COORDINATE-FREE CLASSIC GEOMETRIES
II. CONFORMAL STRUCTURE

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Abstract. We study grassmannian classic geometries in the spirit of the previous paper. The interrelation between a (pseudo-)riemannian projective classic geometry and the conformal structure on its absolute is explained.

1. Introduction

It was illustrated in [AGr1] that the grassmannian $\mathrm{Gr}_{\mathbb{K}}(k,V)$ constitutes a natural habitat for basic geometrical objects of the classic geometry in $\mathbb{P}_{\mathbb{K}}V$. The grassmannian consists of two complementary parts: $\mathrm{Gr}^0_{\mathbb{K}}(k,V)$, formed by nondegenerate subspaces, and the absolute.

A tangent vector at a nondegenerate point p is a linear map $t:p\to p^\perp$ and $\mathrm{tr}(t_1^*t_2)$ defines a hermitian metric on $\mathrm{Gr}^0_{\mathbb{K}}(k,V)$. It seems, however, that there may exist a more adequate way to deal with the geometry on grassmannians. The characteristic polynomial $\mathrm{char}(t_1^*t_2)$, or simply the product $t_1^*t_2$ itself, can be taken in place of the hermitian metric. We can also take tangent vectors t_1,t_2 at distinct points. In the latter case, $\mathrm{tr}(t_1^*t_2)$ looks like a mixture of the usual hermitian structure and parallel displacement; see the formula (3.1), for example.

The product $t_1^*t_2$ partially survives at a degenerate point p. Denote $V_q:=q^\perp/q$, where q stands for the kernel of the hermitian form on p. The points on the absolute with the same q form a fibre of a certain bundle and the product $t_1^*t_2$ is defined for tangent vectors to such fibre because V_q is naturally equipped with a nondegenerate hermitian form. In particular, we obtain a hermitian metric on the fibre. Surprisingly, this hermitian metric provides the conformal (or conformal contact) structure on the absolute $\mathrm{S}\,V$ of $\mathbb{P}_{\mathbb{K}}V$. In other words, the conformal structure is exactly what remains from the metric when we arrive at the absolute.

2. Tangent structure and stratification

Let V be an n-dimensional $\mathbb{K}$-vector space equipped with a nondegenerate hermitian form $\langle -,-\rangle$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. The grassmannian $\mathrm{Gr}_{\mathbb{K}}(k,V)$ of k-dimensional $\mathbb{K}$-vector subspaces in V can be described as follows. Take and fix a $\mathbb{K}$-vector space P such that $\dim_{\mathbb{K}}P=k$. Denote by

$$M:=\big\{p\in\mathrm{Lin}_{\mathbb{K}}(P,V)\mid \mathrm{Ker}\,p=0\big\}$$

the open subset of all monomorphisms in the $\mathbb{K}$-vector space $\mathrm{Lin}_{\mathbb{K}}(P,V)$. The group $\mathrm{GL}_{\mathbb{K}}P$ acts from the right on $\mathrm{Lin}_{\mathbb{K}}(P,V)$ and on M. The grassmannian $\mathrm{Gr}_{\mathbb{K}}(k,V)$ is simply the quotient space

$$\mathrm{Gr}_{\mathbb{K}}(k,V):=M/\mathrm{GL}_{\mathbb{K}}P,\qquad \pi:M\to M/\mathrm{GL}_{\mathbb{K}}P.$$

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We will not distinguish between the notation of points in $\operatorname{Gr}_{\mathbb K}(k,V)$ and of their representatives in M. Moreover, we will frequently write p in place of the image pP and $p^\perp$, in place of the orthogonal $(pP)^\perp$ to that image: in practice, it is impossible to confuse a map and its image. For example, V/p will denote V/pP. So, our notation and convention are similar to those used in [AGr1].

The tangent space $\mathrm T_p M$ is commonly identified with $\operatorname{Lin}_{\mathbb K}(P,V)$ as follows. Given $\varphi : P \to V$, for small ε, the curve $c(\varepsilon) := p+\varepsilon\varphi$ lives in M and φ is identified with $\dot c(0)$. We use a slightly different identification $\mathrm T_p M = \operatorname{Lin}_{\mathbb K}(p,V)$ where p takes the place of P. In this way, $\overline t \in \operatorname{Lin}_{\mathbb K}(p,V)$ is interpreted as the tangent vector $\dot c(0)$, where $c(\varepsilon) := (1+\varepsilon\widetilde t)p$ and $\widetilde t \in \operatorname{Lin}_{\mathbb K}(V,V)$ extends $\overline t$. Note that $\overline t \in \operatorname{Lin}_{\mathbb K}(p,V)$ is tangent at p to the orbit $p\operatorname{GL}_{\mathbb K} P$ if and only if $\overline tp \subset p$ since in this case $(1+\varepsilon\widetilde t)p = pg$ for small ε and suitable $g \in \operatorname{GL}_{\mathbb K} P$. Therefore, we can write $\mathrm T_p \operatorname{Gr}_{\mathbb K}(k,V) = \operatorname{Lin}_{\mathbb K}(p,V/p)$.

