Removable Singularities for the Yang-Mills-Higgs equations in two dimensions

P.D. Smith

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3

West Germany

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1. Introduction

In this paper we prove a removable singularities theorem for the coupled Yang-Mills-Higgs equations over a two dimensional base manifold M .

1.a. Preliminary Definitions

Let M be a domain in R^2 and η be a vector bundle over M with compact structure group $G \subset O(n)$ and Lie algebra $\mathfrak{G}$. Let the metric on $\mathfrak{G}$ be induced by the trace inner product on $O(n)$ and let η have a metric compatible with the action of $\mathfrak{G}$. Let d be exterior differentiation, δ its adjoint, and let [,] denote the Lie bracket in $\mathfrak{G}$.

A connection determines a covariant derivative D which within a local trivialization defines a Lie algebra valued 1-form A by $D = d+A$. On p-forms we have locally $D\omega = d\omega + [A,\omega]$, $D^*\omega = \delta w + *[A,*\omega]$, where D^* is the adjoint of D . We denote the curvature 2-form by F and have $F = dA + \frac{1}{2}[A,A]$ in this local trivialization.

Gauge transformations are sections of Aut η which set on connections and curvature forms according to the transformations:

$$A^g = g^{-1}Ag + g^{-1}dg$$
$$F^g = g^{-1}Fg \ .$$

The pair (A,F) is gauge equivalent to $(\bar{A},\bar{F})$ iff there is a gauge transformation g such that $\bar{A} = A^g$ and $F = F^g$.

We now follow [S62] exactly and define the Higgs field φ using the determinant bundle. We denote by L the determinant bundle raised to $\frac{1}{2}$-power. Sections of this bundle are constant in a fixed co-ordinate system but we have weight 1 under scale transformations.

The Higgs field φ is a section of $\eta \otimes L$. Therefore, in a fixed co-ordinate system φ may be regarded as a matrix-valued function. Under scale charges $y = rx$, $\varphi(y) = \frac{\varphi(x)}{r}$ (cf.: [P] [SB2]).

The Yang-Mills-Higgs equations are:

(YMH1) $$D^*F = [D\varphi,\varphi]$$

(YMH2) $$D^*D\varphi = \frac{\lambda}{2}(|\varphi|^2 - m^2)\varphi \quad ;$$

where λ is a fixed real constant and where m is a section of L and hence constant in a fixed co-ordinate system but having weight 1 under scale changes. Thus under the transformation $y = rx$ we have $m' = m/y$. The equations (YMH1,2) are thus invariant under the scale transformation $y = rx$.

Certain norms are invariant under scale transformations. For example $\|\varphi\|_{L^2}$ is invariant and if ψ is any p-form $\|\psi\|_{L^{2/p}}$ is invariant. We also have an important fact used in [U1].

Fact [U1]

Suppose $\psi \in L^{2/p}$ with $\|\psi\|_{L^{2/p}}$ invariant. Then, given any $\gamma > 0$ there is a metric g_0 conformally equivalent to the Euclidean metric in which on bounded sets in R^2 ; $\int|\psi|^{2/p}\,dx < \gamma$.

This fact follows from conformal invariance and the continuity of the L^p-norms. See [UF] for details.

We now assume that $M = B_4^2 - \{0\}$, where B_4^2 is the 2-ball of radius 4 centered at the origin. We also assume that every connection has some gauge in which it is C^1 over the punctured ball.

1.b. Statement of the Main Theorem

We now state our Main Theorem:

Theorem 10.1

Let $M = B_4^2 - \{0\}$ and let η be as above. Let Λ be a connection on η that satisfies condition H(2), defined in section 1.c. Let F be the connection form of Λ and let F be C^∞ over M. Let (F,φ) satisfy (YMH1) and (YMH2) over M. Let $F \in L^1(B_4^2)$.

If $\lambda \geq 0$ let $\varphi \in H_2^1(B_4^2)$. If $\lambda < 0$ let $\varphi \in L^{2+\varepsilon}(B_4^2)$ and

$\overline{\lim_{t\to 0}} \int_{B_1/B_t} \frac{|\varphi|^2}{|x|^2\log^2\left(\frac{1}{t}\right)} = 0$. Then, there exists a continuous gauge transformation such that (F,φ) is gauge equivalent to a C^∞-pair over B_4^2 and the bundle extends continuously to a bundle over B_4^2.

A theorem of this type was first proved by K. Uhlenbeck for the pure Yang-Mills equations over R^4 in [U1]. Later Parker [P] extended the result to the coupled Yang-Mills-Higgs equations over R^4. Papers of L.M. and R.J. Sibner [SB1], [SB2], [SB3] proved similar theorems for dimension 3 and for all higher dimensions. This paper fills the two-dimensional gap in the literature.

We would like to thank L.M. Sibner for suggesting this problem and C. Taubes for a useful abelian example suggesting that holonomy would be important.

1.c. Auxiliary Gauges

Condition H

We wish to introduce a condition on the connection Λ that insures that the bundle is trivial over the punctured disk M above. This condition is a "holonomy" condition called condition H.

We use the conventions of [KN1] Vol. 1 pg. 71-72. We first define some useful paths.

Definition: Let $\ell_R : [0,1] \to S^1_R$ be given by $\ell_R : t \mapsto (R \cos 2\pi t, R \sin 2\pi t)$ with $S^1_R = \{x \in R^2 \mid |x| = R\}$. We say that ℓ_R is the standard loop for S^1_R. Let $L_\theta : [0,1] \to R$ be given by $L_\theta : t \mapsto (Rt,0)$. We call L_θ the standard line.

Now, choosing the fiber over $(R,0)$ as standard and choosing a point "Q" on this fiber we have a unique Λ-horizontal lift of the standard loop ℓ_R. Parallel transport on this lift is carried by the faithful right action of corresponding elements of the structure group G. We denote the group element that corresponds to the transport of "Q" around the full loop ℓ_R by $g(R)$.

Definition 1.1.: The map $C_R : (0,4] \to G$ given by $R \to g(R)$ is a path denoted by C_R.

Now, we define condition $H(K)$ and condition H.

Definition 1.2.: (condition $H(K)$): If as $R \downarrow 0$ the elements $g(R)$ considered as points on the carrier of the path C_R approach the identity element in the C^K-topology we say the connection satisfies condition $H(K)$.

Theorem 1.1.: The following is equivalent to condition $H(1)$: There exists a trivialization over a small ball $B_{R_0} - \{0\}$, $\exists_{R_0}$, $0 < R_0 \leq 4$ centered at the origin, in which the connection defines a local co-variant derivative $D = d+A$, $A = A_r(r,\theta)d\theta + A_\theta(r,\theta)d\theta$ with $A_r(r,\theta)$, $A_\theta(r,\theta) \in \Gamma(\mathfrak{G} \otimes T^*(B_{R_0} - \{0\}))$ and with $\lim_{r\to 0} A_\theta(r,\theta) = 0$, with the limit taken in the sup-norm topology on $\mathfrak{G}$.

Proof (1→2) Choose an orthonormal framing $\{v_i(r,\theta)\}$ of η over the ray $\{(r,0)|0 \le r \le \varepsilon\}$. Extend this to a framing $\{v_i(r,\theta)\}$ by parallel translation around the circles ℓ_R . Then, $\nabla_\theta v_i = 0$, $v_i(r,\theta) = v_i(r,0)\cdot g(r,\theta)$ for some $g(r,\theta) \in G$. In particular, $v_i(r,2\pi) = v_i(r,0)\cdot g(r)$ for some $g(r) = g(r,2\pi) \in G$. Thy hypothesis imply that for small ε , the element $g(r)$ is close to the identity so that $g(r) = \exp\ (h(r))$ for some $h(r) \in \mathfrak{G}$. Let $\varphi : [0,2\pi] \to [0,1]$ be a smooth function which vanishes near 0 and is 1 near 2π . Then $w_i(r,\theta) = v_i(r,\theta)\cdot \exp(-\varphi(\theta)h(r))$ is a smooth orthonormal framing of η over $B_2-\{0\}$. In this framing the connection form is: $(A_\theta)^i_j = < \nabla_\theta w_i, w_j > = <[\nabla_\theta(v_i\cdot\exp(-\varphi(\theta)h(r))], w_j> = -\varphi'(\theta)h(r)\delta_{ij}$. Hence $|A_\theta| \le c|h(r)| \downarrow 0$ as $r \downarrow 0$. (2→1). This follows from standard O.D.E. estimates on integrating the parallel transport equation for each horizontal lift of ℓ_R .

Q.E.D.

Remark 1.1.: Thus condition H(1) implies that the bundle η is trivial over $B^2_{R_0} - \{0\}$.

Lemma 1.1.: Under the conditions of theorem 1.1, let the connection satisfy condition H(2) . Then, there exists a local trivialization in which the connection induces the local co-variant derivative $D = d+A$, $A := A_r(r,\theta)dr + A_\theta(r,\theta)d\theta$, and we have: $\lim_{r\to 0} A_r(r,0) = 0$, $\lim_{r\to 0} A_\theta(r,\theta) = 0$, $\lim_{r\to 0} \frac{d}{dr}(A_\theta(r,\theta)) = 0$.

Proof: We start with the orthonormal framing v_i over the standard ray L_θ used in the beginning of the proof of theorem 1.1. We use this framing to give a local trivialization for the bundle restricted to have the standard ray as a base space. The connection restricts and we denote the restricted connection by ∇_r . This connection defines $\{\bar{A}_r(r,0)\}^i_j := < \nabla_r v_i(r,0), v_j(r,0) >$. Now we define $\hat{s}(r) \in G$ as the solution to: $\frac{d\hat{s}(r)}{dr} = -\bar{A}_r(r,0)\hat{s}(r)$, $\hat{s}(R_0) = I$, $\exists_{R_0}$, $0 < R_0 < 1$. Now define $\bar{v}_i(r,0) := v_i(r,0)\cdot\hat{s}(r)$. Note that

$$\{\widetilde{A}_r(r,0)\}^i_j := < \nabla_r\bar{v}_i(r,0), \bar{v}_j(r,0) > = \hat{s}^{-1}(r)\bar{A}_r(r,0)\hat{s}(r) + \hat{s}^{-1}(r)\frac{d\hat{s}(r)}{dr}$$

and thus; $\lim_{r\to 0}\{\widetilde{A}(r,0)\}^i_j = 0 = \lim_{r\to 0} < \nabla_r\bar{v}_i(r,0), \bar{v}_j(r,0) >$.

Now carry out the proof of theorem 1.1 with $\{v_i\}$ replaced by $\{\bar{v}_j\}$. Note that in the gauge constructed for which $\lim_{r\to 0} A_\theta(r,\theta) = 0$ we have

$$\lim_{r\to 0} \{A_r(r,0)\}^i_j = \lim_{r\to 0} < \nabla_r w_i(r,0), w_j(r,0) > = \lim_{r\to 0} < \nabla_r \bar{v}_i(r,0), \bar{v}_j(r,0) = 0 .$$

Note also that; $\lim_{r\to 0} \{\frac{d}{dr}(A_\theta(r,\theta))\}^i_j = \lim_{r\to 0} h'(r)\varphi(\theta)\delta_{ij} = 0$ by condition H(2) .

Q.E.D.

Definition 1.3.: We call this gauge the auxiliary gauge.

Lemma 1.2.: Let the conditions of theorem 1.1 hold. Let the connection satisfy condition H(2) . Let the connection be in $L^1(B_{R_0})$. Then in the auxiliary gauge we have: $\int_0^R |A_r(r,\theta)|dr < \infty$, $0 < R < R_0$.

Proof: In the auxiliary gauge we have:

$$\frac{\partial A_r}{\partial\theta} - \frac{\partial A_\theta}{\partial r} + [A_r, A_\theta] = F_{r,\theta} \quad \text{and} \quad \int_0^{2\pi}\int_0^{R_0} \frac{|F_{r,\theta}|}{r} \cdot r dr d\theta = \|F\|_{L^1(B_{R_0})} .$$

Fix R , $0 < R \leqq R_0$ and integrate:

$$A_r(R,\theta) = A_r(R,0) + \int_0^\theta \frac{\partial A_\theta}{\partial r}(R,t)dt - \int_0^\theta [A_r(R,t), A_\theta(R,t)]\, dt - \int_0^\theta F_{r,\theta}(R,t)dt ,$$

$$0 \leqq \theta \leqq 2\pi .$$

Thus:

$$|A_r(R,\theta)| \leqq |A_r(R,0)| + \int_0^\theta \left| \frac{\partial A_\theta(R,t)}{\partial r} \right| dt + \int_0^\theta |F_{r,\theta}(R,t)|dt + 2\int_0^\theta |A_r(R,t)||A_\theta'(R,t)|dt$$

for all R ; $0 < R \leqq R_0$. Let $0 < a < R$. Then:

$$\int_a^R |A_r(r,\theta)|dr \leqq \int_a^R |A_r(r,0|dr + \int_a^R\int_0^\theta \left| \frac{\partial A_\theta(r,t)}{\partial r} \right| dtdr$$

$$+ \int_a^R\int_0^\theta |F_{r\theta}(r,t)|dtdr + 2\int_0^R\int_0^\theta |A_r(r,t)|\ |A_\theta(r,t)|dt .$$

Thus we have: $\int_a^R |A_r(r,\theta)|dr \leqq C(R) + \int_0^\theta [\int_a^R |A_r(r,t)dr] 2|A_\theta(r,t)|dt$, with $C(R) \downarrow 0$ as $R \downarrow a$.

Now we apply Gronwall's inequality, pg. 189 [AMR] to get: $0 < r < R$

$$\int_a^R |A_r(r,\theta)|dr \leqq C(R)\exp\, [\int_0^\theta |A_\theta(r,t)|dt] \leqq KC(R) ,$$

since $\lim_{r\to 0} |A_\theta(r,\theta)| = 0$. Thus, letting $a \downarrow 0$ we have

$$\int_0^R |A_r(r,\theta)|dr \leq \tilde{C}(R) < \infty \text{ , with } \tilde{C}(R)\downarrow 0 \text{ as } R\downarrow 0 \text{ .}$$

Q.E.D.

Definition 1.4.: Let $W_R = \{x \in B_1 | \frac{R}{16} \leq |x| \leq 16R \leq 1\}$.

