ZETA STARS

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Abstract. We present two new families of identities for the multiple zeta (star) values: The first one generalizes the formula $\zeta^\star(\{2\}_n,1)=2\zeta(2n+1)$, where $\{2\}_n$ denotes the n-tuple $(2,2,\dots,2)$, while the second family is a weighted analogue of Euler's formula $\sum_{l=2}^{n-1}\zeta(l,n-l)=\zeta(n)$ $(n\ge3)$.

Introduction

Although the structure of algebraic relations of the multiple zeta values (MZVs)

(1)
$$\zeta(\boldsymbol{s})=\zeta(s_1,\dots,s_n)=\sum_{a_1>\dots>a_n\ge1}\frac{1}{a_1^{s_1}\cdots a_n^{s_n}},\qquad s_1,\dots,s_n\in\{1,2,\dots\},\quad s_1\ge2,$$

got a conjectural description (by means of shuffle and stuffle, or harmonic, relations) and there was a big progress on obtaining the (expected) upper bounds for dimensions of the $\mathbb{Q}$-spaces spanned by MZVs of fixed *weight* $|\boldsymbol{s}|=\mathrm{wt}(\boldsymbol{s})=s_1+\dots+s_n$ (recent works of P. Deligne, A. Goncharov [2] and T. Terasoma [12]), new parametric families of elegant identities for MZVs continue to come. Many of these identities have a simple form for an alternative model of zeta values, so-called multiple zeta star values (MZSVs or zeta stars),[1]

(2)
$$\zeta^\star(\boldsymbol{s})=\zeta^\star(s_1,\dots,s_n)=\sum_{a_1\ge\dots\ge a_n\ge1}\frac{1}{a_1^{s_1}\cdots a_n^{s_n}},\qquad s_1,\dots,s_n\in\{1,2,\dots\},\quad s_1\ge2,$$

see the discussion in [5]. The equivalence of the models of MZVs (1) and MZSVs (2), in the sense that the $\mathbb{Q}$-spaces in $\mathbb{R}$ spanned by MZVs and by MZSVs of fixed weight coincide, is a well-known fact (it is an immediate consequence of Proposition 1 below). In our further discussion we require some standard characteristics of the index $\boldsymbol{s}$ in (1) and (2): the already defined weight $|\boldsymbol{s}|$ and the *depth* (or the *length*) $\ell(\boldsymbol{s})=\mathrm{dep}(\boldsymbol{s})=n$. By an *admissible index* $\boldsymbol{s}=(s_1,\dots,s_n)\in\mathbb{Z}^n$ we mean an index with positive entries and $s_1\ge2$ (the latter condition is necessary for the convergence of the series in (1) and (2)). We also assign these characteristics to a MZV $\zeta(\boldsymbol{s})$ or a MZSV $\zeta^\star(\boldsymbol{s})$ themselves, just speaking about the weight and depth of the multiple zeta (star) value, and calling it admissible if $\boldsymbol{s}$ is an admissible index.

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[1]They also appear as non-strict multiple zeta values in [10], the name which is owed to the anonymous referee of that paper. Note also that the notation $\widetilde{\zeta}$ instead of $\zeta^\star$ was used in the 'Russian' literature [13], [18], [19].

Apart from a simplicity of several identities in terms of MZSVs (a good example is the cyclic sum formula — cf. [11] vs. [8]), these zeta values appear very naturally in the diophantine problems for the values of the classical Riemann zeta function $\zeta(s)$, especially due to the identities

$$\zeta^\star(\{2\}_n, 1) = 2\zeta(2n+1), \tag{3a}$$

$$\zeta^\star(\{2\}_n) = 2(1-2^{1-2n})\zeta(2n), \tag{3b}$$

where $\{2\}_n$ denotes the n-tuple $(2,2,\dots,2)$; the interested reader is referred to the papers [16]–[18] by S. Zlobin (we discuss these aspects at the end of Section 1). Identity (3a) also comes as a special case of the cyclic sum formula [11] while (3b) follows from a generating series argument [1], [18].

The starting goal of our project was to find a general form of identities (3a) and (3b), and on this way we succeeded at least in generalizing (3a) (see the 'two-one formula' accompanied with Theorems 1 and 2 below). Another result (Theorem 3) appearing as an auxiliary identity in our deduction of Theorem 2, which is new and interesting by itself, is a 'weighted' version of Euler's formula [3]

$$\sum_{l=2}^{n-1} \zeta(l, n-l) = \zeta(n), \qquad n \ge 3 \tag{4}$$

(the sum formula of depth 2 in the modern terminology).

In Section 1 we provide formulae for expressing the values (2) in terms of (1) and vice versa, and state our main results (Theorems 1–3). Section 2 is devoted to the proof of Theorem 1, while Theorems 2 and 3 are proved in Section 3.

1. Background and formulae for multiple zeta (star) values

Let us first consider a simple recipe of passing from MZSVs to MZVs and vice versa.

Proposition 1. *For any admissible index $\boldsymbol{s} = (s_1, s_2, \dots, s_n)$, we have the (dual) relations*

$$\zeta^\star(\boldsymbol{s}) = \sum_{\boldsymbol{p}} \zeta(\boldsymbol{p}) \qquad \textit{and} \qquad \zeta(\boldsymbol{s}) = \sum_{\boldsymbol{p}} (-1)^{\sigma(\boldsymbol{p})} \zeta^\star(\boldsymbol{p}), \tag{5}$$

where $\boldsymbol{p}$ runs through all indices of the form $(s_1 \circ s_2 \circ \cdots \circ s_n)$ with '$\circ$' being either the symbol ',' or the sign '+', and the exponent $\sigma(\boldsymbol{p})$ denotes the number of signs '+' in $\boldsymbol{p}$. (The total number of such indices $\boldsymbol{p}$ is 2^{l-1}.)

This statement comes as a special case of the Propositions 2 and 3 in [13] for *generalized polylogarithms*

$$\mathrm{Li}_{\boldsymbol{s}}(z) = \sum_{a_1 > \cdots > a_n \ge 1} \frac{z^{a_1}}{a_1^{s_1} \cdots a_n^{s_n}} \qquad \text{and} \qquad \mathrm{Le}_{\boldsymbol{s}}(z) = \sum_{a_1 \ge \cdots \ge a_n \ge 1} \frac{z^{a_1}}{a_1^{s_1} \cdots a_n^{s_n}} \tag{6}$$

at the point $z = 1$.

The following identity was discovered experimentally when we searched for a generalization of (3a) (which is the particular case $l = 1$ of our finding). It is a weighted analogue the expressions on the right-hand sides in (5).

