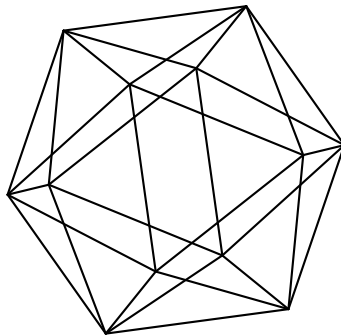# Max-Planck-Institut für Mathematik Bonn

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by

Henrik Bachmann
Tatsushi Tanaka

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Henrik Bachmann
Tatsushi Tanaka

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Graduate School of Mathematics
Nagoya University
Nagoya
Japan

Department of Mathematics
Kyoto Sangyo University
Kyoto
Japan

ROOTED TREE MAPS AND THE KAWASHIMA RELATIONS FOR MULTIPLE ZETA VALUES

HENRIK BACHMANN AND TATSUSHI TANAKA

ABSTRACT. Recently, inspired by the Connes-Kreimer Hopf algebra of rooted trees, the second named author introduced rooted tree maps as a family of linear maps on the noncommutative polynomial algebra in two letters. These give a class of relations among multiple zeta values, which are known to be a subclass of the so-called linear part of the Kawashima relations. In this paper we show the opposite implication, that is the linear part of the Kawashima relations is implied by the relations coming from rooted tree maps.

1. INTRODUCTION

Rooted tree maps were introduced in [T] and they assign to a rooted tree a linear map on the space $\mathfrak{H} = \mathbb{Q}\langle x, y \rangle$ of noncommutative polynomials in x and y. One application of these maps is that any admissible word evaluated at a rooted tree map gives a $\mathbb{Q}$-linear relation between multiple zeta values. To prove this result, one shows that these type of relations follow from a special case, the linear part, of the so called Kawashima relations.

The purpose of this note is to show, that the rooted tree maps relations are actually equivalent to the linear part of the Kawashima relations.

For $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ the multiple zeta values are defined by

$$(1.1) \qquad \zeta(k_1, \ldots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \ldots m_r^{k_r}} .$$

To state the Kawashima relations we need to introduce some notations. Denote by $\mathfrak{H}^1 = \mathbb{Q} + \mathfrak{H}y$ the subspace of words in $\mathfrak{H}$, which end in y. The space $\mathfrak{H}^1$ is spanned by the monomials $z_{k_1} \ldots z_{k_r}$ with $k_1, \ldots, k_r \geq 1$, where $z_k = x^{k-1}y$. On $\mathfrak{H}^1$ one can define recursively the harmonic product $*$ by $1 * w = w * 1 = w$ and

$$z_{k_1} v * z_{k_2} w = z_{k_1}(v * z_{k_2} w) + z_{k_2}(z_{k_1} v * w) + z_{k_1+k_2}(v * w)$$

for $k_1, k_2 \geq 1$ and $v, w \in \mathfrak{H}^1$. Let $\mathfrak{H}^0 = \mathbb{Q} + x\mathfrak{H}y \subset \mathfrak{H}$ be the subspace of admissible words and define the $\mathbb{Q}$-linear map $Z : \mathfrak{H}^0 \to \mathbb{R}$ on a monomial $w = z_{k_1} \ldots z_{k_r}$ by $Z(w) = \zeta(k_1, \ldots, k_r)$. Equipped with the harmonic product $*$ both $\mathfrak{H}^1$ and $\mathfrak{H}^0$ are commutative $\mathbb{Q}$-algebras and it is a well-known fact, that Z is an algebra homomorphism from $\mathfrak{H}^0$ to the algebra of multiple zeta values. Define the automorphism $\varphi \in \mathrm{Aut}(\mathfrak{H})$ (with respect to the concatenation) on the generators by $\varphi(x) = x + y$ and $\varphi(y) = -y$ and define for words $v, w \in \mathfrak{H}^1$ the operator $z_p v \circledast z_q w = z_{p+q}(v * w)$. With this the Kawashima relations can be stated as follows:

Theorem 1.1. *([K, Corollary 5.4]) For all $v, w \in \mathfrak{H}y$ and $m \geq 1$ we have*

$$(1.2) \qquad \sum_{\substack{i+j=m \\ i,j \geq 1}} Z(\varphi(v) \circledast y^i) Z(\varphi(w) \circledast y^j) = Z(\varphi(v * w) \circledast y^m) \,.$$

It is expected that Theorem 1.1 give all $\mathbb{Q}$-linear relations between multiple zeta values after evaluating the product on the left-hand side by the shuffle product formula, whose definition we will omit here. Moreover numerical experiment suggest, that the two cases $m = 1, 2$ are enough to obtain all linear relations.

Rooted tree maps assign to a rooted tree a linear map on the space $\mathfrak{H}$. Evaluated at any admissible word these give also relations between multiple zeta values. Denoting by $\widetilde{\mathcal{H}}_{d \geq 1} \subset \mathrm{End}(\mathfrak{H})$ the space of all rooted tree maps of non-zero degree (see Section 2 for precise definitions), we have the following result proven in [T].

Theorem 1.2. *([T, Theorem 1.3]) For any rooted tree map $f \in \widetilde{\mathcal{H}}_{d \geq 1}$ we have*

$$f(\mathfrak{H}^0) \subset \ker Z.$$

The main result of this work is the following.

Theorem 1.3. *The rooted tree maps relations are equivalent to the linear part of the Kawashima relations, i.e. Theorem 1.2 implies the $m = 1$ case of Theorem 1.1.*

Let R_k be the number of linearly independent rooted tree maps relations (Theorem 1.2) among multiple zeta values in weight k (the number $k_1 + \cdots + k_r$ in (1.1)) and denote by C_k the conjectured number of all linearly independent relations of multiple zeta values in weight k. For small k these are given by

k	2	3	4	5	6	7	8	9	10	11	12	13
R_k	0	1	2	5	10	23	46	98	200	413	838	1713
C_k	0	1	3	6	14	29	60	123	249	503	1012	2032

2. Rooted tree maps

A rooted tree is as a finite graph which is connected, has no cycles, and has a distinguished vertex called the root. We draw rooted trees with the root on top and we just consider rooted trees with no plane structure, which means that we for example do not distinguish between ⋏• and •⋏. A product (given by the disjoint union) of rooted trees will be called a (rooted) forest and by $\mathcal{H}$ we denote the $\mathbb{Q}$-algebra of forests generated by all trees. The unit of $\mathcal{H}$, given by the empty forest, will be denoted by $\mathbb{I}$. Since we just consider trees without plane structure the algebra $\mathcal{H}$ is commutative. Due to the work of Connes and Kreimer ([CK]) the space $\mathcal{H}$ has the structure of a Hopf algebra. To define the coproduct on $\mathcal{H}$ we first define the linear map B_+ on $\mathcal{H}$, which connects all roots of the trees in a forest to a new root. For example we have $B_+ (\Lambda\, \bullet) = \Lambda$. Clearly for every tree $t \in \mathcal{H}$ there exists a unique forest $f_t \in \mathcal{H}$ with $t = B_+(f_t)$, which is just given by removing the root of t. The coproduct on $\mathcal{H}$ can then be defined recursively for a tree $t \in \mathcal{H}$ by

$$\Delta(t) = t \otimes \mathbb{I} + (\mathrm{id} \otimes B_+) \circ \Delta(f_t)$$

and for a forest $f = gh$ with $g, h \in \mathcal{H}$ multiplicatively by $\Delta(f) = \Delta(g)\Delta(h)$ and $\Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I}$. For example we have

$$\Delta(\textstyle\bigwedge) = \textstyle\bigwedge \otimes\, \mathbb{I} + \bullet\bullet \otimes \bullet + 2\, \bullet \otimes \textstyle{\substack{\bullet\\\bullet}} + \mathbb{I} \otimes \textstyle\bigwedge.$$