Lemma 1.3.: Under the hypothesis of theorem 1.1., let the connection satisfy condition H(2) and suppose that $\|F\|_{L^\infty(|x|=R)} \leq \frac{K}{R^2}\|h\|_{L^1(W_R)}$ (where $\|h\|_{L^1}$ is invariant under scale changes) with $0 < 16R < R_0$ (K independent of R). Then in the auxiliary gauge we have: $\int_0^R \left|\frac{dA_r(r,\theta)}{d\theta}\right| dr < \infty$.

Proof: In the auxiliary gauge we have:
$\frac{\partial A_r}{\partial\theta} - \frac{\partial A_\theta}{\partial r} + [A_r,A_\theta] = F_{r\theta}$. Thus, letting $0 < a < R \leq \frac{R_0}{16}$. We have:

$$\int_a^R \left|\frac{\partial A_r}{\partial\theta}\right| dr \leq \int_a^R \left|\frac{\partial A_\theta}{\partial r}\right| dr + \int_a^R 2|A_r|\ |A_\theta|dr$$

$$+ \int_a^R |F_{r\theta}|dr,\ \int_a^R \left|\frac{\partial A_r}{\partial\theta}\right| dr \leq r\,\tilde{c}(r) + 2 \sup_{a<r<R} |A_\theta| \int_a^R |A_r|dr$$

$$+ \int_a^R |F_{r\theta}|dr \text{ with } \tilde{c}(r)\downarrow 0 \text{ as } r\downarrow 0 \text{ ; } \int_a^R \left|\frac{\partial A_r}{\partial\theta}\right| dr \leq r\tilde{c}(r) + D(r)$$

$$+ \int_a^R t\|F(x)\|_{L^\infty(t=|x|)} dt \text{ with } D(r)\downarrow 0 \text{ , } \tilde{c}(r)\downarrow 0 \text{ as } r\downarrow 0 \text{ .}$$

Here we have used $F(x)dxdy = F_{r\theta}drd\theta \Rightarrow F(x) = \frac{F_{r\theta}}{r}$.

Now, fix a and R and define $m \in Z^+$ to be the first positive integer such that $2^{-m}R \leq \frac{a}{2}$. Then:

$$\int_a^R t\|F(x)\|_{L^\infty(t=|x|)} dt =$$

$$\sum_{i=1}^{i=m} \int_{2^{-i}\cdot R}^{2^{-i+1}\cdot R} t\|F(x)\|_{L^\infty(t=|x|)} dt$$

$$\leq \sum_{i=1}^{i=m} \sup_{2^{-i}R\leq\tau\leq 2^{-i+1}\cdot R} (\|F(x)\|_{L^\infty(|x|=\tau)}) \cdot \int_{2^{-i}\cdot R}^{2^{-i+1}\cdot R} t\ dt$$

$$\leq \sum_{i=1}^{i=m} \frac{K}{(2^{-i}R)^2} \int_{\frac{2^{-i}R}{16}}^{16\,2^{-i+1}\cdot R} \int_a^{2\pi} |h(x)|rd\theta dr \left[\frac{(2^{-i+1}\cdot R)^2}{2} - \frac{(2^{-i}\cdot R)^2}{2}\right]$$

$$\leq \sum_{i=1}^{i=m} 16K \int_{\frac{2^{-i}\cdot R}{16}}^{2^{-i+1}\cdot 16\cdot R} |h(x)|\ rd\theta dr$$

$$\leq 16K \; \| h(x) \|_{L^1(B_R)} \; .$$

Thus, finally we have:

$$\int_a^R \left| \frac{\partial A_r}{\partial \theta} \right| dr \leq O(R) + 16K \; \| h(x) \|_{L^1(B_R)} \; .$$

Now let $a \downarrow 0$.

Q.E.D.

Remark: Such an estimate on $\| F \|_{L^\infty}$ is indeed proved (in any smooth gauge over the punctured ball) independently (if A,F are weak solutions of YMH1,2) in section 8. Thus, we may assume the conclusion of this lemma holds in the auxiliary gauge.

2.a. Exponential Gauges

Definition 2.1: Let η be a vector bundle over $B_R - \{0\} \subset R^2$. If there exists a local trivialization in which the connection defines a local covariant derivative $D = d + A_{exp}$ with $A_{exp} \in \Gamma(\mathcal{G} \otimes T^*(B_R - \{0\}))$; $A_{exp} := A_{r,exp}(r,\theta)dr + A_{\theta,exp}(r,\theta)d\theta$ and such that: $A_{\theta,exp}(0,\theta) := \lim_{r\to 0} A_{\theta,exp}(r,\theta) = 0$; as well as:

(a) If $F \in L_1(B_R)$ with $||F||_{L^\infty(|x|=R)} \le \frac{K}{R^2} ||h||_{L^1(W_R)}$ with $||h||_{L^1(W_R)}$ invariant under conformal scaling then $A_{\theta,exp}(r,\theta) = \int_0^r F_{R\theta}(t,\theta)dt$.

(b) If $F \in L_\infty(B_R)$ then $||A_{exp}(x)||_\infty \le \frac{1}{2} |x| \max_{t<|x|} ||F(t,\theta)||_\infty$. (Here $F = dA_{exp} + \frac{1}{2} [A_{exp}, A_{exp}]$),

(then we say that this trivialization is an exponential gauge).

Lemma 2.1: If R is small enough and the connection satisfies condition H(2) , then under condition (a) or (b) above there is an exponential gauge for η .

Proof: First we show that $\lim_{r\to 0} A_{\theta,exp}(r,\theta) = 0$. First we do case (a). Choose R sufficiently small that we have an auxillary gauge and lemmas 1.2, 1.3 apply. Then, by the absolute continuity of Lesbesque integration we note that in this gauge $\int_0^R |A_r(r,\theta)|dr \le Q(R)$ and $\lim_{r\to 0} \int_0^R \left|\frac{dA_r(r,\theta)}{d\theta}\right| dr \le Q(R)$, where $\lim_{R\to 0} Q(R)=0$.

Starting from the auxillary gauge, apply the proof of lemma 2.1 of [U1] and note that

(*) $A_{\theta,exp}(r,\theta) = \sigma^{-1}(r,\theta)A_\theta(r,\theta)\sigma(r,\theta) + \sigma^{-1}(r,\theta)\cdot d_\theta\sigma(r,\theta)$, where $\sigma(r,\theta)$ is the transformation to the exponential gauge constructed in lemma 2.1 of [U1]. Now, we wish to use (*) to show that $\lim_{r\to 0} |A_{\theta,exp}(r,\theta)| = 0$.

We note that the first term satisfies:
$\lim_{r\to 0} |\sigma^{-1}(r,\theta)A_\theta(r,\theta)\sigma(r,\theta)| = \lim_{r\to 0} |A_\theta(r,\theta)| = 0$.

We also note that $|\sigma^{-1}(r,\theta)d_\theta(\sigma(r,0)| = |\frac{d}{d\theta}\sigma(r,\theta)|$.

Now we note that $\sigma(r,\theta)$ is a solution of the differential equation in lemma 2.1 [line 12, pg 14 [U1]]. Since the right side is C^1 in θ for $r > 0$, it follows by the standard theorem for continuous dependence on parameters for ODE solutions [H], using $\sigma(r_0,\theta)$ for $r_0 > 0$ fixed and arbitrary as initial condition for the equation starting at $r = r_0$, we have that $\frac{d\sigma(r,\theta)}{d\theta}$ exists for $r > 0$. We give an improved estimate for our case: Thus we have:

$$\frac{d\sigma(r,\theta)}{dr} = -A_r(r,\theta)\sigma(r,\theta) , \quad \sigma(0,\theta) = I$$

$$\sigma(r,\theta) = I + \int_0^r - A_r(t,\theta)\sigma(t,\theta)dt$$

Let $$\Delta_{h,\theta}f(r,\theta) = \frac{f(r,\theta+h) - f(r,\theta)}{h}$$

$$\Delta_{h,\theta}(\sigma(r,\theta)) = \int_0^r \Delta_{h,\theta}(-A_r(t,\theta))\sigma(t,\theta+h)dt + \int_0^r - A_r(t,\theta)\Delta_{r,\theta}\sigma(t,\theta)dt$$

$$|\Delta_{h,\theta}\sigma(r,\theta)| \leq \int_0^r |\Delta_{h,\theta}(-A_r(t,\theta))|dt + \int_0^r |\Delta_{h,\theta}\sigma(t,\theta)||A_r(t,\theta)|dt.$$

Now, since $\int_0^r \left|\frac{dA_r(t,\theta)}{d\theta}\right| dt < \infty$ we have, if h is small, (using a standard convergence theorem):

$$\int_0^r |\Delta_{h,\theta}(A_r(t,\theta))|dt < \int_0^r 2\left|\frac{dA_r(t,\theta)}{d\theta}\right|dt \quad \text{thus:}$$

$$|\Delta_{h,\theta}\sigma(r,\theta)| \leq 2\int_0^r \left|\frac{dA_r(t,\theta)}{dt}\right| dt + \int_0^r |\Delta_{h,\theta}\sigma(t,\theta)| \, |A_r(t,\theta)|dt$$

$|\Delta_{h,\theta}\sigma(r,\theta| \leq K(r) + \int_0^r |\Delta_{h,\theta}\sigma(r,\theta)| A_r(t,\theta)| dt$ where $K(r)\downarrow 0$ as $r\downarrow 0$.

Now applying Gronwall's inequality we have:

$|\Delta_{h,\theta}(\sigma(r,\theta))| \leq \bar{K}(r) \exp [\int_0^r |A_r(t,\theta)| dt]$ with $\bar{K}(r) \downarrow 0$ as $r \downarrow 0$.

Thus letting $h \downarrow 0$ we have $\left|\frac{d}{d\theta}\sigma(r,\theta)\right| \leq Q(R)$ with $Q(r)\downarrow 0$ as $r\downarrow 0$. Thus $\lim_{r\to 0} A_{\theta,\exp}(r,\theta) = 0$.

(b) The proof is the same noting that:

If $F \in L_\infty(B_R)$ the proofs of lemma 1.2. and lemma 1.3. (mutus mutandis) are greatly simplified. Thus the conclusions of these lemma's hold if we assume $F \in L^\infty$ instead in the hypotheses.

Thus in case a and in case b we have $\lim_{r\to 0} |A_{\theta,\exp}| = 0$.

Remark 2.1: Note that this does not follow just from the estimate (2 - 1 - a) of lemma 2.1 because in case (a) we have not assumed a-priori that F is bounded over the punctured ball. Moreover in our case we need $\lim_{r \to 0} |A_{\theta,exp}| = 0$ to prove the estimates, as we will show.

Now we derive the estimates on A . We do case (a) first. We do the integration in the proof of lemma 2.1 [U1] and obtain $\lim_{\varepsilon \to 0} \int_\varepsilon^r \sum_K x^K F_{ij}(x)dx = rA_{j,exp}(x) - \lim_{r \to 0} rA_{j,exp}(x) = I - II$. We show that II vanishes iff $\lim_{r \to 0} A_{\theta,exp} = 0$. This is necessary since $|A_{j,exp}(x)|$ might be unbounded at $|x| = 0$, at least a-priori.

We note that in the exponential gauge $A_{exp} = A_{1,exp}(x,y)dx + A_{2,exp}(x,y)dy = A_\theta(r,\theta)d\theta$, so writing dx and dy in terms of $d\theta$ and dr we obtain $rA_{1,exp}(x) = A_{\theta,exp}\sin\theta$ and $rA_{2,exp}(x) = A_{\theta,exp}\cos\theta$. Thus $\lim_{r \to 0} rA_{1,exp}(x)=0$ and $\lim_{r \to 0} r\, A_{2,exp}(x) = 0$ iff $\lim_{r \to 0} A_\theta(r,\theta) = 0$. Thus the integral formula of [U1] pg 14 line-2 holds. In polar co-ordinates this implies our estimate.

Another way to see this is to note that since $A_{r,exp} = 0$ we have

$$\frac{\partial A_{r,exp}}{\partial\theta} - \frac{\partial A_{\theta,exp}}{\partial r} + [A_{r,exp}, A_{\theta,exp}] = F_{r,\theta}$$

$$\text{Thus:} \qquad \frac{\partial A_{\theta,exp}}{\partial r} = F_{r\theta}$$

$$A_{\theta,exp}(r,\theta) - A_{\theta,exp}(\varepsilon,\theta) = \int_\varepsilon^r F_{r\theta}dr$$

This is equivalent to the integration in lemma 2.1 [U1]. Thus to get the estimate we need $\lim_{r \to 0} A_{\theta,exp}(r,\theta) = 0$ which we have proved.

(Case b). Once again, we have shown that $\lim_{r \to 0} A_{\theta,exp}(r,\theta) = 0$ which shows that the integral estimate of lemma 2.1 of [U1] holds. Then, since in case b we have bounded curvature, our estimate follows exactly as in lemma 2.1 [U1].

Q.E.D.

2.b. Transverse Gauges

Definition 2.2: $S^1_R = \{x \mid |x| = R,\ x \in R^2\}$

Lemma 2.2.: Let η be a vector bundle over $B_{R_0} - \{0\} \subset R^2$ as described in the introduction. Let Λ_1 be a connection on η satisfying condition $H(2)$, with curvature form F. Let $0 < R < R_0$. Suppose we are given a trivialization on S^1_R of the restricted bundle over S^1_R in which the restriction induces a covariant derivative $D = d + A$. Then, there exists an extension of this trivialization into an inner annular collar neighborhood $U_{r_1,R=r_2} = \{x \in R^2 \mid r_1 \le |x| \le r_2 = R\}$, in which Λ_1 induces a covariant derivative (also denoted, with abuse of notation by $D = d + A$) and moreover:

(a) $A_r = 0$,

(b) $$A_j(x) = \frac{R}{|x|} A_j\left(\frac{x \cdot R}{|x|}\right) + \frac{1}{|x|} \int_R^{|x|} \sum \tau_k \cdot \frac{x^k}{|x|} F_{kj}\left(\frac{x}{|x|} \cdot \tau\right) d\tau ,$$

and over $U_{r_1,r_2=R}$ we have the estimate:

(c) $$|A_j(x)| \le \frac{R}{|x|} \left|A_j\left(R \cdot \frac{x}{|x|}\right)\right| + C|x| \sup_{\substack{\tau \\ r_1<|x|<\tau<R}} |F(\tau x)| .$$

Proof: Since $\frac{\partial}{\partial t}\left(tA_j\left(t, \frac{x^i}{|x|}\right)\right) = \sum_k x^k F_{kj}\left(t, \frac{x^i}{|x|}\right)$ we integrate from $t = |x|$ to $t = R$ and obtain: (Here for clarity we have slightly changed notation).

$$A_j(|x|, x^i/|x|) = \frac{R}{|x|} A_j(R, x^i/|x|) + \frac{1}{|x|} \int_1^{|x|/R} \sum_k R\tau \cdot \left(\frac{x^k}{|x|} F_{kj}\left(R\tau, \frac{x^i}{|x|}\right)\right) R d\tau .$$

Letting $\hat{\tau} = R\tau$ we obtain

$$A_j(|x|, x^i/|x|) = \frac{R}{|x|} A_j(R, x^i/|x|) + \frac{1}{|x|} \int_R^{|x|} \sum_k \tau \cdot \frac{x^k}{|x|} F_{kj}(\tau, x^i/|x|) d\tau$$

which is (b). (c) follows by pulling the sup through the integral.

