ORBIT EQUIVALENCE RIGIDITY AND CERTAIN GENERALIZED BERNOULLI SHIFTS OF THE MAPPING CLASS GROUP

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Abstract. We prove that any ergodic standard action (that is, an essentially free, measure-preserving action on a standard Borel space with a probability measure) of the mapping class group of a compact orientable surface with higher complexity has strong orbital rigidity: if there are ergodic standard actions of the mapping class group and a discrete group which are weakly orbit equivalent, then the two actions are virtually isomorphic. Moreover, we classify certain generalized Bernoulli shifts of the mapping class group up to orbit equivalence. This gives uncountably many examples of ergodic standard actions of the mapping class group which are mutually non-orbit equivalent. We prove similar statements for a finite direct product of mapping class groups.

1. Introduction

This paper is a continuation of [22], in which we established rigidity for the mapping class group in terms of measure equivalence. In this paper, we study the mapping class group from the viewpoint of two central topics in the theory of orbit equivalence, strong orbital rigidity and construction of uncountably many ergodic standard actions which are mutually non-orbit equivalent. Here, a standard action of a discrete group means an essentially free, measure-preserving action on a standard probability space, which means a standard Borel space with a probability measure.

In ergodic theory, it is one of the main problems to study isomorphism classes of ergodic actions of the infinite cyclic group $\mathbb{Z}$.

Definition 1.1. Consider a non-singular action of a discrete group Γ_i on a standard probability space (X_i, μ_i) for $i = 1, 2$. The two actions are said to be isomorphic if there are conull Borel subsets $X_1' \subset X_1$, $X_2' \subset X_2$, a Borel isomorphism $f\colon X_1' \to X_2'$ and an isomorphism $F\colon \Gamma_1 \to \Gamma_2$ such that

(i) the two measures $f_*\mu_1$ and μ_2 are equivalent;
(ii) $f(gx) = F(g)f(x)$ for any $g \in \Gamma_1$ and any $x \in X_1'$.

A Bernoulli shift is a typical example of an ergodic (or more strongly, mixing) action of an infinite discrete group, which is also standard. Given a discrete group Γ and a standard probability space (X, μ), we call the Γ-action on the standard

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probability space $(X,\mu)^\Gamma$ defined by

$$\gamma(x_g)_{g\in\Gamma} = (x_{\gamma^{-1}g})_{g\in\Gamma}$$

for $\gamma \in \Gamma$ and $(x_g)_{g\in\Gamma} \in (X,\mu)^\Gamma$ a Bernoulli shift of Γ. It is a natural question when Bernoulli shifts of $\mathbb{Z}$ arising from two different standard probability spaces are isomorphic. Kolmogorov and Sinaǐ introduced an isomorphism invariant for $\mathbb{Z}$-actions, called entropy, and showed that the entropy of Bernoulli shifts of $\mathbb{Z}$ can be computed in terms of (X,μ) and assumes all non-negative values. In particular, there exist uncountably many isomorphism classes of ergodic $\mathbb{Z}$-actions. As the culmination of the study on this isomorphism problem, Ornstein [26], [27] proved that entropy is a complete invariant for Bernoulli shifts of $\mathbb{Z}$, that is, two Bernoulli shifts of $\mathbb{Z}$ which have the same entropy are isomorphic.

On the other hand, orbit equivalence is a much weaker equivalence for actions of discrete groups on standard probability spaces than isomorphism.

Definition 1.2. Consider a non-singular action of a discrete group Γ_i on a standard probability space (X_i,μ_i) for $i=1,2$. Then the two actions are said to be weakly orbit equivalent if there exist Borel subsets $A_1 \subset X_1$ and $A_2 \subset X_2$ satisfying $\Gamma_1 A_1 = X_1$ and $\Gamma_2 A_2 = X_2$ up to null sets and there is a Borel isomorphism $f\colon A_1 \to A_2$ such that

(i) the two measures $f_*(\mu_1|_{A_1})$ and $\mu_2|_{A_2}$ are equivalent;
(ii) we have $f(\Gamma_1 x \cap A_1) = \Gamma_2 f(x) \cap A_2$ for a.e. $x \in A_1$.

If we can take both A_1 and A_2 to have full measure, then the two actions are said to be orbit equivalent.

Needless to say, two isomorphic actions are orbit equivalent. The study of orbit equivalence was initiated by Dye [5], [6], who treated standard actions of some amenable groups, and Ornstein-Weiss [28] concluded that any ergodic standard actions of any two infinite amenable groups are orbit equivalent. More generally, Connes-Feldman-Weiss [3] showed that amenable discrete measured equivalence relations are hyperfinite, which implies uniqueness of ergodic amenable equivalence relations. These phenomena give rise to a sharp difference between orbit equivalence and isomorphism for ergodic actions of $\mathbb{Z}$.

In contrast, based on Zimmer's pioneering work [35], Furman [10] established strong orbital rigidity for some ergodic standard actions of a lattice in a simple Lie group of higher real rank. Given an ergodic standard action α of a discrete group, we say that α is strongly orbitally rigid if the following holds: let β be any ergodic standard action of another discrete group which is weakly orbit equivalent to α. Then α and β are virtually isomorphic in the following sense:

Definition 1.3. Suppose that we have a non-singular ergodic action of a discrete group Γ_i on a standard finite measure space (X_i,μ_i) for $i=1,2$. Then the two actions are said to be virtually isomorphic if we can find a finite normal subgroup N_i of Γ_i and a finite index subgroup Γ_i'' of $\Gamma_i' = \Gamma_i/N_i$ for $i=1,2$ satisfying the following: let (X_i'',μ_i'') be any one of the at most $[\Gamma_i' : \Gamma_i'']$-many mutually isomorphic Γ_i''-ergodic components. Then the Γ_1''-action on (X_1'',μ_1'') and the Γ_2''-action on (X_2'',μ_2'') are isomorphic.

Monod-Shalom [25] applied the theory of bounded cohomology to the setting of orbit equivalence, and established strong orbital rigidity results for irreducible standard actions of a non-trivial direct product of discrete groups in the class $\mathcal{C}$, which was introduced in [25]. (A non-singular action of a discrete group of the form $\Gamma_1 \times \cdots \times \Gamma_n$ is said to be irreducible if for every j, the subproduct $\prod_{i \neq j} \Gamma_i$ acts ergodically.) This class $\mathcal{C}$ is huge and contains all non-elementary word-hyperbolic groups, the free product of two infinite groups [25, Section 7] and the mapping class group [16]. Recently, Popa [30], [32] discovered that the Bernoulli shift of an infinite Kazhdan group has more amazing rigidity, called von Neumann strong rigidity and superrigidity of embeddings.

For a lattice Γ in a connected simple Lie group G with finite center and real rank at least 2 and a self ME coupling (Ω, ω) of Γ, Furman [9] constructed an essentially unique $\Gamma \times \Gamma$-equivariant Borel map $\Psi\colon \Omega \to \mathrm{Ad}G$, which means that

$$\Psi((\gamma, \gamma')x) = \mathrm{Ad}(\gamma)\Psi(x)\mathrm{Ad}(\gamma')^{-1}$$

for any $\gamma, \gamma' \in \Gamma$ and a.e. $x \in \Omega$. In [10], Furman used this map Ψ to prove strong orbital rigidity of some ergodic standard actions of Γ (e.g., the standard action of $SL(n, \mathbb{Z})$ on $\mathbb{R}^n/\mathbb{Z}^n$ with $n \geq 3$). Moreover, he showed that all other ergodic standard actions of Γ essentially come from the Γ-action on G/Λ for some lattice Λ in G, which is weakly orbit equivalent to the Λ-action on G/Γ. Following Furman's idea and using the construction in [22] of an equivariant Borel map from a self ME coupling of the mapping class group into the automorphism group of the curve complex, we establish strong orbital rigidity for any ergodic standard action of the mapping class group. Furthermore, we show the same rigidity for any ergodic standard action of a direct product of mapping class groups. As mentioned above, it follows from Monod-Shalom's rigidity result that irreducible standard actions of a non-trivial direct product of torsion-free finite index subgroups of mapping class groups have strong orbital rigidity.

Throughout the paper, we assume a surface to be connected, compact and orientable unless otherwise stated. Let $\Gamma(M)^\diamond$ be the extended mapping class group of M, the group of isotopy classes of all diffeomorphisms of M, which contains the mapping class group $\Gamma(M)$ as a subgroup of index 2. We write $\kappa(M) = 3g + p - 4$ when M is a surface of genus g and with p boundary components.

Theorem 1.1. *Let n be a positive integer and let M_i be a surface with $\kappa(M_i) > 0$ for $i \in \{1, \ldots, n\}$. If Γ is a finite index subgroup of $\Gamma(M_1)^\diamond \times \cdots \times \Gamma(M_n)^\diamond$, then any ergodic standard action of Γ on a standard probability space is strongly orbitally rigid.*

In particular, we show the following:

Theorem 1.2. *Let M be a surface with $\kappa(M) > 0$ and let Γ be a normal subgroup of finite index in $\Gamma(M)^\diamond$. If two ergodic standard actions of Γ are weakly orbit equivalent, then they are isomorphic.*

This type of rigidity is also satisfied for certain subgroups of a direct product of mapping class groups (see Corollary 2.2 (i)).

