The quantum region for von Neumann measurements with postselection

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Abstract

It is determined under which conditions a probability distribution on a finite set can occur as the outcome distribution of a quantum-mechanical von Neumann measurement with post-selection, given that the scalar product between the initial and the final state is known as well as the success probability of the postselection. An intermediate von Neumann measurement can enhance transition probabilities between states such that the error probability shrinks by a factor of up to 2.

1 Introduction

The following observation was made in [Fri10]: upon subjecting any quantum system to the procedure,

(a) prepation of some initial state $|\psi\rangle$,

(b) application of a dichotomic von Neumann measurement q,

(c) postselection[1] with respect to some final state $|\phi\rangle$ such that $\langle\psi|\phi\rangle = 0$,

the usual rules of quantum mechanics imply that the two outcomes of q both necessarily occur with a conditional probability of 1/2. This is easiest to see on the level of amplitudes, where it follows from

$$0 = \langle\psi|\phi\rangle = \langle\psi|q|\phi\rangle + \langle\psi|(1-q)|\phi\rangle,$$

so that the two probabilities for measuring $q = 1$ or $q = 0$ are given by, respectively,

$$P(q=1) = \frac{|\langle\psi|q|\phi\rangle|^2}{|\langle\psi|q|\phi\rangle|^2 + |\langle\psi|(1-q)|\phi\rangle|^2} = \frac{1}{2}, \qquad P(q=0) = \frac{|\langle\psi|(1-q)|\phi\rangle|^2}{|\langle\psi|q|\phi\rangle|^2 + |\langle\psi|(1-q)|\phi\rangle|^2} = \frac{1}{2}.$$

Intuitively, this means that a dichotomic von Neumann measurement with postselection which is orthogonal to the inital state is guaranteed to be a perfectly unbiased random number generator.

[1] For an introduction to quantum mechanics with postselection and the counterintuitive effects it gives rise too, see e.g. [AV07].

So when $\langle\psi|\phi\rangle = 0$, there is only a single probability distribution over the outcomes which can arise for an intermediate dichotomic von Neumann measurement. Now the obvious question is, how does this generalize? What if the measurement has n outcomes instead of just 2? What if $|\phi\rangle$ is not orthogonal to $|\psi\rangle$? These are the kind of questions to be answered here.

Note that these questions are of interest in the foundations of quantum mechanics, since they are of the form "under which conditions is it possible to find a quantum-mechanical model for a given set of probabilities?".

Synopsis. Section 2 derives some elementary mathematical results about vectors in $\mathbb{C}^n$. The main result about probability distributions for von Neumann measurements with postselection follows in section 3. Then section 4 discusses some particular special cases of this result and determines to what extent transition probabilities between quantum states can be enhanced by a von Neumann measurement. Finally, section 5 briefly concludes.

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2 Mathematical Preliminaries

We will later need a solution to the following elementary mathematical problem:

> **Question:** Given n non-negative real numbers $x_1, \ldots, x_n$, is it possible to find complex numbers $z_1, \ldots, z_n$ such that $|z_k| = x_k$ and $\sum_k z_k = 0$?

We will see soon that this question can easily be reduced to the following proposition, where now the requirement $\sum_k z_k = 0$ has been replaced by the condition $\sum_k z_k = 1$.

Proposition 2.1. *For given $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$, there exist $z_1, \ldots, z_n \in \mathbb{C}$ with*

$$|z_k| = x_k, \qquad \sum_{k=1}^{n} z_k = 1$$

if and only if the inequalities

$$x_k \leq 1 + \sum_{\substack{j=1 \\ j\neq k}}^{n} x_j, \qquad \sum_{j=1}^{n} x_j \geq 1 \tag{1}$$

hold.

Proof. The necessity of (1) is a direct consequence of the triangle inequality. The burden of the proof lies in showing that these inequalities are sufficient to guarantee the existence of a solution for the z_k. For this, it can be assumed without loss of generality that all the x_k are strictly positive.