Two-one formula. *For* $k=0,1,2,\dots,$ *denote* $\mu_{2k+1}=(\{2\}_k,1)$*. Then for any admissible index* $\boldsymbol{s}=(s_1,s_2,\dots,s_l)$ *with* odd *entries* $s_1,\dots,s_l$ *the following identities are valid:*

$$\zeta^\star(\mu_{s_1},\mu_{s_2},\dots,\mu_{s_l})=\sum_{\boldsymbol{p}}(-1)^{\sigma(\boldsymbol{p})}2^{l-\sigma(\boldsymbol{p})}\zeta^\star(\boldsymbol{p}) \tag{7a}$$

$$=\sum_{\boldsymbol{p}}2^{l-\sigma(\boldsymbol{p})}\zeta(\boldsymbol{p}), \tag{7b}$$

where, as before, $\boldsymbol{p}$ *runs through all indices of the form* $(s_1\circ s_2\circ\cdots\circ s_l)$ *with '*$\circ$*' being either the symbol ',' or the sign '+', and the exponent* $\sigma(\boldsymbol{p})$ *denotes the number of signs '+' in* $\boldsymbol{p}$*.*

Proof of the equality of the right-hand sides in (7a) *and* (7b). By Proposition 1 for the right-hand side in (7a) we have

$$\begin{aligned}&\sum_{\{\square=,\text{ or }+\}}(-1)^{\#\{\square=+\}}2^{l-\#\{\square=+\}}\zeta^\star(s_1\square s_2\square\cdots\square s_l)\\&\quad=\sum_{\{\square=,\text{ or }+\}}\ \sum_{\{\circ=+\text{ or }\square\}}(-1)^{l-\#\{\circ=\square\}-1}2^{\#\{\circ=\square\}+1}\zeta(s_1\circ s_2\circ\cdots\circ s_l)\end{aligned}$$

which in the notation $r=\#\{\circ\;=\;\square\}+1$ turns out to be

$$\begin{aligned}&=\sum_{n=1}^{l}\Biggl(\sum_{r=n}^{l}\binom{l-n}{l-r}(-1)^{l-r}2^r\Biggr)\sum_{l-\#\{\circ=+\}=n}\zeta(s_1\circ s_2\circ\cdots\circ s_l)\\&=\sum_{n=1}^{l}\Biggl(\sum_{m=0}^{l-n}\binom{l-n}{l-n-m}(-1)^{l-n-m}2^{n+m}\Biggr)\sum_{\#\{\circ=+\}=l-n}\zeta(s_1\circ s_2\circ\cdots\circ s_l)\\&=\sum_{n=1}^{l}2^n\sum_{\#\{\circ=+\}=l-n}\zeta(s_1\circ s_2\circ\cdots\circ s_l)\\&=\sum_{\{\circ=,\text{ or }+\}}2^{l-\#\{\circ=+\}}\zeta(s_1\circ s_2\circ\cdots\circ s_l),\end{aligned}$$

which is exactly the right-hand side of (7b). □

In spite of a nicely simple (but somehow unusual) form of the two-one formula we cannot yet prove it in the full generality. The following two particular cases ($l=2$, and $s_1=3$, $s_2=\cdots=s_{n-2}=1$ with $n\ge3$ arbitrary) strongly support the validity of identities (7a), (7b).

Theorem 1. *For any* $n\ge1$ *and* $1\le i\le n$*,*

$$\zeta^\star(\underbrace{2,\dots,2}_{i},1,\underbrace{2,\dots,2}_{n-i},1)=4\zeta^\star(2i+1,2n-2i+1)-2\zeta(2n+2). \tag{8}$$

Theorem 2. *For any $n \ge 3$,*

(9)

$$\zeta^\star(2,\underbrace{1,\dots,1}_{n-2}) = \sum_{\{\circ=,\ or\ +\}} 2^{n-2-\#\{\circ=+\}}\zeta(3\circ\underbrace{1\circ\cdots\circ 1}_{n-3}) = \sum_{i=2}^{n-1}2^{n-i}\sum_{e_1+e_2+\cdots+e_{n-i}=i-2}\zeta(3+e_1,1+e_2,1+e_3,\dots,1+e_{n-i}),$$

where all e_j are non-negative integers.

We deduce Theorem 2 from the following weighted analogue of Euler's formula (4).

Theorem 3 (**Weighted sum formula**). *For any $n \ge 3$,*

$$\sum_{l=2}^{n-1}2^l\zeta(l,n-l) = (n+1)\zeta(n). \tag{10}$$

Before proceeding with our proofs of Theorems 1–3, let us make some comments on the two-one formula.

The formula

$$\zeta^\star(\{2,\{1\}_{m-1}\}_n,1) = (m+1)\zeta((m+1)n+1)$$

for any positive integers m,n is known (two different proofs are given in [18] and [11]). If $m=1$ it is nothing but formula (3a), while if $m \ge 2$ then its left-hand side equals $\zeta^\star(\{\mu_3,\{\mu_1\}_{m-2}\}_n,\mu_1)$. This together with the two-one formula mean that the corresponding right-hand side in (7a) (equivalently, in (7b)) is expected to have a closed-form evaluation by means of the single zeta value $(m+1)\zeta((m+1)n+1)$, where the integers $m \ge 2$ and $n \ge 1$ are arbitrary.

Using the integral representation of MZSVs,

$$\zeta^\star(\boldsymbol{s}) = \int\cdots\int_{[0,1]^{s_1+\cdots+s_l}}\frac{\mathrm{d}x_1\cdots\mathrm{d}x_{s_1+\cdots+s_l}}{\prod_{i=1}^{l}(1-x_1\cdots x_{s_1+\cdots+s_i})}$$

valid for any admissible index $\boldsymbol{s}=(s_1,\dots,s_l)$, we can write the right-hand side of (7a) as follows:

$$2\int\cdots\int_{[0,1]^{s_1+\cdots+s_l}}\frac{\prod_{i=1}^{l-1}(1+x_1\cdots x_{s_1+\cdots+s_i})}{\prod_{i=1}^{l}(1-x_1\cdots x_{s_1+\cdots+s_i})}\mathrm{d}x_1\cdots\mathrm{d}x_{s_1+\cdots+s_l}. \tag{11a}$$

The change of variable $y_j = x_1\cdots x_j$ for $j=1,\dots,s_1+\cdots+s_l$ gives the integral

$$2\int\cdots\int_{1>y_1>\cdots>y_{s_1+\cdots+s_l}>0}\prod_{i=1}^{l-1}\biggl(\prod_{j=s_1+\cdots+s_{i-1}+1}^{s_1+\cdots+s_i-1}\frac{\mathrm{d}y_j}{y_j}\cdot\frac{(1+y_{s_1+\cdots+s_i})\,\mathrm{d}y_{s_1+\cdots+s_i}}{(1-y_{s_1+\cdots+s_i})y_{s_1+\cdots+s_i}}\biggr) \times\prod_{j=s_1+\cdots+s_{l-1}+1}^{s_1+\cdots+s_l-1}\frac{\mathrm{d}y_j}{y_j}\cdot\frac{\mathrm{d}y_{s_1+\cdots+s_l}}{1-y_{s_1+\cdots+s_l}}, \tag{11b}$$

where the empty sum $s_1+\cdots+s_{i-1}$ for $i=1$ is interpreted as 0. Therefore, any of the two integrals in (11a), (11b) may replace the right-hand sides of (7a) or (7b).

