Shifting and Embeddability of Simplicial Complexes

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MPI / 92-51

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We'll take up the notions occuring in the title of this talk[1] in the arabic order, from right to left.

§1. Simplicial complexes.

Simplicial set K: finite set whose members, called its *simplices,* are themselves finite sets. The members of the simplices of K are called K's *vertices*. In the important case when K is closed with respect to $\subseteq$, then it is called a *simplicial complex*.

However we remark that one of the morals of this talk is going to be that it is sometimes useful to also consider simplicial sets which are closed under some partial orders other than inclusion.

Realizations: If K has N vertices, then by thinking of these as the canonical basis vectors of $\mathbf{R}^N$, and of each simplex as the relatively open convex hull of its vertices, one obtains a subspace of $\mathbf{R}^N$, which too will be denoted K.

§2. Embeddability.

The notion of embeddability of an n-complex in m-space (or m-sphere) can be precised (e.g.) in the following four ways, depending on what one means by an embedding $e : K^n \hookrightarrow \mathbf{R}^m$.

(1) LINEAR EMBEDDABILITY: In case some linear map $e : \mathbf{R}^N \to R^m$ happens to be one-one on the realization K, then we will say that e is a *linear embedding* of K in $\mathbf{R}^m$.

(2) PIECEWISE LINEAR EMBEDDABILITY of K in $\mathbf{R}^m$: this means that for some r, the rth derived of K embeds linearly in $\mathbf{R}^m$.

Here, the *derived* K' of any simplicial complex (or even set) K is the simplicial complex whose simplices are sets of nonempty simplices of K which are totally ordered under $\subset$. By mapping each vertex of K' (a simplex of K) to its barycentre, one gets the linear *barycentric* embedding of K' onto K.

(3) TOPOLOGICAL EMBEDDABILITY: weakest notion in which one only requires $e : K^n \hookrightarrow \mathbf{R}^m$ to be continuous.

[1]Given at the Max-Planck-Institut für Mathematik, Bonn, on May 21, 1992.

(4) CONVEX EMBEDDABILITY: strongest notion which requires that $e(K^n)$ occurs as a set of faces within the Schlegel diagram of an $(m+1)$-dimensional simplicial polytope.

Relationships between these and other notions of embeddability is a subject by itself. For example – joint recent work with Ulrich Brehm which disproves a conjecture of Grünbaum – one has

Theorem 1. *For each $n \geq 2$ there is an n-complex which embeds piecewise linearly, but not linearly, in $\mathbf{R}^{2n}$.*

§3. Some history.

Before taking up the last (= first) notion we'll give some motivation for introducing it.

For the case $n = 1$, $m = 2$ (when, by virtue of a theorem of Steinitz, the above 4 notions of embeddability happen to coincide) of *planar graphs* $K^1 \hookrightarrow \mathbf{R}^2$, various interesting theorems are known, e.g. the following one which settled a celebrated conjecture.

Theorem (Kempe-Heawood-...-Heesch-Appel-Haken). *If complex K^1 embeds in $\mathbf{R}^2$, then there exists a function $\chi : vert(K) \to S$, $card(S) = 4$, such that $\chi(a) \neq \chi(b)$ whenever $\{a, b\}$ is a 1-simplex of K.*

At the moment the complete proof of this result is somewhat opaque, but at least the starting point is transparent enough. It is the following important observation which goes back to Kempe and Heawood, and which at once (*resp.* with a little more effort) gives a weaker version of the above theorem with $card(S) = 6$ (*resp.* $= 5$).

$$\boxed{K^1 \hookrightarrow \mathbf{R}^2 \implies f_1(K) < 3 \cdot f_0(K)}$$

Here $f_i(K)$ denotes the number of i-dimensional faces (= simplices) of K.

We will from now on use the generic term *Heawood Inequality* to mean any necessary numerical condition on the face vectors of any class of embeddable simplicial complexes.

The notion of shifting has turned out to be very useful for establishing such inequalities, and also results related to the following graph planarity criterion which appeared in print in 1930.

It is interesting to note that in the years immediately after its publication, the feeling was widespread, that now a proof of the four color theorem was imminent ! It is ironic, in view of this optimism, that there still does not seem to be any direct proof of even $f_1 < 3 \cdot f_0$ as a *corollary* of this criterion !!

