla_{\xi_s}\mathbb{B})(I_ie_a,\xi_i) \\ =-4n\zeta_i(X,,e_a)\mathbb{B}(e_a,\xi_i)-2\rho_k(X,e_a)\mathbb{B}(I_ie_a,\xi_j)+2\rho_j(X,e_a)\mathbb{B}(I_ie_a,\xi_k) \\ +2(\nabla_{\xi_i}\mathbb{B})(X,\xi_i)-2(\nabla_{\xi_j}\mathbb{B})(I_kX,\xi_i)+2(\nabla_{\xi_k}\mathbb{B})(I_jX,\xi_i).$$

$$(5.71)\quad (\nabla^2_{e_a,I_ie_a}\mathbb{B})(X,\xi_i)= \\ -\frac{1}{2}\Big[R(e_a,I_ie_a,X,e_b)\mathbb{B}(e_b,\xi_i)+R(e_a,I_ie_a,\xi_i,\xi_s)\mathbb{B}(X,\xi_s)+8n(\nabla_{\xi_i}\mathbb{B})(X,\xi_i)\Big] \\ =-2n\tau_i(X,e_a)\mathbb{B})(e_a,\xi_i)-4n(\nabla_{\xi_i}\mathbb{B})(X,\xi_i).$$

Next we apply the already established (5.9) and use the condition $L(e_a,I_se_a)=0$ to get

$$(5.72)\quad \Big[(\nabla_XL)(e_a,e_b)-(\nabla_{\mathbf{e}_b}L)(X,e_a)\Big]L(I_ie_b,I_ie_a) \\ =-3\mathbb{B}(e_a,\xi_i)L(X,I_ie_a)+3\mathbb{B}(e_a,\xi_j)L(I_kX,I_ie_a)-3\mathbb{B}(e_a,\xi_k)L(I_jX,I_ie_a).$$

$$(5.73)\quad (\nabla_{\mathbf{e}_b}L)(I_ie_b,e_a)L(X,I_ie_a) \\ =(4n+1)\mathbb{B}(e_a,\xi_i)L(X,I_ie_a)-\mathbb{B}(e_a,\xi_j)L(X,I_je_a)-\mathbb{B}(e_a,\xi_k)L(X,I_ke_a).$$

When we substitute (5.73), (5.72), (5.71), (5.70) in (5.69) we obtain

$$(5.74)\quad (1-2n)\Big[(\nabla_{\xi_i}\mathbb{B})(X,\xi_i)+(\nabla_X\mathbb{B})(\xi_i,\xi_i)\Big]-\Big[(\nabla_{\xi_j}\mathbb{B})(I_kX,\xi_i)+(\nabla_{I_kX}\mathbb{B})(\xi_j,\xi_i)\Big] \\ +\Big[(\nabla_{\xi_k}\mathbb{B})(I_jX,\xi_i)+(\nabla_{I_jX}\mathbb{B})(\xi_k,\xi_i)\Big]=D_{123}(X),$$

where $D_{ijk}(X)$ is defined by

$$D_{ijk}(X) = \Big[2n\zeta_i(X,e_a) + n\tau_i(X,e_a) - (4n-2)L(X,I_ie_a)\Big]\mathbb{B}(e_a,\xi_i) - \Big[\rho_k(X,I_ie_a) + 3L(I_kX,I_ie_a) - L(X,I_je_a)\Big]\mathbb{B}(e_a,\xi_j) + \Big[\rho_j(X,I_ie_a) + 3L(I_jX,I_ie_a) + L(X,I_ke_a)\Big]\mathbb{B}(e_a,\xi_k). \tag{5.75}$$

We also need the next Lemma, showing the symmetry of the vertical tensors $\mathbb{B}$.