Remark 1.1. Thanks to this rigidity, we can construct a new example of an ergodic equivalence relation of type II_1 which cannot arise from any standard action of a discrete group (see Corollary 2.1). The first construction of such an equivalence relation is due to Furman [10], and it solved a longstanding problem formulated by Feldman-Moore [8].

As one direction of the study of orbit equivalence except for seeking rigidity, it is a very interesting and difficult problem to find ergodic standard actions of the same discrete group which are non-orbit equivalent. Using the notion of strong ergodicity of group actions, Connes-Weiss [4] showed that any non-amenable group without Kazhdan's property has at least two non-orbit equivalent actions. The first example of a discrete group which has an uncountable family of mutually non-orbit equivalent ergodic standard actions was constructed by Bezuglyĭ-Golodets [2]. Later, it was shown that such an uncountable family exists for various discrete groups as follows:

- a lattice in a simple Lie group with higher real rank [13];
- an infinite Kazhdan group [17];
- non-trivial finite direct products of torsion-free groups in the class $\mathcal{C}$ [25];
- non-abelian free groups [12].

Popa [31] gives an explicit construction of such a family for all infinite Kazhdan groups and their free products by calculating 1-cohomology groups of certain actions of these groups. Given a discrete group Γ and a countable Γ-space K, we call a measure-preserving action defined by

$$\gamma(x_k)_{k\in K} = (x_{\gamma^{-1}k})_{k\in K}$$

for $\gamma \in \Gamma$ and $(x_k)_{k\in K} \in (X,\mu)^K$ a generalized Bernoulli shift of Γ. Popa-Vaes [33] classified certain quotients of generalized Bernoulli shifts of infinite Kazhdan groups up to orbit equivalence. The reader is referred to [34] for Popa's recent breakthrough rigidity results on Bernoulli shifts of infinite Kazhdan groups.

As an application of Theorem 1.2, we construct an explicit uncountable family of mutually non-orbit equivalent ergodic standard actions of a direct product of mapping class groups. It follows from Theorems 1.1, 1.2 that this problem reduces to the construction of non-isomorphic actions. We classify certain generalized Bernoulli shifts for the mapping class group up to isomorphism and thus, up to orbit equivalence. The main ingredient in the classification is the computation of the space of ergodic components for the restriction of the action to a certain subgroup of the mapping class group, which gives a new insight in construction of non-orbit equivalent actions of the same discrete group. For a surface M with $\kappa(M) > 0$, we denote by $S(M)$ the set of all simplices of the curve complex for M, which is naturally a countable $\Gamma(M)^\diamond$-space.

Theorem 1.3. *Let M be a surface with $\kappa(M) > 0$ and let Γ be a normal subgroup of finite index in $\Gamma(M)^\diamond$. Let $\sigma, \tau \in S(M)$ and consider the countable Γ-spaces $\Gamma\sigma, \Gamma\tau \subset S(M)$. We denote by α, β the generalized Bernoulli shifts of Γ on $(X,\mu)^{\Gamma\sigma}$ and $(Y,\nu)^{\Gamma\tau}$, respectively. Then the two actions α and β are weakly orbit equivalent if and only if the following two conditions hold:*

(i) *we can find $g \in \Gamma(M)^\diamond$ such that $\tau \in g\Gamma\sigma$ or equivalently, $\sigma \in g^{-1}\Gamma\tau$;*
(ii) *the two probability spaces (X,μ) and (Y,ν) are isomorphic.*

Corollary 1.4. *If M is a surface with $\kappa(M) > 0$, then a finite index subgroup of $\Gamma(M)^\diamond$ admits uncountably many ergodic standard actions which are mutually non-weakly orbit equivalent.*

Moreover, we introduce several types of generalized Bernoulli shifts for a direct product of mapping class groups other than the ones in Theorem 1.3. We see that these types of actions belong to mutually different isomorphism classes. One of these constructions can be modified to yield generalized Bernoulli shifts for (a direct product of) non-elementary torsion-free hyperbolic groups. Applying Monod-Shalom's rigidity [25], we obtain an explicit uncountable family of mutually non-weakly orbit equivalent ergodic standard actions of a non-trivial direct product of non-elementary torsion-free hyperbolic groups (see Subsection 3.3).

Remark 1.2. Although in particular, we construct an uncountable family of generalized Bernoulli shifts for non-elementary torsion-free hyperbolic groups which are mutually non-isomorphic, it is unknown whether they are orbit equivalent to each other.

Remark 1.3. It is noteworthy that there seems to be no results about when two Bernoulli shifts for a non-amenable group are isomorphic (or orbit equivalent). Ornstein-Weiss [29] generalized the theory of entropy to the setting of actions of amenable groups, and obtained the classification result of their Bernoulli shifts.

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2. Orbit equivalence rigidity

First, we give a proof of Theorem 1.1. Put

$$G_0 = \Gamma(M_1)^\diamond \times \cdots \times \Gamma(M_n)^\diamond, \quad G = \mathrm{Aut}(C(M_1)) \times \cdots \times \mathrm{Aut}(C(M_n)),$$

where $C = C(M)$ denotes the curve complex for a surface M with $\kappa(M) > 0$. Let us denote by $\pi\colon G_0 \to G$ be the natural homomorphism.

If H, Λ_1, Λ_2 are discrete groups and $\tau_i\colon \Lambda_i \to H$ is a homomorphism for $i = 1, 2$, then we denote by (H, τ_1, τ_2) the Borel space H equipped with the $\Lambda_1 \times \Lambda_2$-action defined by

$$(\lambda_1, \lambda_2)h = \lambda_1 h \lambda_2^{-1}$$

for $h \in H$ and $(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2$.

Proof of Theorem 1.1. We may assume that $M_i \neq M_{1,2}, M_{2,0}$ for each i by [22, Theorem 2.5]. Suppose that ergodic standard actions of Γ and a discrete group Λ on standard probability spaces (X, μ) and (Y, ν), respectively, are weakly orbit equivalent. It follows from [10, Theorem 3.3] that we can construct a ME coupling (Ω, m) of Γ and Λ such that the Γ-actions on X and on Ω/Λ (resp. the Λ-actions on Y and on Ω/Γ) are isomorphic. By [22, Theorem 1.3], we can find a homomorphism $\rho\colon \Lambda \to G$ whose kernel N and cokernel are both finite. Put

$$\Gamma_1 = \pi(\Gamma), \quad \Lambda_1 = \rho(\Lambda),$$

$$(\Omega_1, m_1) = (\Omega, m)/(\ker(\pi) \times N).$$

Note that $\ker(\pi)$ is trivial (e.g., see [19, Corollary 7.5.E]). Then we have the natural $\Gamma_1 \times \Lambda_1$-action on (Ω_1, m_1), which is a ME coupling of Γ_1 and Λ_1.

It follows from [22, Corollary 7.2] that we can find the following:

(a) a bijection t on the set $\{1, \ldots, n\}$;
(b) an isotopy class φ_i of a diffeomorphism $M_{t(i)} \to M_i$ for each $i \in \{1, \ldots, n\}$;

(c) an essentially unique almost $\Gamma_1 \times \Lambda_1$-equivariant Borel map $\Phi\colon \Omega_1 \to (G, \pi, \pi_\varphi)$,

where $\pi_\varphi\colon G_0 \to G$ is the isomorphism defined by

$$\pi_\varphi(\gamma) = (\pi(\varphi_1\gamma_{t(1)}\varphi_1^{-1}), \ldots, \pi(\varphi_n\gamma_{t(n)}\varphi_n^{-1}))$$

for $\gamma = (\gamma_1, \ldots, \gamma_n) \in G_0$.