Theorem (Pontrjagin-Kuratowski-Frink-P.A.Smith ...). *A graph is planar if and only if it does not contain any homeomorph of the graphs*

$$T^1 = \sigma_1^4 \quad \text{or} \quad \sigma_0^2 \cdot \sigma_0^2,$$

i.e., there are only two minimally *nonplanar graphs, viz. these graphs* T^1.

Here, and below, σ_j^i denotes the *j-skeleton of an i-simplex,* and the *join* of disjoint copies of complexes K and L is denoted $K \cdot L$.

We will use the generic term *Kuratowski Theorem* for any result co-relating, in a similar way, the absence of some finite list of forbidden configurations with embeddability.

§4. Shifting.

A *total order* on the set of vertices induces a product partial order on all sets of a given cardinality. A set K of simplices will be called *shifted* if it is closed with respect to these partial orders.

Shifting usually means any operation

$$K \rightsquigarrow \Delta(K)$$

which associates to each set K of simplices (on a totally ordered set of vertices) a shifted set $\Delta(K)$ having the same face vector.

Such combinatorially defined operations were first considered by Erdös and Rado. A different algebraical procedure, called *exterior shifting,* was later introduced by Kalai. We postpone the definition of Kalai's operation to §6 below, where we'll in fact introduce a much more general operation.

Theorem 2. *If K^n embeds topologically in $\mathbf{R}^m$, $n \leq m \leq 2n$, then*

$$f_n(K) < (m - n + 2) \cdot f_{n-1}(K).$$

Restricting the discussion of the proof to the double-dimensional and most important case $m = 2n$, let me point out that the starting point (and also one of the reasons for conjecturing this Heawood Inequality in 1987 or so) is the following easy observation:

Any shifted complex Δ, *with* $f_n(\Delta) \geq (n+2) \cdot f_{n-1}(\Delta)$, *contains* σ_n^{2n+2}.

To see the above, note that the number of n-simplices containing any fixed vertex is less than the the number of $(n-1)$-simplices. So, thanks to the given inequality, Δ must contain an n-simplex not containing any of the first $n+2$ vertices. Being shifted, all simplices less than it, or its faces, in the product partial order, are also in Δ. They provide us with the required subcomplex.

This observation already finishes the proof of Theorem 2 for the *shifted case* because one has

Theorem (Van Kampen - Flores). *The n-skeleton of a $(2n+2)$-simplex, or more generally any of the following $\pi(n+1)$* Kuratowski n-complexes T^n,

$$\sigma_{s_1-1}^{2s_1} \cdot \sigma_{s_2-1}^{2s_2} \cdots, \quad s_1 + s_2 + \cdots = n+1,$$

does not embed in $\mathbf{R}^{2n}$. *Likewise, any* $\sigma_r^r \cdot T^{s-1}$ *with* $r + 2s = m+1$, *does not embed in* $\mathbf{R}^m$.

However the proof of Theorem 2 for the general case is *much* deeper and longer and needs :

(i) some characteristic class theory of free $\mathbf{Z}_2$-complexes which enters into van Kampen's proof of above result, and

(ii) a generalization of exterior shifting which preserves free group actions.

We'll say something about both these theories in §§5 and 6 below.

Using this method we have in fact a forbidden configuration result which has Theorem 2 as an immediate corollary.

Theorem 3. *If K embeds topologically in* $\mathbf{R}^m$, *then its exterior shift* $\Delta(K)$ *does not contain any* $\sigma_r^r \cdot \sigma_{s-1}^{2s}$ *with* $r + 2s = m+1$.

As against Kuratowski's criterion, this result has the advantage that a Heawood inequality *does* follow directly from it, but we won't call Theorem 3 a Kuratowski Theorem because there is no chance of anything resembling a converse:

Examples. (a) $\Delta(U^2 = \sigma_0^1 \cdot \sigma_0^1 \cdot \sigma_0^1)$ contains the non-planar $\sigma_0^2 \cdot \sigma_0^2$.
(b) $\Delta(\sigma_0^2 \cdot \sigma_0^2)$ is planar.

In fact the above two assertions are true for any of the aforementioned *usual* shifting operations.

§5. Van Kampen Theory.

The following definition extends easily to all (or at least all finite) groups G but we'll confine ourselves to the case $G = \mathbf{Z}_2$ which suffices for applications to embeddability.