2.1. Remark. Fix a linear subspace $w \subset V$. Denote by $\operatorname{Gr}_{\mathbb K}(k,w,V) \subset \operatorname{Gr}_{\mathbb K}(k,V)$ the space of all k-dimensional subspaces in V included in w. The identification $\operatorname{Gr}_{\mathbb K}(k,w,V) \simeq \operatorname{Gr}_{\mathbb K}(k,w)$ provides the equality $\mathrm T_p \operatorname{Gr}_{\mathbb K}(k,w,V) = \operatorname{Lin}_{\mathbb K}(p,w/p) \subset \operatorname{Lin}_{\mathbb K}(p,V/p)$.

Dually, let $\operatorname{Gr}_{\mathbb K}(k,V,q) \subset \operatorname{Gr}_{\mathbb K}(k,V)$ denote the space of all k-dimensional subspaces in V containing q, where $q \subset V$ is a d-dimensional subspace. We will show that $\mathrm T_p \operatorname{Gr}_{\mathbb K}(k,V,q) = \operatorname{Lin}_{\mathbb K}(p/q,V/p)$ for all $p \in \operatorname{Gr}_{\mathbb K}(k,V,q)$.

$$\begin{tikzcd} P \arrow[r, "{}_0"] \arrow[d, "p"'] & P_0 \arrow[d, "p_0"] \\ V \arrow[r, "{}_0"] & V/q \end{tikzcd}$$

The identification $i : \operatorname{Gr}_{\mathbb K}(k,V,q) \to \operatorname{Gr}_{\mathbb K}(k-d,V/q)$ can be formally described as follows. Fix an epimorphism ${}_0 : P \to P_0$. The same symbol denotes the canonical map ${}_0 : V \to V/q$. We can think of every $p \in \operatorname{Gr}_{\mathbb K}(k,V,q)$ as of a linear map $p : P \to V$ that sends the kernel of ${}_0 : P \to P_0$ onto q. So, $i : p \mapsto p_0$, where $p_0 : P_0 \to V/q$ is induced by p in the commutative diagram on the left.

$$\begin{tikzcd} P \arrow[r, "{}_0"] \arrow[d, "p"'] & P_0 \arrow[d, "p_0"] \\ p \arrow[r, "{}_0"] \arrow[d, "\overline t"'] & p_0 \arrow[d, "\overline t_0"] \\ V \arrow[r, "{}_0"] & V/q \end{tikzcd}$$

Fix some $p \in \operatorname{Gr}_{\mathbb K}(k,V,q)$. A given linear map $\overline t : p \to V$ satisfying $\overline tq \subset q$ induces a linear map $\overline t_0 : p_0 \to V/q$ in the commutative diagram on the right, where ${}_0 : p \to p_0$ is induced by ${}_0 : V \to V/q$. Fix some $\overline t$ and the induced $\overline t_0$. The curves $c(\varepsilon) := (1_V + \varepsilon\widetilde t)p \in M$ and $c_0(\varepsilon) := (1_{V/q} + \varepsilon\widetilde{t_0})p_0 \in M_0$ have tangent vectors $\dot c(0) = \overline t$ and $\dot c_0(0) = \overline t_0$, where $\widetilde t : V \to V$ and $\widetilde{t_0} : V/q \to V/q$ are extensions of $\overline t$ and $\overline t_0$. Since $\overline tq \subset q$, the map $c(\varepsilon) : P \to V$ sends the kernel of ${}_0 : P \to P_0$ into q. For small ε, it sends that kernel onto q because $c(0) = p$. It is easy to infer from the above diagrams that $i : c(\varepsilon) \mapsto c_0(\varepsilon)$. In other words, i sends the tangent vector $t \in \mathrm T_p \operatorname{Gr}_{\mathbb K}(k,V,q) \subset \operatorname{Lin}_{\mathbb K}(p,V/p)$ corresponding to $\overline t$ to the tangent vector $t_0 \in \mathrm T_{p_0} \operatorname{Gr}_{\mathbb K}(k-d,V/q) = \operatorname{Lin}_{\mathbb K}(p/q,V/p)$ corresponding to $\overline t_0$. Since the $\overline t_0$'s list all linear maps of the form $p/q \to V/q$, we obtain $\mathrm T_p \operatorname{Gr}_{\mathbb K}(k,V,q) = \operatorname{Lin}_{\mathbb K}(p/q,V/p) \subset \operatorname{Lin}_{\mathbb K}(p,V/p)$ ■

There is a stratification

$$\operatorname{Gr}_{\mathbb K}(k,V) = \bigsqcup_d \operatorname{Gr}^d_{\mathbb K}(k,V), \qquad \operatorname{Gr}^d_{\mathbb K}(k,V) := \big\{p \in \operatorname{Gr}_{\mathbb K}(k,V) \mid \dim_{\mathbb K}(p\cap p^\perp) = d\big\}.$$