The case $l=1$ of the two-one formula is identity (3a). The case $l=2$ (Theorem 1) reads as

$$\zeta^\star(\{2\}_{s_1},1,\{2\}_{s_2},1)=2\zeta(2s_1+2s_2+2)+4\zeta(2s_1+1,2s_2+1).$$

In particular, the latter identity implies

$$\begin{aligned}&\zeta^\star(\{2\}_{s_1},1,\{2\}_{s_2},1)+\zeta^\star(\{2\}_{s_2},1,\{2\}_{s_1},1)\\&\quad=4\zeta(2s_1+2s_2+2)+4\zeta(2s_1+1,2s_2+1)+4\zeta(2s_2+1,2s_1+1)\\&\quad=4\zeta(2s_1+1)\zeta(2s_2+1)=\zeta^\star(\{2\}_{s_1},1)\zeta^\star(\{2\}_{s_2},1)\end{aligned}$$

whenever $s_1\ge 1$ and $s_2\ge 1$.

On the right-hand side of (7a) and (7b) we have MZSVs and MZVs of length at most l, while the left-hand side involves a single zeta star attached to an index with entries 2 and 1 only (and the number of 1's is equal to l); the latter circumstance is the reason of dubbing the formula as the two-one formula (with no relations to the 21st century).

We stress that neither the two-one formula nor its special cases treated in Theorems 1 and 2 are specializations of identities for polylogarithms (6) as may be seen from the fact $\operatorname{Le}_{2,1}(z)\ne 2\operatorname{Le}_3(z)$ for $z\ne 1$ (cf. formula (3a) with $n=1$).

Zlobin's theorem [17, Theorem 3] implies that under certain natural restrictions on positive integer parameters $b_i>a_i$, $i=1,\dots,m$, and c_j, $j=1,\dots,l$, and with a choice of integers $2\le r_1<r_2<\dots<r_l=m$ such that $r_{j+1}-r_j$ is 1 or 2 for all $j=1,\dots,l-1$, the multiple integral

$$\int\cdots\int_{[0,1]^m}\frac{\prod_{j=1}^m x_i^{a_i-1}(1-x_i)^{b_i-a_i-1}}{\prod_{j=1}^l(1-zx_1x_2\cdots x_{r_j})^{c_j}}\,\mathrm{d}x_1\cdots\mathrm{d}x_m \tag{12}$$

is a $\mathbb{Q}[z^{-1}]$-linear combination of generalized polylogarithms $\operatorname{Le}_{\boldsymbol{s}}(z)$ with indices $\boldsymbol{s}$ of weight at most m and length at most l whose entries are only 2 and 1. In several cases the method in [17] allows to control the number of 1's in these indices (for instance, the case when only the zeta stars from (3a) occur after specialization $z=1$ is treated in [17] in details). The integrals (12) and formulae (7a), (7b) may therefore become useful in diophantine study of multiple zeta (star) values of depth higher than 1.

2. Depth 2 case of the two-one problem: Proof of Theorem 1

For $a\ge c>0$, we define the harmonic sum

$$H(a,c)=\sum_{\substack{j=1\\ j\ne a}}^{c}\frac{1}{a-j} \tag{13}$$

and interpret $H(\infty,c)$ and $H(a,0)$ as zero.

Lemma 1. *If $B\ge C$, we have*

$$\sum_{\substack{A\ge a\ge B\\ C\ge c\ge D}}\frac{1}{a^2c}=\sum_{C\ge c\ge D}\frac{H(B,c)-H(A+1,c)}{c^2}-\sum_{A\ge a\ge B}\frac{H(a,C)-H(a,D-1)}{a^2}+\delta_{B,C}\frac{2}{B^3}. \tag{14}$$

Proof. It follows that

$$\sum_{\substack{C\ge c\ge D\\ c\ne a}}\frac{1}{a-c}=H(a,C)-H(a,D-1)$$

whenever $a\ge C$, and

$$\sum_{\substack{A\ge a\ge B\\ a\ne c}}\left(\frac{1}{a-c}-\frac{1}{a}\right)=H(B,c)-H(A+1,c)+\delta_{c,B}\frac{1}{c}$$

whenever $c\le B$. Furthermore, for $a\ne c$ the following partial fraction decomposition is valid:

$$\frac{1}{a^2c}=\left(\frac{1}{a-c}-\frac{1}{a}\right)\cdot\frac{1}{c^2}-\frac{1}{a-c}\cdot\frac{1}{a^2}. \tag{15}$$

Thus, under the condition $B\ge C$, we get

$$\begin{aligned}\sum_{\substack{A\ge a\ge B\\ C\ge c\ge D}}\frac{1}{a^2c}&=\sum_{\substack{A\ge a\ge B\\ C\ge c\ge D\\ a\ne c}}\left(\left(\frac{1}{a-c}-\frac{1}{a}\right)\cdot\frac{1}{c^2}-\frac{1}{a-c}\cdot\frac{1}{a^2}\right)+\delta_{B,C}\frac{1}{B^3}\\ &=\sum_{C\ge c\ge D}\frac{H(B,c)-H(A+1,c)}{c^2}\\ &\qquad-\sum_{A\ge a\ge B}\frac{H(a,C)-H(a,D-1)}{a^2}+\delta_{B,C}\frac{2}{B^3}.\end{aligned}$$

which is the desired statement. □

Remark 1. The proof of the cyclic sum theorem in [11] exploits the general forms of identities (15) and (14) which are

$$\sum_{l=1}^{m-1}\frac{1}{a^{m+1-l}c^l}=\left(\frac{1}{a-c}-\frac{1}{a}\right)\cdot\frac{1}{c^m}-\frac{1}{a-c}\cdot\frac{1}{a^m}$$

and

$$\begin{aligned}\sum_{\substack{A\ge a\ge B\\ C\ge c\ge D}}\sum_{l=1}^{m-1}\frac{1}{a^{m+1-l}c^l}&=\sum_{C\ge c\ge D}\frac{H(B,c)-H(A+1,c)}{c^m}\\ &\qquad-\sum_{A\ge a\ge B}\frac{H(a,C)-H(a,D-1)}{a^m}+\delta_{B,C}\frac{m}{B^{m+1}},\end{aligned}$$

respectively, although the function (13) is not used there in an explicit form (it was introduced later by D. Zagier in his unpublished note on the proof in [11]). We are wondering if the two-one formula may be generalized to some kind of 'multiple cyclic formula'.