Lemma 5.8. *The quantities $\mathbb{B}(\xi_i,\xi_j)$ are symmetric,*

$$\mathbb{B}(\xi_s,\xi_t) = \mathbb{B}(\xi_t,\xi_s), \qquad s,t = 1,2,3.$$

Proof. From (5.24) we obtain

$$\mathbb{B}(\xi_i,\xi_j) - \mathbb{B}(\xi_j,\xi_i) = \frac{1}{4n}\Big[(\nabla_{e_a}\mathbb{B})(I_ie_a,\xi_j) - (\nabla_{e_a}\mathbb{B})(I_je_a,\xi_i)\Big]. \tag{5.76}$$

On the other hand, (5.10) imply

$$2(2n+1)(4n-1)\Big[\mathbb{B}(I_iX,\xi_j) - \mathbb{B}(I_jX,\xi_i)\Big] = (4n+1)\Big[(\nabla_{e_a}L)(I_ie_{e_a},I_jX) - (\nabla_{e_a}L)(I_je_{e_a},I_iX)\Big] + 2(\nabla_{e_a}L)(I_ke_a,X). \tag{5.77}$$

Substitute (5.77) into (5.76) to get

$$8n(2n+1)(4n-1)\Big[\mathbb{B}(\xi_i,\xi_j) - \mathbb{B}(\xi_j,\xi_i)\Big] = (4n+1)\Big[(\nabla^2_{e_b,e_a}L)(I_ie_{e_a},I_je_b) - (\nabla^2_{e_b,e_a}L)(I_je_{e_a},I_ie_b)\Big] + 2(\nabla^2_{e_b,e_a}L)(I_ke_a,e_b). \tag{5.78}$$

We calculate using (5.9) and (5.76) that

$$(\nabla^2_{e_b,e_a}L)(I_ke_a,e_b) = (4n+1)(\nabla_{e_a}\mathbb{B})(e_a,\xi_k) + (\nabla_{e_a}\mathbb{B})(I_ie_a,\xi_j) - (\nabla_{e_a}\mathbb{B})(I_je_a,\xi_i) = (4n+1)(\nabla_{e_a}\mathbb{B})(e_a,\xi_k) + 4n\Big[\mathbb{B}(\xi_i,\xi_j) - \mathbb{B}(\xi_j,\xi_i)\Big]. \tag{5.79}$$

The Ricci identities, the symmetry of L and (4.13) imply

$$\Big[(\nabla^2_{e_b,e_a}L)(I_ie_{e_a},I_je_b) - (\nabla^2_{e_b,e_a}L)(I_je_{e_a},I_ie_b)\Big] = \zeta_j(e_b,e_a)L(e_a,I_ie_b) - \zeta_i(e_b,e_a)L(e_a,I_je_b) + 2\omega_s(e_b,e_a)(\nabla_{\xi_k}L)(I_je_a,I_ie_b) = 2\nabla_{\xi_k} tr\, L. \tag{5.80}$$

Substitute (5.80) and (5.79) in (5.78) and apply (5.56) to conclude

$$8n(4n^2+n-1)\Big[\mathbb{B}(\xi_i,\xi_j) - \mathbb{B}(\xi_j,\xi_i)\Big] = (4n+1)\Big[\nabla_{\xi_k} tr\, L + (\nabla_{e_a}\mathbb{B})(e_a,\xi_k)\Big] = 0.$$

□

The second Bianchi identity (3.14) taken with respect to $A=\xi_i, B=\xi_j, C=X, D=e_a, E=I_se_a$ and the formulas described in Theorem 2.4 yield

$$(\nabla_{\xi_i}\rho_s)(\xi_j,X) - (\nabla_{\xi_j}\rho_s)(\xi_i,X) + (\nabla_X\rho_s)(\xi_i,\xi_j) = \rho_s(T(\xi_i,X),\xi_j) - \rho_s(T(\xi_j,X),\xi_i) + \rho_s(e_a,X)\rho_k(I_ie_a.\xi_i) + \frac{tr\, L}{n}\rho_s(\xi_k,X). \tag{5.81}$$

