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SOME REMARKS ON THE TWISTED BURNSIDE-FROBENIUS THEORY FOR INFINITELY GENERATED GROUPS

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Abstract. The TBFT_f conjecture, which is a modification of a conjecture by Fel'shtyn and Hill, says that if the Reidemeister number $R(\phi)$ of an automorphism ϕ of a (countable discrete) group G is finite then it coincides with the number of fixed points of the corresponding homeomorphism $\widehat{\phi}$ of $\widehat{G}_f$ (the part of the unitary dual formed by finite-dimensional representations). The study of this problem for residually finite groups was a subject of a recent activity. We prove here that for infinitely generated residually finite groups there are positive and negative examples for this conjecture. It is detected that the finiteness properties of the number of fixed points of ϕ itself also differ from the finitely generated case.

Introduction

Suppose G is a (countable discrete) group and ϕ is its automorphism. The *Reidemeister number* $R(\phi)$ is the number of its *Reidemeister* or *twisted conjugacy classes*, i.e. the classes of the twisted conjugacy equivalence relation: $g \sim hg\phi(h^{-1})$, $h, g \in G$. Denote by $\{g\}_\phi$ the Reidemeister class of g.

The following two interrelated problems are in the mainstream of the study of Reidemeister numbers.

The first one is the following conjecture by A.Fel'shtyn and R.Hill [8]: $R(\phi)$ is equal to the number of fixed points of the associated homeomorphism $\widehat{\phi}$ of the unitary dual $\widehat{G}$ (the set of equivalence classes of irreducible unitary representations of G), if one of these numbers is finite. The action of $\widehat{\phi}$ on the class of a representation ρ is defined as $[\rho] \mapsto [\rho \circ \phi]$. This conjecture is called TBFT (twisted Burnside-Frobenius theorem). In fact it generalizes to infinite groups and to the twisted case the classical Burnside-Frobenius theorem: the number of conjugacy classes of a finite group is equal to the number of equivalence classes of its irreducible representations.

Some later by A.Fel'shtyn and co-authors the second problem was formulated (see [9] for a historical overview) the problem of description of the class of groups having the R_∞ property. A group has the R_∞ property if $R(\phi) = \infty$ for any automorphism $\phi : G \to G$. Evidently, the second problem is in some sense complementary to the first one: the question about TBFT has no sense for R_∞ groups (formally having a positive answer).

The TBFT conjecture was proved for finite, abelian and abelian-by-finite groups [8, 10]. After that a counter-example was detected in [12]. This counterexample led to the following new version of the conjecture called TBFT_f: if $R(\phi) < \infty$, then it is equal to the number of fixed points of $\widehat{\phi}$ on the subspace $\widehat{G}_f \subset \widehat{G}$ formed by finite-dimensional representations. In [20] we prove that only finite representation (i.e. factorizing through a finite group) can

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be fixed by $\widehat{\phi}$ if $R(\phi) < \infty$. In [11] the TBFT_f conjecture was proved for polycyclic-by-finite groups. Also some counter examples were found among infinite groups with a finite number of usual conjugacy classes. In [19] some steps to the case of general finitely generated residually finite groups were made. In [17] some relations of the TBFT_f and properties of the twisted inner representation were considered. A more general approach to a TBFT-like property was developed in [34].

The property R_∞ was proved and disproved for many groups. Since its study is not a main goal of the present paper, we restrict ourselves to giving reference to several papers and the literature therein: [7, 28, 13, 14, 33, 24, 9, 1, 21, 2, 15, 27, 29, 31, 30, 4, 16, 22, 23, 32, 5, 3, 18]. In some situations the property R_∞ has some direct topological consequences (see e.g. [23]). Some basic theory on relations of Reidemeister numbers with Dynamics can be found in [26, 6].

We start from Section 1 where some necessary facts about Reidemeister numbers are formulated. Also we slightly generalize these facts.

The free finitely generated group F_n ($n \geqslant 2$) has the R_∞ property (in particular, because it is a hyperbolic group [7, 28]). In contrast with this, the infinitely generated free group F_∞ has automorphisms with $R(\phi) < \infty$ [3]. In Section 2 we prove that TBFT_f holds for this example. Also we show that nevertheless the number of fixed points of ϕ itself is infinite in contrast with the finitely generated case [17].