Lemma 2. *For any $i\ge 1$ and $j\ge 0$ we have*

$$\zeta^\star(2i+1,\underbrace{2,\dots,2}_{j},1)=-\sum_{a_0\ge a_1\ge\dots\ge a_j\ge 1}\frac{H(a_0+1,a_j)}{a_0^{2i+1}a_1^2\cdots a_j^2}+2\zeta^\star(2i+1,2j+1). \tag{16}$$

Proof. If $j \geq 1$, we apply Lemma 1 with $a = a_1$ and $c = a_{j+1}$:

$$
\begin{aligned}
(17)\qquad \zeta^\star(2i+1, \underbrace{2, \ldots, 2}_{j}, 1) &= \sum_{a_0 \geq a_1 \geq \cdots \geq a_{j+1} \geq 1} \frac{1}{a_0^{2i+1} a_1^2 \cdots a_j^2 a_{j+1}} \\
&= \sum_{a_0 \geq a_2 \geq \cdots \geq a_{j+1} \geq 1} \frac{H(a_2, a_{j+1}) - H(a_0+1, a_{j+1})}{a_0^{2i+1} a_2^2 \cdots a_j^2 a_{j+1}^2} \\
&\quad - \sum_{a_0 \geq a_1 \geq \cdots \geq a_j \geq 1} \frac{H(a_1, a_j)}{a_0^{2i+1} a_1^2 a_2^2 \cdots a_j^2} + 2 \sum_{a_0 \geq a_1 \geq 1} \frac{1}{a_0^{2i+1} a_1^{2j+1}} \\
&= - \sum_{a_0 \geq a_1 \geq \cdots \geq a_j \geq 1} \frac{H(a_0+1, a_j)}{a_0^{2i+1} a_1^2 \cdots a_j^2} + 2\zeta^\star(2i+1, 2j+1).
\end{aligned}
$$

Although the application of (14) is possible if $j \geq 1$, it is easy to see that the resulting formula (16) trivially holds for $j = 0$, since

$$
\sum_{a_0 \geq 1} \frac{H(a_0+1, a_0)}{a_0^{2i+1}} = \sum_{a_0 > b \geq 0} \frac{1}{a_0^{2i+1}(a_0 - b)} = \sum_{a_0 \geq a_1 \geq 1} \frac{1}{a_0^{2i+1} a_1} = \zeta^\star(2i+1, 1)
$$

in this case. □

We will also need the following formula whose proof is just changing the condition $a_0 \geq a_1$ by $a_0 > a_1$ in (17).

Lemma 3. *For any $i \geq 1$ and $j \geq 0$ we have*

$$
\sum_{a_0 > a_1 \geq \cdots \geq a_{j+1} \geq 1} \left(\frac{1}{a_0 - a_{j+1}} - \frac{1}{a_0} \right) \frac{1}{a_0^{2i-1} a_1^2 \cdots a_{j+1}^2} = 2\zeta(2i+1, 2j+1).
$$

Proof. We proceed as in (17):

$$
\begin{aligned}
&\sum_{a_0 > a_1 \geq \cdots \geq a_{j+1} \geq 1} \frac{1}{a_0^{2i+1} a_1^2 \cdots a_j^2 a_{j+1}} \\
&\quad = - \sum_{a_0 > a_1 \geq \cdots \geq a_j \geq 1} \frac{H(a_0, a_j)}{a_0^{2i+1} a_1^2 \cdots a_j^2} + 2 \sum_{a_0 > a_1 \geq 1} \frac{1}{a_0^{2i+1} a_1^{2j+1}} \\
&\quad = - \sum_{a_0 > a_1 \geq \cdots \geq a_{j+1} \geq 1} \frac{1}{a_0^{2i+1} a_1^2 \cdots a_j^2 (a_0 - a_{j+1})} + 2\zeta(2i+1, 2j+1);
\end{aligned}
$$

again the formula remains valid $j = 0$. Then we use the identity

$$
\frac{1}{a_0^{2i+1} a_{j+1}} + \frac{1}{a_0^{2i+1}(a_0 - a_{j+1})} = \frac{1}{a_0^{2i} a_{j+1}(a_0 - a_{j+1})} = \left(\frac{1}{a_0 - a_{j+1}} - \frac{1}{a_0} \right) \frac{1}{a_0^{2i-1} a_{j+1}^2}
$$

to conclude with the desired claim. □

Lemma 4. *Given $n \geq 1$, for any $1 \leq i \leq n$ the following identity holds:*

$$
\begin{aligned}
(18)\qquad \zeta^\star(\underbrace{2, \ldots, 2}_{i}, 1, \underbrace{2, \ldots, 2}_{n-i}, 1) &= 4\zeta^\star(2i+1, 2n-2i+1) \\
&\quad + \sum_{a_0 \geq \cdots \geq a_n \geq 1} \frac{H(a_0, a_i - 1)}{a_0^2 \cdots a_{n-1}^2 a_n} - 2 \sum_{a_0 \geq a_{i+1} \geq \cdots \geq a_n \geq 1} \frac{H(a_0+1, a_n)}{a_0^{2i+1} a_{i+1}^2 \cdots a_n^2}.
\end{aligned}
$$

Proof. We use Lemma 1 with $a=a_0$ and $c=a_i$:

$$
\begin{aligned}
\zeta^\star(\underbrace{2,\dots,2}_{i},1,\underbrace{2,\dots,2}_{n-i},1) &= \sum_{a_0\ge\dots\ge a_i\ge a_{i+1}\ge\dots\ge a_{n+1}\ge1}\frac{1}{a_1^2\cdots a_{i-1}^2a_ia_{i+1}^2\cdots a_n^2a_{n+1}} \\
&= \sum_{a_1\ge\dots\ge a_{i-1}\ge a_i\ge a_{i+1}\ge\dots\ge a_{n+1}\ge1}\frac{H(a_1,a_i)}{a_1^2\cdots a_{i-1}^2a_i^2\,a_{i+1}^2\cdots a_n^2a_{n+1}} \\
&\quad - \sum_{a_0\ge a_1\ge\dots\ge a_{i-1}\ge a_{i+1}\ge\dots\ge a_{n+1}\ge1}\frac{H(a_0,a_{i-1})-H(a_0,a_{i+1}-1)}{a_0^2a_1^2\cdots a_{i-1}^2\,a_{i+1}^2\cdots a_n^2a_{n+1}} \\
&\quad + 2\sum_{a_0\ge a_{i+1}\ge\dots\ge a_{n+1}\ge1}\frac{1}{a_0^{2i+1}a_{i+1}^2\cdots a_n^2a_{n+1}} \\
&= \sum_{a_0\ge a_1\ge\dots\ge a_{i-1}\ge a_{i+1}\ge\dots\ge a_{n+1}\ge1}\frac{H(a_0,a_{i+1}-1)}{a_0^2a_1^2\cdots a_{i-1}^2\,a_{i+1}^2\cdots a_n^2a_{n+1}} \\
&\quad + 2\zeta^\star(2i+1,\underbrace{2,\dots,2}_{n-i},1) \\
&= \sum_{a_0\ge\dots\ge a_n\ge1}\frac{H(a_0,a_i-1)}{a_0^2\cdots a_{n-1}^2a_n}+2\zeta^\star(2i+1,\underbrace{2,\dots,2}_{n-i},1),
\end{aligned}
$$

and for the latter zeta star term we apply Lemma 2. □

Lemma 5. *We have*

$$\zeta^\star(\underbrace{2,\dots,2}_{n},1,1)=4\zeta^\star(2n+1,1)-2\zeta(2n+2).$$

To prove Lemma 5 we will require an additional identity.

