DEVELOPMENTS IN THE THEORY OF JACOBI FORMS ♣

by

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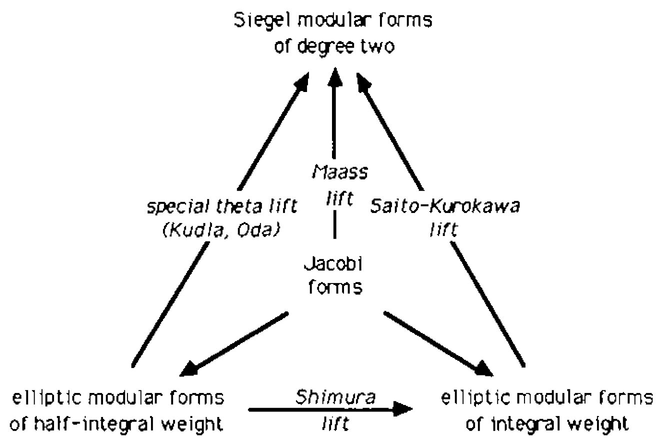
1. Introduction

We would like to begin this survey with some very general remarks about Jacobi forms. These remarks will be as vague as general. But, perhaps, they will give the reader a rough idea of what this survey is dealing with. Later on we shall be as concrete as possible and we shall try to explain everything from scratch.

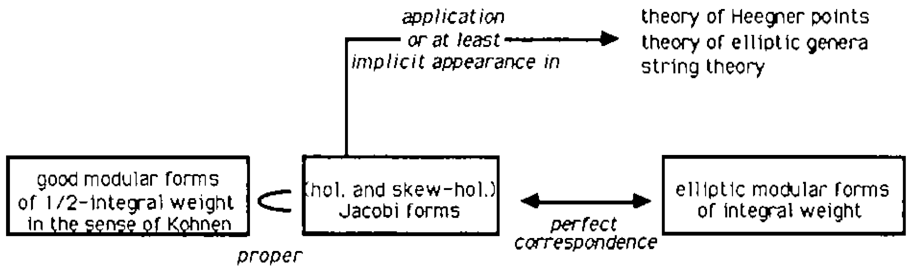
If one would have to explain Jacobi forms by a diagram, then one could possibly give the following one:



The diagram has to be understood in the following sense: there are various well-known connections between those different types of modular forms occuring in the diagram. Key-words for these connections are written at the corresponding connecting arrow. The point is that the diagram is commutative, and that the best way from one type of modular form to the other is the way passing through the center of the triangle. To make the term 'best' a little bit more precise: The connections to the center are quite natural (as natural as Jacobi forms are), and the classical correspondences along the edges can be most easily understood and technically handled when interpreted as the sum of two suitable Jacobi form – modular form correspondences.

However, first of all these Jacobi – modular correspondences had to be discovered, and actually their discovery was historically the starting point for a proper theory of Jacobi forms. There have already been appearances of Jacobi forms in the literature before (although these functions were not called Jacobi forms at that time). Shimura gave a new foundation of the theory of complex multiplication of Abelian functions using these functions (cf. [Sh]). Kuznetsov constructed functions which are almost Jacobi forms from ordinary elliptic modular forms (cf.[Kuz]). Berndt studied the field of Jacobi functions (cf.[B]), and Feingold-Frenkel used them in a paper on Kac-Moody algebras ([F-F]). ♡ Finally, in the early eighties Eichler and Zagier, stimulated by the proof of the Saito-Kurokawa conjecture, developed a systematic theory of Jacobi forms along the lines of Hecke's theory of modular forms. This resulted in the monography [E-Z].

Since then the theory of Jacobi forms has grown quite a bit. There are several beautiful and/or deep results about Jacobi forms. Moreover, Jacobi forms gave and are still giving interesting contributions to other parts of mathematics. And so the above diagram is no longer sufficient to reflect all aspects of Jacobi forms, nor it is necessary to justify their existence. If one takes into account the more recent developments, then a more up-to-date picture could look like this:



Here the modular forms of half-integral weight (to be more precise, Kohnen's '+'-forms) are considered as proper subset of the Jacobi forms. The meaning of 'proper' is roughly as follows: Kohnen's refinements of the Shimura lift are valid only in the case of modular forms of odd, squarefree level. If one wishes to generalize the work of Kohnen to general levels the technical difficulties seem to become overwhelming, it is not even clear how to generalize naturally the definition of Kohnen's '+'-space to higher levels. These difficulties can be overcome very easily by replacing modular forms of half-integral weight throughout by Jacobi forms. This lies at hand since a certain part of the whole variety of Jacobi forms can be considered in a natural manner as Kohnen '+'-forms. If one accordingly establishes the Shimura correspondence for Jacobi forms, as was done in [S-Z], this whole theory turns out to be smooth without any technical (or natural) restrictions.

♡ If a paper is missing here then this is due to the author's ignorance and not to a low opinion of the paper in question.

Unfortunately, at that point, there was a tiny gap left: there have been modular forms of integral weight which did not correspond to Jacobi forms although they should. Meanwhile this gap can be filled by introducing a certain type of non-holomorphic Jacobi forms: this is indicated by the term 'skew-holomorphic' in the above diagram. The resulting, completed correspondence between Jacobi forms and modular forms is now that easy to formulate that it seems to leave no wishes open. Moreover, this correspondence is not merely a tool to study Jacobi forms (although it is), but it is rather a deep (and nevertheless handy) tool for studying elliptic modular forms of integral weight and their arithmetical significance. All this is combined in the adjective 'perfect' in the above diagram.

In the following we shall try to explain the highlights of the theory of Jacobi forms so far obtained. We shall not speak about its applications listed in the diagram. For this the interested reader is referred to [G-K-Z] (for Heegner points), [Z1] (for elliptic genera), [C] (for a sporadic appearance of skew-holomorphic Jacobi forms in string theory). Also, we shall not speak about its applications to the theory of p-adic interpolation and p-adic L-functions (cf. [K1]). And finally, we shall also not speak about results concerning Jacobi forms of higher degree (or genus); for this the reader is referred to [Mu],[Zi]. Instead we have inserted a section to recall the basic features of the Jacobi forms since, to our feeling, Jacobi forms are not really common property yet. Finally, we would like to stress that the list of references at the end is not at all complete, it only reflects what is touched or mentioned in this article.