Now we apply induction on n. In the case $n = 1$, the inequalities state that $x_1 \leq 1$, and $x_1 \geq 1$, so that $x_1 = 1$, which is what is required. For the induction step, given $x_1, \ldots, x_{n+1}$

which satisfy (1), it can be assumed that these numbers are ordered such that $x_1 \leq \ldots \leq x_{n+1}$. Then up to an irrelevant global phase, it is enough to find $z_1, \ldots, z_n \in \mathbb{C}$ such that $|z_k| = x_k$ and $\sum_{k=1}^n z_k = y$ for some freely chosen $y \in [|1 - x_{n+1}|, 1 + x_{n+1}]$, for these are the values of $|1 - z_{n+1}|$ which can be attained by choosing the argument of z_{n+1} with $|z_{n+1}| = x_{n+1}$ appropriately. Using a rescaled version of the induction assumption, this can be done if and only if

$$x_k \leq y + \sum_{\substack{j=1 \\ j \neq k}}^n x_j, \qquad \sum_{j=1}^n x_j \geq y$$

By the assumed ordering of the x_k, the first inequality holds if and only if $y \geq x_n - \sum_{j=1}^{n-1} x_j$. Taking all conditions on y together, the number y has to lie in the interval $[|1 - x_{n+1}|, 1 + x_{n+1}]$, as well as in the interval $[x_n - \sum_{j=1}^{n-1} x_j, \sum_{j=1}^n x_j]$, and also y has to be positive. Therefore, the problem can be solved if and only if these two intervals have a non-empty intersection on the positive real axis. The intervals intersect if and only if the lower endpoint of any one interval is not above the upper endpoint of the other interval; in the present case,

$$|1 - x_{n+1}| \leq \sum_{j=1}^n x_j, \qquad x_n - \sum_{j=1}^{n-1} x_j \leq 1 + x_{n+1}.$$

Now the first inequality holds by the assumption (1), while the validity of the second inequality already follows from the assumed ordering $x_n \leq x_{n+1}$. By $\sum_{j=1}^n x_j > 0$, the intervals even intersect on the positive real axis, so that a consistent choice for y is possible. This finishes the proof. □

Corollary 2.2. *Given non-negative real numbers $x_1, \ldots, x_n$, there exist complex numbers $z_1, \ldots, z_n$ with*

$$|z_k| = x_k, \qquad \sum_{k=1}^n z_k = 0$$

if and only if the inequalities

$$x_k \leq \sum_{\substack{j=1 \\ j \neq k}}^n x_j \tag{2}$$

hold.

Proof. If all x_k vanish, there is nothing to prove. If there is some k with $x_k > 0$, then it suffices to find z_j's for $j \neq k$ with

$$|z_j| = x_j, \;\; j \neq k, \qquad \sum_{\substack{j=1 \\ j \neq k}}^n z_j = x_k.$$

This is possible due to proposition 2.1 rescaled by a factor of x_k^{-1}. □

Lemma 2.3. *Given $z \in \mathbb{C}^n$, there exist $\psi, \phi \in \mathbb{C}^{n+2}$ with*

$$|\psi|^2 = 1 = |\phi|^2, \qquad \overline{\psi}_k \phi_k = \begin{cases} z_k & \text{for } k = 1, \ldots, n \\ 0 & \text{for } k = n+1, n+2 \end{cases}$$

if and only if the inequality

$$\sum_{k=1}^{n} |z_k| \leq 1 \tag{3}$$

holds.

Proof. The necessity of (3) follows from the Cauchy-Schwarz inequality evaluated on ψ and ϕ', where $\phi' \in \mathbb{C}^{n+2}$ is defined by the requirements that firstly, $|\phi'_k| = |\phi_k|$, and that secondly, the argument of ϕ'_k is such that $\overline{\psi}_k \phi'_k = |z_k| \in \mathbb{R}_{\geq 0}$. For the sufficiency of (3), choose any complex square root $\sqrt{z_k}$ for each z_k and consider the two vectors

$$\psi_k = \begin{cases} \overline{\sqrt{z_k}} & \text{for } k = 1, \ldots, n \\ \sqrt{1 - \sum_{j=1}^{n} |z_j|} & \text{for } k = n+1 \\ 0 & \text{for } k = n+2 \end{cases}, \qquad \phi_k = \begin{cases} \sqrt{z_k} & \text{for } k = 1, \ldots, n \\ 0 & \text{for } k = n+1 \\ \sqrt{1 - \sum_{j=1}^{n} |z_j|} & \text{for } k = n+2 \end{cases}.$$