Setting successively $s=1,2,3$ in (5.81), using (5.8) with respect to the function $tr\,L$ and applying Lemma 5.4 and Lemma 5.5, we obtain after some calculations

$$\begin{aligned}
&\Big[(\nabla_{\xi_i}\mathbb{B})(I_iX,\xi_j)-(\nabla_{\xi_j}\mathbb{B})(I_iX,\xi_i)\Big]-\Big[(\nabla_{\xi_i}\mathbb{B})(X,\xi_k)+(\nabla_X\mathbb{B})(\xi_i,\xi_k)\Big]=\alpha_{ijk}(X)\\
&\Big[(\nabla_{\xi_i}\mathbb{B})(I_jX,\xi_j)-(\nabla_{\xi_j}\mathbb{B})(I_jX,\xi_i)\Big]-\Big[(\nabla_{\xi_j}\mathbb{B})(X,\xi_k)+(\nabla_X\mathbb{B})(\xi_j,\xi_k)\Big]=\beta_{ijk}(X)\\
&\Big[(\nabla_{\xi_i}\mathbb{B})(I_kX,\xi_j)-(\nabla_{\xi_j}\mathbb{B})(I_kX,\xi_i)\Big]+\Big[(\nabla_{\xi_i}\mathbb{B})(X,\xi_i)+(\nabla_X\mathbb{B})(\xi_i,\xi_i)\Big]\\
&\qquad\qquad+\Big[(\nabla_{\xi_j}\mathbb{B})(X,\xi_j)+(\nabla_X\mathbb{B})(\xi_j,\xi_j)\Big]=\gamma_{ijk}(X),
\end{aligned}\tag{5.82}$$

where

$$\begin{aligned}
\alpha_{ijk}(X)&=\rho_i(e_a,\xi_i)T(\xi_j,X,e_a)-\rho_i(e_a,\xi_j)T(\xi_i,X,e_a)-\rho_i(e_a,X)\rho_k(I_ie_a,\xi_i)\\
&\qquad+\frac{1}{4n}d(tr\,L)(e_a)T(\xi_j,X,e_a)-\frac{tr\,L}{n}\rho_i(\xi_k,X)\\
\beta_{ijk}(X)&=\rho_j(e_a,\xi_i)T(\xi_j,X,e_a)-\rho_j(e_a,\xi_j)T(\xi_i,X,e_a)-\rho_j(e_a,X)\rho_k(I_ie_a,\xi_i)\\
&\qquad-\frac{1}{4n}d(tr\,L)(e_a)T(\xi_i,X,e_a)-\frac{tr\,L}{n}\rho_j(\xi_k,X)\\
\gamma_{ijk}(X)&=\rho_k(e_a,\xi_i)T(\xi_j,X,e_a)-\rho_k(e_a,\xi_j)T(\xi_i,X,e_a)-\rho_k(e_a,X)\rho_k(I_ie_a,\xi_i)\\
&\qquad+\frac{1}{4n^2}(tr\,L)d(tr\,L)(X)-\frac{tr\,L}{n}\rho_k(\xi_k,X)
\end{aligned}\tag{5.83}$$

Now we can solve the system consisting of (5.75) and (5.82). Indeed, (5.75) and Lemma 5.8 imply

$$\begin{aligned}
&(1-2n)\Big[[(\nabla_{\xi_i}\mathbb{B})(X,\xi_i)+(\nabla_X\mathbb{B})(\xi_i,\xi_i)]+[(\nabla_{\xi_j}\mathbb{B})(X,\xi_j)+(\nabla_X\mathbb{B})(\xi_j,\xi_j)]\Big]\\
&\Big[(\nabla_{\xi_i}\mathbb{B})(I_kX,\xi_j)-(\nabla_{\xi_j}\mathbb{B})(I_kX,\xi_i)\Big]+\Big[(\nabla_{\xi_k}\mathbb{B})(I_jX,\xi_i)+(\nabla_{I_jX}\mathbb{B})(\xi_k,\xi_i)\Big]\\
&\qquad-\Big[(\nabla_{\xi_k}\mathbb{B})(I_iX,\xi_j)+(\nabla_{I_iX}\mathbb{B})(\xi_k,\xi_j)\Big]=D_{123}(X)+D_{231}(X).
\end{aligned}\tag{5.84}$$