2. What are Jacobi forms?

Let k be an integer, and let $M_k(Sp_2(\mathbf{Z}))$ denote the space of Siegel modular forms of degree 2 and weight k on the full Siegel modular group. By definition this is the space of holomorphic functions $F(\tau, z, \tau')$ in three complex variables τ, z, τ' with τ, τ' from the Poincaré upper half plane $\mathfrak{H} = \{\tau = u + iv \in \mathbf{C} \,|\, v > 0\}$ and z from $\mathbf{C}$ such that $\Im(\tau)\Im(\tau') - \Im(z)^2 > 0$, which are periodic in each variable with period 1, which satisfy $F(\frac{-1}{\tau}, \frac{z}{\tau}, \tau' - \frac{z^2}{\tau}) = \tau^k F(\tau, z, \tau')$, and the Fourier expansions of which have the form

$$F(\tau, z, \tau') = \sum_{\substack{n,r,m \in \mathbf{Z}, n,m \geq 0 \\ r^2 - 4mn \leq 0}} A(n, r, m) e^{2\pi i(n\tau + rz + m\tau')},$$

where $A(n, r, m) = A(n', r', m')$ if the quadratic form $[n, r, m]$ $(= nX^2 + rXY + mY^2)$ is equivalent modulo $SL_2(\mathbf{Z})$ to the quadratic form $[n', r', m']$.

In particular, such a function F is periodic with respect to τ', and thus we may consider its Fourier expansion solely with respect to this variable:

$$F(\tau, z, \tau') = \sum_{m \geq 0} \phi_m(\tau, z) e^{2\pi i m \tau'}.$$

This is the so called *Fourier-Jacobi development* of F. Of course, $\phi(\tau, z)$ is nothing else but $\sum A(n,r,m)e^{2\pi i(n\tau + rz)}$. It is clear that the automorphic behaviour of F implies also some automorphic behaviour of the $\phi_m(\tau, z)$. For $m = 0$ it can easily be checked that ϕ_m is independent of z, and, considered as function of τ, it is nothing else than an elliptic modular form of weight k. For positive m the function $\phi = \phi_m$ satisfies the following three conditions:

(i) $\phi(\tau, z)$ is a holomorphic function in $\tau \in \mathfrak{H}$ and $z \in \mathbb{C}$,

(ii) $\phi(\tau, z)$ is periodic in each variable with period 1 and it satisfies the functional equation

$$\phi(\frac{-1}{\tau}, \frac{z}{\tau}) e^{-2\pi i m \frac{z^2}{\tau}} = \tau^k \phi(\tau, z),$$

(iii) the Fourier expansion of ϕ is of the form

$$\phi(\tau, z) = \sum_{\substack{\Delta, r \in \mathbb{Z}, \Delta \leq 0 \\ r^2 \equiv \Delta \bmod 4m}} C(\Delta, r) e^{2\pi i(\frac{r^2 - \Delta}{4m}\tau + rz)}$$

where the Fourier coefficients $C(\Delta, r)$ depend on r only modulo $2m$.

(To identify the latter Fourier development of $\phi = \phi_m$ with the one given above, set $\Delta = r^2 - 4mn$ and $C(\Delta, r) = A(\frac{r^2 - \Delta}{4m}, r, m)$.) The function ϕ is a prototype of what is called a *Jacobi form*. More precisely, any function $\phi(\tau, z)$ which satisfies these three conditions is called a *Jacobi form* of *weight* k and *index* m. The space of all such functions is denoted by $J_{k,m}$.

Of course, in these considerations one does not have to stick to Siegel modular forms on the full modular group. Dropping this restriction and mimicing the above procedure, one is led to the *Jacobi group* $\mathcal{J}(\mathbb{R})$ and to the general notion of Jacobi forms as automorphic forms on this group. The Jacobi group $\mathcal{J}(\mathbb{R})$ is a certain central extension by S^1, the group of complex numbers of modulus one, of the natural semidirect product of $SL_2(\mathbb{R})$ with the group of row vectors $\mathbb{R}^2$:

$$\mathcal{J}(\mathbb{R}) = SL_2(\mathbb{R}) \propto \mathbb{R}^2 \cdot S^1.$$

Identifying $SL_2(\mathbb{R})$, $\mathbb{R}^2$ and S^1 with their canonical images in $\mathcal{J}(\mathbb{R})$ so that any $\eta \in \mathcal{J}(\mathbb{R})$ can uniquely be written as $\eta = A[\lambda, \mu]s$ ($A \in SL_2(\mathbb{R}), (\lambda, \mu) \in \mathbb{R}^2, s \in S^1$), the multiplication law in $\mathcal{J}(\mathbf{R})$ is given by $\eta\eta' = AA'[(\lambda,\mu)A' + (\lambda',\mu')]ss'e^{2\pi i\kappa}$. Here κ denotes the determinant of the two by two matrix with $(\lambda,\mu)A'$ as first and (λ',μ') as second row. The Jacobi group acts on $\mathfrak{H}\times\mathbf{C}$ by $\eta\cdot(\tau,z) = (\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d})$, and on functions ϕ on $\mathfrak{H}\times\mathbf{C}$ by

$$(\phi|_{k,m}\eta)(\tau,z) = \phi(\eta\cdot(\tau,z))(c\tau+d)^{-k}e^{2\pi i m(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z+\lambda\mu)}s^m.$$

Here, as above, $\eta = A[\lambda,\mu]s$ with $A = \left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)$, and k, m is a given pair of integers. For a given subgroup Γ of finite index in $SL_2(\mathbf{Z})$ set $\Gamma^J := \Gamma \propto \mathbf{Z}^2 (\subset \mathcal{J}(\mathbf{R}))$. Then the space $J_{k,m}(\Gamma)$ of *Jacobi forms* of *weight* k and *index* m on Γ is defined to be the space of all holomorphic functions $\phi(\tau,z)$ on $\mathfrak{H}\times\mathbf{C}$, satisfying $\phi|_{k,m}\eta = \phi$ for all $\eta\in\Gamma$ and having for any $A\in SL_2(\mathbf{Z})$ a Fourier expansion $\phi|_{k,m}A = \sum c_A(n,r)e^{2\pi i(n\tau+rz)}$ with $c_A(n,r) = 0$ for $r^2 > 4mn$. (Here the n are in general not integral but rational numbers with bounded denominator depending on A.) For positive m the space $J_{k,m}(\mathrm{SL}_2(\mathbf{Z}))$ coincides with $J_{k,m}$ as defined before, and the spaces $J_{k,m}(\Gamma)$ are the natural generalizations of the $J_{k,m}$.

Note that the really interesting Jacobi forms occur for positive index only: a Jacobi form $\phi(\tau,z)$, considered, for fixed τ as a function of z, is nothing else but a holomorphic theta function on $\mathbf{C}/\mathbf{Z}\tau+\mathbf{Z}$ with $2m$ zeroes; thus, there are no Jacobi forms different from zero for negative m, and a Jacobi form of index $m=0$ does not depend on z and may be considered as a function in τ transforming like an elliptic modular form.