Let $Y_0, X_0 \subset G$ be fundamental domains of Γ_1-, Λ_1-actions on (G, π, π_φ), respectively. We can choose Y_0, X_0 so that $m_0(X_0 \cap Y_0) > 0$, where $m_0 = \Phi_* m_1$. Let $g \in X_0 \cap Y_0$ be an element such that $m_0(\{g\}) > 0$. Then

$$\Gamma_2 = \Gamma_1 \cap g\Lambda_1 g^{-1}, \quad \Lambda_2 = g^{-1}\Gamma_2 g$$

are subgroups of finite index in Γ_1, Λ_1, respectively. Put $Y_1 = \Phi^{-1}(Y_0)$, $X_1 = \Phi^{-1}(X_0)$, which are fundamental domains of the Γ_1-, Λ_1-actions on (Ω_1, m_1), respectively. Moreover, put $X_2 = Y_2 = \Phi^{-1}(\{g\}) \subset X_1 \cap Y_1$. Since the $\Gamma_1 \times \Lambda_1$-action on (Ω_1, m_1) is ergodic and $m_1(X_2) = m_1(Y_2) > 0$, we have

$$\Gamma_1 g\pi_\varphi(\Lambda_1) = X_0\pi_\varphi(\Lambda_1) = \Gamma_1 Y_0 = G$$

up to null sets with respect to m_0. Let us consider the natural actions of Γ_1, Λ_1 on X_1, Y_1, respectively. Then X_2 is invariant for the Γ_2-action on X_1 and is one of at most $[\Gamma_1 : \Gamma_2]$-many mutually isomorphic Γ_2-ergodic components in X_1. Similarly, Y_2 is one of at most $[\Lambda_1 : \Lambda_2]$-many mutually isomorphic Λ_2-ergodic components in Y_1. Moreover, since

$$\Gamma_2 g = \Gamma_1 g \cap g\pi_\varphi(\Lambda_1) = g\pi_\varphi(\Lambda_2),$$

we see that $\Omega_2 = \Gamma_2 X_2 = Y_2\Lambda_2$ and $X_2 = Y_2$ is a fundamental domain of both the Γ_2- and Λ_2-actions on Ω_2.

If we define an isomorphism $F\colon \Gamma_2 \to \Lambda_2$ by $F(\gamma) = \pi_\varphi^{-1}(g^{-1}\gamma g)$ for $\gamma \in \Gamma_2$, then the identity $X_2 \to Y_2$ and F shows that the Γ_2-action on X_2 and the Λ_2-action on Y_2 are isomorphic. □

The following corollary follows from [10, Example 2.9]:

Corollary 2.1. *Let Γ be as above and suppose that we have an ergodic standard Γ-action on a standard probability space (X, μ). We denote by $\mathcal{R}$ the induced discrete measured equivalence relation. Let $A \subset X$ be a Borel subset with $\mu(A)$ positive and irrational. Then the restricted relation $(\mathcal{R})_A = \mathcal{R} \cap (A \times A)$ cannot be generated by any standard action of a discrete group.*

Corollary 2.2. *Let n be a positive number and let M_i be a surface with $\kappa(M_i) > 0$ and $M_i \neq M_{1,2}, M_{2,0}$ for $i \in \{1, \ldots, n\}$. Let Γ be a normal subgroup of finite index in $\Gamma(M_1)^\diamond \times \cdots \times \Gamma(M_n)^\diamond$ such that if t is a bijection on the set $\{1, \ldots, n\}$ and g_i is an isotopy class of a diffeomorphism $M_{t(i)} \to M_i$, then $\pi(\Gamma) = \pi_g(\Gamma)$. Then Γ satisfies the following:*

(i) *Suppose that we have two ergodic standard actions of Γ on (X, μ) and (Y, ν). If they are weakly orbit equivalent, then they are isomorphic.*
(ii) *The fundamental group of an ergodic standard action of Γ is trivial.*

Recall that the fundamental group of an ergodic standard action of a discrete group Λ on (X, μ) is defined to be the subgroup of the multiplicative group $\mathbb{R}_+^*$ of positive real numbers generated by $t \in \mathbb{R}_+^*$ such that for some (or any) Borel subset A of X with $\mu(A)/\mu(X) = t$, we have an isomorphism between the relations $(\mathcal{R})_A$ and $\mathcal{R}$, where $\mathcal{R}$ is the relation on (X, μ) arising from the standard action of Λ.

If Γ is of the form $\Gamma_1 \times \cdots \times \Gamma_n$, where Γ_i is a normal subgroup of finite index in $\Gamma(M_i)^\diamond$ and if Γ_i and Γ_j are isomorphic for any i, j with M_i, M_j diffeomorphic, then Γ satisfies the hypothesis of Corollary 2.2. In particular, if $n = 1$ and Γ is a normal subgroup of finite index in $\Gamma(M)^\diamond$ with $\kappa(M) > 0$, then the hypothesis is satisfied. Theorem 1.2 follows from Corollary 2.2 (i).

Proof of Corollary 2.2. We identify $\Gamma(M_i)^\diamond$ and $\mathrm{Aut}(C(M_i))$ by the natural isomorphism. In the proof of Theorem 1.1, it follows from [22, Corollary 7.2] that $\rho = \pi_g$ for some a bijection t on $\{1, \ldots, n\}$ and an isotopy class g_i of a diffeomorphism $M_{t(i)} \to M_i$. Using the notations as in the proof of Theorem 1.1, the assumption that Γ is a normal subgroup of $\Gamma(M_1)^\diamond \times \cdots \times \Gamma(M_n)^\diamond$ implies that

$$\Gamma = \Gamma_1 = \pi(\Gamma) = \rho(\Gamma) = \Lambda_1.$$

Furthermore, $\Gamma = \Gamma_2 = \Lambda_2$ and $X = X_1 = X_2$, $Y = Y_1 = Y_2$ from the construction. $\square$

Remark 2.1. Note that the assertion (ii) in Corollary 2.2 also follows from the computation of ℓ^2-Betti numbers of the mapping class group due to Gromov [15] and McMullen [24] (see [21, Appendix D]) and Gaboriau's work [11] on the connection between ℓ^2-Betti numbers and orbit equivalence.

3. Classification of certain generalized Bernoulli shifts

3.1. **Preliminaries.** In this subsection, we give a characterization of essential freeness and ergodicity of generalized Bernoulli shifts. For a discrete group Γ and $g \in \Gamma$, let us denote by $\langle g \rangle$ the cyclic group generated by g. Let (X, μ) be a standard probability space with $|\mathrm{supp}(\mu)| \geq 2$.

Lemma 3.1. *Let Γ be a discrete group and let K be a countable Γ-space and assume the following:*

- (i) *if $g \in \Gamma$ has infinite order, then there exists $k \in K$ such that $|\langle g \rangle k| = \infty$;*
- (ii) *if $g \in \Gamma \setminus \{e\}$ has finite order, then there exists a sequence $\{k_n\}_{n \in \mathbb{N}}$ in K such that $|\langle g \rangle k_n| \geq 2$ for each n, and $\langle g \rangle k_n \cap \langle g \rangle k_m = \emptyset$ for each $n \neq m$.*

Then the generalized Bernoulli shift of Γ on $(X, \mu)^K$ is essentially free.

Proof. Put $(Y, \nu) = (X, \mu)^K$ and $Y^g = \{y \in Y : gy = y\}$ for $g \in \Gamma$. For a subset $F \subset K$, let

$$Y(F) = \{(x_k)_{k \in K} \in Y : x_k = x_l \text{ for any } k, l \in F\}.$$

Then

$$\nu(Y(F)) = \sum_{x \in X:\text{atom}} \mu(\{x\})^{|F|},$$

where we let $\mu(\{x\})^{|F|} = 0$ if $|F| = \infty$. Note that $Y^g \subset Y(\langle g \rangle k)$ for each $g \in \Gamma$ and $k \in K$.

If $g \in \Gamma$ has infinite order, then by assumption, there exists $k \in K$ such that $|\langle g \rangle k| = \infty$ and we see that $\nu(Y^g) = 0$.

If $g \in \Gamma \setminus \{e\}$ has finite order, then we can find a sequence $\{k_n\}_{n\in\mathbb{N}}$ in K as in the lemma and $Y^g \subset \bigcap_{n\in\mathbb{N}} Y(\langle g\rangle k_n)$. Moreover, for each $m \in \mathbb{N}$, we have

$$\begin{aligned}\nu\left(\bigcap_{n=1}^{m} Y(\langle g\rangle k_n)\right) &= \prod_{n=1}^{m} \nu(Y(\langle g\rangle k_n))\\ &= \prod_{n=1}^{m}\left(\sum_{x\in X:\mathrm{atom}} \mu(\{x\})^{|\langle g\rangle k_n|}\right)\\ &\leq \prod_{n=1}^{m}\left(\sum_{x\in X:\mathrm{atom}} \mu(\{x\})^{2}\right)\end{aligned}$$

and the right hand side converges to 0 as $m \to \infty$ because $|\mathrm{supp}(\mu)| \geq 2$. Thus, $\nu(Y^g) = 0$. $\square$

Lemma 3.2. *Let Γ be a discrete group and let K be a countable Γ-space and put $(Y, \nu) = (X, \mu)^K$. Then the following four conditions are equivalent:*

(i) *the Γ-action on (Y, ν) is ergodic;*
(ii) *the Γ-action on (Y, ν) is weakly mixing;*
(iii) *for any finite subset $F \subset K$, there exists $g \in \Gamma$ such that $gF \cap F = \emptyset$;*
(iv) *there exist no $k_0 \in K$ such that $|\Gamma k_0| < \infty$.*

Remark 3.1. If G is a locally compact group and there is a measure-preserving action on (X, μ), then the following two conditions are equivalent (see [1, Theorem 2.17]):

(a) the G-action on (X, μ) is weakly mixing;
(b) the diagonal G-action on $(X \times X, \mu \times \mu)$ is ergodic.