The last identity in (5.82) and (5.84) yields

$$\begin{aligned}
&2n\Big[(\nabla_{\xi_i}\mathbb{B})(I_kX,\xi_j)-(\nabla_{\xi_j}\mathbb{B})(I_kX,\xi_i)\Big]+\Big[(\nabla_{\xi_k}\mathbb{B})(I_jX,\xi_i)+(\nabla_{I_jX}\mathbb{B})(\xi_k,\xi_i)\Big]\\
&\qquad-\Big[(\nabla_{\xi_k}\mathbb{B})(I_iX,\xi_j)+(\nabla_{I_iX}\mathbb{B})(\xi_k,\xi_j)\Big]=D_{ijk}(X)+D_{231}(X)+(2n-1)\gamma_{ijk}(X).
\end{aligned}\tag{5.85}$$

The first two equalities in (5.82) together with (5.85) lead to

$$\begin{aligned}
&2(n+1)\Big[(\nabla_{\xi_i}\mathbb{B})(I_kX,\xi_j)-(\nabla_{\xi_j}\mathbb{B})(I_kX,\xi_i)\Big]+\Big[(\nabla_{\xi_k}\mathbb{B})(I_jX,\xi_i)-(\nabla_{\xi_i}\mathbb{B})(I_jX,\xi_k)\Big]+\\
&\qquad\Big[(\nabla_{\xi_j}\mathbb{B})(I_iX,\xi_k)-(\nabla_{\xi_k}\mathbb{B})(I_iX,\xi_j)\Big]=A_{ijk}(X),
\end{aligned}\tag{5.86}$$

where

$$A_{ijk}(X)=D_{ijk}(X)+D_{jki}(X)+(2n-1)\gamma_{ijk}(X)+\alpha_{ijk}(I_jX)-\beta_{ijk}(I_iX).\tag{5.87}$$

Consequently, we derive easily that

$$\begin{aligned}
&2(n+2)(2n+1)\Big[(\nabla_{\xi_i}\mathbb{B})(I_kX,\xi_j)-(\nabla_{\xi_j}\mathbb{B})(I_kX,\xi_i)\Big]\\
&\qquad=(2n+3)A_{ijk}(X)-A_{jki}(X)-A_{kij}(X).
\end{aligned}\tag{5.88}$$

The second equality in (4.13) together with (4.12) and Lemma 5.4 applied to (5.83) and (5.75), after standard calculations, give

$$\begin{aligned}(5.89)\quad \alpha_{ijk}(I_jX)-\beta_{ijk}(I_iX) &= \frac{1}{2}L(X,e_a)\Big[B(I_ie_a,\xi_i)+B(I_je_a,\xi_j)+B(I_ke_a,\xi_k)\Big]\\ &+\frac{1}{2}L(I_iX,e_a)\Big[-B(e_a,\xi_i)-3B(I_ke_a,\xi_j)-2B(I_je_a,\xi_k)\Big]\\ &+\frac{1}{2}L(I_jX,e_a)\Big[3B(I_ke_a,\xi_i)-B(e_a,\xi_j)+2B(I_ie_a,\xi_k)\Big]\\ &+\frac{1}{2}L(I_kX,e_a)\Big[-5B(I_je_a,\xi_i)+5B(I_ie_a,\xi_j)-B(e_a,\xi_k)\Big]+\frac{3}{2n}\,(tr\,L)\,B(I_kX,\xi_k)\end{aligned}$$