One could go even further and define Jacobi forms of half-integral weight. In this wider class of Jacobi forms the simplest of all are those which gave their name to the whole theory, namely the Jacobi theta functions

$$\vartheta_{m,\rho}(\tau,z) = \sum_{\substack{r\in\mathbf{Z},\\ r\equiv\rho \bmod 2m}} e^{2\pi i(\frac{r^2}{4m}\tau+rz)}$$

(m a positive integer, ρ an integer modulo $2m$), and which occur already in the work of C.G.J. Jacobi. In the up-to-date language these Jacobi theta functions are then elements of $J_{\frac{1}{2},m}(\Gamma(4m))$. By the way, these theta functions, or at least combinations of them, occur in the Fourier Jacobi development of the simplest of all Siegel modular forms of degree two, the form $\vartheta^{(2)}(\tau,z,\tau') = \sum_{r,s\in\mathbf{Z}} e^{2\pi i(r^2\tau+2rsz+s^2\tau')}$.

The Jacobi theta functions are not only the simplest of all examples of Jacobi forms but in some sense they are also the most basic Jacobi forms. To explain this in more detail let us consider for simplicity the case of a Jacobi form ϕ of index m and, say for simplicity, on $SL_2(\mathbf{Z})$. The special property of such a form that its Fourier coefficients $C(\Delta, r)$ depend on r only modulo $2m$ can also be stated by saying that any such ϕ can be written as $\phi(\tau, z) = \sum_{\rho=1}^{2m} h_\rho(\tau)\vartheta_{m,\rho}(\tau, z)$ with suitable functions $h\rho(\tau)$ (of course, one has explicitly $h_\rho(\tau) = \sum_\Delta C(\Delta, \rho)e^{2\pi i \frac{|\Delta|}{4m}\tau}$.) It is known that the Jacobi theta functions $\vartheta_{m,\rho}$ are invariant under $SL_2(\mathbf{Z})$ with respect to '$|_{\frac{1}{2},m}$' (to be precise, not each $\vartheta_{m,\rho}$ is fixed by $SL_2(\mathbf{Z})$, but only the space spanned by all the $\vartheta_{m,\rho}$ is). Comparing this transformation law with the one satisfied by ϕ, and observing that the h_ρ are uniquely determined by ϕ, it can be seen that that they transform so to speak dual to the $\vartheta_{m,\rho}$. In fact, the $h_\rho(\tau)$ belong to $M_{k-\frac{1}{2}}(\Gamma(4m))$, the space of modular forms of weight $k - \frac{1}{2}$ on the main congruence subgroup $\Gamma(4m)$. Even more, the correspondence $\phi \mapsto \sum_{\rho=1}^{2m} h_\rho \otimes \vartheta_{m,\rho}$ defines an isomorphism

$$J_{k,m} \xrightarrow{\approx} \Big(M_{k-\frac{1}{2}}(\Gamma(4m)) \otimes span\{\vartheta_{m,\rho} | \rho = 1, \cdots, 2m\}\Big)^{SL_2(\mathbf{Z})}.$$

Here $M_{k-\frac{1}{2}}(\Gamma(4m))$ is considered as $SL_2(\mathbf{Z})$-right module via the usual (projective) action of $SL_2(\mathbf{Z})$ on modular forms of weight $k - \frac{1}{2}$; as mentioned before $SL_2(\mathbf{Z})$ acts (projectively) on the space spanned by the $\vartheta_{m,\rho}$ via '$|_{\frac{1}{2},m}$', thus it acts (in fact, really, not only projectively) on the tensor product of these two spaces, and the right hand side of the above isomorphism denotes the subspace in this tensor product consisting of those elements fixed by $SL_2(\mathbf{Z})$.

This isomorphism is the main key to understand the connection between Jacobi forms and elliptic modular forms of half-integral weight. A closer investigation of this connection was given in [S1]. However, we do not want to go into this here but it may give the reader who is well aquainted with elliptic modular forms an intuitive understanding of the basic features of the theory of Jacobi forms.

These basic features are:

- *For each pair of integers $J_{k,m}$ is finite-dimensional (actually one has $\dim J_{k,m} = \frac{km}{6} + \mathcal{O}(1)$ for $k \to \infty$).*
- *there exists a Hecke theory for Jacobi forms, i.e. for each positive natural number l, relative prime to m, there exists a natural Hecke operator $T(l)$ on $J_{k,m}$, and the space $J_{k,m}$ has a basis consisting of simultaneous eigenforms with respect to all $T(l)$.*
- *there exists a natural notion of Jacobi Eisenstein series and Jacobi cusp forms.*
- *there exists a Petersson scalar product $\langle\phi, \psi\rangle$ on $J_{k,m}^{cusp}$, the space of Jacobi cusp forms (to be precise, $\langle\phi, \psi\rangle = \int_{\mathfrak{F}} \phi(\tau, z)\overline{\psi(\tau, z)}e^{-\frac{4\pi m y^2}{v}} v^k \frac{du\,dv\,dx\,dy}{v^3}$ where*

$\mathfrak{F}$ is a fundamental domain for $\mathfrak{H} \times \mathbf{C}$ modulo $SL_2(\mathbf{Z})^J$, and u, x and v, y denote the the real and imaginary parts of τ and z respectively).

Similar basic facts hold for the more general spaces $J_{k,m}(\Gamma)$ too, of course.

Before ending this short review of basic ingredients of the theory of Jacobi forms it should be noted that Jacobi forms are very concrete objects. Here is an illustration of this fact: If we multiply a Jacobi form by an elliptic modular form we get back a Jacobi form with the same index but, of course, with a different weight; in other words the space $\bigoplus_{k\in\mathbf{Z}} J_{k,m}$ is a module over the ring $\bigoplus_{k\in\mathbf{Z}} M_k(\mathrm{SL}_2(\mathbf{Z})) = \mathbb{C}[E_4, E_6]$ of elliptic modular forms on the full modular group; it is even a free module of rank $2m$. The two generators in the case $m = 1$ are

$$E_k^J(\tau, z) := \sum_{\eta\in SL_2(\mathbf{Z})^J_\infty \backslash SL_2(\mathbf{Z})^J} (1|_{k,1}\eta)(\tau, z)$$

$$= \vartheta_{1,0}(\tau, z) + \gamma_k \sum_{\substack{\Delta, r\in\mathbf{Z}, \Delta<0 \\ r^2\equiv\Delta \bmod 4}} |\Delta|^{k-\frac{3}{2}} \left(\sum_{n\geq 1} \frac{r_\Delta(n)}{n^{k-1}}\right) e^{2\pi i(\frac{r^2-\Delta}{4}\tau + rz)}$$