Lemma 6. *For any positive integers l and m satisfying $l>m$, the following identity is valid:*

$$\sum_{b=1}^{m}\frac{H(l,b)}{l-b}=\sum_{a=l}^{\infty}\biggl(\frac{1}{a-m}-\frac{1}{a}\biggr)H(a+1,m+1). \tag{19}$$

Proof. For the right-hand side in (19) we have

(20)

$$
\begin{aligned}
&\sum_{a=l}^{\infty}\biggl(\frac{1}{a-m}-\frac{1}{a}\biggr)H(a+1,m+1) \\
&\quad= \sum_{a=l}^{\infty}\sum_{d=0}^{m}\biggl(\frac{1}{a-m}-\frac{1}{a}\biggr)\frac{1}{a-d} = \sum_{a=l}^{\infty}\sum_{d=0}^{m}\biggl(\frac{1}{(a-m)(a-d)}-\frac{1}{a(a-d)}\biggr) \\
&\quad= \sum_{a=l}^{\infty}\biggl(\frac{1}{(a-m)^2}+\sum_{d=0}^{m-1}\frac{1}{m-d}\biggl(\frac{1}{a-m}-\frac{1}{a-d}\biggr) \\
&\qquad\quad -\frac{1}{a^2}-\sum_{d=1}^{m}\frac{1}{d}\biggl(\frac{1}{a-d}-\frac{1}{a}\biggr)\biggr) \\
&\quad= \sum_{b=1}^{m}\frac{1}{(l-b)^2}+\sum_{d=0}^{m-1}\frac{1}{m-d}\sum_{a=l}^{\infty}\biggl(\frac{1}{a-m}-\frac{1}{a-d}\biggr)-\sum_{d=1}^{m}\frac{1}{d}\sum_{a=l}^{\infty}\biggl(\frac{1}{a-d}-\frac{1}{a}\biggr)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{b=1}^{m} \frac{1}{(l-b)^2} + \sum_{d=0}^{m-1} \frac{1}{m-d} \sum_{c=d+1}^{m} \frac{1}{l-c} - \sum_{d=1}^{m} \frac{1}{d} \sum_{c=1}^{d} \frac{1}{l-c} \\
&= \sum_{b=1}^{m} \frac{1}{(l-b)^2} + \sum_{b=1}^{m} \frac{1}{b} \sum_{c=m-b+1}^{m} \frac{1}{l-c} - \sum_{b=1}^{m} \frac{1}{b} \sum_{c=1}^{b} \frac{1}{l-c} \\
&= \sum_{b=1}^{m} \frac{1}{(l-b)^2} + \sum_{b=1}^{m} \frac{1}{b} \biggl(\sum_{c=m-b+1}^{m} - \sum_{c=1}^{b} \biggr) \frac{1}{l-c},
\end{aligned}
$$

while the right-hand side can be written as follows:

$$
\begin{aligned}
\sum_{b=1}^{m} \frac{H(l,b)}{l-b} &= \sum_{b=1}^{m} \sum_{c=1}^{b} \frac{1}{(l-b)(l-c)} \\
&= \sum_{b=1}^{m} \frac{1}{(l-b)^2} + \sum_{b=1}^{m} \sum_{c=1}^{b-1} \frac{1}{b-c} \biggl(\frac{1}{l-b} - \frac{1}{l-c} \biggr) \\
&= \sum_{b=1}^{m} \frac{1}{(l-b)^2} + \sum_{d=1}^{m} \sum_{c=1}^{d-1} \frac{1}{d-c} \, \frac{1}{l-d} - \sum_{d=1}^{m} \sum_{c=1}^{d-1} \frac{1}{d-c} \, \frac{1}{l-c} \\
&= \sum_{b=1}^{m} \frac{1}{(l-b)^2} + \sum_{d=1}^{m} \frac{1}{l-d} \sum_{c=1}^{d-1} \frac{1}{d-c} - \sum_{c=1}^{m-1} \frac{1}{l-c} \sum_{d=c+1}^{m} \frac{1}{d-c} \\
&= \sum_{b=1}^{m} \frac{1}{(l-b)^2} + \sum_{d=1}^{m} \frac{1}{l-d} \sum_{b=1}^{d-1} \frac{1}{b} - \sum_{c=1}^{m-1} \frac{1}{l-c} \sum_{b=1}^{m-c} \frac{1}{b} \\
&= \sum_{b=1}^{m} \frac{1}{(l-b)^2} + \sum_{b=1}^{m} \frac{1}{b} \sum_{d=b+1}^{m} \frac{1}{l-d} - \sum_{b=1}^{m} \frac{1}{b} \sum_{c=1}^{m-b} \frac{1}{l-c} \\
&= \sum_{b=1}^{m} \frac{1}{(l-b)^2} + \sum_{b=1}^{m} \frac{1}{b} \biggl(\sum_{c=b+1}^{m} - \sum_{c=1}^{m-b} \biggr) \frac{1}{l-c}.
\end{aligned} \tag{21}
$$

Comparing the right-hand sides of the resulting expressions in (20) and (21) and using

$$
\sum_{c=m-b+1}^{m} - \sum_{c=1}^{b} = \biggl(\sum_{c=1}^{m} - \sum_{c=1}^{m-b} \biggr) - \biggl(\sum_{c=1}^{m} - \sum_{c=b+1}^{m} \biggr) = \sum_{c=b+1}^{m} - \sum_{c=1}^{m-b}
$$

we derive the required identity (19). $\square$

Proof of Lemma 5. We are first interested in the finite multiple sum

$$F_n(a,b)=\sum_{\substack{a_0\ge\cdots\ge a_n\\ a_0\le a,\ a_n>b}}\frac{1}{a_0^2\cdots a_{n-1}^2a_n} \tag{22}$$
$$=\sum_{\substack{a_1\ge\cdots\ge a_n\\ a_1\le a,\ a_n>b}}\frac{H(a_1,a_n)-H(a+1,a_n)}{a_1^2\cdots a_{n-1}^2a_n^2}$$
$$-\sum_{\substack{a_0\ge\cdots\ge a_{n-1}\\ a_0\le a,\ a_{n-1}>b}}\frac{H(a_0,a_{n-1})-H(a_0,b)}{a_0^2a_1^2\cdots a_{n-1}^2}+2\sum_{a\ge a_0>b}\frac{1}{a_0^{2n+1}}$$
$$=\sum_{\substack{a_0\ge\cdots\ge a_{n-1}\\ a_0\le a,\ a_{n-1}>b}}\frac{H(a_0,b)-H(a+1,a_{n-1})}{a_0^2a_1^2\cdots a_{n-1}^2}+2\sum_{a\ge a_0>b}\frac{1}{a_0^{2n+1}}.$$

Multiplying both expressions for the sum in (22) by $1/(a-b)-1/(a-b+1)$, notifying that

$$\sum_{a\ge a_0}\left(\frac{1}{a-b}-\frac{1}{a-b+1}\right)=\frac{1}{a_0-b}$$

and

$$\sum_{b=1}^{a_n-1}\left(\frac{1}{a-b}-\frac{1}{a-b+1}\right)=\frac{1}{a-a_n+1}-\frac{1}{a},$$

and summing over all a and b, $a>b\ge1$, we obtain

$$\sum_{a_0\ge\cdots\ge a_n>b\ge1}\frac{1}{(a_0-b)a_0^2\cdots a_{n-1}^2a_n}=\sum_{a_0\ge\cdots\ge a_{n-1}>b\ge1}\frac{H(a_0,b)}{(a_0-b)a_0^2a_1^2\cdots a_{n-1}^2} \tag{23}$$
$$-\sum_{a\ge a_0\ge\cdots\ge a_{n-1}>1}\left(\frac{1}{a-a_{n-1}+1}-\frac{1}{a}\right)\frac{H(a+1,a_{n-1})}{a_0^2a_1^2\cdots a_{n-1}^2}$$
$$+2\sum_{a_0>b\ge1}\frac{1}{(a_0-b)a_0^{2n+1}}.$$