$$\begin{aligned}(5.90)\quad \gamma_{ijk}(X) &== \frac{1}{2n}\,(tr\,L)\,B(I_kX,\xi_k)\\ &-\frac{5}{2}L(X,e_a)\Big[B(I_ie_a,\xi_i)+B(I_je_a,\xi_j)\Big]-\frac{3}{2}L(I_kX,e_a)\Big[B(I_je_a,\xi_i)-B(I_ie_a,\xi_j)\Big]\end{aligned}$$

$$\begin{aligned}(5.91)\quad D_{ijk}(X)+D_{jki}(X)+(2n-1)\gamma_{ijk}(X) &= \frac{2n+1}{4}L(X,e_a)\Big[B(I_ie_a,\xi_i)+B(I_je_a,\xi_j)\Big]\\ &+\frac{1}{4}L(I_iX,e_a)\Big[-(2n+1)\,B(e_a,\xi_i)+(2n+3)\,B(I_ke_a,\xi_j)+8B(I_je_a,\xi_k)\Big]\\ &+\frac{1}{4}L(I_jX,e_a)\Big[-(2n+3)\,B(I_ke_a,\xi_i)-(2n+1)\,B(e_a,\xi_j)-8B(I_ie_a,\xi_k)\Big]\\ &+\frac{1-10n}{4}L(I_kX,e_a)\Big[B(I_je_a,\xi_i)-B(I_ie_a,\xi_j)\Big]\\ &+\frac{1}{4n}\,(trL)\,\Big[(1-2n)\,\mathbb{B}(I_iX,\xi_i)+(1-2n\,)\,\mathbb{B}(I_jX,\xi_j)+8n\,\mathbb{B}(I_kX,\xi_k)\Big].\end{aligned}$$

A substitution of (5.89), (5.90) and (5.91) in (5.87) shows

$$\begin{aligned}(5.92)\quad A_{ijk}(X) &= \frac{1}{4n}\,(tr\,L)\,[(1-2n)\,\mathbb{B}(I_iX,\xi_i)+(1-2n\,)\,\mathbb{B}(I_jX,\xi_j)+(8n+6)\;\mathbb{B}(I_kX,\xi_k)]\\ &+\frac{1}{4}L(X,e_a)\,[(2n+3)\,B(I_ie_a,\xi_i)+(2n+3)\,B(I_je_a,\xi_j)+2B(I_ke_a,\xi_k)]\\ &+\frac{1}{4}L(I_iX,e_a)\,[-(2n+3)\,B(e_a,\xi_i)+(2n-3)\,B(I_ke_a,\xi_j)+4B(I_je_a,\xi_k)]\\ &+\frac{1}{4}L(I_jX,e_a)\,[-(2n-3)\,B(I_ke_a,\xi_i)-(2n+3)\,B(e_a,\xi_j)-4B(I_ie_a,\xi_k)]\\ &+\frac{1}{4}L(I_kX,e_a)\,[-(10n+9)\,B(I_je_a,\xi_i)+(10n+9)\,B(I_ie_a,\xi_j)-2B(e_a,\xi_k)]\end{aligned}$$

Plugging (5.92) in (5.88) and using (4.12) we obtain

$$\begin{aligned}(5.93)\quad (\nabla_{\xi_i}\mathbb{B})(I_kX,\xi_j)-(\nabla_{\xi_j}\mathbb{B})(I_kX,\xi_i) &= \frac{1}{n}(trL)\,\mathbb{B}(I_kX,\xi_k)\\ &+\Big[2L(I_kX,I_je_a)+T(\xi_j,I_kX,e_a)\Big]\mathbb{B}(e_a,\xi_i)-\Big[2L(I_kX,I_ie_a)+T(\xi_i,I_kX,e_a)\Big]\mathbb{B}(e_a,\xi_j).\end{aligned}$$