In Section 3 we construct examples of infinitely generated residually finite groups such that TBFT_f fails for them.

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1. Preliminaries

The following easy statement is well known:

Proposition 1.1. *Suppose, H is a ϕ-invariant normal subgroup of G and $\overline{\phi} : G/H \to G/H$ is the induced automorphism. Then Reidemeister classes of ϕ map onto Reidemeister classes of $\overline{\phi}$. In particular, $R(\overline{\phi}) \leqslant R(\phi)$.*

The key observation in [11] is as follows.

Proposition 1.2. *The* TBFT_f *conjecture holds for a specific automorphism ϕ if and only if the functions on G of the form $f_\rho(g) = \mathrm{Trace}(\Phi_\rho \circ \rho(g))$ give a basis of the space of that functions which are constant on the Reidemeister classes (i.e. twisted class functions), where ρ runs over $\widehat{\phi}$-fixed points and Φ_ρ is the intertwining operator between ρ and $\rho \circ \phi$, defined uniquely up to scaling.*

The following statement slightly strengths (to the case of infinitely generated groups) considerations in [11] and [17].

Proposition 1.3. *The following properties of $\phi : G \to G$ with $R(\phi) < \infty$ are equivalent:*
1) TBFT_f *holds for the specific ϕ;*
2) *The stabilizer subgroup of any Reidemeister class (under left shifts) is of finite index in G.*

Proof. First of all, the second property evidently is equivalent to the following one: left shifts of twisted class functions form a finite-dimensional space $V \subset \ell^\infty(G)$.

TBFT_f implies that some matrix coefficients of $\widehat{\phi}$-fixed representations form a basis in the space of twisted class functions (by Prop. 1.2). All left shifts of these functions generate, on the one hand, the mentioned above space V. On the other hand, this space is the space of all matrix coefficients coming from a finite collection of irreducible representations (the collection of $\widehat{\phi}$-fixed representations). Thus, V is finite-dimensional and 2) holds.

Conversely, suppose 2). In particular, the intersection of all stabilizers is a subgroup $H \subset G$ of finite index. Thus, $x \in H$ iff for any $g, z \in G$ there exists $h_{g,z} \in G$ such that $xgz\phi(g^{-1}) = h_{g,z} z\phi((h_{g,z})^{-1})$. Then for any $y \in G$, we have

$$yxy^{-1}gz\phi(g^{-1}) = y\Big(xy^{-1}gz\phi(g^{-1})\phi(y)\Big)\phi(y^{-1}) = y\Big(h_{y^{-1}g,z}z\phi((h_{y^{-1}g,z})^{-1})\Big)\phi(y^{-1}).$$

Thus, $yxy^{-1} \in H$ and H is normal. Also, for the same x, we have

$$\phi(x)gz\phi(g^{-1}) = \phi\Big(x\phi^{-1}(g)\phi^{-1}(z)g^{-1}\Big) = \phi(h\phi^{-1}(z)\phi(h^{-1}) = \phi(h)z\phi((\phi(h))^{-1}),$$

where $h = h_{\phi^{-1}(g),\phi^{-1}(z)}$. Thus, H is ϕ-invariant. Then $p : G \to G/H$ gives a bijection of Reidemeister classes, and in particular, TBFT_f for ϕ. Indeed, suppose two classes are mapped to one class. This means that there is an element $h \in H$ which is not in their stabilizers. A contradiction. □

2. A positive example

In this section we revisit an example of an automorphism $\varphi_n : F_\infty \to F_\infty$ with $R(\varphi_n) = n$, constructed in [3] for each positive integer n, where F_∞ is the free group with countable set of generators $\{x_0, x_1, \dots\}$. We will prove that TBFT_f holds for these φ_n.