Applying Lemma 6 with $l=a_0$ and $m=a_{n-1}-1$ we derive from (23) that

$$\sum_{a_0\ge\cdots\ge a_n>b\ge1}\frac{1}{(a_0-b)a_0^2\cdots a_{n-1}^2a_n}=2\sum_{a_0>b\ge1}\frac{1}{(a_0-b)a_0^{2n+1}}.$$

Finally, from Lemma 4 in the case $i=n$ we obtain

$$\zeta^\star(\underbrace{2,\dots,2}_{n},1,1)=4\zeta^\star(2n+1,1)+2\sum_{a_0>b\ge1}\frac{1}{(a_0-b)a_0^{2n+1}}-2\sum_{a_0>b\ge0}\frac{1}{(a_0-b)a_0^{2n+1}}$$
$$=4\zeta^\star(2n+1,1)-2\zeta(2n+2),$$

which is the desired identity. □

Lemma 7. *For $1 \le i < n$ we have*

(24)
$$\sum_{a_0\ge\cdots\ge a_n\ge 1} \frac{H(a_0,a_i-1)-H(a_0,a_{i+1}-1)}{a_0^2\cdots a_{n-1}^2 a_n} = 2\sum_{a_0>a_{i+1}\ge\cdots\ge a_n\ge 1} \frac{H(a_0,a_n)}{a_0^{2i+1}a_{i+1}^2\cdots a_n^2} - \sum_{a_0\ge\cdots\ge a_n\ge 1}\left(\frac{H(a_0,a_n)}{a_0^2\cdots a_{i-1}^2 a_i a_{i+1}^2\cdots a_n^2} - \frac{H(a_0,a_n)}{a_0^2\cdots a_i^2 a_{i+1} a_{i+2}^2\cdots a_n^2}\right).$$

Proof. We use the identity in (22) for $F_i(a, a_{i+1}-1)$ (that is, for n replaced by i and b replaced by $a_{i+1}-1$). Multiplying it by $1/(a_{i+1}^2\cdots a_n^2)$ and summing over $a_{i+1},\dots,a_{n+1}$ satisfying $a_{i+1}\ge\cdots\ge a_n\ge a_{n+1}$ results in the following identity:

$$\begin{aligned}\widetilde{F}_i(a,a_{n+1}) &= \sum_{\substack{a_0\ge\cdots\ge a_i\ge a_{i+1}\ge\cdots\ge a_n\\ a_0\le a,\ a_n\ge a_{n+1}}} \frac{1}{a_0^2\cdots a_{i-1}^2 a_i a_{i+1}^2\cdots a_n^2}\\ &= \sum_{\substack{a_1\ge\cdots\ge a_i\ge a_{i+1}\ge\cdots\ge a_n\\ a_1\le a,\ a_n\ge a_{n+1}}} \frac{H(a_1,a_{i+1}-1)-H(a+1,a_i)}{a_1^2\cdots a_n^2}\\ &\quad + 2\sum_{\substack{a_1\ge a_{i+1}\ge\cdots\ge a_n\\ a_1\le a,\ a_n\ge a_{n+1}}} \frac{1}{a_1^{2i+1}a_{i+1}^2\cdots a_n^2}.\end{aligned}$$

As before, we multiply both sides by $1/(a-a_{n+1})-1/(a-a_{n+1}+1)$ and sum over all $a\ge a_{n+1}\ge 1$ to obtain

$$\begin{aligned}&\sum_{a_0\ge\cdots\ge a_n\ge 1} \frac{H(a_0,a_n)}{a_0^2\cdots a_{i-1}^2 a_i a_{i+1}^2\cdots a_n^2}\\ &= \sum_{\substack{a_0\ge\cdots\ge a_n\ge a_{n+1}\ge 1\\ a_0\ne a_{n+1}}} \frac{1}{(a_0-a_{n+1})a_0^2\cdots a_{i-1}^2 a_i a_{i+1}^2\cdots a_n^2}\\ &= \sum_{\substack{a_1\ge\cdots\ge a_n\ge a_{n+1}\ge 1\\ a_1\ne a_{n+1}}} \frac{H(a_1,a_{i+1}-1)}{(a_1-a_{n+1})a_1^2\cdots a_{n-1}^2 a_n^2}\\ &\quad - \sum_{\substack{a\ge a_1\ge\cdots\ge a_n\ge 1\\ a\ne a_n}} \left(\frac{1}{a-a_n}-\frac{1}{a}\right)\frac{H(a+1,a_i)}{a_1^2\cdots a_{n-1}^2 a_n^2}\\ &\quad + 2\sum_{\substack{a_1\ge a_{i+1}\ge\cdots\ge a_{n+1}\ge 1\\ a_1\ne a_{n+1}}} \frac{1}{(a_1-a_{n+1})a_1^{2i+1}a_{i+1}^2\cdots a_n^2}\end{aligned}$$

(we set $a_0 = a$ in the second sum)

$$
\begin{aligned}
&= \sum_{\substack{a_1\ge\dots\ge a_n\ge a_{n+1}\ge1\\ a_1\ne a_{n+1}}} \frac{H(a_1,a_{i+1}-1)}{(a_1-a_{n+1})a_1^2\cdots a_{n-1}^2a_n^2} \\
&\quad - \sum_{\substack{a_0\ge a_1\ge\dots\ge a_n\ge1\\ a_0\ne a_n}} \left(\frac{1}{a_0-a_n}-\frac{1}{a_0}\right)\frac{H(a_0,a_i-1)+1/a_0}{a_1^2\cdots a_{n-1}^2a_n^2} \\
&\quad + 2\sum_{a_1\ge a_{i+1}\ge\dots\ge a_n\ge1} \frac{H(a_1,a_n)}{a_1^{2i+1}a_{i+1}^2\cdots a_n^2} \\
&= \sum_{\substack{a_0\ge\dots\ge a_n\ge1\\ a_0\ne a_n}} \frac{H(a_0,a_i-1)}{(a_0-a_n)a_0^2\cdots a_{n-1}^2} \\
&\quad - \sum_{\substack{a_0\ge a_1\ge\dots\ge a_n\ge1\\ a_0\ne a_n}} \left(\frac{1}{a_0-a_n}-\frac{1}{a_0}\right)\frac{H(a_0,a_i-1)+1/a_0}{a_1^2\cdots a_{n-1}^2a_n^2} \\
&\quad + 2\sum_{a_0\ge a_{i+1}\ge\dots\ge a_n\ge1} \frac{H(a_0,a_n)}{a_0^{2i+1}a_{i+1}^2\cdots a_n^2}.
\end{aligned}
$$

Using the partial fraction decomposition (15) with $x = a_0$ and $y = a_n$ reduces the latter expression to

