# FUNCTIONAL EQUATIONS FOR HIGHER LOGARITHMS

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Contents




Abstract. Following earlier work by Abel and others, Kummer gave functional equations for the polylogarithm function $\mathrm{Li}_m(z)$ up to $m=5$ in 1850, but no example for larger $m$ was known until recently. We give the first genuine 2-variable functional equation for the 7–logarithm. We investigate and relate identities for the 3-logarithm given by Goncharov and Wojtkowiak and deduce a certain family of functional equations for the 4-logarithm.


## 1. Introduction

1.1. An essential property of the logarithm is its functional equation

$$\log(xy) = \log(x) + \log(y)\,.$$

An essential property of the dilogarithm, which is defined by the power series

$$\mathrm{Li}_2(z) = \sum_{n>0} \frac{z^n}{n^2} \quad , \qquad |z|<1,$$

and can be analytically continued to $\mathbf{C} - [1,\infty)$, is the 5-term equation $\big($note the involutory symmetry $(x,y) \mapsto (\frac{1-x}{1-y^{-1}}, \frac{1-y}{1-x^{-1}})\big)$

$$\mathrm{Li}_2(xy) - \mathrm{Li}_2(x) - \mathrm{Li}_2(y) - \mathrm{Li}_2(\frac{1-x}{1-y^{-1}}) - \mathrm{Li}_2(\frac{1-y}{1-x^{-1}}) = (\text{elementary})\,.$$

Here the right hand side is a sum of products of logarithms.

A similar functional equation is expected for the higher polylogarithms

$$\mathrm{Li}_m(z) = \sum_{n>0} \frac{z^n}{n^m}\,, \qquad |z|<1.$$

The relation of polylogarithms and algebraic $K-$theory, and in particular Zagier's conjecture on special values of Dedekind zeta functions, have made it clear that the question of finding functional equations for higher polylogarithms $\mathrm{Li}_m$ is a central one (cf. e.g. [10]). Kummer gave special functional equations for $m \le 5$ and Goncharov [3] found a basic one for $m=3$ as the key step in his proof of Zagier's conjecture for this case, but almost nothing is known in the general case. In our thesis [1] we gave the first functional equations for $m=6$ and $m=7$ as well as many new examples for $m \le 5$, some of which are contained in the following.

1.2. Recall that the function $\mathrm{Li}_m(z)$ is a many-valued function on $\mathbf{C}-\{0,1\}$ but that one has the one-valued continuous function

$$\mathcal{L}_m(z)=\Re_m\left(\sum_{r=0}^{m}\frac{2^r B_r}{r!}\log^r|z|\,\mathrm{Li}_{m-r}(z)\right),\qquad |z|\le 1,\ z\notin\{0,1\},$$

where $\Re_m$ denotes the real part for $m$ odd and the imaginary part for $m$ even, $B_r$ the $r$-th Bernoulli number ($B_0=1, B_1=-1/2, B_2=1/6,\dots$). For $|z|>1$, $\mathcal{L}_m(z)$ is given by the functional equation $\mathcal{L}_m(z)=(-1)^{m-1}\mathcal{L}_m(1/z)$, while for $z=0$ and $z=1$ one extends the function by continuity. For $m=2$ this is the famous Bloch–Wigner dilogarithm. For $m>2$ one-valued versions of $\mathrm{Li}_m$ were introduced by Ramakrishnan, Wojtkowiak and Zagier. The above function is due to Zagier [9] who denoted it $P_m(z)$. It has the advantage that the corresponding functional equations of $\mathrm{Li}_m$ become "clean" for $\mathcal{L}_m$, e.g. in the 5-term equation stated above the elementary term on the right hand side disappears when we replace $\mathrm{Li}_2$ by $\mathcal{L}_2$.

Note that we can consider $\mathcal{L}_m$ as a function on $\mathbf{C}[\mathbf{C}]$ by extending it linearly.

## 2. A general dilogarithm equation

Let us first recall a general functional equation for the dilogarithm found by Rogers —for a polynomial $\phi$ without constant term—and generalized by Wechsung [6], Wojtkowiak [8] and Zagier [11] to any rational function $\phi$. For $m\in\mathbf{N}$ put

$$\widetilde{\mathcal{L}_m}(x,y,z,w)=\mathcal{L}_m\big(\mathrm{cr}(x,y,z,w)\big)=\mathcal{L}_m\left(\frac{x-z}{x-w}\,\frac{y-w}{y-z}\right),$$

where cr denotes the cross ratio.

**Proposition 2.1.** *Let $\phi:\mathbf{P}^1_{\mathbf{C}}\to\mathbf{P}^1_{\mathbf{C}}$ be a rational function, $\alpha,B,C,D\in\mathbf{P}^1(\mathbf{C})$. Then, putting $A=\phi(\alpha)$, we obtain*

$$\sum_{\beta,\gamma,\delta}\widetilde{\mathcal{L}_2}(\alpha,\beta,\gamma,\delta)=\deg(\phi)\cdot\widetilde{\mathcal{L}_2}(A,B,C,D),$$

*where $\beta$, $\gamma$ and $\delta$ run through the preimages of $B$, $C$ and $D$, respectively, with multiplicities.*

The proposition contains as a special case the five term relation (e.g. if we take $\phi(z)=z(1-z)$ and $B=1$, $C=0$, $D=\infty$) and in fact many other equations. A functional equation which is not covered by the proposition (and presumably the only one, essentially) is the one relating $z$ and $\bar z$, its complex conjugate, via $\mathcal{L}_2(\bar z)=-\mathcal{L}_2(z)$.

## 3. Trilogarithm equations

The following equation for the trilogarithm was found by symmetrizing a functional equation given by Wojtkowiak [8] (which in a similar form had been given earlier by Wechsung, cf. [6], [7]).