$$= \eta(\tau)^{-6} \sum_{\substack{r,s\in\mathbf{Z} \\ r\not\equiv s \bmod 2}} \left\{r^2 E_k(\tau) - \frac{1}{\pi i k}\left(\frac{d}{d\tau} E_k(\tau)\right)\right\} (-1)^s e^{2\pi i(\frac{r^2+s^2}{4}\tau + sz)}$$

for k=4 and k=6. (For the first identity cf.[E-Z], for the second one [S1].) The notations are: $SL_2(\mathbf{Z})^J_\infty$ =subgroup of all $\eta \in \mathrm{SL}_2(\mathbf{Z})^J$ such that $1|_{k,1}\eta = 1$ for even k; furthermore,

$$r_\Delta(n) = \sharp\left\{x \bmod 2n | x^2 \equiv \Delta \bmod 4n\right\}, \quad \gamma_k = \frac{(-1)^{\frac{k}{2}}\pi^{k-\frac{1}{2}}}{2^{k-2}\Gamma(k-\frac{1}{2})\zeta(k-1)}$$

and, as usual,

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n\geq 1} \Big(\sum_{d|n} d^{2k-1}\Big) e^{2\pi i n\tau}, \quad \eta(\tau) = e^{\frac{\pi i\tau}{12}} \prod_{n\geq 1} (1 - e^{2\pi i n\tau}) .$$

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3. Jacobi forms and Siegel modular forms

The starting point for the theory of Jacobi forms here as well historically was the fact that any Siegel modular form has a Fourier-Jacobi development. As indicated above, and as will be further shown in the following sections, the theory is very well developed at present. Thus it lies at hand to set up the following program:

- *Study Siegel modular forms (of degree 2) via their Fourier-Jacobi development.*

To the author's knowledge there are only two main results in this direction. The first stems from the very early beginning of the theory of Jacobi forms.

Theorem ([A-Ma-Z2]). *For each non-negative integer m there exists an operator $V_m : J_{k,1} \longrightarrow J_{k,m}$, given explicitely by*

$$\sum_{n,r} c(n,r)\mathrm{e}^{2\pi i(n\tau+rz)} \mapsto \sum_{n,r}\Big(\sum_{a|(n,r,m)} a^{k-1}c(\frac{mn}{a^2},\frac{r}{a})\Big)\mathrm{e}^{2\pi i(n\tau+rz)},$$

such that the map $\phi \mapsto \phi|V$ with

$$(\phi|V)(\tau,z,\tau') := \sum_{m\geq 0}(\phi|V_m)(\tau,z)\mathrm{e}^{2\pi i m\tau'}$$

defines a Hecke-equivariant embedding $J_{k,1} \hookrightarrow M_k(Sp_2(\mathbf{Z}))$.

(In the formula for $\phi|V_0$ the term $\sum_{a|0} a^{k-1}c(0,0)$ has to be interpreted as $\frac{1}{2}\zeta(1-k)$.) The image of the above embedding is the so-called Maass Spezialschar and the discovery of this embedding was one of the major steps in the proof of the Saito-Kurokawa conjecture.

The second theorem along the lines of the above program is the following one.

Theorem ([K-S]). *Let F and G be two cusp forms in $M_k(Sp_2(\mathbf{Z}))$, let*

$$F(\tau,z,\tau') = \sum_{m\geq 1}\phi_m(\tau,z)\mathrm{e}^{2\pi i m\tau'} \quad G(\tau,z,\tau') = \sum_{m\geq 1}\psi_m(\tau,z)\mathrm{e}^{2\pi i m\tau'}$$

be their Fourier Jacobi developments, and denote by $\langle\phi_m,\psi_m\rangle$ the Petersson scalar product of the Jacobi cusp forms ϕ_m and ψ_m. Then the series

$$D_{F,G}(s) := \zeta(2s-2k+4)\sum_{m\geq 1}\frac{\langle\phi_m,\psi_m\rangle}{m^s}$$

converges absolutely for large $\Re(s)$ (actually for $\Re(s) > k+1$) and it has a meromorphic continuation to $\mathbf{C}$. It is entire if $\langle F, G\rangle = 0$ and otherwise has a simple pole of residue residue $\frac{4^k\pi^{k+2}}{(k-1)!}\langle F,G\rangle$ at $s=k$ as its only singularity. Moreover it satisfies the functional equation

$$D^*_{F,G}(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)D_{F,G}(s) = D^*_{F,G}(2k-2-s).$$

(Here $\langle F,G\rangle$ denotes the Petersson scalar product of F and G, i.e. $\langle F,G\rangle$ equals the integral $\int F(\tau,z,\tau')\overline{G(\tau,z,\tau')}(vv'-y^2)^{k-3}dudxdu'dvdydv'$, where u,x,u' and v,y,v' denote the real and imaginary parts of τ,z,τ', respectively, and where the integral has to be taken over a fundamental domain for the Siegel upper half plane of degree two modulo $Sp_2(\mathbf{Z})$.)

The above theorem obviously shows some anology to the theorem about the Rankin convolution of two elliptic modular forms and, indeed, its proof is essentially an adaption of the Rankin-Selberg method to the case of Siegel modular forms of degree two. What is striking in the above theorem is the fact that for even k the Dirichlet series $D_{F,G}(s)$ satisfies exactly the same functional equation as the Andrianov (or Spinor) zeta function $Z_F(s)$ associated to Siegel-Hecke eigenforms F. At present the exact relation between the series $D_{F,G}(s)$ and the Andrianov zeta functions is not known. There is only a partial result, which, in a different formulation, can already be found in the work of Gritsenko [Gr].

Supplement to the last Theorem. *Let k be even. If G is an element of the Maass Spezialschar and F is a Hecke eigenform then $D_{F,G}(s) = \langle \phi_1, \psi_1 \rangle Z_F(s)$.*

Thus the above theorem give rise to some open problems:

- *Does a Hecke eigenform F in $M_k(Sp_2(\mathbf{Z}))$ necessarily have $\phi_1 \neq 0$? (Note that an affirmative answer would give, via the above theorem and its supplement, a new proof for the analytic continuation and functional equation of the Andrianov zeta function. The answer is of course affirmative for those forms belonging to the Maass Spezialschar, but it is also true for the first Hecke eigenform not belonging to the Spezialschar, which has weight 20 and which has been calculated in [Kur].)*
- *If F is a Hecke eigenform in $M_k(Sp_2(\mathbf{Z}))$, but not in the Maass Spezialschar, what is then the relation between $D_{F,F}(s)$ and the Andrianov zeta functions associated to elements in $M_k(Sp_2(\mathbf{Z}))$, if there exists any? (Note that one has $D_{F,F}(s) \neq Z_F(s)$ since, by the above theorem, $D_{F,F}(s)$ does have a pole whereas, by a result of Evdokimov and Oda, $Z_F(s)$ does have none.) Can it be the case that $D_{F,F}(s)$ provides a counterexample to a so far unproved but exspected converse theorem?*

4. Jacobi forms and elliptic modular forms of integral weight

That what one may consider as the main theorem in the theory of Jacobi forms can be summarized to the following theorem.