In Lemma 3.2, if K satisfies the condition (iii), then the countable Γ-space $K \sqcup K$, the disjoint union of K and its copy, also satisfies it. It follows from the equivalence between (a) and (b) that if we show that (iii) implies (i), then this also shows that (iii) implies (ii).

Proof of Lemma 3.2. Assume the condition (iii). Then it is easy to check that for any Borel subset $A \subset X$ and $\varepsilon > 0$, there exists $g \in \Gamma$ such that

$$|\mu(gA \cap A) - \mu(A)^2| < \varepsilon,$$

which implies (i). It follows from this and Remark 3.1 that (iii) also implies (ii).

It is clear that (ii) implies (i). The equivalence between (iii) and (iv) is due to [20, Lemma 4.4].

We show that (i) implies (iv). If the condition (iv) does not hold, then there exists $k_0 \in K$ such that $|\Gamma k_0| < \infty$. Let A be a Borel subset of X with $0 < \mu(A) < 1$ and define

$$B = \{(x_k)_{k\in K} \in Y : x_k \in A \text{ for } k \in \Gamma k_0\}.$$

Then $\nu(B) = \mu(A)^{|\Gamma k_0|}$ and $0 < \nu(B) < 1$. Since B is invariant for Γ, the condition (i) does not hold. $\square$

For an integer $m \geq 3$, let $\Gamma(M; m)$ be the kernel of the natural $\Gamma(M)^\diamond$-action on the homology group $H_1(M; \mathbb{Z}/m\mathbb{Z})$, which is a torsion-free normal subgroup of finite index in $\Gamma(M)^\diamond$. Let $V(C) = V(C(M))$ be the set of vertices of the curve complex of M and let $S(M)$ be the set of all non-empty finite subsets of $V(C)$ which can be realized disjointly on M at the same time. For $\sigma \in S(M)$, let us denote by

M_σ the surface obtained by cutting along a realization of σ. If $g \in \Gamma(M; m)$ with $m \geq 3$ satisfies $g\sigma = \sigma$ for $\sigma \in S(M)$, then g preserves any component of M_σ by [18, Theorem 1.2]. Therefore, if a subgroup Γ of $\Gamma(M; m)$ and $\sigma \in S(M)$ satisfy $g\sigma = \sigma$ for any $g \in \Gamma$, then we have the natural homomorphism $p_Q \colon \Gamma \to \Gamma(Q)$ for each component Q of M_σ.

Let $g \in \Gamma(M; m)$. By [18, Corollary 1.8], g is pure, that is, the isotopy class of g contains a diffeomorphism F of M satisfying the following condition (P): there exists a closed one-dimensional submanifold C of M such that

- each component of C does not deform on M to a point or to ∂M;
- F is the identity on C, it does not rearrange the components of $M \setminus C$, and it induces on each component of the surface M_C obtained by cutting M along C a diffeomorphism isotopic to either a pseudo-Anosov or the identity diffeomorphism.

We may assume that C does not have superfluous components, that is, we cannot discard any component of C without violating the condition (P). We call the element of $S(M) \cup \{\emptyset\}$ corresponding to C the canonical reduction system (CRS) for g. Note that it coincides with the CRS of the cyclic group $\langle g \rangle$ by [18, Theorem 7.16]. In particular, for each $g \in \Gamma(M; m)$, the isotopy class of C satisfying the condition (P) and having no superfluous components is uniquely determined. It is known that a non-trivial element $g \in \Gamma(M; m)$ is reducible, that is, there exists $\sigma \in S(M)$ such that $g\sigma = \sigma$ if and only if the CRS for g is non-empty (see [18, Corollary 7.12]). We refer the reader to [18, Section 7] for the CRS of a subgroup of $\Gamma(M)$. By applying Lemma 2.2 and Proposition 2.3 in [18], we can prove the following lemma:

Lemma 3.3. *Let $g \in \Gamma(M; m)$ be a non-trivial element, where $m \geq 3$ is an integer. Let $\tau \in S(M) \cup \{\emptyset\}$ be the CRS for g. Then the following two assertions hold:*

(i) *If $\tau \neq \emptyset$ and $\alpha \in V(C)$ satisfies $i(\alpha, \beta) \neq 0$ for some $\beta \in \tau$. Then the orbit $\langle g \rangle \alpha$ consists of infinitely many elements.*

(ii) *If $\tau = \emptyset$, then for any $\alpha \in V(C)$, the orbit $\langle g \rangle \alpha$ consists of infinitely many elements.*

Recall that a pseudo-Anosov element admits the following remarkable dynamics on the Thurston compactification $\overline{\mathcal{T}}$ of the Teichmüller space for M, which is a union of the Teichmüller space and the Thurston boundary $\mathcal{PMF}$ for M:

Lemma 3.4 ([19, Theorem 7.3.A])**.** *Suppose that $\kappa(M) \geq 0$ and $g \in \Gamma(M)$ is pseudo-Anosov. Then there exist two fixed points $F_\pm(g) \in \mathcal{PMF}$ of g such that if K is a compact subset of $\overline{\mathcal{T}} \setminus \{F_-(g)\}$ and U is an open neighborhood of $F_+(g)$ in $\overline{\mathcal{T}}$, then $g^n(K) \subset U$ for all sufficiently large n.*

In the above lemma, the two elements $F_\pm(g)$ is called pseudo-Anosov foliations for g.

Lemma 3.5. *Let M be a surface with $\kappa(M) > 0$ and let Γ be a finite index subgroup of $\Gamma(M; m)$ with an integer $m \geq 3$. Let $\tau \in S(M) \cup \{\emptyset\}$. Then the following assertions hold:*

(i) *We can find $g \in \Gamma$ satisfying the following:*
   - *$g\tau = \tau$ and the isotopy class g contains a diffeomorphism satisfying the condition* (P) *with respect to some closed one-dimensional submanifold C in the class τ. Moreover, C does not have superfluous components. In particular, τ is the CRS for g;*

- *for each component Q of M_τ which is not a pair of pants, the mapping class $p_Q(g)$ is pseudo-Anosov in $\Gamma(Q)$.*

(ii) *The above g satisfies that the orbit $\langle g\rangle\sigma$ has infinitely many elements for any $\sigma \in S(M)$ with $\sigma \setminus \tau \neq \emptyset$.*

Proof. First, we show the assertion (i). We may assume that $\tau \neq \emptyset$. Choose a closed one-dimensional submanifold of M in the class τ and consider its tubular neighborhood N. Each curve $\alpha \in \tau$ corresponds to some component N_α of N.

We define a diffeomorphism on each component of $M \setminus N$ and N. For each component Q of $M \setminus N$, if Q is not a pair of pants, then choose a pseudo-Anosov diffeomorphism on Q fixing each point on ∂Q. If Q is a pair of pants, then consider the identity on Q. Next, we consider a diffeomorphism on each component N_α of N. If $\alpha \in \tau$ is such that any component Q of $M \setminus N$ with $\partial N_\alpha \cap \partial Q \neq \emptyset$ is a pair of pants, then consider a Dehn twist about α on N_α. Otherwise, consider the identity on N_α. Combining these diffeomorphisms on each component of $M \setminus N$ and N, we obtain a desired diffeomorphism on M.

The assertion (ii) in the case of $\tau = \emptyset$ is a consequence of Lemma 3.3 (ii). Suppose that $\tau \neq \emptyset$. It suffices to show the assertion (ii) when $\sigma \setminus \tau$ is contained in

$$\{\beta \in V(C) : i(\alpha, \beta) = 0 \text{ for any } \alpha \in \tau\}$$

or

$$\{\beta \in V(C) : i(\alpha, \beta) \neq 0 \text{ for some } \alpha \in \tau\},$$

respectively. In the former case, each element in $\sigma \setminus \tau$ can be viewed as an element in $V(C(Q))$ for some component Q of M_τ. The assertion (ii) follows from Lemma 3.4. In the latter case, the assertion (ii) follows from Lemma 3.3 (i). $\square$

For a subgroup Γ of $\Gamma(M)^\diamond$ or $\mathrm{Aut}(C)$, let us write $\Gamma_\sigma = \{g \in \Gamma : g\sigma = \sigma\}$ for $\sigma \in S(M)$ and write $\Gamma_\emptyset = \Gamma$ for convenience.