**Proposition 3.1.** *Let $\phi:\mathbf{P}^1_{\mathbf{C}}\to\mathbf{P}^1_{\mathbf{C}}$ be a rational function, $A_i,B_j,C_k,D_l\in\mathbf{P}^1(\mathbf{C})$, $i,j,k,l\in\{1,2\}$. Then*

$$\sum_{i,j,k,l}(-1)^{i+j+k+l}\left(\sum_{\alpha_i,\beta_j,\gamma_k,\delta_l}\widetilde{\mathcal{L}_3}(\alpha,\beta,\gamma,\delta)-\deg(\phi)\cdot\widetilde{\mathcal{L}_3}(A_i,B_j,C_k,D_l)\right)=0,$$

*where $\alpha_i$, $\beta_j$, $\gamma_k$ and $\delta_l$ run through the preimages of $A_i$, $B_j$, $C_k$ and $D_l$, respectively, with multiplicities.*

Wojtkowiak's original equation [8], pp.226-227, is obtained by specializing the equation from the proposition above and can be stated in a simpler form as follows:

**Corollary 3.2.** *Let* $A, B, C \in \mathbf{P}^1(\mathbf{C})$, $\phi : \mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$ *a rational function of degree* $n$ *and* $x$ *an independent variable.*
*Let* $\phi^{-1}(A) = \{\alpha_i\}_{i=1}^n$, $\phi^{-1}(B) = \{\beta_i\}_{i=1}^n$ *and* $\phi^{-1}(C) = \{\gamma_i\}_{i=1}^n$. *Then*

$$\begin{aligned}\widetilde{\mathcal{L}_3}\,(\phi(x), C, B, A) &- \sum_{i,j,k=1}^{n} \widetilde{\mathcal{L}_3}\,(x, \gamma_i, \beta_j, \alpha_k) \\ &- \sum_{i,j,k=1}^{n} \widetilde{\mathcal{L}_3}\,(x, \alpha_i, \alpha_j, \gamma_k) - \sum_{i,j,k=1}^{n} \widetilde{\mathcal{L}_3}\,(x, \beta_i, \beta_j, \gamma_k) \\ &+ \sum_{i,j,k=1}^{n} \widetilde{\mathcal{L}_3}\,(x, \alpha_i, \alpha_j, \beta_k) + \sum_{i,j,k=1}^{n} \widetilde{\mathcal{L}_3}\,(x, \beta_i, \beta_j, \alpha_k) \\ &= \text{const},\end{aligned}$$

*i.e. the expression on the left hand side is independent of* $x$.

Here it is understood that $\alpha_i$, $\beta_j$ and $\gamma_k$ run through all preimages (counted with multiplicity) of $A$, $B$ and $C$, respectively.

## 3.1. **Around Goncharov's equation.**

3.1.1. *The original description.* Goncharov [3] found a beautiful interpretation of functional equations in terms of configuration spaces, and as a by-product he provided an equation for the trilogarithm in three variables $\alpha_i$ for which he gave a threefold symmetry. We reproduce it here using the shorthand $\beta_i = 1-\alpha_i+\alpha_i\alpha_{i-1}$, $i = 1, 2, 3$, where indices are understood modulo 3: put

$$\gamma(\alpha_1, \alpha_2, \alpha_3) = \sum_{i=1}^{3} \Big( [\frac{1}{\alpha_i}] + [\beta_i] + [\frac{\alpha_i\alpha_{i-1}}{\beta_i}] + \Big[\frac{\beta_i}{\beta_{i+1}\alpha_{i+2}}\Big] + \Big[-\frac{\beta_i\alpha_{i+1}}{\beta_{i+1}}\Big] \Big) \tag{3.1}$$

$$+ \Big[-\frac{1}{\alpha_1\alpha_2\alpha_3}\Big] - \sum_{i=1}^{3} \Big( [\frac{\beta_i}{\alpha_{i-1}}] + \Big[\frac{\beta_i}{\beta_{i+1}\alpha_i\alpha_{i-1}}\Big] + [1] \Big). \tag{3.2}$$

**Proposition 3.3.** *With the above notation,*

$$\mathcal{L}_3\big(\gamma(\alpha_1, \alpha_2, \alpha_3)\big) = 0\,.$$

3.1.2. *A more symmetric description.* Since Goncharov's equation plays such a central role for the theory, it seems worthwhile to analyze its structure a bit further. There is actually a much bigger symmetry group $G$ (of order 192) than the cyclic group on three letters acting on the set of arguments and dividing the 22 terms into 2 orbits, one of length 16 (corresponding to the 15 terms in the first sum plus the single term $\big[-\frac{1}{\alpha_1\alpha_2\alpha_3}\big]$), the other one of length 6 (corresponding to the second sum, with the exclusion of the term [1]).
$G$ is generated by two involutions

$$\pi_1 : (\alpha_1, \alpha_2, \alpha_3) \mapsto (\alpha_1, \alpha_2, -\frac{\beta_1}{\alpha_1\beta_3}), \qquad \pi_2 : (\alpha_1, \alpha_2, \alpha_3) \mapsto (\frac{1}{\alpha_1}, \frac{1}{\alpha_3}, \frac{1}{\alpha_2}),$$

together with the obvious symmetry of order 3 (shifting the indices mod 3). The set consisting of the arguments in the sixteen terms of the first orbit is transformed either into itself (e.g. via $\pi_1$) or into the set of inverses (e.g. via $\pi_2$). We give a presentation with more obvious symmetries: let $t_1, \ldots, t_4$ be four variables, subject to the constraint $\prod_i t_i = 1$. Then the following element in $\mathbb{Q}(t_1, t_2, t_3)$ is annihilated by $\mathcal{L}_3$ for each evaluation of $t_1, \ldots, t_3$ in $\mathbb{C}$:

$$\sum_i [t_i] + \sum_{\substack{i,j\\ i\neq j}} \Big[\frac{1-t_i}{1-t_j^{-1}}\Big] - \frac{1}{4}\sum_{\substack{i,j\\ i\neq j}} [t_i\,t_j] - \frac{1}{8} \sum_{\{i,j,k,l\}=\{1,2,3,4\}} \Big[\frac{(1-t_i)(1-t_j)}{(1-t_k^{-1})(1-t_l^{-1})}\Big] - 3[1]\,.$$

The obvious $\mathcal{S}_4$-action on the set of arguments (where we identify $[z]$ and $[1/z]$), together with the involution

$$t_i \mapsto \frac{1-t_i}{1-t_{i+2}^{-1}}, \qquad i \bmod 4,$$

generates the symmetry group $G$ (of order 192) mentioned above.