Q.E.D.

Lemma 2.3.: Let η be a bundle over $B_{R_0} - \{0\} \subset R^2$ with a connection satisfying condition $H(2)$. Let the curvature be in $L^1(B_{R_0} - \{0\})$ and let F satisfy $\sup_{|x|=r} |F| \le \frac{K}{r^2} \int_{B_{2r}} |h|^1 < \infty$ with $\int_{B_{2r}} |h|$ invariant under scale changes.

Let $0 < R_1 < 2R_1 \leq R_0$. In the exponential gauge for η over $B_{R_1} - \{0\}$ let the induced covariant derivative be denoted by $D = d + A_{exp}$. Let D' be the restriction of D to $\eta|_{S^1_{R_1}}$ in the restriction of the exponential gauge.

We have $D' = d_\theta|_{S^1_{R_1}} + A_{exp}|_{S^1_{R_1}}$. Denote $A_{exp}|_{S^1_{R_1}} = A'(x)$. Then,

$A'(x) = A'_\theta(R_1,\theta)d\theta$ and

(a) $|A'_\theta(R_1,\theta)|_\infty < \beta(R_1)$ where $\beta(R_1) \downarrow 0$ as $R_1 \downarrow 0$.

(b) $||A'(R_1,\theta)||_{\infty,S^1_{R_1}} \leq \frac{1}{R_1}\beta(R_1)$

Proof: (a) follows from the estimates on $A_{\theta,exp}$ in the exponential gauge given in the previous section; (b) follows because on $S^1_{R_1}$ we have

$$||A'||_{\infty,S^1_{R_1}} = \sqrt{g^{22}|A'|^2_{\infty,S^1_{R_1}}} = \frac{1}{R_1}|A'|_{\infty,S^1_{R_1}} .$$

Q.E.D.

Lemma 2.4.: (cf. lemma 2.4. in [U1])

Let $0 < 2R_1 \leq R_0 \leq 4$. Let η be a bundle over $U_{r_1,r_2} = \{x \mid r_1 \leq |x| \leq r_2 \leq R_1\}$ with a connection with bounded curvature. Let the bundle and its connection be restrictions of a bundle over $B_{R_0} - \{0\}$ and a connection satisfying condition $H(2)$ as well as condition (a) on curvature in Definition 2.1. Suppose gauges are chosen on $\eta|_{S^1_t}$ for $t = r_1,r_2$ in which the connection restricted to S^1_t defines covariant derivatives $D^t_\theta + \tilde{A}^{t,\theta}$.

Then, there exists a gauge on η over U_{r_1,r_2} in which the connection defines a local covariant derivative $D = d + A$ with $A|_{S^1_t} = \tilde{A}^{t,\theta}$ for $t = r_1,r_2$ and $||A||_{\infty,U_{r_1,r_2}} \leq K \max_{t=r_1,r_2} \{|| \tilde{A}^{t,\theta} ||_{L^\infty,S^1_t}, k\cdot|t|\,||F||_{\infty,U_{r_1,r_2}}\}$

(here K is proportional to $\frac{r_2}{r_1}$ but this is harmless to us).

Proof: Match transverse gauges from the boundaries with exponential gauges exactly as in lemma 2.4. of [U1], and use our lemma 2.2 (c).

Remark 2.2.: We use balls of arbitrary radius to simplify the proof of the estimates in lemma 5.1. condition (g).

3. Application of the Implicit Function Theorem

As in [U1], we are in a position to apply the ordinary Banach space Implicit function theorem to solve the nonlinear system $\delta A = 0 \Leftrightarrow \delta(S^{-1}dS + S^{-1}AS) = 0$ when A is small enough.

We will use annuli of general radius to simplify the proof of lemma 5.1. (g).

Also because S^1 is a 1-manifold and both $[A,A]$ and F are 2-forms that are zero on S^1, the L^p to L^∞ bootstrapping procedure of theorem 2.5. of [U1] breaks down here. However, we substitute another argument.

Let $0 < R < 4$.

Theorem 3.1.:

Let η be a trivial bundle over S^1_R . Let Λ be a connection on η . Suppose in some trivialization Λ defines a covariant derivative by $D_\theta = d_\theta + A^\theta$ where $A^\theta = A^\theta_\theta\,(R,\theta)d\theta$. Then, there exists a trivialization in which $D = d_\theta + \bar{A}$, $\bar{A} = \bar{A}_\theta(R,\theta)d\theta$, $\delta_{S^1_R}[\bar{A}] = 0$ and $||\bar{A}||_{L^\infty,S^1_R} < K||A^\theta||_{L^\infty,S^1_R}$, $|\bar{A}_\theta|_{L^\infty,S^1_R} < K|A^\theta_\theta|_{L^\infty,S^1_R}$, with K independent of R . As in [U1] theorem 2.5, we note this procedure determines the trivialization only up to constant multiplication by an element of G .

Proof: We must solve $\delta_{S^1_R}(S^{-1}dS + S^{-1}A^\theta S) = 0$ for S : Instead, following [U1], we solve $\delta_{S^1_R}(e^{-u}de^u + e^{-u}A^\theta e^u) = 0$ for u .

Now by direct calculation this is equivalent to solving

$$(*) : \qquad \frac{\partial}{\partial\theta}\left[e^{-u}\frac{\partial e^u}{\partial\theta} + e^{-u}A^\theta_\theta(R,\theta)e^u\right] = 0 \,.$$

Now consider the expression:

$$Q(u,B) = \frac{\partial}{\partial\theta}\left[e^{-u}\frac{\partial e^u}{\partial\theta} + e^{-u}B\,e^u\right] .$$

This expression induces a C^∞-map on $u \in C^2(S^1_R,\mathfrak{G})$, $B \in C^1(S^1_R,\mathfrak{G})$, $Q : C^2(S^1_R,\mathfrak{G})\times C^1(S^1_R,\mathfrak{G}) \longrightarrow C^0(S^1_R,\mathfrak{G})$. The image actually lies in:

$$C^{0^\perp}(S^1_R,\mathbb{C}) = \{\xi \in C^0(S^1_R,\mathbb{C}) \mid \langle \xi,u_0\rangle = 0 \ , \ u_0 \in \mathbb{C}\}$$

similarly define:

$$C^{2^\perp}(S^1_R,\mathbb{C}) = \{u \in C^2(S^1_R,\mathbb{C}) \ : \ \int_{S^1_R} u = 0\} \ .$$

Now consider $Q : C^{2^\perp}(S^1_R,\mathbb{C}) \times C^1(S^1_R,\mathbb{C}) \longrightarrow C^{0^\perp}(S^1_R,\mathbb{C})$. Then $d_1 Q_{(0,0)}$ is an isomorphism. Now the ordinary implicit function theorem in Banach spaces tells us we may solve $Q(u,A^\theta_\theta) = 0$ if $|A^\theta_\theta|_{L^\infty,S^1_R}$ is sufficiently small.

Now in order to get our estimates we rewrite this solution method in terms of the inverse function theorem and use a well known estimate of the size of the neighborhoods in the inverse function theorem [AMR Box 2.5, pg 105] .

Now from the form of (*) we note that we may define: $\underline{u}_R(\theta) = u(R,\theta)$ and $\underline{A}^\theta_\theta(\theta) = A^\theta_\theta(R,\theta)$ and solve:

$$(**) : \quad \frac{\partial}{\partial\theta}\left[e^{-u_R(\theta)}\frac{\partial(e^{u_R(\theta)})}{\partial\theta} + e^{-u_R(\theta)}\cdot\underline{A}^\theta_\theta(\theta)\cdot e^{u_R(\theta)}\right] = 0 \ .$$

Let $U \subset C^1(S^1,\mathbb{C})$ be open. Let $V \subset C^{2^\perp}(S^1,\mathbb{C})$ be open. Let $E = C^1(S^1,\mathbb{C})$. Let $H \subset C^{0^\perp}(S^1,\mathbb{C})$ be open. Let $F = C^{0^\perp}(S^1,\mathbb{C})$. Let the norms on E and F be the canonical norms induced by the sup-norm on $\mathbb{C}$. $E \times F$ is a Banach space with norm given as the sup of the norms on E and F .

Define $Q(\underline{u},\underline{B}) = \frac{\partial}{\partial\theta}\left(e^{-\underline{u}(\theta)}\ \frac{\partial(e^{\underline{u}(\theta)})}{\partial\theta} + e^{-\underline{u}(\theta)}\ \underline{B}(\theta)e^{\underline{u}(\theta)}\right]$ and note that $Q(\underline{u},\underline{B})$ is a C^∞-Banach map $Q : C^{2^\perp}(S^1,\mathbb{C}) \times C^1(S^1,\mathbb{C}) \longrightarrow C^{0^\perp}(S^1,\mathbb{C})$.

Define $\Phi : U \times V \longrightarrow E \times H$ by $\Phi : (\underline{B},\underline{u}) \longmapsto (\underline{B},Q(\underline{u},\underline{B}))$. Then Φ is a C^∞ - Banach map and $D\Phi(0,0)(x_1,x_2) = \begin{pmatrix} I & , & 0 \\ \frac{\partial}{\partial\theta} & , & \frac{\partial^2}{\partial\theta^2}\end{pmatrix}\begin{pmatrix} x_1 \\ x_2\end{pmatrix}$ is an isomorphism. We have; $\underline{u}(\theta) = \Phi^{-1}(\underline{A}^\theta_\theta(\theta),0)$.

Now, we apply Corollary 2.5.6, Box(2.5.A) pg 105 of [AMR] with the R in [AM] chosen so that K is fixed. We may do this by the smoothness of the map Φ . Now restrict to the intersection of the R-ball about $(0,0)$ with $U \times V$.

We note that we may write $(D\Phi(0,0))^{-1}$ explicitly by integrating the above definition of $(D\Phi(0,0))$ as a pair of ordinary differential equations and that it follows from sup-norm estimates on this integral and on its derivatives that M is bounded as well. We also see from our formula for $(D\Phi(0,0))$ that L is bounded as well. Thus Φ is a diffeomorphism from the R_2 - Banach ball in $U \times V$ about $(0,0)$ onto the R_3-Banach ball about zero. We choose our $u(\theta) = \Phi^{-1}(A_\theta^\theta(\theta),0)$ to be the unique element of the preimage in this R_2-ball about $(0,0)$. We note that with this domain and co-domain Φ^{-1} is Lipshitz continuous with Lipshitz constant $2L$ as a map from the R_3-Banach ball about zero. Thus if $|A_\theta^\theta(R,\theta)|_{\infty,S_R^1}$ is small enough that $|\underline{A}_\theta^\theta(\theta)|_{\infty,S^1} < \frac{R_3}{2}$, we see that:

$$\sup_{S^1} |\underline{u}(\theta)| + \sup_{S^1}\left|\frac{\partial \underline{u}}{\partial\theta}\right| < K\,|\underline{A}_\theta^\theta(\theta)|_{S^1,\infty} .$$

Now recalling the definition of the $\underline{u}$ and $\underline{A}_\theta^\theta$ we see that:

$$\sup_{S_R^1} |u| + \sup_{S_R^1}\left|\frac{\partial u}{\partial\theta}\right| < K\,|A_\theta^\theta(R,\theta)|_{\infty,S_R^1}$$

with K independent of R . Thus,

$$\sup_{S^1_R} |S| + \sup_{S^1_R} |d_\theta S| < K(G)\,|A_\theta^\theta(R,\theta)|_{\infty,S_R^1} .$$

Now,

$$||\bar{A}||_{L^\infty,S_R^1} = ||S^{-1}dS + S^{-1}A^\theta S||_\infty \le ||dS||_{\infty,S_R^1} + ||A^\theta||_{\infty,S_R^1} \le \frac{1}{R}K(G)\,|A_\theta^\theta(R,\theta)|_{\infty,S_R^1}$$

$$+ ||A^\theta||_{\infty,S_R^1} \le \tilde{K}(G)\,||A^\theta||_{\infty,S_R^1} .$$

Finally

$$|\bar{A}_\theta|_{L^\infty,S_R^1} \le R\,||A^\theta||_{\infty,S_R^1} \le K(G)\cdot|A_\theta^\theta|_{L^\infty,S_R^1} . \qquad \text{Q.E.D.}$$

<u>Corollary 3.1.</u>: Let $0 < R_1 \le 4$. Under the assumptions of lemma 2.3. there exists a gauge on the restriction of η over $S^1_{R_1}$ in which the restriction of the connection defines a covariant derivative given by $\hat{D} = d_\theta|_{S^1_{R_1}} + \bar{A}$ with;

$\bar{A} = \bar{A}_\theta(R,\theta)d\theta, \delta_{S^1_R}[\bar{A}] = 0$ and $|\bar{A}_\theta(R,\theta)|_{\infty,S^1_{R_1}} < \beta(R_1)$ with $\beta(R_1) \downarrow 0$ as $R_1 \downarrow 0$.

Proof: Let A^θ in theorem 2.1. be the restriction of the connection form to η in an exponential gauge from the origin. Noting the estimates of lemma 2.3 apply theorem 3.1.

Q.E.D.