Theorem ([S-Z]). *For each pair of integers $k, m \in \mathbf{Z}$, $m > 0$ the space $J_{k,m}$ of Jacobi forms of weight k and index m is Hecke-equivariantly isomorphic to a certain natural subspace $\mathfrak{M}^-_{2k-2}(m)$ of the space $M_{2k-2}(\Gamma_0(m))$ of all elliptic modular forms of weight $2k-2$ on $\Gamma_0(m)$.*

A first and rough description of the space of modular forms occuring in the theorem is

$$\mathfrak{M}_k(m) = \{\text{all newforms on } \Gamma_0(m)\} \oplus \{\text{a nice choice of oldforms}\}.$$

More precisely, the space $\mathfrak{M}_k(m)$ is spanned by all $f \in M_k(\Gamma_0(m))$ such that the standard L-series $L(f,s) = \sum_{\ell \geq 1} a(\ell)\ell^{-s}$ of $f(\tau) = \sum a(\ell) e^{2\pi i \ell \tau}$ is of the form

$$L(f,s) = \Big(\prod_{p \mid \frac{m}{m'}} Q_p(s)\Big) L(g,s)$$

for some $m'|m$, some new form g on $\Gamma_0(m')$ and polynomials $Q_p(s)$ in p^{-s} satisfying

$$Q_p(s) = p^{t(\frac{k}{2}-s)} Q_p(k-s)$$

for $p^t \| \frac{m}{m'}$. * The symbols $\mathfrak{M}_k^{\pm}(m)$ denote those subspaces which consist of all $f \in \mathfrak{M}_k(m)$ such that $L^*(f,s) := (2\pi)^{-2s} m^{\frac{s}{2}} \Gamma(s) L(f,s) = \pm L^*(f,k-s)$, respectively.

It is left to the reader to contemplate this definition of the space $\mathfrak{M}_k(m)$. But we should add that it may be viewed as the span of all those Hecke eigenforms f on $\Gamma_0(m)$ such that the eigenvalues of f with respect to the various Atkin-Lehner involutions reflect rather an intrinsic property of f and are not just an accident occuring when changing levels. Secondly, one of the most striking features of this space $\mathfrak{M}_k(m)$ is that it has the simplest dimension formula among all subspaces of $M_k(\Gamma_0(m))$ which are still big enough to reflect all various kinds of species occuring on $\Gamma_0(m)$ in weight k (In fact, one has $\dim \mathfrak{M}_k(m) = d(m(k-1)) + \frac{1}{2}a$ where $d(x) = \frac{x}{12} - \frac{1}{3}(\frac{x}{3}) - \frac{1}{4}(\frac{-4}{x})$ ($(\frac{\cdot}{\cdot})$ =Legendre symbols) and a denotes the greatest integer with $a^2|m$). The same remark applies to the formula for the traces of the Hecke operators acting on $\mathfrak{M}_k(m)$. Thus, there are several hints that the spaces $\mathfrak{M}_k(m)$ seem to be very natural and they may deserve further attention (For more arguments the reader is referred to [S-Z].)

* This definition is not strictly precise. First of all the expression 'new form' is usually not applied to Eisenstein series, so a word of explanation is perhaps indispensable: the space of 'new form Eisenstein series' in $M_k(\Gamma_0(m))$ is by definition 0 if m is not a perfect square or $k = 2, m = 1$; otherwise it is spanned by the series $\sum_{\ell \geq 0} \Big(\sum_{d|\ell} d^{k-1} \overline{\chi}(d) \chi(\frac{\ell}{d})\Big) e^{2\pi i \ell \tau}$ where χ runs through all primitive Dirichlet characters modulo $\sqrt{m}$ (with the convention $\sum_{d|0} d^{k-1}\overline{\chi}(d)\chi(\frac{0}{d}) = 0$ or $= \frac{1}{2}\zeta(1-k)$ for $\chi \neq 1$ and $\chi = 1$, respectively). Secondly, if $k = 2$, then $\mathfrak{M}_k(m)$ contains additionally the series $m_1 E_2(m_1\tau) - m_2 E_2(m_2\tau)$ for all decompositions $m = m_1 m_2$. Here E_2 is the (non-holomorphic) modular form $E_2(\tau) = 1 - \frac{3}{\pi \Im(\tau)} - 24 \sum_{\ell \geq 1} \Big(\sum_{d|\ell} d\Big) e^{2\pi i \ell \tau}$.

As mentioned in section 2, Jacobi forms are very closely connected to elliptic modular forms of half-integral weight. In particular, it is possible to show, using the isomorphism described in section 2, that for prime numbers m and even k the map $\phi = \sum_{\rho=1}^{2m} h_\rho \otimes \vartheta_{m,\rho} \mapsto \sum_{\rho=1}^{2m} h_\rho(4m\tau)$ defines a Hecke-equivariant isomorphism of $J_{k,m}$ with that part of Kohnen's '+'-space $M^+_{k-\frac{1}{2}}(m)$ such that this isomorphism, Kohnen's refinement of the Shimura lift (cf. [K2]), and the lifting of the above theorem all together yield a commutative diagram. In that sense the above theorem may be considered as a generalisation of Kohnen's work on the Shimura lifting.

For $m = 1$ the above theorem, together with the theorem of the preceeding section about the Maass lift, yields the Saito-Kurokawa lift, i.e. it establishes a Hecke-equivariant isomorphism between the Maass Spezialschar in $M_k(Sp_2(\mathbf{Z}))$ and the space $M_{2k-2}(SL_2(\mathbf{Z}))$. (Originally, this correspondence was proved to be true using the corresponding Kohnen '+'-spaces.)

Note that the above theorem presents itself in a completely smooth, natural and untechnical manner, once the notion of a Jacobi form is accepted. Even the still slightly technical term 'Hecke-equivariantly' could be eliminated; we shall come back to this in the last section, where we shall reformulate the above theorem. There we shall also give one or two hints that the above theorem can (and should) be read as a theorem about modular forms rather than a theorem about Jacobi forms.

However, for the moment there is one tiny lack of beauty in the above theorem. This is the '$-$'-sign attached to the symbol $\mathfrak{M}_{2k-2}(m)$. In particular, there are so far no Jacobi forms corresponding to $\mathfrak{M}_{2k-2}(1)$ $(= M_k(\mathrm{SL}_2(\mathbf{Z})))$ for k divisible by 4. If there is any hope to fill this gap, then one obviously needs a new type of Jacobi form.