(25)
$$
\begin{aligned}
&\sum_{a_0\ge\dots\ge a_n\ge1} \frac{H(a_0,a_n)}{a_0^2\cdots a_{i-1}^2a_ia_{i+1}^2\cdots a_n^2} \\
&\quad = -\sum_{\substack{a_0\ge\dots\ge a_n\ge1\\ a_0\ne a_n}} \frac{H(a_0,a_i-1)}{a_0^2\cdots a_{n-1}^2a_n} - \sum_{\substack{a_0\ge a_1\ge\dots\ge a_n\ge1\\ a_0\ne a_n}} \left(\frac{1}{a_0-a_n}-\frac{1}{a_0}\right)\frac{1/a_0}{a_1^2\cdots a_{n-1}^2a_n^2} \\
&\qquad + 2\sum_{a_0\ge a_{i+1}\ge\dots\ge a_n\ge1} \frac{H(a_0,a_n)}{a_0^{2i+1}a_{i+1}^2\cdots a_n^2} \\
&\quad = -\sum_{a_0\ge\dots\ge a_n\ge1} \frac{H(a_0,a_i-1)}{a_0^2\cdots a_{n-1}^2a_n} + \sum_{a_0\ge1}\frac{H(a_0,a_0-1)}{a_0^{2n+1}} \\
&\qquad - \sum_{\substack{a_0\ge a_1\ge\dots\ge a_n\ge1\\ a_0\ne a_n}} \frac{1}{(a_0-a_n)a_0a_1^2\cdots a_n^2} + \sum_{a_0\ge a_1\ge\dots\ge a_n\ge1}\frac{1}{a_0^2a_1^2\cdots a_n^2} \\
&\qquad - \sum_{a_0\ge1}\frac{1}{a_0^{2n+2}} + 2\sum_{a_0\ge a_{i+1}\ge\dots\ge a_n\ge1} \frac{H(a_0,a_n)}{a_0^{2i+1}a_{i+1}^2\cdots a_n^2}.
\end{aligned}
$$

Subtracting now the expression in (25) from the one corresponding to the i replaced by $i+1$ we finally obtain

$$\begin{aligned}
&\sum_{a_0\ge\cdots\ge a_n\ge 1}\left(\frac{H(a_0,a_n)}{a_0^2\cdots a_i^2a_{i+1}a_{i+2}^2\cdots a_n^2}-\frac{H(a_0,a_n)}{a_0^2\cdots a_{i-1}^2a_ia_{i+1}^2\cdots a_n^2}\right)\\
&\quad=\sum_{a_0\ge\cdots\ge a_n\ge 1}\frac{H(a_0,a_i-1)-H(a_0,a_{i+1}-1)}{a_0^2\cdots a_{n-1}^2a_n}\\
&\qquad+2\sum_{a_0\ge a_{i+2}\ge\cdots\ge a_n\ge 1}\frac{H(a_0,a_n)}{a_0^{2i+3}a_{i+2}^2\cdots a_n^2}-2\sum_{a_0\ge a_{i+1}\ge\cdots\ge a_n\ge 1}\frac{H(a_0,a_n)}{a_0^{2i+1}a_{i+1}^2\cdots a_n^2},
\end{aligned}$$

which gives us the required formula (24). ☐

Lemma 8. *For $0\le i<n$ we have*

$$\begin{aligned}
&\sum_{a_0\ge\cdots\ge a_n\ge 1}\left(\frac{H(a_0,a_n)}{a_0^2\cdots a_{i-1}^2a_ia_{i+1}^2\cdots a_n^2}-\frac{H(a_0,a_n)}{a_0^2\cdots a_i^2a_{i+1}a_{i+2}^2\cdots a_n^2}\right)\\
&\quad=\zeta^\star(\underbrace{2,\dots,2}_{i+1},1,\underbrace{2,\dots,2}_{n-i-1},1)-2\zeta(2n+2).
\end{aligned}$$

Proof. We apply Lemma 1, this time with $a=a_0$ and $c=a_{n+1}$:

$$\begin{aligned}
\zeta^\star(\underbrace{2,\dots,2}_{i+1},1,\underbrace{2,\dots,2}_{n-i-1},1)&=\sum_{a_0\ge\cdots\ge a_{n+1}\ge 1}\frac{1}{a_0^2\cdots a_i^2a_{i+1}a_{i+2}^2\cdots a_n^2a_{n+1}}\\
&=\sum_{a_1\ge\cdots\ge a_{n+1}\ge 1}\frac{H(a_1,a_{n+1})}{a_1^2\cdots a_i^2a_{i+1}a_{i+2}^2\cdots a_{n+1}^2}\\
&\qquad-\sum_{a_0\ge\cdots\ge a_n\ge 1}\frac{H(a_0,a_n)}{a_0^2\cdots a_i^2a_{i+1}a_{i+2}^2\cdots a_n^2}+2\zeta(2n+2)\\
&=\sum_{a_0\ge\cdots\ge a_n\ge 1}\frac{H(a_0,a_n)}{a_0^2\cdots a_{i-1}^2a_ia_{i+1}^2\cdots a_n^2}\\
&\qquad-\sum_{a_0\ge\cdots\ge a_n\ge 1}\frac{H(a_0,a_n)}{a_0^2\cdots a_i^2a_{i+1}a_{i+2}^2\cdots a_n^2}+2\zeta(2n+2),
\end{aligned}$$

from which the desired identity follows. ☐

We are now ready to prove Theorem 1.

Proof of Theorem 1. We will use the descending induction on $i=n,n-1,\dots,1$. In the case $i=n$ (induction base) the identity of the theorem is already shown in Lemma 5. Therefore, we assume that $i<n$ and that identity (8) is proved with i replaced by $i+1$, that is,

$$\zeta^\star(\underbrace{2,\dots,2}_{i+1},1,\underbrace{2,\dots,2}_{n-i-1},1)=4\zeta^\star(2i+3,2n-2i-1)-2\zeta(2n+2). \tag{26}$$

We substitute expressions

$$\begin{aligned}
&\sum_{a_0>a_{i+1}\ge\dots\ge a_n\ge1}\frac{H(a_0,a_n)}{a_0^{2i+1}a_{i+1}^2\cdots a_n^2}\\
&\quad=\sum_{a_0>a_{i+1}\ge\dots\ge a_n\ge1}\frac{H(a_0+1,a_n)}{a_0^{2i+1}a_{i+1}^2\cdots a_n^2}\\
&\qquad+\sum_{a_0>a_{i+1}\ge\dots\ge a_n\ge1}\biggl(\frac{1}{a_0-a_n}-\frac{1}{a_0}\biggr)\frac{1}{a_0^{2i+1}a_{i+1}^2\cdots a_n^2}\\
&\quad=\sum_{a_0>a_{i+1}\ge\dots\ge a_n\ge1}\frac{H(a_0+1,a_n)}{a_0^{2i+1}a_{i+1}^2\cdots a_n^2}+2\zeta(2i+3,2n-2i-1)
\end{aligned}$$