Theorem 3.2.: (of theorem 2.8. of U1) Under the hypothesis of lemma 2.3. and 2.4., assuming $||h||_{L^1,B_{R_0}} < \gamma$, there exists $r_2^* > 0$ sufficently small, such that if $0 < r_1 < r_2 < r_2^*$ with $\frac{r_2}{r_1} < 100$ there exists a gauge for η over U_{r_1,r_2} in which the connection defines a local covariant derivative $D = d + A$ with: $A = A_r(r,\theta)dr + A_\theta(r,\theta)d\theta, |A_\theta(r_i,\theta)| < \beta(r_2) < \gamma$; $(i = 1,2)$, $\delta_S A_S = 0$ on $r = r_1,r_2$, $\delta A = 0$ in $U_{r_1,r_2}, ||A||_{\infty,U_{r_1,r_2}} \leq \frac{K\gamma}{r_1}$ with K independent of r_1,r_2 .

Proof: By choosing r_2^* small enough it follows from lemma 2.3. that $|A_{\theta,\exp}(r,\theta)| < \beta(r) < \gamma$ for all $r < r_2$. Thus, by the above Corollary 3.1. there exists a gauge on $S^1_{r_i}$ $(i = 1,2)$ in which the restriction of the connection defines a local covariant derivative $\hat{D}_i = d_\theta|_{S^1_{r_i}} + \bar{A}_i$ with $\bar{A}_i = \bar{A}^i_\theta(r_i,\theta)d\theta$ with $\delta_{S^1_{r_i}}[\bar{A}_i] = 0$, $|\bar{A}^i_\theta(r_i,\theta)| < \beta(r_i) < \gamma$. Now, apply lemma 2.4. with $\tilde{A}^{t,\theta} = \bar{A}_1$ for $t = r_1$ and $\tilde{A}^{t,\theta} = \bar{A}_2$ for $t = r_2$. Then, we have a gauge over U_{r_1,r_2} in which the restriction of the connection induces a local covariant derivative $D = d + \tilde{A}$, $\tilde{A} = \tilde{A}_r(r,\theta)dr + \tilde{A}_\theta(r,\theta)\,d\theta$ with : $\tilde{A}_r = 0$, $\tilde{A}_S(r_i,\theta) := \tilde{A}_\theta(r_i,\theta)d\theta = \bar{A}_i(r_i,\theta), |\bar{A}^i_\theta(r_i,\theta)| = |\tilde{A}_\theta(r_i,\theta)| < \beta(r_i) < \gamma$ and with $||\tilde{A}||_{\infty,U_{r_1,r_2}} \leq K \max_{t=r_1,r_2} \{\frac{\gamma}{r_1}, k\cdot|t|\cdot||F||_{\infty,U_{r_1,r_2}}\}$. (We have used $||\tilde{A}_S(r_i,\theta||_{\infty,S^1_{r_i}} = \frac{1}{r_i}|\tilde{A}_\theta(r_i,\theta)|$; note that K is independent of r_1,r_2 because $\frac{r_1}{r_2} < 100$) . Now note that $||F||_{\infty,U_{r_1,r_2}} \leq \frac{K}{r_1^2} \int_{B_{R_0}} |h| \leq \frac{K}{r_1^2}\gamma$. Thus, the second term in the braces, satisfies $k|t|\cdot||F||_{\infty,U_{r_1,r_2}} < \tilde{K}\frac{\gamma}{r_1}$. So, we have

$$||\tilde{A}||_{\infty,U_{r_1,r_2}} \leq \frac{K}{r_1} \quad .$$

Now we apply the argument of [U1] theorem 2.8. exactly to get a new gauge over U_{r_1,r_2} in which $D = d + A$, $\delta A = 0$, $\delta_S A_S = 0$, $\int_{|x|=t} A_r = 0$ for all $t \in [r_1, r_2]$ and our estimates on $\tilde{A}$ and its boundary values then give the required estimates on A . Q.E.D.

Now, we apply a single scale change of the form $x = \lambda y$ (which does not change the value of any scale invariant integrals) to make r_1^* above equal to 4.

As in [U1], at the end of the paper we need to construct a global Hodge gauge over a sufficiently small punctured ball once we know that curvature is bounded. We have:

Theorem 3.3.: (cf. theorem 2.7. of [U1])

Let $0 < R < R_0 \leq 4$. Let η be a bundle over $B_{R_0} - \{0\} \subset R^2$ with a connection satisfying condition H(2) . Let F be the curvature form of this connection and let $||F||_{\infty, B_{R_0}} < \gamma$. Then, there exists $\tilde{R}_1$, $0 < \tilde{R}_1 < R_0$ and a trivialization for η restricted over $B_{\tilde{R}_1} - \{0\}$ such that the connection induces a local covariant derivative $D = d + A$ and $\delta A = 0$ in $B_{\tilde{R}_1}$. Moreover, $\delta_S A_S = 0$ on $S^1_{\tilde{R}_1}$, $||A(x)||_\infty \leq \frac{K}{|x|} \gamma$.

Proof: We use the same arguments as in the previous proof but applied to a ball. First we choose $\tilde{R}_1$ small enough that $|A_{\theta,\exp}(\tilde{R}_1,\theta)| < \beta(r) < \gamma$, then we apply corollary 3.1. on the sphere $S^1_{R_1}$, then we apply the argument of theorem 2.7. of [U1] exactly. Note that in theorem 2.7. of [U1] $\tilde{k}_1 \sim \frac{1}{|x|}$ as well.

Q.E.D.

4. Some Eigenvalue Estimates

In this section all forms are smooth and real valued.

Lemma 4.1.: On $U = \{x | r_1 \le |x| \le r_2\}$, $0 < r_2 \le 4$. Let

$A = \{\omega | \omega$ is a 1-form with $\delta\omega = 0$, $\delta_s \omega_s = 0$ on ∂U, $\int_{|x|=r} (*\omega)_s = 0$

$\forall_{r_1 \le |x| \le r_2}$, $\omega \ne a d\theta$ where a is a nonzero real constant.$\}$

Let $E(\omega) = \frac{\langle d\omega, d\omega \rangle}{\langle \omega, \omega \rangle}$. Then if $\lambda = \inf_{\omega \in A} E(\omega)$, we have $\lambda \ne 0$.

Proof: It is sufficient to minimize $E(\omega)$ with $\langle \omega, \omega \rangle \ne 0$, $\omega \in A$. Assume $\lambda = 0$ and note this implies $\langle d\omega, d\omega \rangle = 0$. Then if we let $\tau \in \operatorname{Ker}\delta \cap C_0^\infty(U)$ we see by calculation that the Euler-Lagrange equation gives $\langle d\omega, d\tau \rangle = 0$. Since this holds for all τ with compact support in U we have $\delta d\omega = 0$.

Let $\omega_s = T\omega$ be the tangential part of ω on each boundary sphere of ∂U and let $\omega_N = N\omega$ be the corresponding normal component (cf:[Mo pg. 302]). Since ω_s is a 1-form on the outer boundary $\partial^+ U$ of U we have that $d\omega_s = 0$. Thus ω_s is a harmonic 1-form on $\partial^+ U$ and thus $\omega_s = c\,d\theta$ for some constant c.

Now, consider $\sigma = *\omega$. Since $\delta d\omega + d\delta\omega = \delta d\omega = 0$ by assumption, we have $*d*d*\sigma = 0$ inside U. Thus in U, $d\delta\sigma = 0$. On the other hand, $\delta\omega = 0$ inside U implies that $d\sigma = 0$ inside U and thus $(d\delta + \delta d)\sigma = 0$ inside U. Thus σ is a harmonic 1-form inside U.

Since $d\sigma = 0$ and $\int_{|x|=r} *\omega = \int_{|x|=r} \sigma = 0$ we have $\sigma = df$ for some function $f(r,\theta)$ inside U. Since σ is harmonic in U it follows that $d\delta\sigma = d\delta df = 0$ in U. Thus f is biharmonic in U.

It follows from $\int_{|x|=r} \sigma = 0$ that $\int_{|x|=r} \frac{\partial f}{\partial \theta} d\theta = 0$ and thus f is periodic in θ. Moreover $\omega_s : \omega_\theta = T\omega = T(*\sigma) = \frac{\partial f}{\partial r} r d\theta$. Recall, we have just shown that $\omega_s|_{\partial^+ U} = c\,d\theta$. Since r is constant on $\partial^+ U$, we see that $\frac{\partial f}{\partial r} = c$ on $\partial^+ U$.

Now separate variables and solve $\Delta^4 f = 0$, $\frac{\partial f}{\partial r} = c$ on $\partial^+ U$. We observe that $f = (a \log r + br^2 \log r + cr^2 + D)(g(\theta))$ with a,b,c,D as constants. Since $\frac{\partial f}{\partial r} = c$ on $\partial^+ U$ we have $g(\theta)$ is a constant, at no loss of generality absorbed in the constants a,b,c,D.

Since $\sigma = df$ we obtain $\sigma = (\frac{a}{r} dr + 2rb \log r + br + 2cr)dr$.

$$*\sigma = (a + 2br^2 \log r + br^2 + 2cr^2)d\theta$$

$$d*\sigma = ((4br \log r) + 4br + 4cr)drd\theta$$

$$*d*\sigma = 4b \log r + 4b + 4c \text{ , since } *drd\theta = \frac{1}{\sqrt{g}} = \frac{1}{r} .$$

Thus $d\delta\sigma = \frac{4b}{r} = 0 \Rightarrow b = 0$ thus $\sigma = (\frac{a}{r} + 2cr)dr$, $\omega = (a + 2cr^2)d\theta$. Now since $\lambda = 0$ we have $0 = \frac{\langle d\omega, d\omega\rangle_{\bar{U}}}{\langle\omega,\omega\rangle_{\bar{U}}}$ with $\langle\omega,\omega\rangle \neq 0$, thus $d\omega = 0$, which implies $c = 0$. Thus $\omega = ad\theta$ which is impossible by hypothesis. Thus we have a contradiction which implies $\lambda \neq 0$.

Q.E.D.

Lemma 4.2.: Let $U = \{x | r_1 \leq |x| \leq r_2\}$. There exists $\alpha_0 > 0$ such that: if $\omega \in A$, where $A = \{\omega | \omega$ is a 1-form on U , $\int_{|x|=r} (*\omega)_s = 0$, $r \in [r_1, r_2]$, $\sup_{\partial U} \|\omega\|_{L^\infty} < \alpha_0\}$, and $\lambda = \operatorname{Inf}_A \frac{\langle d\omega, d\omega\rangle_{\bar{U}}}{\langle\omega,\omega\rangle_{\bar{U}}}$ then $\lambda > \frac{14\cdot 4}{r_2^2}$.

Proof: At no loss of generality we may assume $\langle\omega,\omega\rangle_{\bar{U}} = 1$. Since ω is a co-closed 1-form in U we have:

$$\frac{\langle d\omega, d\omega\rangle_{\bar{U}}}{\langle\omega,\omega\rangle_{\bar{U}}} = \frac{\langle\delta d\omega, \omega\rangle_{\bar{U}}}{\langle\omega,\omega\rangle_{\bar{U}}} + \frac{\int_{\partial U} *d\omega\wedge\omega}{\langle\omega,\omega\rangle_{\bar{U}}} = \frac{I}{\langle\omega,\omega\rangle} + \frac{II}{\langle\omega,\omega\rangle} .$$

$$= \tilde{I} + \tilde{II} .$$

Now, lets estimate $|II|$. Letting $\| \quad \|$ denote the pointwise norm on forms we have: (estimating $|*d\omega\wedge\omega|$ by Hölder pointwise).

$$|II| \leq \|\omega_s\|_{L^\infty(\partial U)} \cdot \int_{\partial U} \|*d\omega\| \leq \alpha_0 \int_{\partial U} \|*d\omega\| .$$

Now, since $*d\omega$ is a function in $\bar{U}$ so $\delta(*d\omega) = 0$, we apply the trace inequality for $H_{1,2}$ functions to obtain:

$$\left|\int_{\partial U} \|*d\omega\|\right| \leq K_1\left[\int_U \|d*d\omega\|\right] + \frac{K_2}{r_2}\left[\int_U \|*d\omega\|\right] \leq$$

$$K_1\left[\int_{\bar{U}} \|\delta d\omega\|\right] + K_3\langle d\omega, d\omega\rangle \quad \text{(Hölder)}$$

$$\leq K_1\langle\delta d\omega, \delta d\omega\rangle + K_3\langle d\omega, d\omega\rangle .$$

Thus:

$$|II| \leq \alpha_0\left[\langle\delta d\omega, \delta d\omega\rangle + K_3\langle d\omega, d\omega\rangle\right]$$

(Note that $\delta d\omega = \Delta\omega$ since $\delta\omega = 0$).

Now, let $\square$ denote the Laplacian on co-closed 1-forms on U with zero tangential boundary values on ∂U . Note that $\square$ possesses a complete set of smooth orthonormal eigenforms because of the spectral theorem for self-adjoint compact operators and elliptic regularity. Expanding ω in these eigenfunctions, choosing α_0 small enough and using elementary arithmetic, it follows that the quantity $\frac{\langle d\omega, d\omega\rangle_{\bar{U}}}{\langle\omega,\omega\rangle_{\bar{U}}}$ is bounded from below by $(1+\varepsilon(\alpha))\lambda_1$, where λ_1 is the first positive eigenvalue of $\square$ on co-closed 1-forms ω with $\omega_s = 0$ on ∂U . We now estimate λ_1 essentially by constructing the eigenfunctions of $\square$ using classical special functions.

Thus, we must find the first positive λ for which $\delta d\omega = \lambda\omega$ for some co-closed form on U with vanishing boundary values. We assume at no loss of generality that $\langle\omega,\omega\rangle_{\bar{U}} = 1$.

First we write $\delta d\omega = \lambda\omega$ on co-closed 1-forms in local polar co-ordinates. We obtain by elementary computations that if $\omega = P dR + Q d\theta$, then

$$0 = \delta d\omega - \lambda\omega =$$
$$\left[-\lambda Q - (Q_{RR}) + \frac{Q_R}{R} - \frac{P_\theta}{R} + P_{QR}\right] \cdot d\theta +$$
$$\left[-\lambda P + \frac{1}{R^2}(Q_{R\theta}) - \frac{1}{R^2}(P_{\theta\theta})\right] dR .$$

Thus gives us the system of equations:

(a) $\frac{1}{R^2}(Q_{R\theta}) - \frac{1}{R^2}(P_{\theta\theta}) - \lambda P = 0$

(b) $P_R + \frac{P}{R} + \frac{Q_\theta}{R^2} = 0$

(c) $Q_{RR} - \frac{Q_R}{R} + \frac{P_\theta}{R} - P_{\theta R} + \lambda Q = 0$.