Lemma 3.6. *Let M be a surface with $\kappa(M) > 0$ and let Γ be a finite index subgroup of $\Gamma(M)^\diamond$ such that the natural map $\pi\colon \Gamma(M)^\diamond \to \mathrm{Aut}(C)$ is injective on Γ. Let $\sigma \in S(M)$. Then the following assertions hold:*

(i) *If $g \in \Gamma$ has infinite order, then there exists $\sigma_0 \in \Gamma\sigma$ such that the orbit $\langle g\rangle\sigma_0$ has infinitely many elements.*

(ii) *If $g \in \Gamma \setminus \{e\}$ has finite order, then we can find a sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset \Gamma\sigma$ such that $|\langle g\rangle\tau_n| \geq 2$ for each n, and $\langle g\rangle\tau_n \cap \langle g\rangle\tau_m = \emptyset$ for each $n \neq m$.*

(iii) *If $\tau \in S(M) \cup \{\emptyset\}$ is such that there exist no $\sigma' \in \Gamma\sigma$ with $\sigma' \subset \tau$, then for any $\sigma' \in \Gamma\sigma$, the orbit $\langle g\rangle\sigma'$ has infinitely many elements for some $g \in \Gamma_\tau$.*

Proof. Let $g \in \Gamma$ be an element of infinite order. Choose a positive integer $n \in \mathbb{N}$ such that $g^n \in \Gamma(M; 3)$. Let $\tau \in S(M) \cup \{\emptyset\}$ be the CRS for g^n. If $\tau = \emptyset$, then the assertion (i) follows from Lemma 3.3 (ii). Thus, we assume that $\tau \neq \emptyset$. Let $\gamma \in \Gamma \cap \Gamma(M)$ be a pseudo-Anosov element and $F_\pm(\gamma) \in \mathcal{PMF}$ be its pseudo-Anosov foliations. Note that

$$\{F \in \mathcal{PMF} : i(\beta, F) = 0\} \cap \{F_\pm(\gamma)\} = \emptyset$$

for any $\beta \in V(C)$ and that $\gamma^m\alpha \to F_+(\gamma)$ as $m \to +\infty$ for any $\alpha \in V(C)$ by Lemma 3.4. Hence, there exists $m \in \mathbb{N}$ such that

$$\gamma^m\alpha \notin \{F \in \mathcal{PMF} : i(\beta, F) = 0\}$$

for any $\alpha \in \sigma$ and $\beta \in \tau$. It follows from Lemma 3.3 (i) that for any $\alpha \in \sigma$, the orbit $\langle g^n \rangle(\gamma^m \alpha)$ has infinitely many elements and thus, so does $\langle g^n \rangle(\gamma^m \sigma)$. For the proof of the assertion (i), it is enough to choose $\sigma_0 = \gamma^m \sigma$.

Let $g \in \Gamma \setminus \{e\}$ be an element of finite order and put

$$\mathrm{Fix}(g) = \{F \in \mathcal{PMF} : gF = F\},$$

which is a proper closed subset of $\mathcal{PMF}$ because we assume that π is injective on Γ. We can find a pseudo-Anosov element $\gamma \in \Gamma \cap \Gamma(M)$ such that $F_+(\gamma) \notin \mathrm{Fix}(g)$ since the set of all pseudo-Anosov foliations is dense in $\mathcal{PMF}$ (see [23, Section 5, Example 1]). It follows from the dynamics of γ on $\mathcal{PMF}$ (see Lemma 3.4) that for any $\tau \in S(M)$, we see that $\gamma^n \tau \notin \mathrm{Fix}(g)$ for all sufficiently large $n \in \mathbb{N}$. If we let $\tau = \sigma$, then we can find a sequence $\{\tau_n\}$ in the assertion (ii) as a subset of $\{\gamma^n \sigma\}$.

The assertion (iii) follows from Lemma 3.5. $\square$

3.2. **First example.** In this subsection, suppose that any surface M satisfies $\kappa(M) > 0$ and $M \neq M_{1,2}, M_{2,0}$, and that (X, μ) and (Y, ν) denote standard probability spaces with $|\mathrm{supp}(\mu)|, |\mathrm{supp}(\nu)| \geq 2$. For $i \in \{1, \ldots, n\}$, let M_i be a surface and Γ_i a finite index subgroup of $\Gamma(M_i)^\diamond$. For $\sigma \in S(M_i)$, let us write

$$\Gamma_{i,\sigma} = \{g \in \Gamma_i : g\sigma = \sigma\}.$$

and put $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$. Let $\sigma_i \in S(M)$ and put $K_i = \Gamma_i \sigma_i$. Let $K = K_1 \times \cdots \times K_n$ be a countable Γ-space by the coordinatewise action and put $(X^*, \mu^*) = (X, \mu)^K$, on which we can consider the generalized Bernoulli shift of Γ naturally. We call this action α^*.

Lemma 3.7. *In the above notation,*

(i) *the action α^* is ergodic and essentially free.*

(ii) *let $\tau_i \in S(M_i)$ and suppose that for some $i \in \{1, \ldots, n\}$, there exist no $\sigma' \in K_i$ such that $\sigma' \subset \tau_i$. Then the restriction of α^* to $\Gamma_{1,\tau_1} \times \cdots \times \Gamma_{n,\tau_n}$ is ergodic.*

(iii) *let $\rho_i \in K_i$ for $i \in \{1, \ldots, n\}$. Then the space of ergodic components for the restriction of α^* to $\Gamma_\rho = \Gamma_{1,\rho_1} \times \cdots \times \Gamma_{n,\rho_n}$ is isomorphic to (X, μ).*

Proof. The assertions (i), (ii) follow from Lemmas 3.1, 3.2 and Lemma 3.6 (i), (ii). Let $\rho_i \in K_i$. Note that

$$K_i \setminus \{\rho_i\} \subset \{\sigma' \in S(M) : \sigma' \setminus \rho_i \neq \emptyset\}$$

and the left hand side is invariant for the Γ_{i,ρ_i}-action. It follows from Lemmas 3.2, 3.6 (iii) that the Γ_ρ-action on $(X, \mu)^{K \setminus \{(\rho_1, \ldots, \rho_n)\}}$ is ergodic. Then the assertion (iii) follows. $\square$

Let M_i, Γ_i, Γ as above. Let $\sigma_i, \tau_i \in S(M)$ for $i \in \{1, \ldots, n\}$ and put

$$K = \Gamma_1 \sigma_1 \times \cdots \times \Gamma_n \sigma_n, \quad L = \Gamma_1 \tau_1 \times \cdots \times \Gamma_n \sigma_n$$

as above. We construct the standard probability Γ-spaces

$$(X^*, \mu^*) = (X, \mu)^K, \quad (Y^*, \nu^*) = (Y, \nu)^L,$$

and call these actions α^*, β^*, respectively.

Theorem 3.8. *In the above notation, the two actions α^* and β^* are isomorphic if and only if the following two conditions are satisfied:*

(i) *we can find a bijection on the set* $\{1, \ldots, n\}$ *and an isotopy class* g_i *of a diffeomorphism* $M_{t(i)} \to M_i$ *such that*
 (a) $g_i \Gamma_{t(i)} g_i^{-1} = \Gamma_i$*;*
 (b) *the orbits* $\Gamma_{t(i)}\sigma_{t(i)}$ *and* $\Gamma_i \tau_i$ *are equal via* g_i*;*
 for any $i \in \{1, \ldots, n\}$*;*
(ii) *the two probability spaces* (X, μ) *and* (Y, ν) *are isomorphic.*

Proof. It is clear that if the conditions (i), (ii) are satisfied, then α^* and β^* are isomorphic. We show the converse. We can find an isomorphism $F\colon \Gamma \to \Gamma$, conull Γ-invariant Borel subsets $X_1^* \subset X^*$, $Y_1^* \subset Y^*$ and a Borel isomorphism $f\colon X_1^* \to Y_1^*$ such that

- $f_*\mu^* = \nu^*$;
- $F(g)f(x) = f(gx)$ for any $g \in \Gamma$ and $x \in X_1^*$.

It follows from [22, Corollary 7.3] that we can find a bijection t on the set $\{1, \ldots, n\}$ and an isotopy class g_i of a diffeomorphism $M_{t(i)} \to M_i$ such that for any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma$, we have

$$F(\gamma) = (g_1 \gamma_{t(1)} g_1^{-1}, \ldots, g_n \gamma_{t(n)} g_n^{-1}).$$

Thus, $\Gamma_i = g_i \Gamma_{t(i)} g_i^{-1}$ and if we regard $g_i \sigma_{t(i)} \in S(M_i)$ and put

$$\Gamma_\sigma = \Gamma_{1,\sigma_1} \times \cdots \times \Gamma_{n,\sigma_n}, \quad \Gamma_{g\sigma} = \Gamma_{1,g_1\sigma_{t(1)}} \times \cdots \times \Gamma_{n,g_n\sigma_{t(n)}},$$

then $F(\Gamma_\sigma) = \Gamma_{g\sigma}$.