□

3 Von Neumann measurements with postselection

Suppose now that we have our quantum system prepared in some initial state $|\psi\rangle$, apply a von Neumann measurement of some observable E having finite spectrum with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and spectral projectors $E_1, \ldots, E_n$,

$$E = \sum_{k=1}^{n} \lambda_k E_k$$

and postselect with respect to some final state $|\phi\rangle$. Under these conditions, the probability—conditional with respect to successful postselection—of getting the outcome λ_k for the measurement of E is given by (see e.g. [AV07])

$$P(k) = \frac{|\langle\psi|E_k|\phi\rangle|^2}{\sum_{j=1}^{n} |\langle\psi|E_j|\phi\rangle|^2}, \tag{4}$$

where the normalization factor

$$S \equiv \sum_{j=1}^{n} |\langle\psi|E_j|\phi\rangle|^2$$

stands for the (unconditional) probability of successful postselection. Without loss of generality, we will label the outcomes by $1, \ldots, n$ instead of the eigenvalues $\lambda_1, \ldots, \lambda_n$ as the labels for measurement outcomes; this is entirely for notational convenience only.

Question 3.1. Given the transition amplitude $A = |\langle\psi|\phi\rangle|$, which probability distributions $P(\cdot)$ on $\{1, \ldots, n\}$ can occur in this way for which values of the success probability S?

Note that it is no loss of generality to ask this question only for pure states, since a mixed state can always be purified by adding an ancilla to the system with which it is entangled. Furthermore, just like the quantities $P(k)$ and S, the transition amplitude A also has an operational interpretation as the success probability of a kind of "postselection", namely postselection in the case when the intermediate measurement is not present. Therefore, one may think of all the quantities $P(k)$, S and A as given in terms of experimental data, and the question then is whether it is possible to find a quantum-mechanical model reproducing these particular values, without specifying the Hilbert space dimension or anything else in advance.

The case $A = 0$, $n = 2$ and $S \neq 0$ of this question has been treated in section 2 of [Fri10], where it was found that, surprisingly, the only possibility is given by $P(1) = P(2) = 1/2$. Using the elementary mathematical results derived in the previous section, we are now ready to answer this question in complete generality.

Theorem 3.2. *A given $P(\cdot)$ with given A and S can occur in this way if and only if the following inequalities hold:*

$$\sqrt{P(k)} \leq \frac{A}{\sqrt{S}} + \sum_{j\neq k} \sqrt{P(j)}, \qquad \frac{A}{\sqrt{S}} \leq \sum_{k=1}^{n} \sqrt{P(k)} \leq \frac{1}{\sqrt{S}} \tag{5}$$

Proof. The main idea here is to use the completeness relation $\sum_k E_k = \mathbb{1}$ in order to obtain an identity for amplitudes

$$\langle\psi|\phi\rangle = \sum_{k=1}^{n} \langle\psi|E_k|\phi\rangle$$

and then translate this into conditions on the probabilities (4). To this end, we can apply 2.2 to

$$z_k = \langle\psi|E_k|\phi\rangle, \;\; k = 1, \ldots, n, \qquad z_{n+1} = -\langle\psi|\phi\rangle.$$

Then upon setting $x_k \equiv \sqrt{P(k)S} = |\langle\psi|E_k|\phi\rangle|$ for $k = 1, \ldots, n$, and defining $x_{n+1} = A$, it follows that the first inequalities of (5) are necessary, as well as the first inequality of the second formula. In the case that the E_k are rank-one projectors—so that they define an orthonormal basis of the Hilbert space—the remaining inequality is implied by lemma 2.3 applied to $z_1, \ldots, z_n$. In general, we can choose an orthonormal basis $\{|j\rangle\}_j$ in which all the E_k are diagonal, and apply an argument analogous to the proof of lemma 2.3 as follows:

$$\sum_{k=1}^{n} |z_k| = \sum_{k=1}^{n} |\langle\psi|E_k|\phi\rangle| \leq \sum_{j} |\langle\psi|j\rangle\langle j|\phi\rangle|.$$

Now let $|\phi'\rangle$ be the vector which has the components $\langle j|\phi'\rangle$ such that $\langle\psi|j\rangle\langle j|\phi'\rangle = |\langle\psi|j\rangle\langle j|\phi\rangle|$. It follows that

$$\sum_{k=1}^{n} |z_k| \leq \sum_{j} \langle\psi|j\rangle\langle j|\phi'\rangle = \langle\psi|\phi'\rangle \leq 1,$$

as was to be shown.