Hence, (5.62) follows. Substituting (5.62) in the first equality of (5.82) we obtain (5.64). Inserting (5.64) in (5.74) we see (5.65). □

5.5. **Case 6, $A,B,C\in V$. Integrability conditions (5.95) and (5.97).**
Case 6_a, $\xi_k,\xi_i,\xi_j\in V$. In this case the Ricci identity (5.6) becomes

$$\begin{aligned}(5.94)\quad \nabla^2du(\xi_k,\xi_i,\xi_j)-\nabla^2du(\xi_i,\xi_k,\xi_j) &= -R(\xi_k,\xi_i,\xi_j,du)-\nabla du(T(\xi_k,\xi_i),\xi_j)\\ &= 2du(\xi_i)\rho_k(\xi_k,\xi_i)-2du(\xi_k)\rho_i(\xi_k,\xi_i)+\rho_j(I_ke_a,\xi_k)\nabla du(e_a,\xi_j)+\frac{1}{n}tr\,L\,\nabla du(\xi_j,\xi_j).\end{aligned}$$

After some calculations we see that (5.94) is equivalent to

$$(\nabla_{\xi_i}\mathbb{B})(\xi_j,\xi_k)-(\nabla_{\xi_k}\mathbb{B})(\xi_j,\xi_i)+\mathbb{B}(I_ke_a,\xi_j)\mathbb{B}(e_a,\xi_i)-\mathbb{B}(I_ie_a,\xi_j)\mathbb{B}(e_a,\xi_k)$$
$$-\rho_i(I_ke_a,\xi_k)\mathbb{B}(e_a,\xi_i)+\rho_k(I_ie_a,\xi_i)\mathbb{B}(e_a,\xi_k)-\rho_j(I_ke_a,\xi_k)\mathbb{B}(e_a,\xi_j)$$
$$+\frac{1}{2n}\Big[trL\,\mathbb{B}(\xi_i,\xi_i)-trL\,\mathbb{B}(\xi_k,\xi_k)-2trL\,\mathbb{B}(\xi_j,\xi_j)\Big]=0.$$

Using Lemma 5.4 and the above equation shows that the integrability condition in this case is

$$(\nabla_{\xi_i}\mathbb{B})(\xi_k,\xi_j)-(\nabla_{\xi_k}\mathbb{B})(\xi_i,\xi_j)=\frac{1}{2n}(tr\,L\,)\,[\mathbb{B}(\xi_i,\xi_i)-2\mathbb{B}(\xi_j,\xi_j)+\mathbb{B}(\xi_k,\xi_k)]$$
$$+2\mathbb{B}(e_a,\xi_i)\mathbb{B}(I_je_a,\xi_k)+\mathbb{B}(e_a,\xi_i)\mathbb{B}(I_ke_a,\xi_j)+\mathbb{B}(I_ie_a,\xi_k)\mathbb{B}(e_a,\xi_j). \tag{5.95}$$

Case 6_b, $\xi_k,\xi_j,\xi_j\in V$. Here, equation (5.6) reads

$$\nabla^2du(\xi_k,\xi_j,\xi_j)-\nabla^2du(\xi_j,\xi_k,\xi_j)=-R(\xi_k,\xi_j,\xi_j,du)-\nabla du(T(\xi_k,\xi_j),\xi_j) \tag{5.96}$$
$$=2du(\xi_i)\rho_k(\xi_k,\xi_j)-2du(\xi_k)\rho_i(\xi_k,\xi_j)-\rho_i(I_ke_a,\xi_k)\nabla du(e_a,\xi_j)-\frac{1}{n}(tr\,L)\,\nabla du(\xi_i,\xi_j).$$