5. Skew-holomorphic Jacobi forms

The simplest of all examples of Jacobi forms are, as pointed out before, the Jacobi theta series $\vartheta_{m,\rho}(\tau, z) = \sum_{r\equiv\rho \bmod 2m} e^{2\pi i(\frac{r^2}{4m}\tau + rz)}$. They may be considered – at least with respect to their analytic features – as prototypes of the Jacobi forms. In particular, they are holomorphic in τ and z. But moreover they satisfy the heat equation

$$(8\pi i m \frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2})\,\vartheta_{m,\rho}(\tau, z) = 0.$$

The latter is not true for the Jacobi forms so far considered. Thus it is reasonable to ask whether there are further natural examples of automorphic forms on the Jacobi group $\mathcal{J}(\mathbf{R})$ satisfying the heat equation. To find such examples it lies at hand to look at Jacob theta series attached to quadratic forms which, of course, must not necessarily be definite. Non-holomorphic elliptic modular forms attached to quadratic forms have been constructed systematically in [V]. By mimicing the method of this paper it is possible to generalize the results stated loc.cit. from elliptic modular forms to Jacobi forms.

Theorem ([S3]). *Let F be a symmetric, non-singular, integral $n \times n$ matrix with even diagonal. Let $p(X)$ be a function on $\mathbf{R}^n$ such that $p(X)e^{-\pi X^t FX}$ is a Schwartz function and $\left(\frac{-1}{4\pi}\nabla' F^{-1}\nabla + X^t \cdot \nabla\right) p = (k - \frac{n}{2})p$ for some $k \in \mathbf{Z}$. Finally, let $X_0 \in \mathbf{Z}^n$, and set*

$$m = \frac{1}{2} X_0^t F X_0, \quad D = (-1)^{\frac{n}{2}} det(F), \quad \ell = \text{ level of} F.$$

Then the series

$$\vartheta(\tau, z) := \sum_{X \in \mathbf{Z}^n} p(\sqrt{v}[X + \frac{y}{v} X_0])e^{\pi i(X^t FX\tau + 2X^t FX_0 z)} \qquad (v = \operatorname{Im}\tau,\ y = \operatorname{Im} z)$$

satisfies

$$\vartheta\left(\tfrac{a\tau+b}{c\tau+d}, \tfrac{z+\lambda\tau+\mu}{c\tau+d}\right)(c\tau+d)^{-k}|c\tau+d|^{k-\frac{n}{2}} e^{2\pi i m\left(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z\right)} = \left(\frac{D}{d}\right)\vartheta(\tau, z)$$

for all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\ell)$, and all $\lambda, \mu \in \mathbf{Z}$.

(The notations are: '$.^t$'=transposition, $\nabla^t = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ for $X = (x_1, \ldots, x_n)$, thus $X^t \cdot \nabla = \sum_{r=1}^n x_r \frac{\partial}{\partial x_r}$; recall that the level of F is the smallest positive integer ℓ such that ℓF^{-1} is integral with even diagonal entries. Note that n has necessarily to be even (as can be shown using the assumption '$k \in \mathbf{Z}$').)

For positive definite F and $p(X) \equiv 1$ the theorem produces in a known way holomorphic Jacobi forms. But we also find such functions as we are looking for: consider a matrix F with signature $(1, n-1)$, n even, pick a vector $X_0 \in \mathbf{Z}^n$ with $X_0^t F X_0 > 0$, choose a vector $\Xi \in \mathbf{C}^n$ and a non-negative integer d, and set $p(X) = (\Xi^t F X_\perp)^d e^{2\pi X_\perp{}^t F X_\perp}$ (for any n-vector X, we use $X_\perp := X - \frac{X^t F X_0}{X_0^t F X_0} X_0$). If $d \geq 2$ we assume $\Xi_\perp{}' F \Xi_\perp = 0$. Then $p(X)$ satisfies the assumptions of the theorem (with $k = 1 - d - \frac{n}{2}$). Moreover, it is easily verified, differentiating term by term, that $v^{-\frac{d}{2}}\vartheta(\tau, z)$ with the corresponding ϑ constructed from these data satisfies the same heat equation as the Jacobi theta functions do.

We can look at these examples as the prototypes for a new kind of Jacobi form. Thus, after a closer examination of these examples we would be led to the following: for a subgroup Γ of finite index in $SL_2(\mathbf{Z})$, we define $J^*_{k,1}(\Gamma)$, the space of *skew-holomorphic Jacobi forms of weight k and index m on Γ*, to be the space of all smooth functions $\phi(\tau, z)$

on $\mathfrak{H} \times \mathbf{C}$, which are holomorphic in z and satisfy $(8\pi i m\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2})\phi(\tau, z) = 0$, which satisfy $\phi|^*_{k,m}\eta = \phi$ for all $\eta \in \Gamma \propto \mathbf{Z}^2$, and which, for each $A \in SL_2(\mathbf{Z})$, have a Fourier development of the form $\phi(\tau, z) = \sum c_A(n, r)e^{2\pi i(n\tau + \frac{r^2-4mn}{2m}iv + rz)}$ with $c_A(n, r) = 0$ for $r^2 < 4mn$. Here, for any pair of integers k, m, the slash operator '$|^*_{k,m}$' is defined by almost the same formula as '$|_{k,m}$' with the only difference that the factor $(c\tau + d)^k$ has to be replaced by $(c\overline{\tau} + d)^{k-1}|c\tau + d|$.

However, we are mainly interested in Jacobi forms on $SL_2(\mathbf{Z})$, and here, as in the holomorphic case and for positive m, the above definition is easily seen to be equivalent to the following one. The space $J^*_{k,m}$ of *skew-holomorphic Jacobi forms of weight k and index m (on $SL_2(\mathbf{Z})$)* is the space of all functions $\phi(\tau, z)$ satisfying:

(i) $\phi(\tau, z)$ is a smooth function in $\tau \in \mathfrak{H}$, and holomorphic in $z \in \mathbf{C}$,

(ii) $\phi(\tau, z)$ is periodic in each variable with period 1 and it satisfies the functional equation

$$\phi(\frac{-1}{\tau}, \frac{z}{\tau})e^{-2\pi i m\frac{z^2}{\tau}} = \overline{\tau}^{k-1}|\tau|\phi(\tau, z),$$

(iii) the Fourier expansion of ϕ is of the form

$$\phi(\tau, z) = \sum_{\substack{\Delta, r \in \mathbf{Z}, \Delta \geq 0 \\ r^2 \equiv \Delta \bmod 4m}} C(\Delta, r)e^{2\pi i\left(\frac{r^2-\Delta}{4m}u + \frac{r^2+|\Delta|}{4m}iv + rz\right)} \quad (\tau = u + iv)$$

where the Fourier coefficients $C(\Delta, r)$ depend on r only modulo $2m$.