Solving for Q_θ in (b) and using this in (a) we obtain:

$$P_{RR} + \frac{3}{R} P_R + \frac{P_{\theta\theta}}{R^2} + \frac{P}{R^2} + \lambda P = 0 .$$

Now, let $P = e^{im\theta} f(r)$. For P to be well-defined, we require m to be an integer. Since, $\int_{|x|=r} (*\omega)_s = 0$, we obtain $m \neq 0$. Substituting in the above differential equation for P we get:

$$\left[f(R)\right]_{RR} + \frac{3}{R}\left[f(R)\right]_R + \left[\frac{1-m^2}{R^2} + \lambda\right] f(R) = 0 \ .$$

Letting $w = f(R)$, using the above equation for $f(R)$ we obtain:

$$(*) \quad w_{RR} + \frac{1}{R} w_R + \left[\lambda - \frac{m^2}{R^2}\right] w = 0 \quad \text{(Bessels equation)} \ .$$

Since w vanishes on $|x| = r_1$ and $|x| = r_2$, we have $w(r_1) = 0$, $w(r_2) = 0$. If instead we solve (*) with $w(0) = 0$, $w(r_2) = 0$ we do not increase the first positive λ for which there is a nonzero solution of (*). Thus we solve (*) with the boundary conditions $w(0) = 0$, $w(r_2) = 0$ for expository simplicity. The solutions are $\bar{w} = cJ_{|m|}(\sqrt{\lambda}\, x)$ with $J_m(\sqrt{\lambda}\, r_2) = 0$. The smallest positive value of λ is bounded below by λ_0 where $\sqrt{\lambda_0}\, r_2 = Z$ and Z is the first positive zero of $J_1(x) = 0$ (note that $m \neq 0$). Thus $\lambda > \lambda_0 > \frac{14 \cdot 4}{r_2^2}$. Finally, we have $\lambda_1 > \frac{14 \cdot 4}{r_2^2}$ if α_0 is small enough.

5. Broken Hodge Gauges

We now state the properties of special gauges - the Broken Hodge Gauges - constructed from previous gauges by matching by rotations by constant elements of G. These gauges where first used in [U1].

Definition 5.1.: Let $U^i = \{x | \frac{1}{\tau^i} \leq |x| \leq \frac{1}{\tau^{i-1}}\}$ where $1 \leq \tau \leq 2$ and $i = 0,1,2,3\ldots$; and let $S^i = \{x | \ |x| = \frac{1}{\tau^i}\}$.

Lemma 5.1.: (cf. lemma 4.5 [U1] and lemma 4.5 [Sb2]) (Broken Hodge Gauges). Let η_i be a bundle over U^i obtained as the restriction of a bundle over $B_2 - \{0\}$. Let Λ_i be the restriction to U^i of a connection Λ_0 on η_0. Let η_0 and Λ_0 satisfy the conditions of definition 2.1. Then, there exist local trivializations (gauges) for η_i over U^i such that in these gauges Λ_i induces a local covariant derivative $D^i = d + A^i$, and curvature form F_i with:

(a) $\delta A^i = 0$

(b) $\delta_s A^i_s = 0$ on ∂U^i

(c) $\int (*A)_s = 0$ on absolute cycles

(d) $\sup |A^i(x)| < \gamma_3 \tau^i$

(e) $\int_{U^i} |A^i(x)|^2 dx \leq \frac{1}{(\lambda - \gamma_3(\tau^i)^2)} \int_{U^i} |F^i(x)|^2 dx$

(e') for any $\alpha > 1$, $\int_{U^i} |x|^\alpha |A^i(x)| dx \leq \frac{\tau^\alpha}{(\lambda - \gamma_3(\tau^i)^2)} \int_{U^i} |x|^\alpha |F^i(x)|^2 dx$

(Here λ is the λ of lemma 4.1 and $\gamma_3 = k\gamma$ in the conclusion of Theorem 3.2).

(f) $\lim_{i \to \infty} A^i_\theta(x) \Big|_{\partial U^i} = 0$

(g) $\int_{U^i} |A^i|^2 dx \leq \frac{1/\tau^{2i}}{4.5 - \gamma_3} \int_{U^i} |F^i(x)|^2 dx$, if τ is close enough to one

(g') $\int_{U^i} |x|^\alpha |A^i|^2 dx \leq \frac{\tau^\alpha/\tau^{2i}}{4.5 - \gamma_3} \int_{U^i} |x|^\alpha |F^i|^2 dx$, if τ is close enough to one

(h) $A^i(x) = A^{i+1}(x)$ on S^i.

Proof: a → d follow by our implicit function theorem results, theorem 3.2, and by matching gauges by a constant element of G as in the proof of lemma 4.5 pg. 25 of [U1]. Note that we produce (h) by this construction. Thus the A^i match to form a 1-form A, continuous on $B_2 - \{0\}$.

(f) follows because of the estimate in Theorem 3.2.

Sublemma: For each i, A^i is not of the form $cd\theta$ for some constant c .

Proof of Sublemma: By (h) this constant must be independent of i and thus $A = cd\theta$. But, by (f) we have $c = 0$. (e) follows as in Corollary 2.9 of [U1], estimating $\int_{U^i} |[A^i, A^i]|^2 dx \le \sup|A^i|^2 \int_{U^i} |A^i|^2 dx$ as in lemma 4.5 of [SB2] (cf. also corollary 2.6 in [U1]), and noting that because of the sublemma above we may minimize the functional of Corollary 2.9 [U1] over 1-forms with the additional condition that they are not of the form $cd\theta$, c constant, so that the zero eigenvalue os not taken on because of lemma 4.1.

(e') follows from (e) by estimating the weights from below and pulling them through the integrals.

To prove (g) we use theorem 3.2 with $r_1 = \frac{1}{\tau^i}$ and $r_2 = \frac{1}{\tau^{i-1}}$. Since r_1 and r_2 are less than four, noting we have done the dilation following theorem 3.2 in the text so that $r_2^* = 4$, it follows from theorem 3.2 that $\sup_{\partial U^i} |A_\theta(r_i, \theta)| < k\gamma = \gamma_3$. Choosing $\gamma_3 < \alpha_0$ (α_0 defined in the hypothesis of lemma 4.2) it follows from lemma 4.1 and lemma 4.2 that $\lambda > 14(\tau^{i-1})^2 > \frac{14}{\tau^2}(\tau^i)^2 > 4\cdot 5\,(\tau^i)^2$ if τ is close enough to 1. Now, noting (d) we see that (g) follows from (e).

(g') follows from (g) in the same way that (e') follows from (e).

6. Some Improvements on Morrey's Theorem

In this section we state some improved versions of Morrey's theorem in 2-dimensions that will be used later.

First we state Morrey's theorem in 2-dimensions.

Theorem 6.1. (Morrey's Theorem in 2-dimensions) [MO]. Let $u \in H_2^1(\Omega)$ with $u \geq 0$ and suppose that: Ω is a locally Lipshitz domain in R^2, and $\int_\Omega \nabla u \nabla \xi + f\, u\, dx \leq 0$ for all non-negative $\xi \in C_0^\infty(\Omega)$. Let f satisfy the Morrey Condition:

$$\int_{B_R \subset \Omega} |f|^{1+\varepsilon} dx \leq c\, R^\beta \quad \text{for all} \quad B_R \subset \Omega \quad \text{and some} \quad \varepsilon, \beta > 0 \quad \text{then}$$

$$\sup_{B(x_0,\rho)} |u(x)|^2 \leq \frac{c}{a^2} \int_{B(x_0,\rho+a)} |u(y)|^2 dy \quad \text{for all} \quad B(x_0,\rho) \subset B(x_0,\rho+a) \subset \Omega .$$

Proof: Identical to the proof of Theorem 5.3.1 of [MO], pg. 137, except that we need our somewhat stronger Morrey Condition because the inequality $\int g|w|^2 \leq c_n[\int |\nabla w|^2 dx + \int |g|^{n/2} dx]$ fails in 2-dimensions due to critical Sobolov exponents.

We would now like to note that if $u \in C^\infty(\Omega)$ we can state an improvement of Morrey's estimate involving $\frac{K}{a^2} \int_{B(x_0,\rho+a)} |u(y)| dy$. This improvement follows from an iteration argument of E. Bombieri. See [BO], pg. 66.

Theorem 6.2 (Bombieri). Let Ω be compact. Let $u \in C^\infty$ in Ω and let $u \geq 0$. Let u satisfy:

$$\sup_{B_\rho} (u(x))^2 \leq \frac{c}{(R-\rho)^2} \int_{B_R} u^2 dx \quad \text{for all concentric} \quad B_R, B_\rho \subset \Omega , \quad 0 < \rho < R .$$

Then

$$\sup_{B_\rho} u(x) \leq \frac{c}{(R-\rho)^2} \int_{B_R} u\, dx \quad \text{where } B_R \text{ and } B_\rho \text{ are as above.}$$

Proof: Use the iteration at the top of pg. 66 of [BO]. Q.E.D.

We will also need an improved Morrey theorem based on the Alexanderov--Bakelman estimates and due to Trudinger [TR].

Theorem 6.3. Let Ω be a compact domain in R^n. Let $u \in W^{2,n}(\Omega)$ weakly satisfy: $\Delta u + au \leqq 0$ in Ω, with $a > 1$ and $u \geqq 0$. Then, for any $p \in (0,n]$ and $\sigma \in (0,1)$,, we have for all concentric balls $B_{\sigma R}$ and B_R in Ω with $R < 1$ that:

$$\sup_{B_{\sigma,R}} u \leqq c R^{-n/p} ||u||_{p,B_R}$$ where c is independent of R and u.

Proof: This follows from the more general estimate of Theorem 2.1, pg. 5 of [TR] with c independent of R by the remark after Corollary 2.3 of [TR] with $h_R = 1 + b_2 R^2 < 1 + b_2$ with $b_2 = a > 1$. The theorem in [TR] is stated for weak solutions; however, its proof shows it is also valid for weak subsolutions.

Q.E.D.

7. A Regularity Theorem for the Higgs Field

In this section we assume that the Higgs field is a C^∞ solution of the field equation:

$$\text{(YMH2)}\quad D^*D\phi = \frac{\lambda}{2}\,(|\phi|^2 - m^2)\phi$$

in the punctured unit ball $B^2 - \{0\}$. As in [Sb2] the assumptions on ϕ near the origin depend on the sign of λ.

Because of the criticality of the Sobolev exponent $\frac{2n}{n-2}$ for L_2 functions in 2-dimensions, we require several technical changes from the argument in [SB2]. This is where we use the estimates of section 6.

The main result of this section is:

Theorem 7.1. Let ϕ be a C^∞ solution of (YMH2) in $B^2 - \{0\}$ in R^2. We assume:

(a) $\varphi \in H_2^1(B^2)$ if $\lambda > 0$

(b) $\varphi \in H_2^1(B^2)$ if $\lambda = 0$

(c) $\varphi \in L^{2+\varepsilon}(B^2)$ for some $\varepsilon > 0$ and $\overline{\lim_{t\to 0}} \int_{B_1/B_t} \frac{|u|^2}{|x|^2 \log^2(\frac{1}{t})} = 0$, if $\lambda < 0$.

Remark 7.1: That condition (c) is natural follows by considering the case when the sructure group is commutative (i.e., the real numbers) and looking at the scalar inequality

$$\Delta u + u^3 \gtrsim 0 .$$

Then, $u = \ln r - r$ is an unbounded function satisfying the above inequality and $\ln r - r$ is in all L^p except for $p = \infty$.

Also note that our condition (c) is weaker than $\varphi = o(\log|x|)$ and that $\varphi \in O(\log|x|)$ is stronger than (c).

Similarly, we see that conditions (b) and (a) are natural by considering $\Delta u = 0$ in $B^2 - \{0\}$. Then $u = \ln |x|$ is an unbounded solution of $\Delta u + u^3 = 0$ with $u \notin H_2^1(B^2)$.

To prove 7.1 we make strong use of the fact that $u = |\varphi|$ is a weak solution in $B^2 - \{0\}$ of: $(\Delta|\varphi|) \geq \frac{\lambda}{2}(|\varphi|^2 - m^2)|\varphi|$, where Δ is the ordinary Laplacian on functions. This follows from Weitzenblock-like identities and details may be found in [Sb2] (formula 2 and lemma 1.2.).

At no loss of generality we assume $u \geq 1$ so that $||u||_{B^2 - \{0\}} \geq 1$.

For example, in case (b) the function $|\phi|$ is subharmonic. First we dispose of case (b).

<u>Proof</u> (case (b)): Consider a sub-ball $B(x.r)$ in $B^2 - \{0\}$ with $0 < r < \frac{1}{8}$ (MAX(dist(x,0), dist(x,∂B^2))) and thus by Morrey's theorem [MO, pg. 137] we have:

$$\sup_{B(x,r)}(u) \leq \frac{K}{r}\left[\int_{B(x,2r)} u^2 dx\right]^{1/2} = J .$$

If $r > r_0$ we have $J \leq K(R_0)||u||_{H_2^1(B^2 - \{0\})}$. We may choose r_0. Now there exists a sequence of test functions $\eta_i \in C_0^\infty(B^2)$ with $\eta_i = 0$ for $|x| \leq E_i$, that tend to 1 as E_i tends to zero and such that $\int|\nabla\eta_i|^2 dx \to 0$ as $i \uparrow \infty$ [SB2]. Choose r_0 close enough to zero that for any fixed $r \leq r_0$ we can choose $i(r)$ such that $B_{0,E_i(r)} \cap B_{x,2r} = \phi$, $\eta_{i(r)}|_{B_{x,2r}} \geq 1/2$, meas $(B_{x,4r} \cap B_{0,E_i}) \geq \frac{1}{10}$ meas $B_{x,4r}$ and such that $\int_{B_{x,4r}} |\nabla\eta_i|^2 dx \leq K_0$, where K_0 will be chosen below.