A small calculation shows that (5.96) is equivalent to

$$(\nabla_{\xi_j}\mathbb{B})(\xi_k,\xi_j)-(\nabla_{\xi_k}\mathbb{B})(\xi_j,\xi_j) \tag{5.97}$$
$$=-B(I_ke_a,\xi_j)\mathbb{B}(e_a,\xi_j)+3\mathbb{B}(I_je_a,\xi_k)\mathbb{B}(e_a,\xi_j)+\frac{3}{2n}(tr\,L)\,\mathbb{B}(\xi_i,\xi_j)=0.$$

Proposition 5.9. *If $W^{qc}=0$ then the conditions 5.95, 5.97 hold.*

Proof. Differentiate (5.64) and take the corresponding trace to get

$$(\nabla^2_{e_a,\xi_i}\mathbb{B})(I_ke_a,\xi_j)+(\nabla^2_{e_a,I_ke_a}\mathbb{B})(\xi_i,\xi_j)= \tag{5.98}$$
$$2(\nabla_{e_b}L)(I_ke_b,I_je_a)\mathbb{B}(e_a,\xi_i)+2L(I_ke_b,I_je_a)(\nabla_{e_b}\mathbb{B})(e_a,\xi_i)$$
$$-(\nabla_{e_b}T)(\xi_i,I_ke_b,e_a)\mathbb{B}(e_a,\xi_j)-T(\xi_i,I_ke_b,e_a)(\nabla_{e_b}\mathbb{B})(e_a,\xi_j)$$
$$+\frac{1}{2n}d(tr\,L)(e_a)\mathbb{B}(I_ke_a,\xi_k)+\frac{1}{2n}(tr\,L)(\nabla_{e_a}\mathbb{B})(I_ke_a,\xi_k).$$

On the other hand, the Ricci identities, (5.24), (2.6) and (4.13) yield

$$(\nabla^2_{e_a,I_ke_a}\mathbb{B})(\xi_i,\xi_j)=-4n(\nabla_{\xi_k}\mathbb{B})(\xi_i,\xi_j)+4(tr\,L)\mathbb{B}(\xi_j,\xi_j)-4(tr\,L)\mathbb{B}(\xi_i,\xi_i). \tag{5.99}$$

$$(\nabla^2_{e_a,\xi_i}\mathbb{B})(I_ke_a,\xi_j)=(\nabla^2_{\xi_i,e_a}\mathbb{B})(I_ke_a,\xi_j)+4n\zeta_k(\xi_i,e_a)\mathbb{B}(e_a,\xi_j) \tag{5.100}$$
$$-2\rho_i(e_a,\xi_i)\mathbb{B}(I_ke_a,\xi_k)+2\rho_k(e_a,\xi_i)\mathbb{B}(I_ke_a,\xi_i)+T(\xi_i,e_a,e_b)(\nabla_{e_b}\mathbb{B})(I_ke_a,\xi_j)$$
$$=4n(\nabla_{\xi_i}\mathbb{B})(\xi_j,\xi_k)-2(\nabla_{\xi_i}L)(e_a,e_b)(L(I_je_a,I_ke_b)+4n\zeta_k(\xi_i,e_a)\mathbb{B}(e_a,\xi_j)$$
$$-2\rho_i(e_a,\xi_i)\mathbb{B}(I_ke_a,\xi_k)+2\rho_k(e_a,\xi_i)\mathbb{B}(I_ke_a,\xi_i)+T(\xi_i,e_a,e_b)(\nabla_{e_b}\mathbb{B})(I_ke_a,\xi_j)$$