3.1.3. *A yet more symmetric description.* Consider the finite group $G'$ (of order 96) of automorphisms of $\mathbb{Q}(y_1, y_2, y_3, z_1, z_2, z_3)$, generated by $G' = \langle g, h \rangle$, where

$$g : (y_1, y_2, y_3, z_1, z_2, z_3) \mapsto (\frac{1}{y_1}, z_2, z_3, z_1, y_2, y_3), \tag{3.3}$$

$$h : (y_1, y_2, y_3, z_1, z_2, z_3) \mapsto (y_2, y_3, y_1, z_2, z_3, z_1). \tag{3.4}$$

The orbits of $y_1$ and of $\dfrac{y_1-z_3}{1-y_1z_2}\dfrac{y_2-z_1}{1-y_2z_3}\dfrac{y_3-z_2}{1-y_3z_1}$ under $G'$ have order 12 and 32, respectively. As in the previous subsection, we introduce "parametrization variables" $t_1, \dots, t_4$, subject to the constraint $\prod_{j=1}^4 t_j = 1$, and form 6 arguments

$$\{A_i\}_{i=1,2,3} = \{t_i t_4\}_i\,, \quad \{B_i\}_{i=1,2,3} = \left\{ \frac{1-t_j^{-1}}{1-t_i}\frac{1-t_k^{-1}}{1-t_4} \;\middle|\; \{i,j,k\} = \{1,2,3\}, j<k \right\}.$$

The automorphism induced by

$$(t_1, t_2, t_3, t_4) \mapsto \big(\iota(t_1,t_4), \iota(t_4,t_1), \iota(t_2,t_3), \iota(t_3,t_2)\big)$$

where $\iota(x,y) = \dfrac{1-y^{-1}}{1-x}$, acts on $(A_1, A_2, A_3, B_1, B_2, B_3)$ like the automorphism $g$ in (3.3) above while the automorphism induced by $(t_1, t_2, t_3) \mapsto (t_2, t_3, t_1)$ acts like $h$ in (3.4).

If we put $y_i$ resp. $z_i$ equal to $A_i$ resp. $B_i$ in the above, the second orbit contains only 16 different terms up to inversion of the arguments—obviously, the first orbit contains only 6 terms up to inversion—and the resulting 16+6 arguments coincide precisely with the ones occurring in Goncharov's equation.

3.1.4. *Other descriptions.* A different way to use the above parametrization in order to exhibit the symmetries among the 22 terms is obtained by putting $\alpha_i = -\sqrt{A_i}$, $\beta_i = \sqrt{B_i}$. Then the four elements $\alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2}\alpha_3^{\varepsilon_3}$ with $\varepsilon_j = \{\pm 1\}$ and $\prod \varepsilon_j = -1$, together with the elements $\beta_1^{\varepsilon_1}\beta_2^{\varepsilon_2}\beta_3^{\varepsilon_3}$ with $\varepsilon_j = \{\pm 1\}$ and $\prod \varepsilon_j = 1$, form the 16 arguments in the second orbit above, while the $\alpha_i^2$ and $\beta_i^2$ give the 6 arguments in the first one.

Yet another description in terms of equations instead of parametrizations is given by the following 9 (redundant) equations where we put $q(x,y) = \dfrac{x-y}{1-xy}$:

$$\begin{aligned}
B_i^{-1} &= \frac{q(B_{i-1}, B_{i+1})}{q(A_{i+1}, A_{i-1})}\,,\\
A_i^{-1} &= q(A_{i+1}, \frac{1}{B_{i-1}})\; q(A_{i-1}, \frac{1}{B_{i+1}})\,,\\
\frac{A_{i+1}(1-A_i)^2}{A_i(1-A_{i+1})^2} &= \frac{B_{i+1}(1-B_i)^2}{B_i(1-B_{i+1})^2}\,,
\end{aligned}$$

where $i = 1, 2, 3$ and the indices are taken modulo 3. (Obviously, we can break the symmetry and reduce this to a system of 7 equations in only 4 variables by eliminating, say, $A_1$ and $B_1$. The system of equations is no complete intersection, though.)

3.2. **Relating Goncharov's and Wojtkowiak's equations.** In order to compare the equations resulting from Goncharov's and Wojtkowiak's approach, we propose to study an "intermediate" relation, with 34 terms, which allows an interpretation for both situations.

1. In Wojtkowiak's equation, the "intermediate" relation occurs when we consider the rational function

$$\phi(x) = \frac{x-a}{x-c^{-1}} \cdot \frac{x-b}{x-abc} = \operatorname{cr}(x,0,a,c^{-1})\operatorname{cr}(x,0,b,abc)\,,$$