Now: $u(x) \leq \sup_{B(x,r)}(u) \leq \frac{K}{r}\left[\int_{B(x,2r)} u^2 dx\right]^{1/2} \leq \frac{2K}{r}\left[\int_{B(x,2r)} (\eta_i u)^2 dx\right]^{1/2}$

$$\leq \frac{2K}{r}\left[\int_{B(x,4r)} (\eta_i u)^2 dx\right]^{1/2} \leq \tilde{K}\left[\int_{B(x,4r)} |\nabla(\eta_i u)|^2 dx\right]^{1/2}, \quad \text{(Poincaré ineq.)}$$

$$\leq \tilde{\tilde{K}}\left[[\int_{B(x,4r)} |\nabla\eta_i|^2 dx][\int_{B(x,4r)} u^2 dx] + \int_{B(x,4r)} |\nabla u|^2 dx\right]^{1/2} = I$$

$\leq K||u||_{H_2^1(B^2 - \{0\})}$, with K depending on K_0. Now choose K_0 small enough so that $K < 1$ and we have $I \leq ||u||_{H_2^1(B^2 - \{0\})}$. Thus:

$$\sup_{B^2-\{0\}} (u) \leq K\left[||u||_{H_2^1(B^2 - \{0\})}\right] < \infty .$$

Q.E.D.

We now dispose of Case (a).

Proof: (Case (a)). In Case (a) we have that $u = |\phi|$ solves $\Delta u \geq \frac{\lambda}{2}(u^2 - m^2)u$ with $\lambda > 0$. Thus: $\Delta u \geq \frac{\lambda}{2}(u^2 - m^2)u$. Now consider the two sets.

$$A = \{x \in B^2 - \{0\} \text{ such that } u \leq m\} ,$$

$$B = \{x \in B^2 - \{0\} \text{ such that } u > m\} .$$

These sets are pairwise disjoint. Now, because $u \in C^\infty$ on $B^2 - \{0\}$, the set B is open.

Cover B by a countable collection of small balls, each contained in B . Then on any such small ball in B we have $\Delta u \geq 0$ and by the estimate above used in the proof of case (b) we obtain:

$$\sup_B u \leq K \, ||u||_{H_2^1(B^2 - \{0\})} .$$

Now on A , u is bounded above by m . Hence u is bounded on $B^2 - \{0\}$.

Q.E.D.

We now prove case (c). This requires some work because the proof of Proposition 2.3 of [Sb2] fails in 2-dimensions. The main problem is that when $n = 2$ inequality (1.14), page 7 of [Sb2], fails since $\frac{2n}{n-2} = \infty$ and $c_n = \infty$ when $n = 2$. Nevertheless we establish the same estimate as in the conclusion of Proposition 2.3 of [Sb2] using a modified technique.

First we prove the following propositon.

Proposition 7.1 (cf. Prop. 2.3 of [Sb2]). If condition (c) is satisfied, either we have:

$$\int_{B^2} \eta^2 |\nabla u|^2 dx \leq K \int_{B^2} |\nabla \eta|^2 u^2 \, dx$$

for all test functions η in $C_0^\infty(B^2)$ or u is bounded.

Proof: We use a sequence η_K of test functions that vanish for $|x| \leq \varepsilon_K$, tend to 1 as ε_K tends to zero and such that $\int |\nabla \eta_K|^2 dx \longrightarrow 0$ as $K \longrightarrow \infty$. These are defined cf. [G] pg. 547 bottom, by:

$$\bar{\eta}_K = \bar{\eta}^{\varepsilon_K}(|x|) = \left\{ \begin{array}{ll} 0 & \text{for } |x| \leq \varepsilon_K \\ 1 & \text{for } |x| \geq 1 \\ \dfrac{1}{\log\left(\frac{1}{\varepsilon_K}\right)} \cdot \log\left(\dfrac{|x|}{\varepsilon_K}\right) & \text{for } \varepsilon_K < |x| < 1 \end{array} \right\}$$

Remark 7.2: Note that our growth condition in case (c) is chosen exactly to insure that $\int_{B_2} |u|^2 |\nabla \bar{\eta}_K| \longrightarrow 0$ as $K \longrightarrow \infty$.

Now let η be C_0^∞ and let $\bar{\eta}$ be a C^∞ function vanishing in a neighborhood of the origin. Use the test function $\tau = (\eta\bar{\eta})^2(u)$ as ξ in: $\int \nabla u \cdot \nabla \xi \, dx \leq \int h u \xi \, dx$ for all non-negative $\xi \in C_0^\infty(B^2 - \{0\})$, where $h = -\frac{\lambda}{2}(|\varphi|^2 - m^2)$ and $u = |\varphi|$. We get:

$$K \int (\eta\bar{\eta})^2 |\nabla u|^2 dx \leq \int |2\eta\bar{\eta}\nabla u| \, |\nabla(\eta\bar{\eta})u| \, dx + \int (\eta\bar{\eta}) \, h u^2 \, dx = I_1 + I_2 .$$

Now, $I_1 \leq \mu \int (\eta\bar{\eta})^2 |\nabla u|^2 dx + C(\mu) \int |\nabla(\eta\bar{\eta})|^2 |u|^2 dx$ and the first term on the right may be absorbed into the left hand side. Also, $\int |\nabla(\eta\bar{\eta})|^2 u^2 dx \leq K[\int |\nabla \bar{\eta}|^2 u^2 dx + \int |\nabla \bar{\eta}|^2 u^2 dx]$. Note that $\int |\nabla \bar{\eta}|^2 dx \longrightarrow 0$ if we set $\bar{\eta} = \eta_k$ and let $k \longrightarrow \infty$. Do this. Thus, in the limit as $k \longrightarrow \infty$, $I_1 \leq \int |\nabla \eta|^2 |u|^2 dx$. Now, $I_2 = \int (\eta\bar{\eta})^2 h u^2 dx \leq \int_{\operatorname{supp} \eta \cap \operatorname{supp} \bar{\eta}} (\eta\bar{\eta})^2 h u^2 dx$. Since $\lambda \leq 0$ we have

$$h = \frac{-\lambda}{2}(|\phi|^2 - m^2) \leq \frac{-\lambda}{2}(|\phi|^2), \quad I_2 \leq K \int_{\operatorname{supp} \eta \cap \operatorname{supp} \bar{\eta}} (\eta\bar{\eta})^2 |\phi|^2 |u|^2 dx = J_2 .$$

We now estimate J_2:

Remark: The estimate of I_2 in the proof of proposition 2.3, pg. 11 of [Sb2], is based on the inequality: $\int gw^2dx \le C_n ||g||_{n/2} \int |\nabla w|^2dx$ which is proved using Sobolev's inequality. This inequality estimates I_2 from above by a sum of terms, the first of which is proportional to $||\phi||_{L^2}$. Then use is made of conformal scaling to make $||\phi||_{L^2}$ small.

In two dimensions however, the Sobolev estimate has a critical exponent and constant c_n corresponding to this exponent is infinite. Thus we need a new argument.

This new estimate is contained in the proof of the following sublemma.

Sublemma 7.1. Let $B^2 - \{0\} \supset \Omega \supset \text{supp}\ \eta \cap \text{supp}\ \bar{\eta}$. Then: $J_2 \le C[\int_\Omega |\phi|^2dx] \cdot [\int_\Omega (\eta\bar{\eta}u)^2dx + \int_\Omega |\nabla(\eta\bar{\eta}u)|^2dx]$.

Remark: The idea of the proof is that $V = |\phi|^2$ is a weak sub-solution (in fact a C^∞ solution) of an elliptic equation on $\text{supp}\ \eta \cap \text{supp}\ \bar{\eta}$. Thus by a Morrey--like estimate (Bombieri's lemma) we can estimate $\sup_{B(R)\subset\Omega} |\phi| \le \frac{C}{R} [\int_{B(2R)\subset\Omega} \phi^2dx]^{1/2}$.

Then by simple estimates we get a "Reverse Holder inequality" with $||\phi||_{2+\varepsilon,B(R)\subset\Omega}$ estimated from above by $||\phi||_{2,B(2R)\subset\Omega}$. The sublemma then follows from a covering theorem. We do it now.

Let $V = \phi^2$, let all balls $B(r)$ be contained in Ω. Let $\Omega_o = \text{supp}\ \eta \cap \text{supp}\ \bar{\eta} \subset \Omega$. Choose the balls B_R so that $B_R \subset B_{2R} \subset \Omega_o$ and $\text{meas}(B_{4R} \cap \Omega_o^c) \ge \frac{1}{100}\ \text{meas}\ B_{4R}$. Then Ω_o is covered by a number of such balls. Since u is C^∞ in Ω_o we can at no loss of generality assume that $u \ge 1$ on Ω_o. (If no such Ω_o exists then u is bounded.) Recall that $u = |\phi|$ is a subsolution of $\Delta u \ge \frac{\lambda}{2}(|u|^2 - m^2)|u| \ge \frac{\lambda}{2}(|u|^2)|u|$ in Ω_o since $\lambda < 0$. Thus $\Delta u - \frac{\lambda}{2}|u|^3 \ge 0$ in Ω_o. Now since $u \ge 1$, $u \in C^\infty$ on Ω_o, we have: $\Delta(|u|^2) = 2u\Delta u + 2|\nabla u|^2 \ge \Delta u$. Thus $V = |u|^2$ is a C^∞ subsolution in Ω_o of $\Delta V + (\frac{-\lambda}{2}|u|)V \ge 0$. Note that $(\frac{-\lambda}{2}|u|)$ is in $L_{1+\varepsilon,\exists_\varepsilon,\varepsilon>0}$ (by our growth assumption $\phi \in L_{2+\varepsilon}(B^2)$). We now apply Theorem 6.1 (Morrey's theorem in 2-dimensions) and theorem 6.2 (Bombieri's lemma) to get

$$\sup_{B(R)\subset\Omega_o} V \leq \frac{C}{R^2} \int_{B(2R)\subset\Omega_o} |V|^1\, dx \quad , \quad \forall_{B(R),B(2R)} \text{ concentric in } \Omega_o \quad .$$

Thus

$$\sup_{B(R)\subset\Omega_o} \phi \leq \frac{C}{R}\left[\int_{B(2R)\subset\Omega_o} \varphi^2\, dx\right]^{1/2} .$$

We now use the above inequality and Holder's inequality to achieve our estimate of J_2 . Using Holder's inequality with $p = 1+\frac{\varepsilon}{2}$, $q = \frac{2+\varepsilon}{\varepsilon}$ we get:

$$J_2 \leq \left|\sup_{B(R)\subset\Omega_o} \phi\right|^{\frac{2\varepsilon}{2+\varepsilon}} \left[\int_{B(R)\subset\Omega_o} (\eta\bar{\eta}u)^{2\cdot(\frac{2+\varepsilon}{\varepsilon})}\right]^{(\frac{\varepsilon}{2+\varepsilon})} \cdot \left[\int_{B(R)\subset\Omega_o} \varphi^2\right]^{\frac{2+\varepsilon}{\varepsilon}} = J_3 .$$

Now extend $\eta\bar{\eta}u$ to B_{4R} with the extension equal to zero on $\Omega_o^c \cap B_{4R}$ and call the extension $E(\eta\bar{\eta}u)$. We have

$$J_3 \leq \left[\int_{B_R} \sup \phi\right]^{\frac{2\varepsilon}{2+\varepsilon}} \left[\int B_R \phi^2\right]^{\frac{2}{2+\varepsilon}} \left[\int_{B_{4R}} E(\eta\bar{\eta}u)^{2(\frac{2+\varepsilon}{\varepsilon})}\right]^{(\frac{\varepsilon}{2+\varepsilon})\frac{1}{2}\cdot 2} .$$

Now use Sobolev's inequality in the form:

$$\left[\int_{B_{4R}} u^t dx\right]^{1/t} \leq CR^{2/t}\left[\int_{B_{4R}} |\nabla u|^2 dx\right]^{1/2} \quad \text{where} \quad t \geq 2$$

for $u \in H_2^1(B_{4R})$ with $u = 0$ on $B_{4R} \cap \Omega_0^c$. We let $u = E(\eta\bar{\eta}u)$ and $t = (2(2+\varepsilon))/\varepsilon$ to get:

$$\left[\int_{B_{4R}} |E(\eta\bar{\eta}u)|^{\frac{2(2+\varepsilon)}{\varepsilon}}\right]^{\frac{\varepsilon}{2+\varepsilon}} \leq c\, R^{\frac{2\varepsilon}{2+\varepsilon}} \left[\int_{B_{4R}} |\nabla E(\eta\bar{\eta}u)|^2 dx\right]$$

and thus

$$\int_{B_R\subset\Omega_0} \phi^2\eta^2\bar{\eta}^2 u^2 dx \leq (\sup_{B_R}\phi)^{\frac{2\varepsilon}{2+\varepsilon}} \cdot \left[\int_{B_{4R}} |\nabla E(\eta\bar{\eta}u)|^2 dx\right]\cdot\left[\int_{B_R} \phi^2 dx\right]^{\frac{2}{2+\varepsilon}} \left[CR^{\frac{2\varepsilon}{2+\varepsilon}}\right] .$$

Recall that $\sup_{B_R}\phi \leq (K/R)[\int_{B_{2R}} \phi^2 dx]^{1/2}$. Thus combining all our estimates we get:

$$\int_{B_R\subset\Omega_0} \phi^2\eta^2\bar{\eta}u^2 \leq \left[\frac{C}{R}\left(\int_{B_{4R}} \phi^2 dx\right)^{1/2}\right]^{\frac{2\varepsilon}{2+\varepsilon}} \cdot \left[C\, R^{\frac{2\varepsilon}{2+\varepsilon}}\right]\cdot\left[\int_{B_R} \phi^2 dx\right]^{\frac{2}{2+\varepsilon}}\cdot\left[\int_{B_{4R}} |\nabla(E\eta\bar{\eta}u)|^2 dx\right].$$

But $\int_{B_{4R}} |\nabla E(\eta\bar{\eta}u)|^2 dx \le \int_{\Omega_0 \cap B_{4R}} |\nabla E(\eta\bar{\eta}u)|^2 dx$ so we have:

$$\int_{B_R \subset \Omega_0} \phi^2 \eta^2 \bar{\eta}^2 u^2 dx \le K\Big[\int_{B_{2R}} \phi^2 dx\Big]\Big[\int_{B_R} (\eta\bar{\eta}u)^2 dx + \int_{\Omega_0 \cap B_{4R}} |\nabla(\eta\bar{\eta}u)|^2 dx\Big] .$$

Now using Besocovitch's covering lemma and changing constants appropriately we have $\int_\Omega \phi^2 (\eta\bar{\eta})^2 \le C[\int_\Omega \phi^2][\int_\Omega (\eta\bar{\eta}u)^2 dx + \int_\Omega |\nabla(\eta\bar{\eta}u)|^2 dx$. This completes the proof of the sublemma.