Substituting (5.99) and (5.100) in (5.98) we come to

$$4n\Big[(\nabla_{\xi_i}\mathbb{B})(\xi_j,\xi_k)-(\nabla_{\xi_k}\mathbb{B})(\xi_i,\xi_j)\Big] \tag{5.101}$$
$$=2(\nabla_{e_b}L)(I_ke_b,I_je_a)\mathbb{B}(e_a,\xi_i)+2\Big[(\nabla_{e_b}\mathbb{B})(e_a,\xi_i)+(\nabla_{\xi_i}L)(e_b,e_a)\Big]L(I_ke_b,I_je_a)$$
$$-\mathbb{B}(e_a,\xi_j)\Big[4n\zeta_k(\xi_i,e_a)+(\nabla_{e_b}T)(\xi_i,I_ke_b,e_a)\Big]+\Big[2\rho_i(e_a,\xi_i)+\frac{1}{2n}d(tr\,L)(e_a)\Big]\mathbb{B}(I_ke_a,\xi_k)$$
$$-2\rho_k(e_a,\xi_i)\mathbb{B}(I_ke_a,\xi_i)+T(\xi_i,I_ke_a,e_b)\Big[(\nabla_{e_b}\mathbb{B})(e_a,\xi_j)-(\nabla_{e_a}\mathbb{B})(e_b,\xi_j)\Big]$$
$$+\frac{1}{2n}(tr\,L)(\nabla_{e_a}\mathbb{B})(I_ke_a,\xi_k)-4(tr\,L)\mathbb{B}(\xi_j,\xi_j)+4(tr\,L)\mathbb{B}(\xi_i,\xi_i)$$

With the help of (5.51), the symmetry of L, and the divergence formulas (5.52) and (5.53) we find

$$(5.102)\quad 4n\zeta_k(\xi_i,e_a)=(4n+1)\mathbb{B}(I_ke_a,\xi_i)-\mathbb{B}(e_a,\xi_j)+\mathbb{B}(I_ie_a,\xi_k)+\frac{1}{4n}d(trL)(I_je_a)$$
$$=-\frac{1}{4}\left[\nabla L(e_b,I_ie_a,I_ke_b)+\nabla L(e_b,e_a,I_je_b)+\nabla L(e_b,I_ke_a,I_ie_b)+\nabla L(e_b,I_je_a,e_b)\right].$$

It follows from (4.12) that

$$(5.103)\quad (\nabla_{e_b}T)(\xi_i,I_ke_b,e_a)=\frac{1}{4}(\nabla_{e_b}L)(I_je_b,e_a)-\frac{3}{4}(\nabla_{e_b}L)(I_ke_b,I_ie_a)+\frac{1}{4}(\nabla_{e_b}L)(e_b,I_je_a)$$
$$+\frac{1}{4}(\nabla_{e_b}L)(I_ie_b,I_ke_a)-\frac{1}{4n}d(trL)(I_je_a).$$

Adding the last two equations we see

$$4n\zeta_k(\xi_i,e_a)+(\nabla_{e_b}T)(\xi_i,I_ke_b,e_a)=4nB(I_ke_a\xi_i)-4nB(I_ie_a\xi_k).$$

Using in (5.101) the above identity, Lemma 5.4, (5.49), (5.23), together with $L(\mathbf{e}_b,I_se_b)=0$, a long calculation gives

$$4n(\nabla_{\xi_i}\mathbb{B})(\xi_k,\xi_j)-4n(\nabla_{\xi_k}\mathbb{B})(\xi_i,\xi_j)=-4(trL)B(\xi_j,\xi_j)+2(trL)B(\xi_i,\xi_i)+2(trL)B(\xi_k,\xi_k)$$
$$8nB(I_je_a,\xi_k)B(e_a,\xi_i)+4nB(e_a,\xi_i)B(I_ke_a,\xi_j)+4nB(I_ie_a,\xi_k)B(e_a,\xi_j).$$

Hence (5.95) is proven.

The other integrability condition in this case, (5.97), can be obtained similarly using (5.66) and the Ricci identities. □

The proof of Theorem 1.1 is complete.

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(Stefan Ivanov) University of Sofia, Faculty of Mathematics and Informatics, blvd. James Bourchier 5, 1164, Sofia, Bulgaria

and Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany
E-mail address: ivanovsp@fmi.uni-sofia.bg

(Dimiter Vassilev) University of California, Riverside, Riverside, CA 92521

and Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany
E-mail address: dvassilev@math.ucr.edu