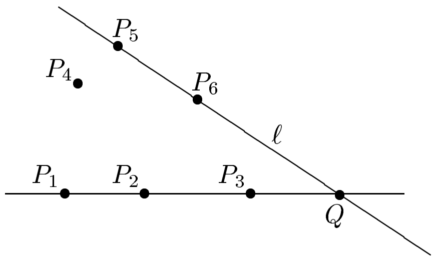
(for $x=1$ this is a $B_i$-type term above) and $(A,B,C)=(\infty,0,1)$, and subtracting two different specializations for $x$ of the ensuing terms from (3.2)—each specialization providing 17 terms—we are left with 34 terms which form a functional equation in three variables. The generic 17 terms are given, in factored form, as follows:

$$\begin{aligned}
&\Big[\frac{(1-ct)a}{a-t}\Big] + \Big[\frac{(1-ct)b}{b-t}\Big] + \Big[\frac{(1-ct)c}{t-a}\Big] + \Big[\frac{(1-ct)c}{t-b}\Big] \\
+&\Big[\frac{abc-t}{(a-t)bc}\Big] + \Big[\frac{abc-t}{(b-t)ac}\Big] + \Big[\frac{abc-t}{a-t}\Big] + \Big[\frac{abc-t}{b-t}\Big] \\
+&\Big[\frac{(a-t)b(ac-1)}{(b-t)a(bc-1)}\Big] + \Big[\frac{(t-a)(1-bc)}{(t-b)(1-ac)}\Big] + \Big[\frac{(tc-1)b(ac-1)}{(abc-t)(bc-1)}\Big] + \Big[\frac{(tc-1)a(bc-1)}{abc-t)(ac-1)}\Big] \\
-&\Big[\frac{(1-ct)abc}{abc-t}\Big] - \Big[\frac{(1-ct)}{(abc-t)c}\Big] - \Big[\frac{t-a}{t-b}\Big] - \Big[\frac{b(a-t)}{a(b-t)}\Big] - \Big[\frac{(b-t)(a-t)c}{(abc-t)(1-ct)}\Big]\,.
\end{aligned}$$

2. Let us recall that Goncharov [3] has defined a triple ratio for 6 points in the projective plane in such a way that it respects a 7-term relation: for any 7 distinct points $P_1,\dots,P_7$ in $\mathbb{P}^2$ one has

$$\sum_{i=1}^{7} \text{triple ratio}(P_1,\dots,\widehat{P_i},\dots,P_7) = 0\,.$$

In the setting of configurations, the intermediate 34-term relation mentioned above encodes the well-definedness of the triple ratio: take a configuration of 6 points $P_i$, $i=1,\dots,6$, in the plane where $P_1$, $P_2$ and $P_3$ lie on a line:




Goncharov reduces this configuration with the help of the 7-term relation to more degenerate configurations by introducing the intersection point $Q$ of the line through the three points $P_1, P_2, P_3$ and a line $\ell$ passing through 2 of the other 3 points (here $P_5$ and $P_6$). He uses the fact that he has already defined the triple ratio for the more degenerate configurations which result by leaving out any other of the 6 original points. But there are different possibilities for choosing $\ell$—e.g. one could take $\ell$ to be the line through $P_4$ and $P_5$—and one verifies (this is not explicitly done in [3]) that the difference of two such possibilities essentially results in two sets of 17 terms which correspond precisely to the 34 terms in question. (Here "essentially" alludes to the fact that we argue "modulo" the inversion relation $\mathcal{L}_3(x)=\mathcal{L}_3(1/x)$ and the 3-term relation $\mathcal{L}_3(x)+\mathcal{L}_3\big(1/(1-x)\big)+\mathcal{L}_3(1-1/x)=\mathcal{L}_3(1)$.)

3.3. **Relating the 34-term and 22-term equation.** If we take the difference of the specializations of the terms in (3.2) to $x = 1$ and $x = 0$, respectively, then a number of terms degenerate and we are left with the sum of a Kummer-Spence relation and a 22-term relation. (Again, we work up to inversion and 3-term relation.) Similarly, the difference of the specializations to $x = a$ and $x = 0$ give a version of the Kummer-Spence relation itself. Conversely, the following transformation $(t_1, t_2, t_3) \mapsto \big(\mathrm{cr}(t, 0, c^{-1}, a), \mathrm{cr}(t, 0, b, c^{-1}), \mathrm{cr}(t, 0, a, abc)\big)$ maps the terms $A_i, B_i$, $i = 1, 2, 3$, in the 22-term relation above to the "generic 17 terms" as in 3.2.1 together with five terms which are independent of $t$. This shows immediately that the 22-term relation implies the 34-term relation.

3.4. **A 21-term equation.** Yet another related equation of possible interest is a sum of four 22-term relations which combines to a functional equation in 3 variables but with less, viz. only 21, different arguments (up to inversion), but with coefficients in $\{\pm 1, \pm 2\}$: with the notation in 3.1, consider

$$\Gamma(x,y,z) = \gamma\Big(\frac{1}{1-x}, \frac{1-x}{1-xy}, 1-z\Big) + \gamma\Big(1-\frac{1}{x}, \frac{1-xy}{y(1-x)}, \frac{1}{1-z^{-1}}\Big)$$

and note that its symmetrization in the first 2 arguments has the following form—we introduce a "symmetrizing variable" $z_2 = (x_1 x_2 z_1)^{-1}$ and $j(t,u) = \frac{1-u^{-1}}{1-t}$ :

$$\begin{aligned}\Gamma(x_1,x_2,z_1)+\Gamma(x_2,x_1,z_1) = {} & -2[x_1\,x_2] - 2[1] \\ & +2\sum_{i=1}^{2}\Big([x_i] + [j(x_i,x_{3-i})] + [z_i] + [j(z_i,z_{3-i})] \\ & \quad -[x_i\,j(z_1,z_2)] - [j(x_i,x_{3-i})\,j(z_1,z_2)] - [x_i z_1] - [j(x_i,x_{3-i})\,z_1]\Big) \\ & +\sum_{i=1}^{2}\Big([x_i\,z_1\,j(x_i,x_{3-i})\,j(z_1,z_2)] + [x_i\,z_1\,/\,j(x_i,x_{3-i})\,j(z_1,z_2)]\Big)\,.\end{aligned}$$

## 4. 4-logarithm equations

It is in general a tedious job to verify even a single functional equation for the polylogarithms. Therefore it seems worthwhile finding a rather short (and elementary) way to verify a whole family of them. We propose to do this for a family of equations for the 4-logarithm, using essentially only one (and actually trivially checked) polynomial equation. One can proceed similarly (albeit in a somewhat more complicated fashion) with other families (cf. [2], Thm 4.4 and Thm. 4.9), even up to $n = 6$. Let us emphasize that not only had there been no example for $n \geq 6$ previously known (apart from the trivial distribution relations) but Wechsung [6] even *proved* that the type of equation (called "Kummer-type") which gave the only examples for $n = 4, 5$ is not "good enough" for $n \geq 6$. The following example is not of Kummer-type. For a proof see the appendix.