Since the space of ergodic components for the restriction of α^* to Γ_σ is isomorphic to (X, μ) by Lemma 3.7 (iii), so is that of β^* to $\Gamma_{g\sigma}$. It follows from Lemma 3.7 (ii) that for each i, there exists $\tau_i' \in L_i$ such that $\tau_i' \subset g_i \sigma_{t(i)}$. Similarly, considering F^{-1}, we see that for each i, there exists $\sigma_{t(i)}' \in K_{t(i)}$ such that $\sigma_{t(i)}' \subset g_i^{-1}\tau_i$. It follows that $\tau_i' = g_i \sigma_{t(i)}$ and $\sigma_{t(i)}' = g_i^{-1}\tau_i$ for each i. By Lemma 3.7 (iii), the spaces (X, μ) and (Y, ν) are isomorphic. □

Theorem 1.3 can be shown by combining Corollary 2.2 (i) and Theorem 3.8.

3.3. **Second example and hyperbolic groups.** In this subsection, let (X, μ) and (Y, ν) be standard probability spaces with $|\mathrm{supp}(\mu)|, |\mathrm{supp}(\nu)| \geq 2$.

For a hyperbolic group Λ_0 and an element $g \in \Lambda_0$ of infinite order, let $\sigma_\pm(g) \in \partial\Lambda_0$ be the two fixed points of g on the boundary $\partial\Lambda_0$ such that $g^n x \to \sigma_+(g)$ for $x \in \partial\Lambda_0 \setminus \{\sigma_-(g)\}$ and $g^{-n}y \to \sigma_-(g)$ for $y \in \partial\Lambda_0 \setminus \{\sigma_+(g)\}$ as $n \to +\infty$. We denote by $\Sigma = \Sigma(\Lambda_0)$ the set consisting of all subsets $\{\sigma_\pm(g)\}$ of $\partial\Lambda_0$ for $g \in \Lambda_0$ of infinite order, which is naturally a countable Λ_0-space and satisfies

$$h\sigma_\pm(g) = \sigma_\pm(hgh^{-1})$$

for $g \in \Lambda_0$ of infinite order and any $h \in \Lambda_0$. For $\sigma \in \Sigma$, let us write

$$\Lambda_{0,\sigma} = \{\lambda \in \Lambda_0 : \lambda\sigma = \sigma\}.$$

We refer the reader to [14, Chapitre 8] for these fundamental facts about the dynamics of the action of a hyperbolic group on its boundary.

Lemma 3.9. *Suppose that* Λ_0 *is non-elementary, that is, the boundary* $\partial\Lambda_0$ *consists of infinitely many elements or equivalently,* Λ_0 *is non-amenable. If* $g \in \Lambda_0$ *is an element of infinite order and* $\sigma \in \Sigma$ *with* $\sigma \neq \{\sigma_\pm(g)\}$*, then the orbit* $\langle g \rangle\sigma$ *has infinitely many elements.*

Proof. Let $x \in \sigma \setminus \{\sigma_\pm(g)\}$. Then g^n does not fix x for any non-zero integer n because $\sigma_\pm(g^n) = \sigma_\pm(g)$ if n is positive, and $\sigma_\pm(g^n) = \sigma_\mp(g)$ if n is negative. This means that the orbit $\langle g\rangle x$ consists of infinitely many elements. $\square$

Let n be a positive integer and for $i \in \{1,\dots,n\}$, let Λ_i be a torsion-free non-elementary hyperbolic group. Put $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$.

Lemma 3.10. *If $\theta\colon \Lambda \to \Lambda$ is an automorphism, then we can find a bijection t on the set $\{1,\dots,n\}$ and an isomorphism $\theta_i\colon \Lambda_{t(i)} \to \Lambda_i$ such that*

$$\theta(\lambda) = (\theta_1(\lambda_{t(1)}),\dots,\theta_n(\lambda_{t(n)}))$$

for any $\lambda = (\lambda_1,\dots,\lambda_n) \in \Lambda$.

Proof. Consider the self ME coupling $(\Lambda, \mathrm{id}, \theta)$ of Λ. It follows from [25, Proposition 5.1] that we can find a bijection t on the set $\{1,\dots,n\}$ and $g \in \Lambda$ such that $g\theta(\Lambda_{t(i)}) \subset \Lambda_i$ for each i. In particular, $g \in \bigcap_i \Lambda_i$, that is, $g = e$. Thus, $\theta(\Lambda_{t(i)}) \subset \Lambda_i$ for each i. Repeating the above process for θ^{-1}, we see that $\theta(\Lambda_{t(i)}) = \Lambda_i$ for each i. $\square$

The following lemma can easily be deduced from Lemmas 3.1, 3.2 and 3.9:

Lemma 3.11. *Let $\sigma_i \in \Sigma(\Lambda_i)$ and put $K_i = \Lambda_i\sigma_i$, $K = K_1 \times \cdots \times K_n$, which is a countable Λ-space by the coordinatewise action. Let us denote by α^* the generalized Bernoulli shift of Λ on $(X,\mu)^K$. Then the following assertions hold:*

(i) *The action α^* is ergodic and essentially free.*
(ii) *Let Λ_i' be a subgroup of Λ_i. If there exists i such that $\Lambda_i' \not\subset \Lambda_{i,g\sigma_i}$ for any $g \in \Lambda_i$, then the restriction of α^* to $\Lambda_1' \times \cdots \times \Lambda_n'$ is ergodic.*
(iii) *If $\rho_i \in K_i$, then the space of ergodic components for the restriction of α^* to $\Lambda_{1,\rho_1} \times \cdots \times \Lambda_{n,\rho_n}$ is isomorphic to (X,μ).*

Theorem 3.12. *Let $\sigma_i, \tau_i \in \Sigma(\Lambda_i)$ for $i \in \{1,\dots,n\}$ and put*

$$K = \Lambda_1\sigma_1 \times \cdots \times \Lambda_n\sigma_n, \quad L = \Lambda_1\tau_1 \times \cdots \times \Lambda_n\sigma_n$$

as above. We construct the standard probability Λ-spaces

$$(X^*,\mu^*) = (X,\mu)^K, \quad (Y^*,\nu^*) = (Y,\nu)^L,$$

and call these actions α^*, β^*, respectively. Then the two actions α^* and β^* are isomorphic if and only if the following two conditions are satisfied:

(i) *there exist a bijection t on the set $\{1,\dots,n\}$ and an isomorphism $\theta_i\colon \Lambda_{t(i)} \to \Lambda_i$ such that for each i, we have $\theta_i(\Lambda_{t(i),\sigma_{t(i)}}) = \Lambda_{i,g_i\tau_i}$ for some $g_i \in \Lambda_i$.*
(ii) *the two probability spaces (X,μ) and (Y,ν) are isomorphic.*

Proof. It is clear that if the conditions (i), (ii) are satisfied, then α^* and β^* are isomorphic. We show the converse. Suppose that α^* and β^* are isomorphic and let $F\colon \Lambda \to \Lambda$ be an isomorphism arising from the isomorphism. It follows from Lemma 3.10 that we can find a bijection t on the set $\{1,\dots,n\}$ and an isomorphism $\theta_i\colon \Lambda_{t(i)} \to \Lambda_i$ such that

$$F(\lambda) = (\theta_1(\lambda_{t(1)}),\dots,\theta_n(\lambda_{t(n)}))$$

for any $\lambda = (\lambda_1,\dots,\lambda_n) \in \Lambda$. By Lemma 3.11 (ii), (iii), for each i, there exists $g_i \in \Lambda_i$ such that $\theta_i(\Lambda_{t(i),\sigma_{t(i)}})$ is contained in $\Lambda_{i,g_i\tau_i}$. Repeating the above process for θ_i^{-1}, we have $\theta_i(\Lambda_{t(i),\sigma_{t(i)}}) = \Lambda_{i,g_i\tau_i}$ for each i. Comparing the spaces of ergodic components for the restrictions to them, we see that (X, μ) and (Y, ν) are isomorphic. □

The following theorem follows from [25, Theorem 2.17]:

Theorem 3.13. *In Theorem 3.12, if $n \geq 2$, then the two actions α^* and β^* are weakly orbit equivalent if and only if they satisfy the conditions (i), (ii) in Theorem 3.12.*

Therefore, the above example gives uncountably many standard actions of $\Lambda_1 \times \cdots \times \Lambda_n$ with $n \geq 2$ which are mutually non-weakly orbit equivalent. Moreover, all the actions satisfy that for every i, the factor Λ_i acts ergodically.

The above argument on generalized Bernoulli shifts for a direct product of non-elementary hyperbolic groups can be applied to the case of a direct product of mapping class groups by using the set of pseudo-Anosov foliations instead of Σ. We collect results on such actions.