In [T] the second named author uses the coproduct Δ to assign to a forest $f \in \mathcal{H}$ a $\mathbb{Q}$-linear map on the space $\mathfrak{H} = \mathbb{Q}\langle x, y \rangle$, called a rooted tree map, by the following:

Definition 2.1. *For any non-empty forest $f \in \mathcal{H}$, we define a $\mathbb{Q}$-linear map on $\mathfrak{H}$, also denoted by f, recursively: For a word $w \in \mathfrak{H}$ and a letter $u \in \{x, y\}$ we set*

(2.1) $$f(wu) := M(\Delta(f)(w \otimes u)),$$

where $M(w_1 \otimes w_2) = w_1 w_2$ denotes the multiplication on $\mathfrak{H}$. This reduces the calculation to $f(u)$ for a letter $u \in \{x, y\}$, which is defined by the following:

  i) If $f = \bullet$, then $f(x) := xy$ and $f(y) := -xy$.
  ii) For a tree $t = B_+(f)$ we set $t(u) := R_y R_{x+2y} R_y^{-1} f(u)$, where R_v is the linear map given by $R_v(w) = wv$ $(v, w \in \mathfrak{H})$.

iii) If $f = gh$ is a forest with $g, h \neq \mathbb{I}$, then $f(u) := g(h(u))$.

The rooted tree map of the empty forest $\mathbb{I}$ is given by the identity.

By $\widetilde{\mathcal{H}} \subset \mathrm{End}(\mathfrak{H})$ we denote the space spanned by all rooted tree maps and by $\widetilde{\mathcal{H}}_d$ the space spanned by rooted tree maps of degree d (number of vertices). Note that for any $f \in \widetilde{\mathcal{H}}_d$ with $d \geq 1$ we have $f(1) = 0$, which follows by induction on the degree of f from (2.1) with $w = 1$. In [BT] it was shown that the derivation ∂_n on $\mathfrak{H}$, defined for $n \geq 1$ by $\partial_n(x) = x(x+y)^{n-1}y$ and $\partial_n(y) = -x(x+y)^{n-1}y$ can be written in terms of rooted tree maps. In particular the rooted tree map $\bullet = \partial_1$ is a derivation, which will be used for various calculations in the remaining parts of this work.

3. The space $\mathcal{F}$

Denote by $\mathcal{H}_d \subset \mathcal{H}$ the subspace spanned by all rooted forests of degree d. In this section we will consider a subspace of $\mathcal{H}_d$ defined recursively by $\mathcal{F}_1 = \mathbb{Q} \cdot \bullet$ and for $d \geq 2$ by

$$\mathcal{F}_d = B_+ (\mathcal{F}_{d-1}) + \bullet \, \mathcal{F}_{d-1} \,.$$

For example for $d = 2, 3, 4$ the $\mathcal{F}_d$ are given by

$$\mathcal{F}_2 = \mathbb{Q}\, \vcenter{\hbox{\raisebox{0pt}{$\mathbin{\vcenter{\hbox{$\bullet$}}}$}}} + \mathbb{Q} \,\bullet\bullet, \qquad \mathcal{F}_3 = \mathbb{Q}\, \vcenter{\hbox{$\bullet$}} + \mathbb{Q}\, \wedge + \mathbb{Q}\, \vcenter{\hbox{$\bullet$}}\bullet + \mathbb{Q}\, \bullet\bullet\bullet,$$

$$\mathcal{F}_4 = \mathbb{Q}\, \vcenter{\hbox{$\bullet$}} + \mathbb{Q}\, \wedge + \mathbb{Q}\, \vcenter{\hbox{$\bullet$}} + \mathbb{Q}\, \wedge + \mathbb{Q}\, \vcenter{\hbox{$\bullet$}}\bullet + \mathbb{Q}\, \wedge\bullet + \mathbb{Q}\, \vcenter{\hbox{$\bullet$}}\bullet\bullet + \mathbb{Q}\, \bullet\bullet\bullet\bullet.$$