followed from Lemma 3 and

$$\begin{aligned}
&\sum_{a_0\ge\dots\ge a_n\ge1}\biggl(\frac{H(a_0,a_n)}{a_0^2\cdots a_{i-1}^2a_ia_{i+1}^2\cdots a_n^2}-\frac{H(a_0,a_n)}{a_0^2\cdots a_i^2a_{i+1}a_{i+2}^2\cdots a_n^2}\biggr)\\
&\quad=4\zeta^\star(2i+3,2n-2i-1)-4\zeta(2n+2)=4\zeta(2i+3,2n-2i-1)
\end{aligned}$$

followed from Lemma 8 and (26) into the identity of Lemma 7 to get

$$\begin{aligned}
&\sum_{a_0\ge\dots\ge a_n\ge1}\frac{H(a_0,a_i-1)-H(a_0,a_{i+1}-1)}{a_0^2\cdots a_{n-1}^2a_n}=2\sum_{a_0>a_{i+1}\ge\dots\ge a_n\ge1}\frac{H(a_0+1,a_n)}{a_0^{2i+1}a_{i+1}^2\cdots a_n^2}\\
&\quad=2\sum_{a_0\ge a_{i+1}\ge\dots\ge a_n\ge1}\frac{H(a_0+1,a_n)}{a_0^{2i+1}a_{i+1}^2\cdots a_n^2}-2\sum_{a_0\ge a_{i+2}\ge\dots\ge a_n\ge1}\frac{H(a_0+1,a_n)}{a_0^{2i+3}a_{i+2}^2\cdots a_n^2}.
\end{aligned}$$

The last identity may be written as

$$\begin{aligned}
&\sum_{a_0\ge\dots\ge a_n\ge1}\frac{H(a_0,a_i-1)}{a_0^2\cdots a_{n-1}^2a_n}-2\sum_{a_0\ge a_{i+1}\ge\dots\ge a_n\ge1}\frac{H(a_0+1,a_n)}{a_0^{2i+1}a_{i+1}^2\cdots a_n^2}\\
&\quad=\sum_{a_0\ge\dots\ge a_n\ge1}\frac{H(a_0,a_{i+1}-1)}{a_0^2\cdots a_{n-1}^2a_n}-2\sum_{a_0\ge a_{i+2}\ge\dots\ge a_n\ge1}\frac{H(a_0+1,a_n)}{a_0^{2i+3}a_{i+2}^2\cdots a_n^2},
\end{aligned}$$

where the right-hand side equals $-2\zeta(2n+2)$ by Lemma 4 applied to $i+1$ instead of i and (26), so does the left-hand side:

(27)
$$\sum_{a_0\ge\dots\ge a_n\ge1}\frac{H(a_0,a_i-1)}{a_0^2\cdots a_{n-1}^2a_n}-2\sum_{a_0\ge a_{i+1}\ge\dots\ge a_n\ge1}\frac{H(a_0+1,a_n)}{a_0^{2i+1}a_{i+1}^2\cdots a_n^2}=-2\zeta(2n+2).$$

Finally, from (18) and (27) we obtain identity (8) for the given i, just completing the proof of Theorem 1. □

3. Weighted sum formula: Proof of Theorems 2 and 3

We first prove the weighted sum formula.

Proof of Theorem 3. Recall two standard ways of computing the product of two zeta values [19]: the shuffle product (coming from the representation of MZVs by iterated integrals [14])

$$\zeta(m)\zeta(n-m)=\sum_{l=2}^{n-1}\left(\binom{l-1}{m-1}+\binom{l-1}{n-m-1}\right)\zeta(l,n-l) \tag{28a}$$

and the harmonic (or stuffle) product (originated by the series representation [7])

$$\zeta(m)\zeta(n-m)=\zeta(n)+\zeta(m,n-m)+\zeta(n-m,m); \tag{28b}$$

here $m>1$ and $n>m+1$ are arbitrary integers.[2] Summing up the right-hand sides in (28a) and (28b) over $m=2,3,\dots,n-2$ we get

$$\begin{aligned}\sum_{l=2}^{n-1}\sum_{m=2}^{n-2}\left(\binom{l-1}{m-1}+\binom{l-1}{n-m-1}\right)\zeta(l,n-l)\\ =(n-3)\zeta(n)+\sum_{m=2}^{n-2}\big(\zeta(m,n-m)+\zeta(n-m,m)\big)\end{aligned}$$

which reduces to

$$2\sum_{l=2}^{n-1}\sum_{m=2}^{\min\{l,n-2\}}\binom{l-1}{m-1}\zeta(l,n-l)=(n-3)\zeta(n)+2\sum_{m=2}^{n-2}\zeta(m,n-m).$$

Adding $2\zeta(n-1,1)$ to the both sides of the latter equality we obtain

$$2\sum_{l=2}^{n-1}\sum_{m=2}^{l}\binom{l-1}{m-1}\zeta(l,n-l)=(n-3)\zeta(n)+2\sum_{m=2}^{n-1}\zeta(m,n-m)$$

and using the binomial theorem for the left-hand side results in

$$2\sum_{l=2}^{n-1}(2^{l-1}-1)\zeta(l,n-l)=(n-3)\zeta(n)+2\sum_{m=2}^{n-1}\zeta(m,n-m).$$

It remains to apply Euler's formula (4) to the both sides, to arrive at the desired formula (10). □

Proof of Theorem 2. For $i=2,\dots,n-1$, the sum

$$\sum_{e_1+e_2+\dots+e_{n-i}=i-2}\zeta(3+e_1,1+e_2,1+e_3,\dots,1+e_{n-i})$$

can be written as

$$\sum_{e_1+e_2=i-2}\zeta(n-i+1+e_1,1+e_2)=\sum_{l=1}^{i-1}\zeta(n-l,l)$$

by the identity in [9]. Therefore,

$$\begin{aligned}2^{n-2}\zeta(3,\underbrace{1,\dots,1}_{n-3})+2^{n-3}\sum_{e_1+e_2+\dots+e_{n-3}=1}\zeta(3+e_1,1+e_2,\dots,1+e_{n-3})+\cdots\\ +2^{n-i}\sum_{e_1+e_2+\dots+e_{n-i}=i-2}\zeta(3+e_1,1+e_2,\dots,1+e_{n-i})+\cdots+2\zeta(n)\end{aligned}$$

[2]A general form of these relations, known as double shuffle relations, is discussed in [6].

$$= \sum_{i=2}^{n-1} 2^{n-i} \sum_{l=1}^{i-1} \zeta(n-l,l) = \sum_{l=1}^{n-2} \sum_{i=2}^{n-l-1} 2^i \zeta(n-l,l)$$
$$= \sum_{l=1}^{n-2} (2^{n-l}-2)\zeta(n-l,l).$$

Applying Euler's formula (4), the latter sum becomes

$$\sum_{l=1}^{n-2} 2^{n-l}\zeta(n-l,l) - 2\zeta(n)$$

which turns out to be $(n-1)\zeta(n)$ by Theorem 3. Finally, we use the formula

$$(n-1)\zeta(n) = \zeta^\star(2, \underbrace{1,1,\dots,1}_{n-2})$$

which is a special case of the sum formula due to A. Granville [4] and D. Zagier [15]. This implies (9) and completes our proof of Theorem 2. □

As one can see from the above proofs of Theorems 2 and 3, these theorems are equivalent modulo the algebraic relations from [9] (that also contain the sum formula as a particular case).

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