To see that the inequalities (5) taken together are also sufficient for the existence of a quantum-mechanical model, we again set x_k to be given by the square roots of the unnormalized probabilities as $x_k \equiv \sqrt{P(k)S}$ for $k = 1, \ldots, n$, and again define $x_{n+1} = A$. Then again by 2.2, some corresponding z_k's with $\sum_{k=1}^{n+1} z_k = 0$ can now assumed to be given, and they also satisfy $\sum_{k=1}^{n} |z_k| = \sum_{k=1}^{n} x_n \leq 1$ by the assumption (5). Now one can use lemma 2.3 to obtain the states on $\mathbb{C}^{n+2}$ which are given by

$$|\psi\rangle = \sum_{k=1}^{n+2} \psi_k |k\rangle, \qquad |\phi\rangle = \sum_{k=1}^{n+2} \phi_k |k\rangle$$

in conjunction with $E_k = |k\rangle\langle k|$ for $k = 1, \dots, n$. The remaining two rank-one projections

$$|n+1\rangle\langle n+1|, \qquad |n+2\rangle\langle n+2|$$

can be added to any one or two of the E_k, so that one obtains a complete set of projectors. Then $\sqrt{P(k)S} = |\langle\psi|E_k|\phi\rangle|$ and $A = |\langle\psi|\phi\rangle|$ both hold by construction. The requirement $S = \sum_{k=1}^n |\langle\psi|E_k|\phi\rangle|^2$ is automatic by normalization of the probability distribution $P(\cdot)$. $\square$

It is possible to rewrite the inequalities (5) in a slightly more convenient form. Since the first inequality holds for all k if and only if it holds for that k for which $P(k)$ is largest, it is enough to require

$$2\sqrt{\max_k P(k)} \leq \frac{A}{\sqrt{S}} + \sum_{k=1}^n \sqrt{P(k)}$$

Now using the definitions of "moments"

$$M_\infty \equiv \max_k P(k), \qquad M_{1/2} \equiv \sum_{k=1}^n \sqrt{P(k)} \tag{6}$$

we can see that the inequalities (5) are in fact equivalent to

$$\boxed{2\sqrt{M_\infty} - M_{1/2} \leq \frac{A}{\sqrt{S}} \leq M_{1/2} \leq \frac{1}{\sqrt{S}}} \tag{7}$$

so that the dependence on the distribution $P(\cdot)$ is only through the dependence on the quantities M_∞ and $M_{1/2}$.

Remark 3.3. (a) The proof of the theorem shows that it is sufficient to employ Hilbert spaces of dimension at most $n+2$. It is unclear whether the conditions (5) also guarantee the existence of a quantum-mechanical model using a Hilbert space of dimension n or $n+1$.

(b) One can also reformulate (5) in terms of the min-entropy and the Rényi 1/2-entropy

$$H_{1/2} = 2\log M_{1/2}, \qquad H_\infty = -\log M_\infty$$

where it means that

$$2\log\left(2e^{-H_\infty/2} - e^{H_{1/2}/2}\right) \leq \log\frac{A^2}{S} \leq H_{1/2} \leq \log\frac{1}{S}$$

Intuitively, the last inequality in this chain means that the more information one wants to obtain about the postselected ensemble, the lower the optimal probability for conducting a successful measurement with postselection is going to be. And by the second inequality in the chain, higher information gain for given success probability also implies lower transition amplitude from $|\psi\rangle$ to $|\phi\rangle$.

4 Discussion

Let us now look at some specific cases of this result.

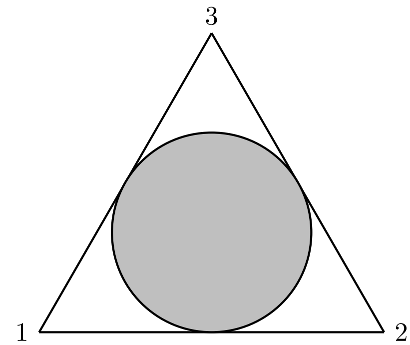


Figure 1: The quantum-mechanical region for $A = 0$ (orthogonal postselection), $n = 3$, and arbitrary success probability S (ternary plot).