Notice that the space $\mathcal{F} = \bigoplus_{d \geq 1} \mathcal{F}_d \subset \mathcal{H}$ is a subspace of $\mathcal{H}$, but not a subalgebra since for example $\vcenter{\hbox{$\bullet$}} \in \mathcal{F}_2$ but $\vcenter{\hbox{$\bullet$}}\vcenter{\hbox{$\bullet$}} \notin \mathcal{F}_4$. By definition we have $\dim_{\mathbb{Q}} \mathcal{F}_d = 2^{d-1}$. Denote by $\widetilde{\mathcal{F}}_d \subset \mathrm{End}(\mathfrak{H})$ the space spanned by all rooted tree maps corresponding to the rooted trees in $\mathcal{F}_d$ and set $\widetilde{\mathcal{F}} = \bigoplus_{d \geq 1} \widetilde{\mathcal{F}}_d \subset \widetilde{\mathcal{H}}$. The main goal of this section is to prove the following.

Theorem 3.1. *We have an isomorphism of $\mathbb{Q}$-vector spaces*

$$\Theta : \widetilde{\mathcal{F}} \longrightarrow x\mathfrak{H}y$$
$$f \longmapsto f(x) \,.$$

Before we can prove Theorem 3.1 at the end of this section, we need to introduce some notation and prove some Lemma. Define the $2^{n-1} \times 2^n$-matrix A_n by $A_1 = (1\ 1)$ and for $n \geq 2$ recursively by

$$A_n := \left(\begin{array}{c|cc} A_{n-1} & E_{2^{n-2}} & 0 \\ \hline 0 & E_{2^{n-2}} & A_{n-1} \end{array} \right) =: \left(A_n^{(1)} \mid A_n^{(2)} \right) ,$$

where E_n denotes the $n \times n$ identity matrix. The matrices $A_n^{(j)}$ are both $2^{n-1} \times 2^{n-1}$ square matrices. Denote for $d \geq 2$ by $w_d \in (\mathfrak{H}_d^0)^{2^{d-2}}$ the vector of all 2^{d-2} monomials in $\mathfrak{H}_0$ of degree d, ordered in lexicographical order $(x < y)$ from the top to the bottom. For example we have for $d = 2, 3, 4$

$$w_2 = \begin{pmatrix} xy \end{pmatrix}, \qquad w_3 = \begin{pmatrix} x^2 y \\ xy^2 \end{pmatrix}, \qquad w_4 = \begin{pmatrix} x^3 y \\ x^2 y^2 \\ xyxy \\ xy^3 \end{pmatrix}.$$

For a rooted forest $f \in \mathcal{H}$ we denote by $f(w_d) \in (\mathfrak{H}^0)^{2^{d-2}}$ the component-wise evaluation of the rooted tree map f on the entries of w_d.

In the following we denote for $v, w \in \mathfrak{H}$ by R_v and L_v the linear maps given by $R_v(w) = wv$ and $L_v(w) = vw$.

Lemma 3.2. *For all $d \geq 2$ we have $\bullet(w_d) \equiv A_{d-1} w_{d+1} \mod 2$.*

Proof. We prove this statement by induction on d. For $d = 2$ this follows immediately, since $\bullet(w_2) = (\bullet(xy)) = (xyy - xxy)$ and $A_1 w_3 = (xxy + xyy)$. By definition of w_d one can check that

$$(3.1) \qquad w_d = \begin{pmatrix} L_x(w_{d-1}) \\ L_{xy} L_x^{-1}(w_{d-1}) \end{pmatrix} = \begin{pmatrix} L_{x^2}(xw'_{d-2}) \\ L_{x^2}(yw'_{d-2}) \\ L_{xy}(xw'_{d-2}) \\ L_{xy}(yw'_{d-2}) \end{pmatrix},$$

where we write $w'_{d-2} := L_x^{-1}(w_{d-2})$. Using that $\bullet$ is a derivation with $\bullet(x) = xy$ and $\bullet(y) = -xy$ yields