**Theorem 4.1.** *Let $F$ be a field of characteristic 0, $\phi(x) = x^{n-1}(x-1)$ for some $n \in \mathbf{N}$. For some $t, u \in F$ let $\{p_i\}_{i=1}^n = \phi^{-1}(t)$ resp. $\{q_i\}_{i=1}^n = \phi^{-1}(u)$ be the sets of preimages in some finite extension field $F'$ of $F$. Then the following element in $\mathbf{Z}[F']$ is annihilated by $\mathcal{L}_4$:*

$$\begin{aligned}(4.1)\quad n(n-2)\Big[\frac{\prod_i p_i}{\prod_j q_j}\Big] - (n-1)^2 \sum_{i,j=1}^{n}\Big[\frac{1-p_i^{-1}}{1-q_j^{-1}}\Big] + n^2 \sum_{i,j=1}^{n}\Big[\frac{1-p_i}{1-q_j}\Big] \\ - n^2(n-1)^2 \sum_{i,j=1}^{n}\Big[\frac{p_i}{q_j}\Big] + n(n-1)^2 \sum_{i=1}^{n}\Big(\big[1-\frac{1}{p_i}\big] - \big[1-\frac{1}{q_i}\big]\Big)\,.\end{aligned}$$

## 5. A 7-logarithm equation

In this section we give a 2-variable functional equation for the 7–logarithm. This functional equation consists of 274 terms, but by using an appropriately symmetrized notation we can write it in a more digestible form.

5.1. Let $F$ be the function field $\mathbf{Q}(t,u)$. Let

$$f_1(z)=\frac{-z}{1-z+z^2},\quad f_2(z)=\frac{z-1}{1-z+z^2},\quad f_3(z)=\frac{z(1-z)}{1-z+z^2}$$

$$\text{and}\quad f(z)=-f_1(z)f_2(z)f_3(z)=\frac{z^2(1-z)^2}{(1-z+z^2)^3}.$$

Notice that the group $G\cong\mathcal{S}_3$ generated by $z\to 1/z$ and $z\to 1-z$ permutes the $f_i$ and leaves $f$ invariant. For $a,b,c,d\in\mathbf{Z}$ define elements $\{a,b;c,d\}_0(t,u)$ and $\{a,b;c,d\}$ in $\mathbf{C}[\mathbf{C}]$ by

$$\{a,b;c,d\}_0(t,u)=\sum_{i,j=1}^{3}\left[\frac{f(t)^a f_i(t)^{b-a}}{f(u)^c f_j(u)^{d-c}}\right]\quad\text{and}$$

$$\begin{aligned}\{a,b;c,d\}&=\{a,b;c,d\}_0(t,u)+\{c,d;a,b\}_0(t,u)\\&=\{a,b;c,d\}_0(t,u)+\{a,b;c,d\}_0(u,t),\end{aligned}$$

where the latter equation holds only up to inversions of the arguments (which, of course, does not affect the functional equation to be presented below). Finally, set

$$\begin{aligned}\xi_7^{(3)}=&-\tfrac{1}{18}\cdot\tfrac{609}{4}\{-1,-1;-1,-1\}-\tfrac{1}{3}\cdot 35\{-1,-1;-2,1\}+\tfrac{1}{3}\cdot\tfrac{105}{8}\{-1,-1;3,-5\}\\&-\tfrac{1}{3}\cdot 21\{-1,-1;-1,4\}-\tfrac{1}{3}\cdot 15\{-1,-1;-2,5\}+\tfrac{1}{3}\cdot 15\{-1,-1;3,-4\}\end{aligned}$$

$$\begin{aligned}\xi_7^{(2)}=&+\tfrac{1}{2}\cdot 700\{1,0;1,0\}+\tfrac{1}{2}\cdot\tfrac{175}{4}\{1,-3;1,-3\}+\tfrac{1}{2}\cdot 28\{-2,3;-2,3\}\\&-35\{1,-3;-2,3\}-140\{-2,3;1,0\}+175\{1,0;1,-3\}\end{aligned}$$

$$\begin{aligned}\xi_7^{(1)}=&+\tfrac{1}{2}\cdot 700\{1,-2;-1,2\}+3150\{0,1;1,-1\}+\tfrac{1}{2}\cdot 1575\{-1,1;1,-1\}\\&-2100\{1,-2;0,-1\}+\tfrac{1}{2}\cdot 6300\{0,1;0,-1\}-1050\{-1,2;-1,1\}\\&-\tfrac{1}{2}\cdot 700\{-1,2;-1,2\}-\tfrac{1}{2}\cdot 1575\{-1,1;-1,1\}-\tfrac{1}{2}\cdot 6300\{0,1;0,1\}\\&+1050\{-1,2;1,-1\}+2100\{0,-1;-1,2\}-3150\{1,-1;0,-1\}.\end{aligned}$$

**Theorem 5.1.** *The element* $\xi_7=\xi_7^{(1)}+\xi_7^{(2)}+\xi_7^{(3)}\in\mathbf{Q}[F]$ *is a functional equation for* $\mathcal{L}_7$, *i.e.*

$$\mathcal{L}_7\bigl(\xi_7(t,u)\bigr)=0$$

*for any* $t,u\in\mathbf{C}$ *where* $f_j(t),f_j(u)\notin\{0,\infty\}$, $j=1,2,3$.