Deleted joins K_*. For any simplicial complex choose two (i.e. one for each element of $\mathbf{Z}_2$) disjoint copies: K, the positive copy, and $\overline{K}$, the negative copy. The process of changing the sign of a vertex from positive to negative or conversely will be called *conjugation,* and denoted by overlining. We define

$$K_* = \{\sigma \cup \overline{\theta} : \sigma \in K, \theta \in K, \sigma \cap \theta = \emptyset\},$$

and equip this simplicial complex with the free simplicial $\mathbf{Z}_2$-action given by conjugation.

This construction, which associates to each simplicial complex a free $\mathbf{Z}_2$-simplicial complex, has the following important *multiplicative property* with respect to joins.

$$\boxed{(K \cdot L)_* \cong K_* \cdot L_*}$$

This shows e.g. that the deleted join of the closed simplex with N vertices coincides with the the N-fold join

$$U = \mathbf{Z}_2 \cdot \mathbf{Z}_2 \cdots \mathbf{Z}_2.$$

Note that U is a maximal or, in Milnor's terminology, *universal free* $\mathbf{Z}_2$*-simplicial complex* on N vertex-pairs. Geometrically, $U = U^{N-1}$ is an *octahedral sphere* of dimension $N - 1$.

The equivariant (i.e., since we are working with $\mathbf{Z}_2$, symmetric or skewsymmetric) cohomology classes of U restrict to the *characteristic classes* of any free $\mathbf{Z}_2$-subcomplex $E \subseteq U$, $E = \overline{E}$.

Since the equivariant cohomology of U coincides with the (possibly twisted) cohomology of its orbit space, and since the latter happens to be a projective space for the case $G = \mathbf{Z}_2$ under consideration, one needs to consider just one characteristic class in each dimension k, and this will be denoted by $\mathfrak{o}^k$, or just $\mathfrak{o}$, if k is understood.

We recall that if field coefficients F are being used, then these classes $\mathfrak{o}$ are zero, unless $char(\mathsf{F}) = 2$ (when of course one also has symmetry = skewsymmetry).

Theorem (Van Kampen). *A complex K embeds in $\mathbf{R}^m$ (or S^m) only if the $(m+1)$th characteristic class $\mathfrak{o}$ of its deleted join K_* is zero.*

The proof of this result of 1932 is (now) not hard at all:
Any embedding of K in $\mathbf{R}^m$ induces a continuous $\mathbf{Z}_2$-map, from the deleted join, into the complement in $\mathbf{R}^m \cdot \mathbf{R}^m$ of the diagonal. But this complement has the $\mathbf{Z}_2$-homotopy type of an antipodal m-sphere, so its $(m+1)$th characteristic class, and thus also its pull-back in the deleted join, must be zero.

If integer coefficients are being used, as e.g. in the strictly double dimensional result stated below, then we'll use the distinguishing notation $\tilde{\mathfrak{o}}$ for these characteristic classes. We recall that $\tilde{\mathfrak{o}}^k$ is a lift of the mod 2 class $\mathfrak{o}^k$, and is symmetric or skewsymmetric depending on the parity of k.

Theorem (Van Kampen-Wu-Shapiro). *The vanishing, $\tilde{\mathfrak{o}} = 0$, of the $(2n+1)$th integral characteristic class, of the deleted join K_* of K^n, is also sufficient for the piecewise-linear embeddability of K^n in $\mathbf{R}^{2n}$, provided $n \neq 2$.*

The key points, of the proof of the above result, were already in van Kampen's paper of 1932, but the argument was completed only in the mid-1950's.
The case $n = 2$ is open, but it appears likely that Freedman's methods would imply at least the topological embeddability of such a K^2 in $\mathbf{R}^4$.
The above theorem is very satisfying because it gives a *purely combinatorial characterization, viz.* $\tilde{\mathfrak{o}} = 0$, for the embeddability of any K^n in the double dimensional space $\mathbf{R}^{2n}$.

It is doubly so, in view of the fact that the obvious attempt to generalize Kuratowski's criterion to higher dimensions fails:
For each $n \geq 2$, there exist, besides the $\pi(n+1)$ Kuratowski n-complexes T^n, infinitely many non-homeomorphic K^n's, which too are minimally non-embeddable in $\mathbf{R}^{2n}$.
However the use of deleted joins led me to a nice *uniform characterization* of the $\pi(n+1)$ Kuratowski n-complexes T^n, which showed that they remain important for $n \geq 2$.