$$(3.2) \qquad \begin{aligned} \bullet(L_x(w_{d-1})) &= \bullet(x^2 w'_{d-1}) = \bullet(x)xw'_{d-1} + x\bullet(x)w'_{d-1} + x^2\bullet(w'_{d-1}) \\ &= L_{xy}(w_{d-1}) + L_{x^2}(L_y + \bullet)(w'_{d-1}) \end{aligned}$$

and

$$(3.3) \qquad \begin{aligned} \bullet(L_{xy} L_x^{-1}(w_{d-1})) &= \bullet(xyw'_{d-1}) = \bullet(x)yw'_{d-1} + x\bullet(y)w'_{d-1} + xy\bullet(w'_{d-1}) \\ &\equiv L_{xy}(L_y + \bullet)(w'_{d-1}) + L_{x^2}(yw'_{d-1}) \mod 2. \end{aligned}$$

Now by assumption we have

$$A_{d-2} w_d \equiv \bullet(w_{d-1}) = \bullet(xw'_{d-1}) = L_{xy}(w'_{d-1}) + L_x \bullet(w'_{d-1}) \mod 2.$$

This together with (3.2) and (3.3) yields

$$\bullet(L_x(w_{d-1})) \equiv A_{d-2}(L_x(w_d)) + L_{xy}(w_{d-1}) \mod 2,$$

$$\bullet(L_{xy}L_x^{-1}(w_{d-1})) \equiv A_{d-2}(L_{xy}L_x^{-1}(w_d)) + L_{x^2}L_y(w'_{d-1}) \quad \mathrm{mod}\ 2\,.$$

This concludes the statement in the Lemma, since by definition of A_{d-1} and (3.1) the right-hand side gives the entries of $A_{d-1}w_{d+1}$. $\qquad\square$

Define for $d \geq 1$ the $2^d \times 2^{d-1}$-matrices

$$\mathcal{E}_d^{(1)} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ 0 & 1 & & \vdots \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \qquad \mathcal{E}_d^{(2)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & & \vdots \\ 0 & 0 & & \vdots \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & \cdots & 1 \end{pmatrix}.$$

Lemma 3.3. *We have for all $d \geq 2$*

$$(3.4) \qquad\qquad w_{d+1} = \mathcal{E}_{d-1}^{(2)} R_y w_d + \mathcal{E}_{d-1}^{(1)} R_{xy} R_y^{-1} w_d\,.$$

Proof. This follows from the fact that if one orders the monomials $m_*^{(d)}$ in $x\mathfrak{H}y$ of degree d lexicographically by $m_1^{(d)} < m_2^{(d)} < \cdots < m_{2^{d-2}}^{(d)}$, then $m_{2j-1}^{(d+1)} = R_{xy}R_{y^{-1}}m_j^{(d)}$ and $m_{2j}^{(d+1)} = R_y m_j^{(d)}$ for $j = 1, \ldots, 2^{d-2}$. $\qquad\square$

Define on $\mathfrak{H}$ the anti-automorphism τ by $\tau(x) = y$ and $\tau(y) = x$. Clearly we have $\tau(x\mathfrak{H}y) = x\mathfrak{H}y$ which we can use to define for $d \geq 1$ the unique permutation matrix $T_d \in M_{2^{d-2}}(\mathbb{Z})$ satisfying

$$T_d w_d = \tau(w_d)\,.$$

Lemma 3.4. *We have for all $d \geq 2$*

$$(3.5) \qquad\qquad \det\left(A_{d-1}\mathcal{E}_{d-1}^{(2)}\right) \equiv 1 \quad \mathrm{mod}\ 2\,.$$

Proof. By Lemma 3.2 and 3.3 we have

$$(3.6) \qquad \begin{aligned} \bullet(w_d) &\equiv A_{d-1}w_{d+1} \quad \mathrm{mod}\ 2 \\ &= A_{d-1}\mathcal{E}_{d-1}^{(2)} R_y(w_d) + A_{d-1}\mathcal{E}_{d-1}^{(1)} R_{xy} R_y^{-1}(w_d)\,. \end{aligned}$$