Note that the shape of the Fourier development (iii) implies that $\phi(\tau, z)$ satisfies the heat equation, whereas it would imply that $\phi(\tau, z)$ is holomorphic in τ if we would have to sum over non-positive Δ. Thus, if we replace in (ii) the expression $\overline{\tau}^{k-1}|\tau|$ by τ^k and the condition $\Delta \geq 0$ in (iii) by $\Delta \leq 0$ we reobtain exactly the definition of the holomorphic Jacobi forms. (This is the reason that we superfluously wrote $|\Delta|$ for Δ in (iii).)

With regard to this formal analogy in the definition it will not be unexspected that skew-holomorphic Jacobi forms exhibit essentially the same basic features as the holomorphic ones, e.g. $J^*_{k,m}$ is finite dimensional, one has a natural Hecke theory, the notion of Eisenstein series, cusp forms, Petersson scalar product, connection with elliptic modular forms of half-integral weight etc..

To give a completely explicit example of those theta functions which led to the definition of skew-holomorphic Jacobi forms apply the above theorem to $F = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, i.e. set

$$T(\tau, z) = \sum_{s,t \in \mathbf{Z}} e^{2\pi i(st\tau + \frac{(s-t)^2}{2}iv + (s+t)z)}$$

or, more generally,

$$T_{k;m_1,m_2}(\tau,z) = \sum_{s,t\in\mathbf{Z}} (m_1 s - m_2 t)^{k-1} e^{2\pi i (st\tau + \frac{(m_1 s - m_2 t)^2}{2m} iv + (m_1 s + m_2 t)z)},$$

where $k = 1, 2$ and $m = m_1 m_2$ is any given decomposition of a given positive integer m. From the above theorem (or by a simple, direct application of Poisson summation) it is easily verified that $T_{k;m_1,m_2} \in J^*_{k,m}$. Starting with these relatively simple functions it is easy to construct more examples: multiplication of a skew-holomorphic $\phi(\tau,z)$ by $f(-\overline{\tau})$, where $f(\tau)$ is an elliptic modular form, yields again a skew-holomorphic form with the same index; using this, the space $\bigoplus_{k\in\mathbf{Z}} J^*_{k,1}$ becomes a free $\bigoplus_{k\in\mathbf{Z}} M_k(SL_2(\mathbf{Z}))$-module of rank two; it is possible to show that

$$J^*_{k,1} = M_{k-1}(SL_2(\mathbf{Z})) \cdot T(\tau,z) \oplus M_{k-3}(SL_2(\mathbf{Z}))) \cdot U(\tau,z)$$

where

$$T(\tau,z) = \sum_{\substack{d,r\in\mathbf{Z} \\ r^2 \equiv d^2 \bmod 4}} e^{2\pi i \left(\frac{r^2-d^2}{4}u + \frac{r^2+d^2}{4} iv + rz\right)}$$

as above, and

$$U(\tau,z) := \frac{\partial}{\partial\overline{\tau}} T(\tau,z) + \frac{\pi i}{12} T(\tau,z) E_2(-\overline{\tau}).$$

6. The Main Theorem of the theory of Jacobi forms

If we denote by $C(\Delta,r)$ the Δ, r-th Fourier coefficient of the skew-holomorphic $T_{2;m_1,m_2}$ $\in J^*_{2,m_1m_2}$ introduced in the foregoing paragraph, i.e. if we set

$$C(\Delta,r) = \sum_{\substack{s,t\in\mathbf{Z} \\ (m_1 s - m_2 t)^2 = \Delta \\ m_1 s + m_2 t = r}} (m_1 s - m_2 t),$$

then it is easily checked that

$$m_1 - m_2 - 24 \sum_{\ell\geq} \left(\sum_{d|\ell} C(d^2,d) \right) e^{2\pi i \ell\tau} = m_1 E_2(m_1\tau) - m_2 E_2(m_2\tau).$$

Recall that $m_1 E_2(m_1\tau) - m_2 E_2(m_2\tau)$ defines an element of $\mathfrak{M}_2^+(m_1 m_2)$. These examples are special (and the simplest of all) cases of the following general theorem.

Main Theorem. *For any pair of positive integers k, m, $k \geq 2$ the space $J_{k,m} \oplus J^*_{k,m}$ is Hecke-equivariantly isomorphic to $\mathfrak{M}_{2k-2}(m)$. More precisely, one has, for any fundamental discriminant Δ and any integer r such that $r^2 \equiv \Delta$ mod $4m$, a map*

$$\mathcal{S}_{\Delta,r} : J_{k,m} \oplus J^*_{k,m} \longrightarrow \mathfrak{M}_{2k-2}(m),$$

given explicitely by

$$\sum_{\substack{r,\Delta\in\mathbb{Z} \\ r^2\equiv\Delta \bmod 4m}} C(\Delta,r) e^{2\pi i\left(\frac{r^2-\Delta}{4m}u+\frac{r^2+|\Delta|}{4m}iv+rz\right)} \mapsto \sum_{\ell\geq 0}\left(\sum_{a|\ell} a^{k-2}\left(\frac{\Delta}{a}\right) C\left(\tfrac{\ell^2}{a^2}\Delta, \tfrac{\ell}{a}r\right)\right) e^{2\pi i \ell\tau}.$$

It commutes with all Hecke operators and maps $J_{k,m}$ and $J^*_{k,m}$ to $\mathfrak{M}^-_{2k-2}(m)$ and $\mathfrak{M}^+_{2k-2}(m)$ respectively. Some linear combination of the maps $\mathcal{S}_{\Delta,r}$ defines an isomorphism.

(The expression $\sum_{a|\ell} a^{k-2}\left(\frac{\Delta}{a}\right) C_\phi\left(\frac{\ell^2}{a^2}\Delta, \frac{\ell}{a}r\right)$ for $\ell = 0$ has to be interpreted as the value of $\frac{1}{2}C_\phi(0,r)\sum_{a\geq 1}\left(\frac{\Delta}{a}\right)a^{-s}$ at $s = 2-k$.)

Note that the description of Hecke operators for Jacobi forms is implicit in the theorem since it explains what a Jacobi Hecke eigenform has to look like: a Jacobi form ϕ is a Hecke eigenform if and only if its image under all $\mathcal{S}_{\Delta,r}$ is a Hecke eigenform, or, what is equivalent by some formal manipulations, if and only if the Dirichlet series

$$\sum_{\substack{\ell\geq 1 \\ (\ell,m)=1}} C_\phi\left(\ell^2\Delta, \ell r\right)\ell^{-s}$$

has an Euler product for all r and all fundamental Δ such that $r^2 \equiv \Delta$ mod $4m$. Thus, in order to understand or to apply the theorem, it is not necessary to study Hecke operators for Jacobi forms.