*Proof.* Functional equations for polylogarithms can be characterized by an algebraic criterion (cf. [9], Proposition 1), so the proof can be reduced to checking this criterion for $\xi_7$ which was done with a computer program. The statement applies, of course, to any functional equation; the difficulty lies in finding functional equations, not in checking their validity. □

5.2. **Remark.** In $\xi_7$ the denominator of the first factor of a coefficient (which we suppress if it is 1) gives the multiplicity of an argument in the sum $\{a,b;c,d\}$ (up to inverse, i.e. we identify $[z]$ and $[1/z]$).
We associate a weight to each $\{a,b;c,d\}$ by $\mathrm{wt}(\{a,b;c,d\}) = \mathrm{wt}(a,b)$ in the following way: setting $\mathrm{wt}(a,b) = \frac{1}{2}(|a|+|a+b|+|2a+b|)$ we find that in all terms $\{a,b;c,d\}$ of $\xi_7$ we have $\mathrm{wt}(a,b) = \mathrm{wt}(c,d)$.
This will subdivide the 24 terms into three blocks of weights 3, 2 and 1, as indicated in the formatting above, namely into the first 6, the second 6 and the final 12 terms. We can observe another operation on these blocks which even includes the coefficients. We want to explain this in more detail.

5.3. **More symmetric form of $\xi_7$.** Let $(\mathbf{Z}^3)_0 = \{(a,b,c) \in \mathbf{Z}^3 \mid a+b+c=0\}$ and $(\mathbf{Z}^3)_1 = \{(a,a,b) \in \mathbf{Z}^3\}$. We consider the map

$$\Theta : \mathbf{Z}^3 \to \mathbf{Z}^3 , \tag{5.1}$$

$$(a,b,c) \mapsto (a, -b-c, b-a) \tag{5.2}$$

which sends $(\mathbf{Z}^3)_0$ to $(\mathbf{Z}^3)_1$, and a second one (the $f_i$ are taken as in (5.1))

$$\Phi : \mathbf{Z}^3 \to \{\phi : \mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}} \mid \phi \text{ rational}\} , \tag{5.3}$$

$$\alpha = (\alpha_1,\alpha_2,\alpha_3) \mapsto \Big(\phi_\alpha : z \mapsto \big(-f_1(z)\big)^{\alpha_1} f_2(z)^{\alpha_2} f_3(z)^{\alpha_3}\Big) . \tag{5.4}$$

The symmetric group $\mathcal{S}_3$ operates on $(\mathbf{Z}^3)_0$ and $\mathbf{Z}^3$ via permutation.
For each $k \in \mathbf{Z}$ we have the element $(k,-1,1-k) \in (\mathbf{Z}^3)_0$.
We set

$$A_k = \Theta\big(\mathcal{S}_3 \cdot (k,-1,1-k)\big) , \qquad k = 1,2,3.$$

Then we have

$$\begin{aligned}
A_1 &= \{\pm(1,1,-2), \pm(-1,-1,1), \pm(0,0,1)\} ,\\
A_2 &= \{(2,2,-3), (-1,-1,3), (-1,-1,0)\} ,\\
A_3 &= \{(-1,-1,-1), (-1,-1,4), (-2,-2,5), (-2,-2,1), (3,3,-4), (3,3,-5)\} .
\end{aligned}$$

We will denote by $\delta$ the $\mathcal{S}_3$-invariant element $(-1,-1,-1)$ of $A_3$ which will play a special role. Finally we define for $\alpha = (\alpha_1,\alpha_2,\alpha_3) \neq 0$ a "weight"

$$\omega(\alpha) = \frac{1}{\alpha_1 - \alpha_3} .$$

Then the theorem can be restated by saying that

$$60\,\xi_7 = \sum_{i=1}^{3} \xi^{(i)}$$

with

$$\begin{aligned}
\xi^{(1)} &= -\frac{29}{20}\Big[\frac{\phi_\delta(t)}{\phi_\delta(u)}\Big] + \sum_{\alpha \in A_3 \setminus \{\delta\}} \omega(\alpha) \sum_{\sigma \in \mathcal{S}_3/\mathcal{S}_2} \Big(\Big[\frac{\phi_{\sigma\alpha}(t)}{\phi_\delta(u)}\Big] + \Big[\frac{\phi_\delta(t)}{\phi_{\sigma\alpha}(u)}\Big]\Big)\\
\xi^{(2)} &= \frac{20}{3} \sum_{\alpha,\beta \in A_2} \omega(\alpha)\omega(\beta) \sum_{\sigma,\tau \in \mathcal{S}_3/\mathcal{S}_2} \Big[\frac{\phi_{\sigma\alpha}(t)}{\phi_{\tau\beta}(u)}\Big]\\
\xi^{(3)} &= -\,30 \sum_{\alpha,\beta \in A_1} \omega(\alpha)\omega(\beta) \sum_{\sigma,\tau \in \mathcal{S}_3/\mathcal{S}_2} \Big[\frac{\phi_{\sigma\alpha}(t)}{\phi_{\tau\beta}(u)}\Big] .
\end{aligned}$$

Altogether we get $1+30+81+162 = 274$ different arguments (up to inverse) since in the last sum (over $A_1$) the arguments associated to $(\alpha,\beta)$ and $(-\alpha,-\beta)$ are inverse to each other.

## 6. Appendix: a proof of Theorem 4.1

**Proof.** Using Zagier's criterion (cf. [9], Proposition 1) alluded to earlier, we need only verify that the above combination lies in the kernel of the map

$$\begin{aligned}\beta_4 :\ \mathbf{Z}[F'] &\longrightarrow \mathrm{Sym}^2 F'^{\times} \otimes \bigwedge{}^2 F'^{\times} \\ [x] &\longmapsto x^{\odot 2} \otimes \big(x \wedge (1-x)\big)\,.\end{aligned}$$

Our strategy is as follows: a few preliminary considerations (steps 0.-2.) allow to rewrite the $\beta_4$-images in a more convenient way, so that the theorem is essentially reduced to Claim (6.1) (in step 3.).

**Part I: Reformulations.** 0. Note that $\phi(x_i) - \phi(z)$ is a polynomial in (the variable) $z$ with roots equal to $\{x_i\}_i$, therefore

$$\phi(x_i) - \phi(z) = \lambda \prod_i (x_i - z)$$

for some constant $\lambda$. Thus, for any fixed $l$ and $m$,

$$t - u = \phi(x_l) - \phi(y_m) = \lambda \prod_i (x_i - y_m) = \mu \prod_j (x_l - y_j)$$

for some constants $\lambda$ and $\mu$ (which actually turn out to be equal to $\pm 1$ and will be neglected in the following).