Since for all $w \in \mathfrak{H}$ we have $\bullet(\tau(wy)) = xy\tau(w) + x\bullet(\tau(w))$ and $\tau(\bullet(wy)) = x\tau(\bullet(w)) - xy(\tau(w))$ the operators $\bullet$ and τ commute modulo 2. Together with the definition of the matrix T_d, Lemma 3.2, $\tau L_x = R_y \tau$ and $\tau L_{xy} = R_{xy}\tau$ we get

$$\begin{aligned} T_d \bullet(w_d) = \bullet(\tau(w_d)) &\equiv \tau(\bullet(w_d)) \equiv \tau(A_{d-1}w_{d+1}) \\ &= \tau\left(A_{d-1}^{(1)} L_x(w_d) + A_{d-1}^{(2)} L_{xy}L_x^{-1}(w_d)\right) \\ &= A_{d-1}^{(1)} T_d R_y(w_d) + A_{d-1}^{(2)} T_d R_{xy} R_y^{-1}(w_d) \quad \mathrm{mod}\ 2\,. \end{aligned}$$

Since the entries in the first summand of this expression end all in y^2 and the entries of the second expression end all in xy, we get by (3.6) the identities $A_{d-1}\mathcal{E}_{d-1}^{(2)} \equiv T_d A_{d-1}^{(1)} T_d$

mod 2 and $A_{d-1}\mathcal{E}_{d-1}^{(1)} \equiv T_d A_{d-1}^{(2)} T_d \mod 2$. Since $A_{d-1}^{(1)}$ is an upper triangle matrix with 1 on the diagonal we have $\det(A_{d-1}^{(1)}) = 1$ which together with $\det(T_d)^2 = 1$ gives (3.5). $\qquad\square$

Now define for $d \geq 1$ the matrices $B_d \in M_{2^{d-1}}(\mathbb{Z})$ by $B_1 = (1)$ and for $d \geq 2$ by

$$B_d = \left(\begin{array}{c} B_{d-1}(\mathcal{E}_{d-1}^{(1)})^t \\ \hline B_{d-1}A_{d-1} \end{array} \right) .$$

Notice that $B_{d-1}(\mathcal{E}_{d-1}^{(1)})^t$ is just B_{d-1} with zero columns added at the even places, i.e. if $B_{d-1} = (b_1, \ldots, b_{2^{d-2}})$ then $B_{d-1}(\mathcal{E}_{d-1}^{(2)})^t = (b_1, 0, b_2, 0, \ldots, b_{2^{d-2}}, 0)$.

Lemma 3.5. *For all $d \geq 1$ we have $\det(B_d) \equiv 1 \mod 2$.*

Proof. With the matrix $\mathcal{E}_d := (\mathcal{E}_d^{(1)} \mid \mathcal{E}_d^{(2)}) \in M_{2^{d-1}}(\mathbb{Z})$ one checks that

$$B_d \mathcal{E}_d = \left(\begin{array}{c|c} B_{d-1} & 0 \\ \hline B_{d-1}A_{d-1}\mathcal{E}_{d-1}^{(1)} & B_{d-1}A_{d-1}\mathcal{E}_{d-1}^{(2)} \end{array} \right) .$$

The result now follows inductively together with Lemma 3.4. $\qquad\square$

Before we can finally prove Theorem 3.1 we will now give the connection of the map $\Theta : \mathcal{F} \to x\mathfrak{H}y$ given by $f \mapsto f(x)$ and the matrix B_d. For this we define the vector $f_d \in \mathcal{F}_d^{2^{d-1}}$ by $f_1 = (\bullet)$ and

$$f_d = \left(\begin{array}{c} B_+(f_{d-1}) \\ \hline \bullet\, f_{d-1} \end{array} \right) .$$

So in particular f_d contains all generators of $\mathcal{F}_d$. By $f_d(x) \in (x\mathfrak{H}y)^{2^{d-1}}$ we will denote the vectors obtained by evaluating all entries at x.