Denote by $\theta : \{x_0, x_1, \dots\} \to F_\infty$ a surjective map (not unique), such that $\theta(x_i)$ is a word containing $x_0, \dots, x_{i-1}$ and their inverses. Now fix a positive integer n. For each $i = 0, \dots, n-1$ denote by $J_i \subset \{x_0, x_1, \dots\}$ the subset formed by those x_k that the sum of powers of generators in $\theta(x_k)$ is equal to i modulo n. Denote $W_i := \theta(J_i)$. Thus, W_i is formed by those elements of F_∞ for which the sum of powers of generators is equal to i modulo n and we have

$$\{x_0, x_1, \dots\} = J_0 \sqcup J_1 \sqcup \dots \sqcup J_{n-1}, \qquad F_\infty = W_0 \sqcup W_1 \sqcup \dots \sqcup W_{n-1}.$$

Define $\varphi_n : F_\infty \to F_\infty$ on generators by the formula $\varphi_n(x_k) := (\theta(x_k))^{-1} x_k (x_0)^i$, where i is the number of that J_i which contain x_k. For any $w \in F_\infty$ there exists m such that $\theta(x_m) = w$. Let $k \in \{0, \dots, n-1\}$ be such that $x_m \in J_k$, i.e., the exponent sum of w is k modulo n. Then

$$\begin{aligned} w = x_m(x_0)^k(x_0)^{-k}(x_m)^{-1}\theta(x_m) &= x_m(x_0)^k((\theta(x_m))^{-1}x_m(x_0)^k)^{-1} \\ &= x_m(x_0)^k(\varphi_n(x_m))^{-1} \in \{(x_0)^k\}_{\varphi_n}. \end{aligned}$$

On the other hand, all elements from one Reidemeister class have the same exponent sum modulo n (denote it by ES_n), because for any $w \in F_\infty$ and any generator $x_k \in J_i$ one has by the definition of J_i

$$\begin{aligned} ES_n(x_k w(\varphi_n(x_k))^{-1}) &= ES_n(x_k w(\theta(x_k)^{-1}x_k(x_0)^i)^{-1}) \\ &= ES_n(w) + ES_n(\theta(x_k)) - ES_n((x_0)^i) = ES_n(w) + i - i = ES_n(w). \end{aligned}$$

Thus, each class contains $(x_0)^i$ and they are pairwise distinct. The details can be found in [3].

Now, we can observe that in fact these classes are W_i, i.e., a subgroup W_0 and its cosets. Also the epimorphism

$$ES_n : F_\infty \to \mathbb{Z}_n$$

is equivariant (for the identity automorphism induced on $\mathbb{Z}_n$) and is a bijection of Reidemeister classes by Prop. 1.1 since the numbers are equal. Thus, TBFT_f holds in this situation.

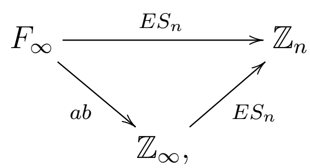
Also, we are interested to calculate the number of fixed points of φ_n. This is important for various arguments in the field (see e.g. [25, 17]).

Consider the case of φ_1. Then by definition $\varphi_1(x_k) = \theta(x_k)x_k$. There exist a number $s = s(k)$ such that $\theta(x_s) = (\theta(x_k)x_k)^{-1}x_k$. Thus

$$\varphi_1(x_k x_s) = (\theta(x_k))^{-1}x_k(\theta(x_s))^{-1}x_s = (\theta(x_k))^{-1}x_k((\theta(x_k)x_k)^{-1}x_k)^{-1}(x_k)^{-1}x_k x_s = x_k x_s.$$

So, the number of fixed points is infinite.

By the universal property of the abelianization we have a commutative equivariant diagram of epimorphisms



which give bijections of Reidemeister classes. For the automorphism induced by φ_1 the images of the same points $x_k x_s$ are fixed after abelianization, and they are distinct. Thus, we still have an infinite set of fixed points.

So, in contrast with the finitely generated case [17, Prop. 3.4], an automorphism φ with $R(\varphi) < \infty$ of a general residually finite (even Abelian) group can have infinitely many fixed points.

3. A negative example

Consider the following example of an infinitely generated residually finite group G and its automorphism ϕ with $R(\phi) < \infty$. The examples of such form naturally arise as invariant quotients of general infinitely generated residually finite groups (in particular, some invariant subgroups of finitely generated residually finite groups).

Let F be a finite non-trivial group and $G = \oplus_{i\in\mathbb{Z}} F_i$, $F_i \cong F$, i.e.

$$G = \{g = (\dots, g_{-1}, g_0, g_1, g_2, \dots) \mid g_i \in F_i,\ g_i \neq e \text{ only for a finite number of } i\}.$$

Evidently G is an infinitely generated residually finite group.

Suppose ϕ is the right shift, i.e. $\phi(g)_i = g_{i-1}$, $i \in \mathbb{Z}$.