Let M be a surface with $\kappa(M) > 0$. We denote by $\Phi = \Phi(M)$ be the set consisting of all subsets $\{F_\pm(g)\}$ of $\mathcal{PMF}$ for a pseudo-Anosov element $g \in \Gamma(M)^\diamond$, which is naturally a countable $\Gamma(M)^\diamond$-space and satisfies

$$hF_\pm(g) = F_\pm(hgh^{-1})$$

for a pseudo-Anosov element $g \in \Gamma(M)^\diamond$ and any $h \in \Gamma(M)^\diamond$. For $\phi \in \Phi$, let us write

$$\Gamma_\phi = \{\gamma \in \Gamma : \gamma\phi = \phi\}.$$

Lemma 3.14. *Let M be a surface with $\kappa(M) > 0$ and let Γ be a finite index subgroup of $\Gamma(M)^\diamond$ such that the natural map $\pi\colon \Gamma(M)^\diamond \to \mathrm{Aut}(C)$ is injective on Γ. Let $\phi \in \Phi$. Then the following assertions hold:*

(i) *If $g \in \Gamma$ has infinite order, then there exists $\psi \in \Gamma\phi$ such that the orbit $\langle g \rangle \psi$ has infinitely many elements.*

(ii) *If $g \in \Gamma \setminus \{e\}$ has finite order, then we can find a sequence $\{\psi_n\}_{n \in \mathbb{N}} \subset \Gamma\phi$ such that $|\langle g \rangle \psi_n| \geq 2$ for each n, and $\langle g \rangle \psi_n \cap \langle g \rangle \psi_m = \emptyset$ for each $n \neq m$.*

(iii) *If $g \in \Gamma(M)$ is a pseudo-Anosov element such that $\{F_\pm(g)\} \neq \phi$, then the orbit $\langle g \rangle \phi$ has infinitely many elements.*

Proof. As in Lemma 3.9, the assertion (iii) follows from Lemma 3.4.

We show the assertion (i). Let $g \in \Gamma$ be an element of infinite order. If g is reducible, then g fixes no points in Φ by [7, Exposés 9, 11]. Therefore, the orbit $\langle g \rangle \phi$ consists of infinitely many elements. Next, assume that g is pseudo-Anosov. Let $\gamma \in \Gamma \cap \Gamma(M)$ be a pseudo-Anosov element such that $\{F_\pm(\gamma)\} \cap \{F_\pm(g)\} = \emptyset$. We can find such γ because Γ is a finite index subgroup of $\Gamma(M)^\diamond$. By Lemma 3.4, it is easy to see that $\gamma^n \phi \neq \{F_\pm(g)\}$ for sufficiently large n. The assertion (i) then follows from the assertion (iii).

The assertion (ii) can be shown by the same idea as in Lemma 3.6 (ii) as follows: let $g \in \Gamma \setminus \{e\}$ be an element of finite order and put

$$\mathrm{Fix}(g) = \{F \in \mathcal{PMF} : gF = F\},$$

which is a proper closed subset of $\mathcal{PMF}$. We can find a pseudo-Anosov element $\gamma \in \Gamma \cap \Gamma(M)$ such that $\{F_\pm(\gamma)\} \cap \phi = \emptyset$ and $F_\pm(\gamma) \notin \mathrm{Fix}(g)$. It follows from Lemma 3.4 that $\gamma_1^n \phi \notin \mathrm{Fix}(g)$ for sufficiently large $n \in \mathbb{N}$. We can find a sequence $\{\psi_n\}$ in the assertion (ii) as a subset of $\{\gamma^n \phi\}$. □

Let n be a positive integer and for $i \in \{1, \ldots, n\}$, let M_i be a surface with $\kappa(M_i) > 0$ and $M_i \neq M_{1,2}, M_{2,0}$ and let Γ_i be a subgroup of finite index in $\Gamma(M_i)^\diamond$. Put $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$. The following lemma is a consequence of Lemmas 3.1, 3.2 and 3.14:

Lemma 3.15. *Let $\phi_i \in \Phi(M_i)$ and put $K_i = \Gamma_i\phi_i$, $K = K_1 \times \cdots \times K_n$, which is a countable Γ-space by the coordinatewise action. Let us denote by α^* the generalized Bernoulli shift of Γ on $(X,\mu)^K$. Then the following assertions hold:*

(i) *The action α^* is ergodic and essentially free.*
(ii) *Let Γ_i' be a subgroup of Γ_i. If there exists i such that Γ_i' contains a pseudo-Anosov element g_i with $\{F_\pm(g_i)\} \notin K_i$, then the restriction of α^* to $\Gamma_1' \times \cdots \times \Gamma_n'$ is ergodic.*
(iii) *If $\rho_i \in K_i$, then the space of ergodic components for the restriction of α^* to $\Gamma_{1,\rho_1} \times \cdots \times \Gamma_{n,\rho_n}$ is isomorphic to (X,μ).*

Using the same idea as in Theorem 3.8, we can show the following:

Theorem 3.16. *Let $\phi_i, \psi_i \in \Phi(M_i)$ for $i \in \{1, \ldots, n\}$ and put*

$$K = \Gamma_1\phi_1 \times \cdots \times \Gamma_n\phi_n, \quad L = \Gamma_1\psi_1 \times \cdots \times \Gamma_n\psi_n$$

as above. We construct the standard probability Γ-spaces

$$(X^*, \mu^*) = (X,\mu)^K, \quad (Y^*, \nu^*) = (Y,\nu)^L,$$

and call these actions α^*, β^*, respectively. Then the two actions α^* and β^* are isomorphic if and only if the following two conditions are satisfied:

(i) *we can find a bijection t on the set $\{1, \ldots, n\}$ and an isotopy class g_i of a diffeomorphism $M_{t(i)} \to M_i$ such that*
  (a) *$g_i\Gamma_{t(i)}g_i^{-1} = \Gamma_i$;*
  (b) *the orbits $\Gamma_{t(i)}\phi_{t(i)}$ and $\Gamma_i\psi_i$ are equal via g_i;*

  for any $i \in \{1, \ldots, n\}$;
(ii) *the two probability spaces (X,μ) and (Y,ν) are isomorphic.*

3.4. **Third example.** Also in this subsection, we consider only surfaces M satisfying $\kappa(M) > 0$ and $M \neq M_{1,2}, M_{2,0}$. Let n be a positive integer and let (X_i,μ_i) and (Y_i,ν_i) be standard probability spaces with $|\mathrm{supp}(\mu_i)|, |\mathrm{supp}(\nu_i)| \geq 2$ for $i \in \{1, \ldots, n\}$. Put

$$(X,\mu) = \prod_{i=1}^n (X_i,\mu_i), \quad (Y,\nu) = \prod_{i=1}^n (Y_i,\nu_i).$$

Let M_i be a surface and let Γ_i be a finite index subgroup of $\Gamma(M_i)^\diamond$. Put $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$.

Lemma 3.17. *Suppose that we have ergodic standard actions α_i, β_i of Γ_i on (X_i,μ_i), (Y_i,ν_i), respectively. If the standard Γ-actions α, β on (X,μ), (Y,ν) defined by the coordinatewise actions are isomorphic, then there exists a bijection t on the set $\{1, \ldots, n\}$ such that $\alpha_{t(i)}$ and β_i are isomorphic for any i.*

Proof. We can find an isomorphism $F\colon \Gamma \to \Gamma$, conull Γ-invariant Borel subsets $X' \subset X$, $Y' \subset Y$ and a Borel isomorphism $f\colon X' \to Y'$ such that

- $f_*\mu = \nu$;
- $F(g)f(x) = f(gx)$ for any $g \in \Gamma$ and $x \in X'$.

It follows from [22, Corollary 7.3] that we can find a bijection t on the set $\{1, \ldots, n\}$ such that $F(\Gamma_{t(i)}) = \Gamma_i$ for any i. If we compare the spaces of ergodic components for the restrictions of α, β to $\Gamma'_{t(i)} = \prod_{j\neq t(i)} \Gamma_j$, $F(\Gamma'_{t(i)}) = \Gamma'_i$, respectively, then we see that $\alpha_{t(i)}$ and β_i are isomorphic and the lemma follows. □

Let $\sigma_i, \tau_i \in S(M_i)$ and put $K_i = \Gamma_i\sigma_i$, $L_i = \Gamma_i\tau_i$. One can construct the standard probability Γ-spaces

$$(X^+, \mu^+) = \prod_{i=1}^{n}(X_i, \mu_i)^{K_i}, \quad (Y^+, \nu^+) = \prod_{i=1}^{n}(Y_i, \nu_i)^{L_i}.$$

We call these actions α^+, β^+, respectively. It is easy to prove the following:

Theorem 3.18. *The two actions α^+ and β^+ are isomorphic if and only if there exists a bijection t on the set $\{1, \ldots, n\}$ and an isotopy class g_i of a diffeomorphism $M_{t(i)} \to M_i$ such that*

(i) $g_i\Gamma_{t(i)}g_i^{-1} = \Gamma_i$;
(ii) *the orbits $\Gamma_{t(i)}\sigma_{t(i)}$ and $\Gamma_i\tau_i$ are equal via g_i;*
(iii) *the two probability spaces $(X_{t(i)}, \mu_{t(i)})$ and (Y_i, ν_i) are isomorphic*

for any $i \in \{1, \ldots, n\}$.