Q.E.D. (Sublemma)

Now we return to the main proof and use the sublemma. We have, using the sublemma and recalling that conformal invariance implies that we may choose $[\int \phi^2 dx]^{1/2} < \gamma$ (where γ may chosen small) that:

$$\text{(II)} \quad K\int_\Omega (\eta\bar{\eta}_k)^2 |\nabla u|^2 dx \le \int_\Omega |\nabla\eta|^2 u^2 dx + g(k) + c(\gamma)[(\int_\Omega |\nabla(\eta\bar{\eta}_k u)|^2 dx)$$

$$+ C(\int_\Omega |(\eta\bar{\eta}_k u)|^2 dx)] \quad \text{with} \quad c(\gamma) \downarrow 0 \ \text{if} \ \gamma \to 0 \ \text{and} \ \lim_{k\to\infty} g(k) = 0 .$$

Note that:

$$\text{(III)} \quad \int_\Omega |\nabla(\eta\bar{\eta}_k u)|^2 dx \le 2\int_\Omega \eta^2 \bar{\eta}_k^2 |\nabla u|^2 dx + 2\int_\Omega |\nabla(\eta\bar{\eta}_k)|^2 u^2 dx$$

so from (II) and (III) we obtain

$$\text{(IV)} \quad K\int_\Omega (\eta\bar{\eta}_k)^2 |\nabla u|^2 dx \le \int_\Omega |\nabla\eta|^2 u^2 dx + g(k) + 2C(\gamma)[\int_\Omega \eta^2 \bar{\eta}_k^2 |\nabla u|^2 dx$$

$$+ 2\int_\Omega |\nabla\eta\bar{\eta}_k|^2 u^2 dx] + C(\gamma)[\tilde{K}\int_\Omega (\eta\bar{\eta}_k u)^2 dx] .$$

Now choosing γ small enough we absorb the term $2C(\gamma)\int_\Omega |\eta\bar{\eta}_k|^2 |\nabla u|^2 dx$ in the left hand side of (IV) and we get:

$$\text{(V)} \quad K\int_\Omega |\eta\bar{\eta}_k|^2 |\nabla u|^2 dx \le g(k) + \int |\nabla\eta|^2 u^2 dx + C(\gamma)[2\int_\Omega |\nabla\eta\bar{\eta}_k|^2 u^2 + K\int_\Omega (\eta\bar{\eta}_k u)^2 dx \ldots$$

Now using growth condition c, we have

$$\int_\Omega |\nabla(\eta\bar{\eta}_k)|^2 u^2 dx \le 2\int |\nabla\eta|^2 u^2 dx + 2\int \eta^2 (|\nabla\bar{\eta}_k|)^2 u^2 dx \le \hat{2}\int_\Omega |\nabla\eta|^2 u^2 dx + 2\int |u^2| \nabla\bar{\eta}_k|^2$$

$$\cdot \sup_\Omega \eta \le 2\int |\nabla\eta|^2 u^2 dx + h(k)$$

where $h(k) \downarrow 0$ as $k \to \infty$. Using this in (V) we obtain

$$\text{(VI)} \quad K\int_\Omega |\eta\bar{\eta}_k|^2 |\nabla u|^2 dx \le h(k) + g(k) + \tilde{K}\int_\Omega |\nabla\eta|^2 u^2 dx + KC(\gamma)\int_\Omega (\eta\bar{\eta}_k u)^2 dx$$

with $C(\gamma) \downarrow 0$ if $\gamma \to 0$. But

$$\int_\Omega (\eta\bar{\eta}_k u)^2 dx = \int_{B^2} (\eta\bar{\eta}_k)^2 u^2 dx \le 2\int_{B^2} |\nabla(\eta\bar{\eta}_k)|^2 u^2 dx + 2\int_{B^2} \eta^{2-2}\eta_k |\nabla u|^2 dx .$$

Thus

$$\text{(VII)} \quad K\int_{B^2} |\eta\bar{\eta}_k|^2 |\nabla u|^2 dx \le h(k) + g(k) + \tilde{K}\int_{B^2} |\nabla\eta|^2 |u|^2 dx$$

$$+ 2K(\gamma)\int_{B^2} |\nabla(\eta\bar{\eta}_k)|^2 u^2 dx + 2K(\gamma)\int_{B^2} \eta^2 \bar{\eta}_k^2 |\nabla u|^2 dx$$

(again with $K(\gamma) \downarrow 0$ as $\gamma \downarrow 0$.)

Now choose γ small enough and absorb the last right hand term on the left hand side.

$$\text{(VIII)} \quad K\int_{B^2} |\eta\bar{\eta}_k|^2 |\nabla u|^2 dx \le h(k) + g(k) + \tilde{K}\int_{B^2} |\nabla\eta|^2 u^2 dx + 2K(\gamma)\int_{B^2} |\nabla(\eta\bar{\eta}_k)|^2 u^2 dx .$$

But, $\int_{B^2} |\nabla(\eta\bar{\eta}_k)|^2 u^2 dx = \int_\Omega |\nabla(\eta\bar{\eta}_k)|^2 u^2 dx = A$ (we have already shown that $A \le 2\int_\Omega |\nabla\eta|^2 u^2 dx + h(k)$, with $h(k) \downarrow 0$ as $k \to \infty$. Thus combining terms we obtain

$$\text{(IX)} \quad K\int_{B^2} |\eta\bar{\eta}_k|^2 |\nabla u|^2 dx \le m(k) + \tilde{K}\int_{B^2} |\nabla\eta|^2 u^2 dx$$

with $m(k) \downarrow 0$ if $k \to \infty$.

Now, let $k \to \infty$ and we get:

$$\text{(X)} \quad \int_{B^2} |\eta|^2 |\nabla u|^2 dx \le K\int_{B^2} |\nabla\eta|^2 u^2 dx$$

with K independent of u.

Q.E.D.

Now we prove Theorem 7.1.

Proof: Theorem 7.1 now follows from De-Georgi iteration, pg. 76 [LU] which uses the estimate of Proposition 7.1 as its basic inequality.

Q.E.D.

We now conclude this section with a final corollary.

Corollary 7.1. Under the hypothesis of Theorem 7.1, $D\phi$ is in $L^2(B^2)$.

Proof: This is the same as the proof of Corollary 2.4 of [Sb2].

Q.E.D.

8. A Subelliptic Estimate for (F,ϕ)

In this section we assume that (F,ϕ) is a smooth solution in $B^2-\{0\} \subset R^2$ of (YMH1) and (YMH2) and that F and $D\phi$ belong to $L_1(B^2)$. We define the total field $h(x) = |F| + |D\phi| + |\phi|^2$.

The main result of this section is a preliminary growth estimate on $h(x)$. Because we are in two dimensions the argument based on Lemma 3.4 of [Sb2] fails completely. In fact, Lemma 3.4 is false in our setting. We substitute an argument based on the estimates of Section 6. Our main estimate will then follow by conformal scaling.

Denote by $V_\rho = \{x|\rho/2 \leq |x| \leq 2\rho\}$ the reference ring about the puncture. We require that $\|h\|_{L_1(B_4)} < \gamma < \hat{\gamma}$ for $\hat{\gamma}$ chosen small enough.

Theorem 8.1. There is a constant C such that for $|x| = r$, $|x|^2 h(x) \leq C\|h\|_{L_1(V_r)}$. This is true in all smooth gauges.

To prove Theorem 8.1 we consider solutions of the Yang-Mills-Higgs equations in a bundle over the unit reference ring $V_1 = \{y|1/2 \leq |y| \leq 2\}$. We obtain a bound on the L^∞ norm of the total field h which we state as:

Proposition 8.1. Let h be the total field of the smooth pair (F,ϕ) in a bundle over V_1 . If $\|h\|_1 < \gamma_2$, then there is a constant C such that $h(y) \leq C\|C\|_{L^1(V_1)}$ for y belonging to the unit sphere in $V_1(\|y\| = 1)$. Before proving Proposition 8.1 we show that Proposition 8.1 implies Theorem 8.1.

Proof: Map the reference ring V_t onto V_1 by the scale transformation $y = x/r$. The field equations are invariant under this transformation. By assumption and using norm invariance $\|h\|_{L^1(V_1)} = \|h\|_{L^1(V_r)} \leq \gamma < \gamma_2$. Therefore in y coordinates F, ϕ , and h satisfy the hypothesis of Proposition 8.1. Pulling back to V_r and using the fact that $h(y) = r^2 h(x)$, the inequality above becomes our conclusion.

Q.E.D.

We need Lemma 8.1 of [Sb2] which follows in any dimension from the Weitzenblock identity.

Lemma 8.1. The scalar function h is a solution of the subelliptic inequality $\Delta h + (ah+b)h \geq 0$ where $a = 10 + 2|\lambda|$ and $b = |\lambda|m^2$.

Proof: This is the same as Lemma 3.3 of [Sb2].

We now prove a preliminary estimate from which a Morrey condition will follow later. We have

Lemma 8.2. Let h be as above in V_1. Then on any ball $\widetilde{B}_{1/4}$ of radius $1/4$ centered on $|y| = 1$ in V_1 we have: Let $H = h+b+1$. Then $\sup_{\widetilde{B}_{1/4}} H \leq K\int_{\widetilde{B}_{1/2}} H\,dx$ where $\widetilde{B}_{1/2}$ is the doubling of $\widetilde{B}_{1/4}$ in V_1, where b is the constant defined above, and where K is independent of h. (In fact, K depends on a).

Proof: Recall that $h \geq 0$ and h satisfies $\Delta h + ah^2 + bh \geq 0$ in V_1. Now let $H = h+b+1$ and notice that $H \geq 1$. Now, elementary computations imply $\Delta(H^2) + aH^2 \geq 0$ in V_1, since $\Delta(H^2) \geq 2H\Delta H \geq \Delta H$ and $a,b \geq 0$.

Now we apply Theorem 6.3 of section 6 to this equation, with $p = 1/2$ and $R = 1/2$. We get

$$\sup_{\widetilde{B}_{1/4}} (H^2) \leq (1/2)^{-4} K\Big[\int_{\widetilde{B}_{1/2}\subset V_1} H dx\Big]^2 , \quad \sup_{\widetilde{B}_{1/4}} (H) \leq \widetilde{K}\int_{\widetilde{B}_{1/2}\subset V_1} H\,dx .$$

Now since $H = h+b+1$ we are done. Note that $\widetilde{K}$ depends on a and not on m. Moreover $\int_{B\subset R^2} b = |\lambda|\int_{B\subset R^2} m^2\, r dr d\theta$ and since m has conformal weight one this integral is scale invariant. Recall $\int h$ is scale invariant. Noting that in theorem 6.3 that $\widetilde{K}$ has conformal weight two, we see choosing $\widetilde{K} > 1$ that this estimate is scale invariant.

Q.E.D.

Now, as promised, we use Lemma 8.2 to get a Morrey-type condition on h.

Lemma 8.3. Let $\widetilde{B}_\rho$ be any ball of radius ρ centered on $|y| = 1$ with $\rho \leq 1/4$. Then, if h is defined as above, $[\int_{B_\rho} h^{1+\varepsilon} dy]^{1/\varepsilon} \leq K\rho^\beta$, $\varepsilon > 0$ $\beta > 0$.

Proof:

$$\sup_{\widetilde{B}_\rho} h \leq \sup_{\widetilde{B}_{1/4}} h \leq \sup_{\widetilde{B}_{1/4}} H \leq \widetilde{K}\int_{\widetilde{B}_{1/2}} H \leq \widetilde{K}\int_{\widetilde{B}_{1/2}} h+b+1 \leq \widetilde{K}\int_{V_1} h + \widetilde{K}\int_{\widetilde{B}_{1/2}} (b+1) \leq K_1\gamma + K_2 ,$$

where K_2 depends on b. Note, the above integral estimate is scale invariant as in lemma 8.2. Thus, since K_1 and K_2 have conformal weight two the pointwise estimate is scale invariant. Thus $[\int_{\widetilde{B}_\rho} h^{1+\varepsilon} dy]^{\frac{1}{1+\varepsilon}} \leq K_3(\gamma,b)\rho^{\frac{2}{1+\varepsilon}}$. Here $K_3(\gamma,b)$ has conformal weight two. Note, that this estimate is also scale invariant. Q.E.D.

Remark: Note, this is the appropriate Morrey condition for Morrey's theorem in 2-dimensions - - Theorem 6.1 of Section 6.

Now we finally prove Proposition 8.1.

<u>Proof of Proposition 8.1</u>. Since h satisfies a Morrey condition on the above balls $\tilde{B}_\rho$ centered on $|y| = 1$ and $0 < \rho < 1/4$, the function $ah + b$ also satisfies a Morrey condition on these balls. Now recall $\Delta h + (ah+b)h \geq 0$ in V_1. Thus applying Theorem 6.1 of Section 6 we get $\sup_{\tilde{B}_\rho} h \leq C[\int_{\tilde{B}_{2\rho} \subset V_1} h^2 dy]^{1/2}$ where $\tilde{B}_\rho$ and $\tilde{B}_{2\rho}$ are both centered on $|y| = 1$. Note by theorem 6.1 the constant C has conformal weight one so that this estimate is scale invariant.

Now we apply Bomberi's improvement on Morrey's Theorem -- Theorem 6.2 of Section 6 with $p = 1$, to obtain $\sup_{\tilde{B}_\rho} h \leq \tilde{C}\int_{\tilde{B}_{4\rho}} h\, dy$, where $\tilde{B}_{4p} \subset V_1$ and $\tilde{B}_\rho \subset B_{4\rho}$ are concentric and are centered on $|y| = 1$. Thus by a covering argument we have $\sup_{|y|=1} h \leq C\int_{V_1} h$. Note that by theorem 6.2 $\tilde{C}$ and C have conformal weight two so that this estimate is also scale invariant.