Theorem (S). *A simplicial n-complex K^n, $n \neq 2$, is* critically non-embeddable in $\mathbf{R}^{2n}$ *if and only if its deleted join K_* is a homogenously $(2n+1)$-dimensional pseudomanifold, and this happens if and only if K^n is isomorphic to a Kuratowski n-complex T^n.*

Here, by *critically non-embeddable in* $\mathbf{R}^{2n}$ we mean that, for any pair of points $\{x, y\}$, belonging to disjoint simplices of K^n, one can choose, for each $\{x', y'\}$ sufficiently near $\{x, y\}$, a continuous map $e_{x'y'} : K^n \longrightarrow \mathbf{R}^{2n}$, depending continuously on $\{x', y'\}$, and having just one double point, namely $\{x', y'\}$.

Note that the second half of the above theorem is a *purely combinatorial classification.*This generalizes, but becomes somewhat more involved, and still hasn't been worked out in full, if one only demands that K_* be *any* , i.e. not necessarily $(2n+1)$-dimensional, pseudomanifold. For example

$(\mathbf{R}P^2_6)_*$, *the deleted join of the 6-vertex real projective plane, is an antipodal 12-vertex 4-sphere.*

Likewise

$(\mathbf{C}P^2_9)_*$, *the deleted join of the 9-vertex complex projective plane, is an antipodal 18-vertex 7-sphere.*

It seems in fact that, whenever K_* is a pseudomanifold, then it must necessarily be an antipodal sphere.

§6. Equivariant shifting theory.

The following definitions also extend easily to all (or at least all finite) groups G, but once again, we'll confine ourselves to the case $G = \mathbf{Z}_2$, and work with a universal free $\mathbf{Z}_2$-simplicial complex U.

Since a simplex σ of the octahedral sphere U contains at most one member of each vertex pair, it follows that if *we fix a total ordering of the N vertex-pairs* $\{v_1, \overline{v_1}\}, \ldots, \{v_N, \overline{v_N}\}$, then each $\sigma \in U$ gets equipped with an induced total ordering of its vertices.

Type $|\sigma|$ of a simplex $\sigma \in U$. By this we mean the function,

$$\alpha : \{1, 2, \ldots, card(\sigma)\} \rightarrow \mathbf{Z}_2,$$

determined by the factorization of σ into alternately positive and negative faces under the aforementioned total ordering.

Note that the set $U_\alpha \subset U$ of simplices of a fixed type α thus has a natural lexicographic order, as well as a natural product partial order.

A simplicial subset of U will be called *type-shifted* if it is closed with respect to the aforementioned product partial orders in each type.

The object of this section is to define an operation $E \rightsquigarrow \Delta(E)$ commuting with conjugation, which associates to each simplicial subset $E \subseteq U$ a type-shifted simplicial set $\Delta(E) \subseteq U$, and which preserves the number of simplices in each type.

This operation will generalize Kalai's shifting operation: in case $E = K$ has all vertices *positive*, $\Delta(K)$ will coincide with the *exterior shift* of K.

Just as for exterior shifting, the operation depends on the choice of some field F which is *big*, i.e. has transcendence degree at least N over some subfield.

Let $L(U)$ denote the F-vector space spanned by the (always ordered as above) simplices of U.

As against exterior shifting, in which the next step would be to identify such a vector space with the underlying vector space of an exterior F-algebra generated by the vertices, we will identify our $L(U)$ with the underlying space of an algebra which is *not sign commutative*.

Star algebra $(\Omega, *)$. By this we mean the associative F-algebra with unity, generated by the $2N$ vertices, subject to the relations

$$v * w = -w * v,\ v * \overline{w} = -w * \overline{v},\ \overline{v} * w = -\overline{w} * v,\ \overline{v} * \overline{w} = -\overline{w} * \overline{v},$$

for all positive vertices v and w.

Because of the bilinearity of $*$, note that the relations

$$x * y = -y * x,\ x * \overline{y} = -y * \overline{x},\ \overline{x} * y = -\overline{y} * x,\ \overline{x} * \overline{y} = -\overline{y} * \overline{x},$$

are valid even when the letters x and y denote *any* elements of Ω_1, the subspace spanned by the positive vertices.

The *grading of* Ω will be by type: one has the required

$$\Omega_\alpha * \Omega_\beta \subseteq \Omega_{\alpha * \beta},$$

with $\alpha * \beta$ obtained by juxtaposing the sequences of group elements α and β.