Lemma 3.6. *For all $d \geq 1$ we have $f_d(x) \equiv B_d w_{d+1} \mod 2$.*

Proof. We will again use induction on d. For $d = 1$ the statement can be checked easily. Now for $d \geq 2$ we get by assumption, the definition of rooted tree maps and Lemma 3.2

$$f_d(x) = \left(\begin{array}{c} B_+(f_{d-1})(x) \\ \hline \bullet\, f_{d-1}(x) \end{array} \right) \equiv \left(\begin{array}{c} R_y R_x R_y^{-1} f_{d-1}(x) \\ \hline \bullet\, f_{d-1}(x) \end{array} \right)$$

$$\equiv \left(\begin{array}{c} B_{d-1} R_y R_x R_y^{-1} w_d \\ \hline B_{d-1}\bullet(w_d) \end{array} \right) \equiv \left(\begin{array}{c} B_{d-1} R_y R_x R_y^{-1}(w_d) \\ \hline B_{d-1}A_{d-1}w_d \end{array} \right) \mod 2 .$$

Together with $B_{d-1}R_y R_x R_y^{-1}(w_d) = B_{d-1}(\mathcal{E}_{d-1}^{(1)})^t w_{d+1}$, which follows from the lexicographical ordering of the w_d, we obtain the desired result. $\qquad\square$

Proof of Theorem 3.1. That Θ is an isomorphism follows now directly from Lemma 3.2 and 3.6, which imply that every monomial in $x\mathfrak{H}y$ of degree $d+1$ can be written as some $f(x)$ with $f \in \mathcal{F}_d$. Therefore the map Θ is surjective on the degree graded parts $\mathcal{F}_d$ and $(x\mathfrak{H}y)_{d+1}$, which both have dimension 2^{d-1}. $\qquad\square$

4. KAWASHIMA RELATIONS

In this section we want to give the proof of Theorem 1.3, i.e. that the rooted tree map relations imply the linear part of the Kawashima relation. Recall that with the automorphism $\varphi \in \operatorname{Aut}(\mathfrak{H})$ defined by $\varphi(x) = x + y$ and $\varphi(y) = -y$ the linear part of the Kawashima relations, i.e. the $m = 1$ case in Theorem 1.1, states that

$$(4.1) \qquad \varphi(v * w) \circledast y = L_x \varphi(v * w) \in \ker Z$$

for any $w, v \in \mathfrak{H}y$. In [T] it was shown, that the rooted tree maps relations follow from the linear part of the Kawashima relations (see proof of [T, Corollary 4.7]), i.e. we have in particular

$$\{f(w) \mid f \in \widetilde{\mathcal{F}},\ w \in \mathfrak{H}^0\} \subset L_x \varphi(\mathfrak{H}y * \mathfrak{H}y)\,.$$

We will now show that also the opposite inclusion holds.

Theorem 4.1. *We have*

$$L_x \varphi(\mathfrak{H}y * \mathfrak{H}y) \subset \{f(w) \mid f \in \widetilde{\mathcal{F}},\ w \in \mathfrak{H}^0\}\,.$$

Proof. Due to Lemma 4.2 in [T] we know that

$$(1 - \tau)(x\mathfrak{H}y) \subset L_x \varphi(\mathfrak{H}y * \mathfrak{H}y)\,.$$

So we have $(1 - \tau)(x\mathfrak{H}y) = \tau(1 - \tau)(x\mathfrak{H}y) \subset \tau L_x \varphi(\mathfrak{H}y * \mathfrak{H}y)$ and hence

$$L_x \varphi(\mathfrak{H}y * \mathfrak{H}y) = ((1 - \tau) + \tau)L_x \varphi(\mathfrak{H}y * \mathfrak{H}y) \subset (1 - \tau)(x\mathfrak{H}y) + \tau L_x \varphi(\mathfrak{H}y * \mathfrak{H}y)$$
$$\subset \tau L_x \varphi(\mathfrak{H}y * \mathfrak{H}y)\,.$$