Q.E.D.

9. An Elliptic Estimate

In this section we improve our results to obtain a final growth condition on the curvature F and on $D\phi$. We assume that $F \in L_1$, that ϕ is bounded and hence that $D\phi \in L_2$ by Corollary 7.1. Since integration by parts is essential in these arguments, we must work in an L_2 setting. Just as in [Sb2] this forces us to use weighted L_2 norms.

Our first aim in this section is to obtain a growth condition on the Higgs field. This will be used in the next section to estimate the total curvature.

Theorem 9.1. $\int_{B_\rho} |D\phi| dx \leq K\rho$. $0 < \rho < \tau < 2$.

Proof: Since $D\rho \in L^2$ apply Hölders inequality.

Q.E.D.

Remark: Note that the integral on the left hand side is scale invariant so that by scaling we can shrink the ball and decrease the left hand side. The right hand side , which came from the differential equations, picks up a scale factor that quantifies this decrease.

This improvement on conformal scaling estimates is key in the method of [U1]. For similar estimates with the same scaling behavior cf: estimate (4.7) in (Sb2) , the estimate pg. 28 line 16 [Uh1].

Theorem 9.2.

$$\int_{|x|\leq\tau} |x|^2|F(x)|^2 dx \leq C_1\int_{|x|\leq\tau} |x|^2(|D\phi|^2+|\phi|^4)dx + C_2\int_{|x|=\tau} |F|^2 dS .$$

We will prove Theorem 9.2 at the end of this section. But we first have:

Corollary 9.1. $\int_{|y|\leq\rho} |y|^2|F(y)|^2 dy \leq C\rho^1$, $0 < \rho < \tau$.

Proof: Starting with the inequality of the conclusion of Theorem 9.2 we change scale to obtain $\int_{|y|\leq\rho} |y|^2|F(y)|^2 dy \leq \rho^2\int|D\phi|^2 dy + \rho^2\int_{|y|\leq\rho} |\phi|^4 dy + C\rho^3\int_{|y|=\rho} |F|^2 dS_y$. Recall that we have just proved that $\int_{|y|\leq\rho} |D\phi|^2 dy \leq K_4$. Using this fact, and since ϕ is bounded, we get $\int_{|y|\leq\rho} |y|^2|F(y)|^2 dy \leq K_5\rho^2 + C_2\rho^3(\rho^{-2})\int_{|y|=\rho} |y|^2|F|^2 dS_y$. Letting $f(\rho) = \int_{|y|\leq\rho} |y|^2|F(y)|^2 dy$ we obtain $f(\rho) \leq K_S\rho^2 + C_2\rho f'(\rho)$. We integrate this inequality from $\rho = \tau$ to $\rho = r$ to obtain $f(\rho) \leq C \cdot \rho^1$.

Q.E.D.

We now prove Theorem 9.2. We do this by working in the broken Hodge gauges of Lemma 5.1.

Proof of Theorem 9.2. Assume we are working in broken Hodge gauges on $\eta|_{U_i}$ for each i . Note that if $\tau > 1$ is taken sufficiently close to 1 and if γ_4 is chosen small enough, then we have $(\tau^2/4.5-\gamma_4)^{1/2}(2+(\gamma_4/2)) < 1$.

This arithmetical fact is the reason for all the estimates of eigenvalues in Section 4 and Lemma 5.1. Thus we now assume $\gamma < \gamma_3 < \gamma_4$. We integrate by parts to obtain $\int_{U_i} |x|^2|F^i(x)|^2dx =$

$$\int_{U^i}(A^i,D^*(|x|^2F^i)) - \int_{U^i}(1/2[A^i,A^i],\ |x|^2F^i) + \int_{S^{i-1}} - \int_{S^i} A_S^i \Lambda |x|^2(*F)_S = I_1 + I_2 +$$

boundary terms. Now, using the field equations, we get

$$I_1 \le \int_{U^i}(A^i,|x|^2[D\phi,\phi]) + 2\int_{U^i}|x|^{2-1}|A^i||F^i|dx$$

$$\le (\tau^2/(4.5-\gamma_4))^{1/2}(2+(\gamma_4/2)\int_{U^i}|x|^2|F^i|^2dx + \int_{U^i}|x|^2(|D\phi|^2 + |\phi|^4)dx\ .$$

Since the coefficient of the right hand side is less than one, we obtain by subtraction $(1-\varepsilon')\int_{U^i}|x|^2|F^i(x)|^2dx \le \int_{U^i}|x|^2(|D\phi|^2+|\phi|^4)dx + \int_{S^{i-1}} - \int_{S^i} A_S^i\Lambda(*F)_S|x|^2$.

Adding the integrals over each U^i , $i = 1,2,3,\ldots$, we see that intermediate boundary integrals cancel out.

Recall that $|F|_{|x|=r} \le K/r^2\int_{B_2-\{0\}}|F|dx \le \hat{\gamma}/r^2$. Thus

$$\left|\int_{S^i}|x|^2A_S^i\Lambda(*F^i)_S\right| \le K\int_{S^i}|x|^2|A_S^i||F_S^i||x|d\theta \le Kr\ \sup_{S^i}|A_S^i|\int_{S^i}(\gamma/r^2)r^2d\theta \le Kr\ \sup_{S^i}|A_A^i| \le Kr\ .$$

$\sup_{S^i}(1/r)|A_\theta^i| \le K\ \sup_{S^i}|A_\theta^i|$ where $A_S^i = A_\theta^i d\theta$.

Note that in broken Hodge gauges we worked hard to get $\lim_{i\to\infty}\sup_{S^i}|A_\theta^i| = 0$. Thus

$\lim_{i\to\infty}\left|\int_{S^i}|x|^2A_S^i\Lambda(*F)_S^i\right| = 0$.

Now consider the outer boundary term: $\int_{S^0}A_S^1\Lambda(*F_S^1)$. We have :

$\left|\int_{S^0}A_S^1\Lambda(*F_S)\right| \le K(\int_{S^0}|A_S^1|^2dx)^{1/2}(\int_{S^0}|F_S^1|^2dx)^{1/2}$. We would like to use an estimate of the form

$$(*) \qquad \int_{S^0}|A_S^1|^2dx \le K\int_{S^0}|F_S^1|^2dx\ .$$

However, because the Laplacian on all co-closed 1-forms on $S^0 = \{x\,|\,|x| = \tau\}$ is zero, we do not have this inequality. We will use instead the inequality:

$$(**) \qquad \int_{S^0}|A_S^1\Lambda(*F^1)_S| \le \varepsilon\int_{U^i}|F^1|^2dV + \frac{K}{\varepsilon}\int_{S^0}|F^1|^2dx \quad \text{for } \varepsilon > 0\ .$$

Now we prove (**).

Lemma 9.1. (**) is valid.

Proof: Let $T_1 = \{x | \frac{1}{\tau} \le x \le \tau\}$. We have $|\int_{S^o} A_S^1 \Lambda (*F^1)_S| \le \int_{S^o} |A_S^1 \Lambda (*F^1)_S| \le \varepsilon \int_{S^o} |A_S|^2 dS + \frac{K}{\varepsilon} \int_{S^o} |F^1|^2 dS \ \forall\ \varepsilon > 0$. But $\int_{S^o} |A_S^1|^2 dS \le \int_{S^o} |A_S^1|^2 dS + \int_{\partial B_{\frac{1}{\tau}}} |A_S^1|^2 dS = \int_{\partial T_1} |A_S^1|^2 dS$. Since A is in $H_2^1(T_1)$ and since $\delta A^1 = 0$, we have from the trace inequality for Sobolev functions that $\int_{S^o} |A_S^1|^2 dS \le C_1 \int_{T_1} |A^1|^2 + C \int_{T_1} |dA^1|^2$ so that,

$$\int_{S^o} |A_S^1|^2 dS \le \int_{S^o} |A_S^1|^2 dS + \int_{\partial B_{\frac{1}{\tau}}} |A_S^1|^2 dS$$

$$= \int_{\partial T_1} |A_S^1|^2 dS \le C_1 \int_{T_1} |A^1|^2 dV + C_2 \int_{T_1} |dA^1|^2 \le C_1 \int_{U^1} |A^1|^2 dV + C_2 \int_{U^1} |dA^1|^2 dV .$$

Now, in our broken Hodge gauge we have also that $\int_{U^1} |A^1|^2 \le K \int_{U^1} |F^1|^2 dV$ and $\sup_{U^1} |A^1| \le K\tau^1$ which implies that

$$|dA^1|^2 \le |dA^1 + 1/2[A^1,A^1] - 1/2[A^1,A^1]|^2 \le |F^1|^2 + C|A^1|^4 \le |F^1|^2 + K\tau^2 |A^1|^2$$

and thus $\int_{S^o} |A_S^1|^2 dS \le \tilde{K} \int_{U^1} |F^1|^2 dV + \tilde{\tilde{K}} \int_{U^1} |A^1|^2 dV$ so that $\int_{S^o} |A_S^1|^2 dS \le K \int_{U^1} |F^1|^2$ and thus $\int_{S^o} |A_S^1 \Lambda (*F^1)_S| \le \varepsilon K \int_{U^1} |F^1|^2 dV + \frac{K}{\varepsilon} \int_{S^o} |F_S^1|^2 dS$.

Q.E.D.

Now we have, using Lemma 9.1, that

$$|\int_{S^o} A_S^1 \Lambda (*F^1)_S| \le \int_{S^o} |A_S^1 \Lambda (*F^1)| \le \varepsilon K(\tau) \int_{U^i} |x|^2 |F^1|^2 dV + \frac{K}{\varepsilon} \int_{S^o} |F^1|^2 dS .$$

Now we return to our task of estimating $\int_{U^i} |x|^2 |F^i(x)|^2 dx$. We have $(1-\varepsilon') \int_{U^i} |x|^2 |F^i(x)|^2 dx \le \int_{U^i} |x|^2 (|D\phi|^2 + |\phi|^4) dx + \int_{S^o} A_S^1 \Lambda (*F^1)_S$. We now apply our estimate from Lemma 9.1 of the boundary terms to the above inequality. We obtain

$$(1-\varepsilon') \int_{B_\tau} |x|^2 |F(x)|^2 dx \le \int_{B_\tau} |x|^2 (|D\phi|^2 + |\phi|^2) dx + \varepsilon \int_{U^1} |F^1| dv + \frac{K}{\varepsilon} \int_{B_\tau} |F_S^1| dS .$$

Since $U^1 \subset B_\tau$ we choose ε small and subtract the second right hand term from the left hand side. We obtain

$(1-\varepsilon'') \int_{B_\tau} |x|^2 |F(x)|^2 dx \leq \int_{B_\tau} |x|^2 (|D\phi|^2 + |\phi|^2) dx + K\int_{S^o} |F|^2 dS$. Since $\sup_{S^o} |F| \leq K\int_{V_1} |F| \leq K\gamma$, the term $K\int_{S^o} |F|^2 dS$ is bounded. Thus we have proved Theorem 9.2.

Q.E.D.

10. Statement and Proof of the Removable Singularities Theorem

In this section we finally prove our main theorem on removable singularities, Theorem 10.1.

First, we combine all our previous estimates to obtain

Proposition 10.1. For some $\delta > 0$, $|x|^{2-\delta}(|F(x)| + |D\phi(x)|) \leq C$, $0 \leq |x| \leq \tau/2$.

Proof: Let $V_\rho = \{|\frac{\rho}{2} \leq |x| < 2\rho\}$, $0 < \rho < \tau/2$. We have already shown that if h is the total curvature, then $\sup |h(x)|\big|_{|x|=r} \leq \frac{K}{|r|^2}\int_{V_r}|h|dx$, $0 \leq r \leq 1$. Thus $|r|^2(|F(x) + |D\phi|)\big|_{|x|=r} \leq C||h||_{L_1(V_r)} \leq C_1||F||_{L_1(V_r)} + C_2||D\phi||_{L_2(V_r)} + C_3||\phi^2||_{L_1(V_r)}$. Since ϕ is bounded, $||\phi^2||_{L_1(V_r)} \leq Cr^2$. Since $\int_{B_r}|D\phi|^2 \leq C$, $0 < r \leq \tau$, it follows from Holder's inequality that $\int_{B_r}|D\phi| \leq Kr$, $\forall_r$, $0 < r \leq \tau$. Thus $\int_{V_r}|D\phi| \leq Kr$, $0 < r \leq \tau/2$.

We also have; $(0 < r \leq \tau/2)$, $\int_{V_r}|x|^2|F(x)|^2dx \leq K\rho$. Thus by Holder's inequality we obtain $\int_{V_r}|F(x)|dx \leq [\int_{V_r}|x|^{-2}dx]^{1/2}[\int_{V_r}|x|^2|F(x)|^2dx]^{1/2} \leq [\int_0^\pi \int_{\rho/2}^\rho \frac{1}{|x|^2}|x|d|x|d\theta]^{1/2}[\int_{V_r}|x|^2|F(x)|^2dx]^{1/2} \leq K[\int_{V_r}|x|^2|F(x)|^2dx]^{1/2} \leq K\rho^{1/2}$.
Thus $\int_{V_r}|F(x)|dx \leq K\rho^{1/2}$, $0 < r \leq \tau/2$.

We have shown that $|r|^2(|F(x)| + |D\phi|)\big|_{|x|=r} \leq Kr^{1/2} + Kr + Kr^2$, $0 < r \leq \tau/2$.

Thus if $0 < \delta < 1/4$ we have $|r|^{2-\delta}(|F(x)|+|D\phi|)\big|_{|x|=r} \leq K\rho^{1/2-\delta} + Kr^{1-\delta} + Kr^{2-\delta} \leq \tilde{K}$.

Q.E.D.

Corollary 10.1. The curvature F is in L^p for $1 \leq p < \frac{2}{2-\delta}$.

Proof: Elementary arithmetic.

Corollary 10.2. (F,ϕ) is a weak solution of the field equations in the full ball B_4^2 .

Proof: Elementary using Corollary 10.1. Compare Corollary 5.3 of [Sb2].

Proof of Theorem 10.1: This follows from Corollary 10.1 by exact repetition of the last two pages of [Sb1].

Q.E.D.

We are finished!

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