Definition of $E \rightsquigarrow \Delta(E)$: *The following construction can be summarized by saying that the graded* canonical basis $E \subseteq U$, *of the graded F-vector space $L(E)$ spanned by E, is going to be replaced by a lexicographically first graded* generic basis $\Delta(E)$:

Choose, for the subspace Ω_1 of $L(U)$, a new ordered basis $x_1, \ldots, x_N$, related to the positive vertices by

$$x_j = \sum_i (\xi_j)^i \cdot v_i,$$

where the N field elements ξ_j are algebraically independent over some subfield of F. The new vertices will be from these x_i's and their conjugates: thus we are actually going to construct $\Delta(E)$ as a simplicial subset of the octahedral sphere U_x on the N totally ordered *letter-pairs* $\{x_1, \overline{x_1}\}, \ldots, \{x_N, \overline{x_N}\}$, but of course this identifies with the original octahedral sphere $U = U_v$ in the obvious way.

Notice that any *word* in these letters determines, by $*$-multiplying its letters in order, a homogenous element of Ω of the same type as the word. So, thinking of $L(E)$ as a quotient vector space of $L(U)$ in the obvious way, it also determines a homogenous element of $L(E)$.

These elements obviously constitute a graded spanning set of $L(E)$, but certainly *not* a basis. For example, if a letter repeats, even with a change of overlining, the element determined is zero. And more generally, if we permute and change the overlining of the individual letters in such a way that the type remains same, then the element is unchanged upto sign.

To obtain a basis from this spanning set we now *seive* these words *à la* Eratosthenes: in each type cross out words which (as elements of $L(E)$) depend linearly on the lexicographically preceding words: what remains is a set of words with letters strictly increasing, and is the required generic graded basis $\Delta(E)$ of the graded vector space $L(E)$. It remains only to check the following.

$\Delta(E)$ *is type-shifted.*

Proof. Since the N elements ξ_j of the field F are algebraically independent over a subfield, the *symmetric group* Σ_N of all permutations of these field elements identifies with a subgroup of the group $Aut(\mathsf{F})$ of all field automorphisms of F.

Furthermore, each element of $Aut(\mathsf{F})$ identifies with a graded F_p-linear algebra automorphism of Ω which commutes with conjugation. Here F_p denotes the minimal (or characteristic) subfield of F, $p = char(\mathsf{F})$.

Next, note that such a graded algebra automorphism, which arises from an element of Σ_N, i.e. from a permutation of the field elements ξ_j, involves the corresponding permutation of the positive letters x_j.

So it follows, by using a shuffle permutation which throws a strictly increasing word σ of a certain type, onto another strictly increasing word θ of the same type, which is bigger than it in the product partial order, that σ could have been seived out in the above process, only if θ too was seived out. *q.e.d.*

Remark. Though we have'nt developed this viewpoint here, it is quite useful also to 'visualize' the shifting process, *à la* §1, by thinking of the positive vertices v as the canonical basis vectors of F^N, and thus the positive letters x as points on the *moment curve* $(t, t^2, \ldots, t^N)$ of F^N.

For the proof of Theorem 3 we develop further properties of equivariant shifting. These allow, if the conclusion of this theorem were not valid, the construction of an equivariant cochain map,

$$C(K_*) \to C((\sigma_r^r \cdot \sigma_{s-1}^{2s})_*),$$

which images o^{2n+1} of K_* to that of $(\sigma_r^r \cdot \sigma_{s-1}^{2s})_*$. But this is not possible because the former is zero since K embeds topologically in R^m, while the latter is not since $(\sigma_r^r \cdot \sigma_{s-1}^{2s})_*$ is an antipodal $(m+1)$-sphere.

Bibliographical note. The classification theorem for n-complexes critically non-embeddable in R^{2n} is given in [5].

Complete proofs of the three theorems announced in this talk will be given in [1] and [6]. The latter paper will also discuss the very interesting connection between Theorem 3, some unpublished work of Kalai, and McMullen's g-conjecture for simplicial spheres.

These and other related topics also constitute the subject matter of Chapters IV (on "Linear Embeddability") and V (on "Heawood Inequalities") of [7], a book under preparation.

Grünbaum's conjecture occurs on the first page of [3]. Exterior shifting was introduced in [4]. The seminal ideas of Van Kampen Theory were introduced in [8], see also Flores [2].

ACKNOWLEDGEMENTS

It is a pleasure to thank Thomas Bier for many stimulating conversations, including some which enabled me to find the method of equivariant shifting described above.

Also, I would like to thank Professor Hirzebruch, and the Max-Planck-Institut für Mathematik, for their warm hospitality throughout this very pleasant and fruitful year.

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See also the references of [5].