The opposite inclusion $\tau L_x \varphi(\mathfrak{H}y * \mathfrak{H}y) \subset L_x \varphi(\mathfrak{H}y * \mathfrak{H}y)$ was shown in the proof of Corollary 4.7 in [T] and therefore we have

$$(4.2) \qquad L_x \varphi(\mathfrak{H}y * \mathfrak{H}y) = \tau L_x \varphi(\mathfrak{H}y * \mathfrak{H}y)\,.$$

Following [T] we define $\chi_x := \tau L_x \varphi$. Due to Corollary 4.5. in [T] there exist for any $f \in \widetilde{\mathcal{F}}$ a $w \in \mathfrak{H}y$ such that

$$(4.3) \qquad f\chi_x = \chi_x H_w\,,$$

where $H_w(v) = w * v$ for $w, v \in \mathfrak{H}y$. The element w is uniquely determined by $w = \chi_x^{-1} f(y)$. Since we have for any rooted tree map $f \in \widetilde{\mathcal{F}}$ that $f(y) = -f(x)$, we get $-\chi_x(w) = f(x)$. In Theorem 3.1 we proved that the map $\Theta : \mathcal{F} \to x\mathfrak{H}y$ with $f \mapsto f(x)$ is an isomorphism and hence for an arbitrary $w \in \mathfrak{H}y$ the rooted tree map $f = -\Theta^{-1}(\chi_x(w)) \in \mathcal{F}$ satisfies (4.3).

Now let w, v be arbitrary elements in $\mathfrak{H}y$. To proof the statement of the theorem we want to show that we can find a rooted tree map $f \in \widetilde{\mathcal{F}}$ and a $u \in \mathfrak{H}^0$, such that $L_x\varphi(w * v) = f(u)$. Because of (4.2) it suffices to show that we can find such an f and u with $\tau L_x\varphi(w * v) = \chi_x H_w(v) = f(u) = f\chi_x(\chi_x^{-1}u)$. By the above discussion we can choose $u = \chi_x(v)$ and $f = -\Theta^{-1}(\chi_x(w))$. $\qquad\square$

Remark 4.2. *The space $\mathcal{F}$ is a proper subset of $\mathcal{H}$, since $\vcenter{\hbox{⁚}} \in \mathcal{H}$ but $\vcenter{\hbox{⁚}} \notin \mathcal{F}$. Due to Theorem 4.1 all rooted tree maps relations are already obtained by considering just the rooted trees in $\mathcal{F}$, that is the space $\widetilde{\mathcal{F}}$. Due to numerical experiments we actually expect that the space $\widetilde{\mathcal{F}}$ is the whole space $\widetilde{\mathcal{H}}$, i.e. the recursively defined elements at the beginning of Section 3 might give a basis of the space of rooted tree maps and in particular $\dim_{\mathbb{Q}} \widetilde{\mathcal{H}}_d = 2^{d-1}$. Since the number of rooted tree maps in degree d is larger than 2^{d-1} for $d \geq 4$, this would give linear relations between rooted tree maps. For example we expect that we have for all $w \in \mathfrak{H}$*

$$\vcenter{\hbox{⁚}}(w) = 2\,\mathord{\scriptstyle\triangle\!\cdot}(w) + \mathord{\scriptstyle\wedge\!\cdot}\,\!\cdot(w) - \mathord{\scriptstyle\triangle\!\triangle}(w) - \mathord{\scriptstyle\curlywedge}(w)\,.$$

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GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, NAGOYA, JAPAN.
E-mail address: henrik.bachmann@math.nagoya-u.ac.jp

DEPARTMENT OF MATHEMATICS, KYOTO SANGYO UNIVERSITY, KYOTO, JAPAN.
E-mail address: t.tanaka@cc.kyoto-su.ac.jp