1. As a first preliminary step, we express $\alpha$ and $1-\alpha$ in terms of the factors $x_i$ and $y_j$ for each of the arguments $\alpha$ in (4.1). In order to save indices, we put $x = x_\ell$ and $y = y_m$ for some fixed $\ell$ and $m$. We obtain

$$\frac{1-x}{1-y} = \frac{y^{n-1}X}{x^{n-1}Y} \qquad \text{and} \qquad \frac{1-x^{-1}}{1-y^{-1}} = \frac{y^n X}{x^n Y}\,,$$

where we have set $X = \prod_i x_i$, $Y = \prod_j y_j$. A further decomposition we need is

$$1 - \frac{1-x}{1-y} = \frac{y^{n-1}}{Y}(x-y) \qquad \text{and} \qquad 1 - \frac{1-x^{-1}}{1-y^{-1}} = \frac{y^{n-1}}{xY}(x-y)\,.$$

With these preparations, we can write the corresponding images under $\beta_4$ as

$$\beta_4\left(\left[\frac{1-x}{1-y}\right]\right) = \left(\frac{y^{n-1}X}{x^{n-1}Y}\right)^{\otimes 2} \otimes \frac{y^{n-1}X}{x^{n-1}Y} \wedge \frac{y^{n-1}}{Y}(x-y) \tag{6.1}$$

and

$$\beta_4\left(\left[\frac{1-x^{-1}}{1-y^{-1}}\right]\right) = \left(\frac{y^n X}{x^n Y}\right)^{\otimes 2} \otimes \frac{y^n X}{x^n Y} \wedge \frac{y^{n-1}}{xY}(x-y)\,. \tag{6.2}$$

These two expressions decompose "naturally" into two parts, one of which contains the factor $(x-y)$ in the last tensor factor. E.g., the part in $\bigwedge^2 F'^{\times}$ in the first expression (6.1) factors as

$$\frac{y^{n-1}X}{x^{n-1}Y} \wedge \frac{y^{n-1}}{Y}(x-y) = \frac{X}{x^{n-1}} \wedge \frac{y^{n-1}}{Y} \ +\ \frac{y^{n-1}X}{x^{n-1}Y} \wedge (x-y) \tag{6.3}$$

where the first summand on the right has been reduced using the defining property of $\wedge$.

2. As a second preliminary step, we "switch" to an additive notation—formally, we put $\xi_i =$ "$\log x_i$", $\eta_j =$ "$\log y_j$", and introduce $\zeta_{lm} =$ "$\log(x_l - y_m)$" as well as the shorthands $\xi = \sum_i \xi_i$, $\eta = \sum_j \eta_j$; then the term (6.1) resp. (6.2) is written, using the decomposition (6.3), as

$$\begin{aligned}&\big((n-1)(\eta_m - \xi_l) + \xi - \eta\big)^3 \wedge \zeta_{lm} \\ &\quad + \big((n-1)(\eta_m - \xi_l) + \xi - \eta\big)^2 \cdot \Big(\xi - (n-1)\xi_l\Big) \wedge \Big(-\eta + (n-1)\eta_m\Big)\end{aligned} \tag{6.4}$$

resp.

$$(6.5)\quad \big(n(\eta_m-\xi_l)+\xi-\eta\big)^3\wedge\zeta_{lm} \\ +\big(n(\eta_m-\xi_l)+\xi-\eta\big)^2\cdot\big(\xi-(n-1)\xi_l+\eta_m\big)\wedge\big(-\eta+(n-1)\eta_m-\xi_l\big)\,.$$

Introducing the shorthands $S=\xi-\eta$ and $s_{lm}=\xi_l-\eta_m$ , we rewrite the latter two equations as

(6.6)

$$\big(S-(n-1)s_{lm}\big)^3\wedge\zeta_{lm}+\big(S-(n-1)s_{lm}\big)^2\cdot\big(\xi-(n-1)\xi_l\big)\wedge\big(-\eta+(n-1)\eta_m\big)$$

resp.

(6.7)

$$\big(S-ns_{lm}\big)^3\wedge\zeta_{lm}+\big(S-ns_{lm}\big)^2\cdot\big(\xi-(n-1)\xi_l+\eta_m\big)\wedge\big(-\eta+(n-1)\eta_m-\xi_l\big)\,.$$

**Part II: Calculations.** 3. So far, we have only reformulated the objects under consideration. Now we can proceed to the two actual calculations involved, the first of which follows easily from inserting the previous expressions: With our conventions from 1. and 2., the $\beta_4$-image of $n^2\Big[\frac{1-x_l}{1-y_m}\Big]-(n-1)^2\Big[\frac{1-x_l^{-1}}{1-y_m^{-1}}\Big]$ becomes

$$\begin{aligned}
&\beta_4\bigg(n^2\Big[\frac{1-x_l}{1-y_m}\Big]-(n-1)^2\Big[\frac{1-x_l^{-1}}{1-y_m^{-1}}\Big]\bigg)\\
&=\big((2n-1)S^3-3n(n-1)S^2s_{lm}+n^2(n-1)^2s_{lm}^3\big)\wedge\zeta_{lm}\\
(6.8)\quad &-\Big(n^2\big(S-(n-1)s_{lm}\big)^2-(n-1)^2(S-ns_{lm})^2\Big)\bigg(\sum_{i,j}(2-n)^{\delta_{il}+\delta_{jm}}\ \xi_i\wedge\eta_j\bigg)\\
&+(n-1)^2(S-ns_{lm})^2\cdot\big(\eta_m\wedge\xi_l\big)\\
&+(n-1)^2(S-ns_{lm})^2\cdot\big(\xi\wedge\xi_l\ +\ \eta_m\wedge\eta\big)\,.
\end{aligned}$$

Here $\delta_{ij}$ denotes the usual "Kronecker-$\delta$".