Lemma 3.1. *$R(\phi)$ is equal to $|F|$, in particular it is finite.*

Proof. Let $a, g \in G$. Then

$$(ga\phi(g^{-1}))_i = g_i a_i (g_{i-1})^{-1}.$$

First, let us detect when two elements of the form

$$\alpha_0 = x, \quad \alpha_i = e \text{ for } i \neq 0, \qquad \beta_0 = y, \quad \beta_i = e \text{ for } i \neq 0,$$

are twisted conjugate. The condition is:

$$g_0 x (g_{-1})^{-1} = y, \quad g_i (g_{i-1})^{-1} = e \text{ for } i \neq 0.$$

Thus,

$$g_0 = g_1 = \dots, \quad g_{-1} = g_{-2} = \dots.$$

Since $g_i = e$ for large i, $g = (\dots, e, e, e, \dots)$. Thus, α and β are twisted conjugate if and only if they coincide.

Now let us show that any element $a = (\dots, a_i, \dots)$, $a_i = e$ for $i < -m$ and $i > n$, is twisted conjugate to some element of the same form as α (with $x = a_n \dots a_{-m}$). The condition is:

$$g_0 x (g_{-1})^{-1} = a_0, \quad g_i (g_{i-1})^{-1} = a_i \text{ for } i \neq 0.$$

Thus

$$g_1 = a_1 g_0, \quad g_2 = a_2 a_1 g_0, \quad g_3 = a_3 a_2 a_1 g_0, \quad \dots$$

$$g_{-1} = a_0^{-1} g_0 x, \quad g_{-2} = a_{-1}^{-1} g_{-1} = a_{-1}^{-1} a_0^{-1} g_0 x, \quad g_{-3} = a_{-2}^{-1} a_{-1}^{-1} a_0^{-1} g_0 x, \quad \dots$$

We have a unique restriction: $g_i = e$ for large i. Hence,

$$a_n a_{n-1} \cdots a_1 g_0 = e, \qquad e = a_{-m}^{-1} \cdots a_0^{-1} g_0 x.$$

Thus g_0 should be $(a_n a_{n-1} \cdots a_1)^{-1}$ and $x = a_n \dots a_{-m}$ satisfies the restriction. □

Lemma 3.2. *Suppose, F has a trivial center. Then* TBFT_f *fails for G.*

Proof. As it is known (see Prop. 1.3), TBFT_f is equivalent to the property: the stabilizer subgroup of any Reidemeister class (under left shifts) is of finite index in G (for ϕ with $R(\phi) < \infty$).

In our case the stabilizer of $\{e\}_\phi$ is trivial ($= \{e\}$). Indeed, let

$$a = (\dots, e, \dots, e, a_{-m}, \dots, a_n, e, \dots, e, \dots)$$

be a non-trivial element from the stabilizer, i.e., $a_{-m} \neq e$. Since $ae \in \{e\}_\phi$, then $a \in \{e\}_\phi$ and $a_{-m} \cdots a_n = e$. Since a_{-m} is not in the center of F, there exists $b \in F$ such that $b a_{-m} b^{-1} (a_{-m})^{-1} \neq e$. Consider $\beta \in G$ with all components trivial, except of $b_{-m-1} := b$ and $b_{-m} := b^{-1}$. In particular, $\beta \in \{e\}_\phi$. In the same time $a\beta \notin \{e\}_\phi$, because the product of its components is

$$b a_{-m} b^{-1} a_{-m+1} \cdots a_n = b a_{-m} b^{-1} (a_{-m})^{-1} a_{-m} a_{-m+1} \cdots a_n = b a_{-m} b^{-1} (a_{-m})^{-1} \neq e.$$

A contradiction. □

Remark 3.3. Evidently, the argument can be extended to a more general center.

Remark 3.4. Of course, in the opposite case, when F is Abelian, G is Abelian too and the TBFT should be true (see [11] for details). In the present case one can write down the desired fixed (1-dimensional) representations explicitly:

$$(\rho_i)^{\otimes \infty}, \quad i = 1, \dots, |F|, \text{ where } \{\rho_1, \dots, \rho_{|F|}\} = \hat{F}.$$

But in a more general case these invariant representations will be infinite dimensional. Even a more bad fact is that there is only as many of them as $\#\hat{F}$. This is strictly less than $|F| = R(\phi)$ for a non-Abelian F.

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