Half of the theorem, namely the part concerning the holomorphic Jacobi forms, was proved in [S-Z]. To be honest, the other part is not yet completely proved. (Its proof depends on a comparison of the traces of Hecke operators acting on Jacobi forms and modular forms. This can be done without doubt completely analoguous to the case of the holomorphic Jacobi forms but it is not completely written up yet.)

The theorem is, at the first glance, clearly a theorem about the arithmetical structure of the spaces of Jacobi forms. This is, of course, not of any interest for those who are not at all interested in Jacobi forms. But it is also (or even, rather) a theorem about elliptic modular forms: first of all, it links, as explained in the introduction and by theorems cited or indicated above, elliptic modular forms to other types of modular forms. Secondly, and this is perhaps the most important point, it provides a new tool for the study of elliptic modular forms in $M_k(\Gamma_0(N))$. If we consider, say, a new-Hecke eigenform on $\Gamma_0(m)$, then, by the Main Theorem, there is a unique Jacobi form (up to multiplication by constants) of index m corresponding to it via the above theorem. This Jacobi form does not only carry explicit information about the Fourier coefficients of the modular form (via the explicit description of the above $\mathcal{S}_{\Delta,r}$), but, as can be shown, also the values of the L-series of the given modular form at the integer points in the critical strip, and more generally the periods of that modular form, are explicitly given by the Fourier coefficients of this Jacobi form. And this Jacobi form links both informations. The result about the special values is obtained by computing the adjoint maps of the $\mathcal{S}_{\Delta,r}$ with respect to the Petersson scalar products (and after restriction to cusp forms, of course). For details and more information in this direction (at least in the case of holomorphic Jacobi forms) the interested reader is referred to [G-K-Z].

To conclude this overview, we give some examples of Jacobi forms which are more subtle than those so far considered. Fix a matrix $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in SL_2(\mathbf{Z})$ and set

$$\Theta(\tau,z;\eta) = v^{\frac{1}{2}} \sum_{\substack{r,a,b,c\in\mathbf{Z}\\ b\equiv r \bmod 2m}} \left(a'\eta - \frac{c'}{\eta}\right) \mathrm{e}^{-\frac{\pi v}{m}\left(b^2-4mac+\left(a'\eta+\frac{c'}{\eta}\right)^2\right)} \mathrm{e}^{2\pi i\left(\frac{r^2-b^2+4mac}{4m}\tau+rz\right)}.$$

Here $\eta \in \mathbf{R}$, and, as always, $\tau \in \mathfrak{H}$, $z \in \mathbf{C}$, $v = \Im(\tau)$. Moreover, for each triple a, b, c we use

$$a' = ma\alpha^2 + b\alpha\gamma + c\gamma^2,\ c' = ma\beta^2 + b\beta\delta + c\delta^2.$$

By the theorem of section 5 this transforms like a skew-holomorphic Jacobi form of index m and weight 2 in τ, z, although it is none. But if we set

$$\phi_A(\tau,z) = m^{-\frac{1}{2}} \int_0^\infty \Theta(\tau,z;\eta)\,\frac{d\eta}{\eta}$$

then this still retains the transformation laws satisfied by $\Theta(\tau,z;\eta)$ and by a simple computation

$$\phi_A(\tau,z) = \sum_{\substack{\Delta,r\in\mathbf{Z},\Delta>0\\ r^2\equiv\Delta \bmod 4m}} c_A(\Delta,r)\mathrm{e}^{2\pi i\left(\frac{r^2-\Delta}{4m}u+\frac{r^2+|\Delta|}{4m}iv+rz\right)},$$

where

$$c_A(\Delta,r) = \sharp\{(a,b,c)\in\mathcal{L}(\Delta,r) | a'>0,\ c'<0\} - \sharp\{(a,b,c)\in\mathcal{L}(\Delta,r) | a'<0,\ c'>0\}$$

$$- \sum_{\substack{(a,b,c)\in\mathcal{L}(\Delta,r)\\ 0<a'<m,c'=0}} B_1(\frac{a'}{m}) + \sum_{\substack{(a,b,c)\in\mathcal{L}(\Delta,r)\\ a'=0,0<c'<m}} B_1(\frac{c'}{m}).$$

Here

$$\mathcal{L}(\Delta,r) = \{(a,b,c)\in\mathbf{Z} | b^2-4mac=\Delta,\ b\equiv r \bmod 2m\},$$

$B_1(x) = x - \frac{1}{2}$ is the first Bernoulli polynomial, and a', c' have the same meaning as above. Thus $\phi_A(\tau, z)$ is a skew-holomorphic Jacobi form of weight 2 and index m.

Note that the two sums in the last formula vanish unless Δ is a perfect square. By the Main Theorem (and the facts that ϕ_A is a cusp form and that $\mathcal{S}_{\Delta,r}$ maps cusp forms to cusp forms if $\Delta\neq 1$), we thus obtain for example, for any m such that $\mathfrak{M}_2^+(m)$ contains only one cusp form, say $f(\tau) = \sum_{\ell\geq 1} a_f(\ell)e^{2\pi i\ell\tau}$ and $a_f(1)=1$, and for any r,Δ, $r^2\equiv\Delta \bmod 4m$, $\Delta\neq 1$ fundamental, the identity

$$\sum_{\ell\geq 1}\left\{\sum_{\substack{(a,b,c)\in\mathcal{L}(\Delta\ell^2,r\ell)\\ a'c'<0}} \mathrm{sign}(a')\right\}\ell^{-s}$$

$$= \left\{\sum_{\substack{(a,b,c)\in\mathcal{L}(\Delta,r)\\ a'c'<0}} \mathrm{sign}(a')\right\} \times \left(\sum_{\ell\geq 1}\left(\frac{\Delta}{\ell}\right)\mu(\ell)\ell^{-s}\right)\left(\sum_{\ell\geq 1} a_f(\ell)\ell^{-s}\right).$$

(The reader may verify that the first sum on the right hand side of the last identity for $m=11$ and $A=\left(\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right)$, $\Delta=5, r=7$ equals -1; thus, in general, the given identities really involve the modular forms f.)

The method used here to produce the Jacobi forms ϕ_A can be generalized. It is even possible to construct in this way, for any arbitrarily given index m, a set of Jacobi forms of weight 2, the Fourier coefficients of which can be described as explicitly and effective as those of ϕ_A, and of which it can be shown that they span the whole space of (holomorphic and skew-holomorphic) Jacobi forms of index m. Via the Main Theorem we then obtain as well a simple arithmetical rule to generate explicitly the space of all modular forms of weight 2 on $\Gamma_0(m)$ for any given m. For details the reader is referred to [S2].

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