Summing the latter expression over all $l$ and $m$, we obtain

**Claim 6.1.**

$$\begin{aligned}
\beta_4\bigg(n^2\Big[\frac{1-x_l}{1-y_m}\Big]-(n-1)^2\Big[\frac{1-x_l^{-1}}{1-y_m^{-1}}\Big]\bigg)&=n(n-2)\bigg(S^3\wedge\Big(\frac{1}{n}\sum_{l,m}\zeta_{lm}\Big)+S^2\sum_{l,m}\xi_l\wedge\eta_m\bigg)\\
&-n^2(n-1)^2\bigg(\sum_{l,m}s_{lm}^3\wedge\zeta_{lm}\ +\ \sum_{l,m}s_{lm}^2(\xi_l\wedge\eta_m)\bigg)\\
&+\sum_i f_1(\xi_i)+\sum_j f_2(\eta_j)
\end{aligned}$$

*for some functions $f_k$ $(k=1,2)$ depending on only one of the variables $t$ and $u$.*

*Proof.* We will need the following immediately verified identities:

$$(6.9)\qquad Z:=\frac{1}{n}\sum_{i,j}\zeta_{ij}=\sum_i\zeta_{i,m}=\sum_j\zeta_{lj}\,,\quad\forall l\,,\ \forall m\,,$$

$$(6.10)\qquad S=\sum_i\xi_i-\sum_j\eta_j=\frac{1}{n}\sum_{i,j}s_{i,j}=\sum_l T_{l,l+k}\,,\quad\forall k\,,$$

and the distribution properties (for fixed $l$, $m$):

$$(6.11)\qquad \sum_{i,j}(2-n)^{\delta_{il}+\delta_{jm}}=\sum_i(2-n)^{\delta_{il}}=1\,,$$

as well as some simple consequence of the above

$$\sum_{l,m} s_{lm}(2-n)^{\delta_{il}+\delta_{jm}} = S-(n-1)s_{ij}\,. \tag{6.12}$$

[Proof: write $s_{lm} = \xi_l - \eta_m$ and use successively the equations in (6.11).]

We observe that, by expanding $(S-ns_{lm})^2 = \big((\xi - n\xi_l)-(\eta-\eta_m)\big)^2$, we obtain 3 terms: one purely in the $\xi_i$, a second one purely in the $\eta_j$, while the third one cancels after summing over $l$ and $m$:

$$\sum_{l,m}(\xi-n\xi_l)(\eta-\eta_m)(\xi\wedge\xi_l \;+\; \eta_m\wedge\eta) = \quad n\textstyle\sum_l(\xi-n\xi_l)(\eta-\eta_m)(\xi\wedge\xi_l)$$
$$-n(\xi-n\xi_l)(\eta-\eta_m)(\eta_m\wedge\eta)\,,$$

both terms of which are obviously zero.

So it will suffice to concentrate on summing the first three lines in (6.8) over all $l$ and $m$ and identifying the result with the terms given in the first two lines of the claim. We obtain for the sum over all $l,m$ of the first three lines in (6.8):

$$\begin{aligned} = \quad & (2n-1)S^3\wedge\sum_{l,m}\zeta_{lm} - 3n(n-1)S^2\sum_{l,m}s_{lm}\wedge\zeta_{lm} + n^2(n-1)^2\sum_{l,m}s_{lm}^3\wedge\zeta_{lm} \\ & -\sum_{l,m}\Big((2n-1)S^2-2n(n-1)Ss_{lm}\Big)\sum_{i,j}(2-n)^{\delta_{il}+\delta_{jm}}\;\xi_i\wedge\eta_j \qquad (6.13) \\ & +(n-1)^2\sum_{l,m}\big(S-ns_{lm}\big)^2\eta_m\wedge\xi_l \end{aligned}$$

(6.14)

which can be simplified using (6.9)-(6.12) as follows: the first two terms combine to

$$-n(n-2)S^3\wedge Z\,, \tag{6.15}$$

the fourth term is divided up into two sums, the first of which is

$$-(2n-1)S^2\sum_{i,j}\sum_{l,m}(2-n)^{\delta_{il}+\delta_{jm}}\;\xi_i\wedge\eta_j \;= -(2n-1)S^2\sum_{i,j}\xi_i\wedge\eta_j\,, \tag{6.16}$$

by virtue of (6.11), while the second one is equal to

$$2n(n-1)S\sum_{l,m}s_{lm}\sum_{i,j}(2-n)^{\delta_{il}+\delta_{jm}}\;\xi_i\wedge\eta_j = \tag{6.17}$$
$$2n(n-1)S^2\sum_{i,j}\xi_i\wedge\eta_j \;-\; 2n(n-1)^2S\sum_{i,j}s_{ij}\big(\xi_i\wedge\eta_j\big)$$

where we have used (6.12). Multiplying out the fifth term in (6.13) gives us three further sums, one being quadratic, one linear and one constant in $S$; the quadratic one combines with (6.16) and the first term in (6.17) to $n(n-2)S^2\sum_{i,j}\xi_i\wedge\eta_j$ , the linear one cancels with the second term in (6.17), and the constant one is given by $-n^2(n-1)^2\sum_{l,m}s_{lm}^3\wedge\zeta_{lm}$ . This proves the claim.

4. We finish by observing that the first two terms on the right hand side of Claim 6.1 correspond precisely to the images under $\beta_4$ of $n(n-2)\big[\prod p_i/\prod q_j\big]$ and $-n^2(n-1)^2\sum_{i,j}\big[p_i/q_j\big]$, respectively, while the last line can be recognized readily—after more careful investigation of the terms "purely in $\xi_i$ or $\eta_j$"— as the image under $\beta_4$ of $n^2(n-1)\big(\sum_i\big[1-x_i^{-1}\big]-\sum_i\big[1-y_i^{-1}\big]\big)$ .

□

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