Case $A = 0$, S arbitrary. This was studied for $n = 2$ in [Fri10]. As long as we allow the success probability S to be arbitrarily small, all that remains are the inequalities

$$\sqrt{P(k)} \leq \sum_{j \neq k} \sqrt{P(j)} \tag{8}$$

For $n = 2$, this reads $\sqrt{P(1)} \leq \sqrt{P(2)}$ and $\sqrt{P(2)} \leq \sqrt{P(1)}$, implying that $P(1) = P(2) = 1/2$. Hence a dichotomic measurement with postselection which is orthogonal to the initial state is guaranteed to be a perfectly unbiased random number generator. Generally, the $\sqrt{P(k)}$ which satisfy (8) lie in the convex cone spanned by the rays of the form

$$\sqrt{P(k)} = \delta_{kl} + \delta_{km}$$

for some pair of indices $l \neq m$. The $n = 3$ case is illustrated in figure 1; one obtains a circular region in probability space, due to the fact that its boundary is then given by quadratic equations. Just as it should due to the result for the $n = 2$ case, this region intersects with any side of the triangle in exactly the middle of that side.

Case $A \neq 0$, S arbitrary. Here, it is possible for any $P(\cdot)$ to find some appropriately small success probability S such that all inequalities in (7) hold, so no constraints abound. This is one reason why it is important to always consider S as an additional parameter.

Case $n = 2$, general. Here, the two probability values $P(1)$ and $P(2)$ determine each other uniquely, so let us write $P(1) = p$ and $P(2) = 1 - p$. Then the inequalities are

$$\left|\sqrt{p} - \sqrt{1-p}\right| \leq \frac{A}{\sqrt{S}} \leq \sqrt{p} + \sqrt{1-p} \leq \frac{1}{\sqrt{S}} \tag{9}$$

The projection of this into the p-S-plane, where only the last inequality is relevant, is shown in figure 2. For fixed S, some sections of the quantum region are graphed in figure 3. The first two inequalities of (9) define the upper and lower boundary curves in this case, while the third inequality ledas to vertical cuts whenever $S > 1/2$.

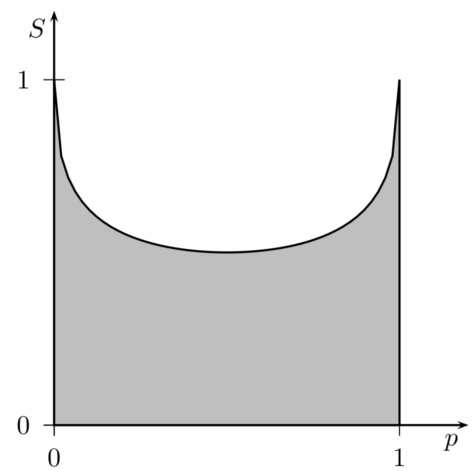
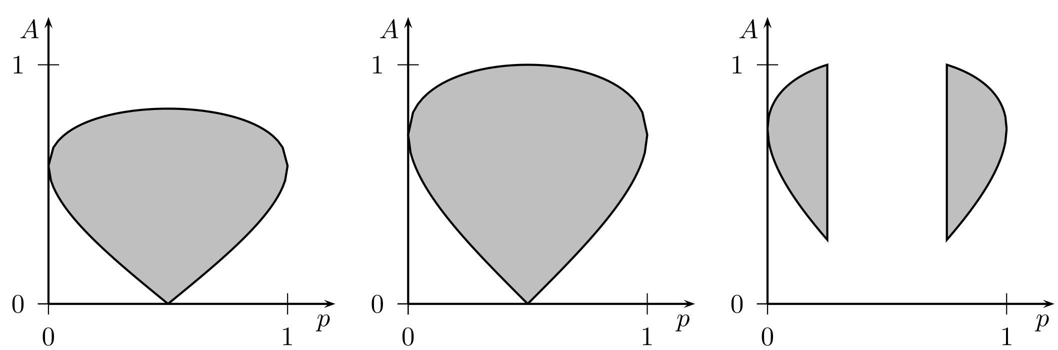


Figure 2: For $n = 2$, the possible quantum-mechanical success probabilities as a function of p.



(a) $S = 1/3$.

(b) $S = 1/2$.

(c) $S = 2/(2 + \sqrt{3}) \approx 0.536$.

Figure 3: Again $n = 2$. These plots show the quantum-mechanical region for (A, p) for some values of S. For bigger S than those shown, according to figure 2, the region rapidly shrinks down to the $p = 0$ and $p = 1$ lines.