3.5. **Fourth example.** Also in this subsection, we consider a surface M satisfying $\kappa(M) > 0$ and $M \neq M_{1,2}, M_{2,0}$. Let Γ be a finite index subgroup of $\Gamma(M)^\diamond$. Let (X, μ) and (Y, ν) be standard probability spaces with $|\mathrm{supp}(\mu)|, |\mathrm{supp}(\nu)| \geq 2$. In this subsection, we consider the $\Gamma \times \Gamma$-action on Γ defined by

$$(\gamma_1, \gamma_2)\gamma = \gamma_1\gamma\gamma_2^{-1}$$

for $\gamma, \gamma_1, \gamma_2 \in \Gamma$.

Lemma 3.19. *In the above notation, the following assertions hold:*

(i) *If $(\gamma_1, \gamma_2) \in \Gamma \times \Gamma$ has infinite order, then there exists $g \in \Gamma$ such that the set $\{\gamma_1^n g\gamma_2^{-n}\}_{n\in\mathbb{Z}}$ consists of infinitely many elements.*
(ii) *If $\gamma = (\gamma_1, \gamma_2) \in \Gamma \times \Gamma \setminus \{e\}$ has finite order, then there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ in Γ such that $|\langle\gamma\rangle g_n| \geq 2$ for each n, and $\langle\gamma\rangle g_n \cap \langle\gamma\rangle g_m = \emptyset$ for each $n \neq m$.*

Proof. We show the assertion (i). Assume that the orbit $\{\gamma_1^n\gamma_2^{-n}\}_{n\in\mathbb{Z}}$ has finitely many elements, that is, there exists $n_0 \in \mathbb{Z} \setminus \{0\}$ such that $\gamma_1^{n_0} = \gamma_2^{n_0}$. Put $\gamma_0 = \gamma_1^{n_0} = \gamma_2^{n_0}$, which is an element of Γ of infinite order.

First, we assume that γ_0 is pseudo-Anosov. Let $g \in \Gamma$ be a pseudo-Anosov element such that $\{F_\pm(g)\} \cap \{F_\pm(\gamma_0)\} = \emptyset$. Then $F_+(\gamma_0^n g\gamma_0^{-n}) = \gamma_0^n F_+(g)$ for any $n \in \mathbb{Z}$. It follows from the dynamics of γ_0 on $\mathcal{PMF}$ (see Lemma 3.4) that the set $\{F_+(\gamma_0^n g\gamma_0^{-n})\}_{n\in\mathbb{Z}}$ consists of infinitely many elements and so does the set $\{\gamma_0^n g\gamma_0^{-n}\}_{n\in\mathbb{Z}}$. Then the set $\{\gamma_1^n g\gamma_2^{-n}\}_{n\in\mathbb{Z}}$ has infinitely many elements as well.

Secondly, we assume that γ_0 is reducible. Let k be a positive integer such that $\gamma_0^k \in \Gamma(M; 3)$. Choose a pseudo-Anosov element $g \in \Gamma$. Then $F_+(\gamma_0^{nk} g\gamma_0^{-nk}) = \gamma_0^{nk} F_+(g)$ for any $n \in \mathbb{Z}$. It follows from [18, Lemma 3.3] that $\{F_+(\gamma_0^{nk} g\gamma_0^{-nk})\}_{n\in\mathbb{Z}}$ consists of infinitely many elements, and so does the set $\{\gamma_0^{nk} g\gamma_0^{-nk}\}_{n\in\mathbb{Z}}$. Then the same holds true for the set $\{\gamma_1^n g\gamma_2^{-n}\}_{n\in\mathbb{Z}}$.

Next, we show the assertion (ii). We may assume that both γ_1 and γ_2 are non-trivial. Let $g \in \Gamma$ be a pseudo-Anosov element such that

$$F_+(g) \notin \mathrm{Fix}(\gamma_1) = \{x \in \mathcal{PMF} : \gamma_1 x = x\}.$$

We show that $g^n\gamma_2 \neq \gamma_1 g^n$ for all sufficiently large n. If this is shown, then we can find a sequence $\{g_n\}$ as in the assertion (ii) which is a subset of $\{g^n\}$. Let x be a point in the Teichmüller space of M fixed by γ_2. Since $F_+(g) \notin \mathrm{Fix}(\gamma_1)$, there exists an open neighborhood U of $F_+(g)$ in the Thurston compactification such that $\gamma_1(U) \cap U = \emptyset$. It follows from the dynamics of g on the Thurston compactification (see Lemma 3.4) that $g^n\gamma_2 x \in U$ and $\gamma_1 g^n x \in \gamma_1(U)$ for all sufficiently large n, which implies $g^n\gamma_2 \neq \gamma_1 g^n$. □

Lemma 3.20. *Let α be the generalized Bernoulli shift of $\Gamma \times \Gamma$ on $(X,\mu)^\Gamma$ defined by*

$$(\gamma_1, \gamma_2)(x_g)_{g\in\Gamma} = (x_{\gamma_1^{-1}g\gamma_2})_{g\in\Gamma}$$

for $\gamma_1, \gamma_2 \in \Gamma$ and $(x_g)_{g\in\Gamma} \in (X,\mu)^\Gamma$. Then the following assertions hold:

(i) *The action α is essentially free.*

(ii) *The restriction α_1 (resp. α_2) of α to $\Gamma \times \{e\}$ (resp. $\{e\} \times \Gamma$) is mixing.*

(iii) *Let Δ be the diagonal subgroup of $\Gamma \times \Gamma$ and define an action α_0 of Δ on $(X,\mu)^{\Gamma\setminus\{e\}}$ by*

$$\gamma(x_g)_{g\in\Gamma\setminus\{e\}} = (x_{\gamma^{-1}g\gamma})_{g\in\Gamma\setminus\{e\}}$$

for $\gamma \in \Gamma$ and $(x_g)_{g\in\Gamma\setminus\{e\}} \in (X,\mu)^{\Gamma\setminus\{e\}}$. Then α_0 is ergodic. In particular, the space of ergodic components for the restriction of α to Δ is isomorphic to (X,μ).

Proof. The assertion (i) follows from Lemmas 3.1, 3.19. The assertion (ii) is well known. The assertion (iii) follows easily from Lemma 3.2 and [22, Theorem 2.6]. □

Theorem 3.21. *Let α, β be the generalized Bernoulli shifts of $\Gamma \times \Gamma$ on $(X,\mu)^\Gamma$, $(Y,\nu)^\Gamma$, respectively, defined as above. Suppose that the two actions α and β are isomorphic and let $F\colon \Gamma\times\Gamma \to \Gamma\times\Gamma$ be an isomorphism arising from the isomorphism. Let t be a bijection on the set $\{1,2\}$ and $g_1, g_2 \in \Gamma(M)^\diamond$ such that*

$$F(\gamma_1,\gamma_2) = (g_1\gamma_{t(1)}g_1^{-1}, g_2\gamma_{t(2)}g_2^{-1})$$

for each $(\gamma_1,\gamma_2) \in \Gamma\times\Gamma$. Then $g_1g_2^{-1} \in \Gamma$ and the two probability spaces (X,μ) and (Y,ν) are isomorphic.

Remark that any isomorphism $F\colon \Gamma\times\Gamma \to \Gamma\times\Gamma$ can be described as in the above theorem by [22, Corollary 7.3].

Proof. Note that

$$F(\Delta) = \{(g_1\gamma g_1^{-1}, g_2\gamma g_2^{-1}) \in \Gamma\times\Gamma : \gamma \in \Gamma\}$$

and for $\gamma' \in \Gamma$, we have

$$\begin{aligned} &|\{(g_1\gamma g_1^{-1})\gamma'(g_2\gamma^{-1}g_2^{-1}) \in \Gamma : \gamma \in \Gamma\}| \\ =&|\{\gamma(g_1^{-1}\gamma' g_2)\gamma^{-1} \in \Gamma(M)^\diamond : \gamma \in \Gamma\}|. \end{aligned}$$

If $g_1g_2^{-1} \notin \Gamma$, then the right hand side is infinite for any $\gamma' \in \Gamma$ by [22, Theorem 2.6]. This implies that the restriction of β to $F(\Delta)$ is ergodic by Lemma 3.2, which is a contradiction because the space of ergodic components for the restriction of α to

Δ is isomorphic to (X, μ). Thus, $g_1 g_2^{-1} \in \Gamma$ and for any $\gamma' \in \Gamma \setminus \{g_1 g_2^{-1}\}$, the orbit $\{(g_1 \gamma g_1^{-1})\gamma'(g_2 \gamma^{-1} g_2^{-1}) \in \Gamma : \gamma \in \Gamma\}$ is infinite. It follows from Lemma 3.2 that the generalized Bernoulli shift of $F(\Delta)$ on $(Y, \nu)^{\Gamma \setminus \{g_1 g_2^{-1}\}}$ is ergodic. Therefore, the space of ergodic components for the restriction of β to $F(\Delta)$ is isomorphic to (Y, ν). $\square$

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