The A-S-region. How does the transition amplitude relate in general to the probability of successful postselection? To study this, it is best to consider the inequalities in the form (7). Figure 10 shows an illustration of the following proposition.

Proposition 4.1. *For a given number of outcomes n, a pair of values (A, S) can appear in quantum theory if and only if*

$$\boxed{\frac{A^2}{n} \leq S \leq \frac{A^2+1}{2}} \tag{10}$$

Proof. Again it is first shown that these inequalities are necessary. Since $M_{1/2} \leq \sqrt{n}$, the second inequality in (7) implies that

$$A^2 \leq nS.$$

Now consider the expression

$$\frac{A^2}{S} + \frac{1}{S} \overset{(7)}{\geq} 4M_\infty + 2M_{1/2}^2 - 4\sqrt{M_\infty}M_{1/2} = 2\left[M_\infty + (M_{1/2} - \sqrt{M_\infty})^2\right]$$

and assume without loss of generality that $P(n) = \max_k P(k)$, so that

$$\frac{A^2}{S} + \frac{1}{S} \geq 2\left[P(n) + \left(\sum_{k=1}^{n-1}\sqrt{P(k)}\right)^2\right] \geq 2\left[P(n) + \sum_{k=1}^{n-1} P(k)\right] = 2,$$

as was to be shown.

For checking sufficiency of (10), consider first the case that $S \leq A^2$. Then since $2\sqrt{M_\infty} - M_{1/2}$ can at most be 1, it follows that the first inequality of (7) holds automatically. Now the possible values for $M_{1/2}$ are given by the closed interval $[1, \sqrt{n}]$, so that it is possible to find some value for $M_{1/2}$ in this interval which also satisfies (7) whenever $\frac{1}{\sqrt{S}} \geq 1$, which holds trivially, and $\frac{A}{\sqrt{S}} \leq \sqrt{n}$, which is true by assumption.

It remains to prove sufficiency when $A^2 \leq S \leq \frac{A^2+1}{2}$. Here, it is in fact enough to consider probability distributions $P(\cdot)$ supported on two elements, which brings us effectively down to the dichotomic case $n = 2$ from equation (9). By $\frac{A}{\sqrt{S}} \leq 1$, the middle inequality is automatic, so one only needs to take care of the remaining two. A direct calculation finally shows that when solving the equation $\sqrt{p} + \sqrt{1-p} = \sqrt{S}$ for p, then the equation

$$\left|\sqrt{p} - \sqrt{1-p}\right| = \frac{A}{\sqrt{S}}$$

holds for the maximal allowed amplitude $A = \sqrt{2S-1}$. □

This result 4.1 states in particular that transition probabilities between quantum states can be enhanced by an appropriate intermediate von Neumann measurement, the outcome of which can be discarded. This constitutes a (rather weak) kind of measurement-based quantum control. (10) shows that when using such a procedure, the error probability—i.e. the probability that the desired transition does not happen—can be reduced by a factor of up to 2.

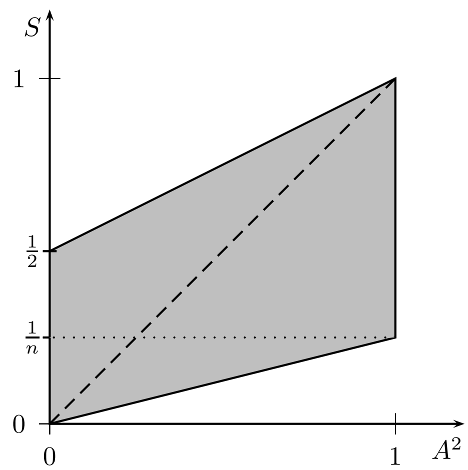


Figure 4: The quantum region of transition probabilities: A^2 is the transition probability without measurement, while S is the transition probability with n-ary von Neumann measurement. All points above the dashed line $S = A^2$ represent a measurement-enhanced transition probability.

5 Conclusion

In this paper, we have determined when a probability distribution over a finite number of measurement outcomes can appear for some quantum-mechanical postselected ensemble, given that the transition amplitude between the initial and final states is known, as well as the success probability of the postselection. The ensuing conditions are inequalities which depend on the probability distribution only through its Rényi 1/2-entropy and its min-entropy.

Finally, it was found that a von Neumann measurement can enhance the transition probability between the initial and the final state. The maximal enhancement is independent of the number of outcomes and is such that the error probability decreases by a factor of 2.

References

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