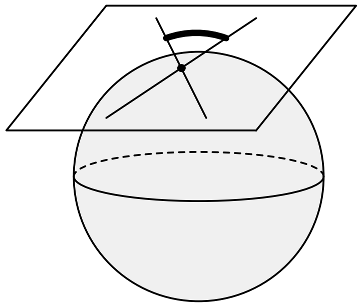
The subspaces of a given signature form an $\mathrm U V$-orbit. Therefore, every strata is the disjoint union of a finite number of such orbits, hence, a manifold. Associating to each p the kernel of the hermitian form on p, we get the $\mathrm U V$-equivariant fibre bundle

$$\pi_d : \operatorname{Gr}^d_{\mathbb K}(k,V) \to \operatorname{Gr}^d_{\mathbb K}(d,V), \qquad \pi_d : p \mapsto p\cap p^\perp.$$

The fibre $\pi_d^{-1}(q)$ can be naturally identified with $\operatorname{Gr}^0_{\mathbb K}(k-d,V_q)$, where $V_q := q^\perp/q$ is equipped with a natural nondegenerate hermitian form and $\dim_{\mathbb K} V_q = n-2d$. By Remark 2.1, $\mathrm T_p\,\pi_d^{-1}(q) = \operatorname{Lin}_{\mathbb K}(p/q,q^\perp/p)$ for all $p \in \pi_d^{-1}(q)$ because $\operatorname{Gr}^0_{\mathbb K}(k-d,V_q)$ is open in $\operatorname{Gr}_{\mathbb K}(k-d,V_q)$.

Let $p \in \pi_d^{-1}(q)$. Clearly, $q = p\cap p^\perp$, $q^\perp = p+p^\perp$, and $V_q = p_0 \oplus p_0^\perp$, where $p_0 := p/q$ and $p_0^\perp = p^\perp/q$. Denote by $\pi'[p_0]$ and $\pi[p_0]$ the corresponding orthogonal projectors. We have $\mathrm T_p\,\pi_d^{-1}(q) = \operatorname{Lin}_{\mathbb K}(p_0,p_0^\perp)$.

Being extended by zero, a tangent vector $t_0 : p_0 \to p_0^\perp$ can be viewed as a linear map $t : V_q \to V_q$. We found it reasonable to interpret every linear map $t : V_q \to V_q$ as a *footless* tangent vector: composed with the projectors, i.e., *observed* from p_0, the map t becomes $t_{p_0} := \pi[p_0]t\pi'[p_0]$, a usual tangent vector to $\pi_d^{-1}(q)$ at p_0.



In the case of $d = 1$, the map π_1 shows how the generic part $\operatorname{Gr}^1_{\mathbb{K}}(k, V)$ of the absolute of the grassmannian $\operatorname{Gr}_{\mathbb{K}}(k, V)$ is fibred over the absolute $\operatorname{S} V = \operatorname{Gr}^1_{\mathbb{K}}(1, V)$ of the projective classic geometry $\mathbb{P}_{\mathbb{K}} V = \operatorname{Gr}_{\mathbb{K}}(1, V)$. Fibres are easy to visualize: they correspond to subspaces 'rotating' about their common unidimensional kernel, hence, forming a grassmannian. If $k = 2$, each fibre carries the structure of a projective classic geometry.

2.2. Remark. Let $\mathbb{K} = \mathbb{R}$. The tangent space to the absolute $\operatorname{S} V$ has the form $\operatorname{T}_q \operatorname{S} V = \operatorname{Lin}_{\mathbb{K}}(q, V_q)$. Indeed, take $u \in V$ such that $\operatorname{S} V \subset \mathbb{P}_{\mathbb{K}} V$ is locally given by the equation $f(x) = 0$ in a neighbourhood of q, where $f(x) := \dfrac{\langle x, x\rangle}{\langle x, u\rangle\langle u, x\rangle}$. Let $t : q \to V/q$ be a tangent vector to $\mathbb{P}_{\mathbb{K}} V$ at q and let $\widetilde{t} : V \to V$ be a lift of t. Then t is tangent to $\operatorname{S} V$ if and only if $tf = 0$, i.e.,

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{\langle q + \varepsilon\widetilde{t}q, q + \varepsilon\widetilde{t}q\rangle}{\langle q + \varepsilon\widetilde{t}q, u\rangle\langle u, q + \varepsilon\widetilde{t}q\rangle} = \frac{2\operatorname{Re}\langle\widetilde{t}q, q\rangle}{\langle q, u\rangle\langle u, q\rangle} = \frac{2\operatorname{Re}\langle tq, q\rangle}{\langle q, u\rangle\langle u, q\rangle} = 0.$$

(In the formula, $0 \neq q \in V$, $\langle q, q\rangle = 0$, is an element representing the subspace q.) Thus, the bundle $\pi_1 : \operatorname{Gr}^1_{\mathbb{R}}(k, V) \to \operatorname{Gr}^1_{\mathbb{R}}(1, V)$ is the grassmannization of the tangent bundle of $\operatorname{S} V$.

Let $\mathbb{K} = \mathbb{C}$. The above calculus shows that $\operatorname{T}_q \operatorname{S} V = \big\{t : q \to V/q \mid \operatorname{Re}\langle tq, q\rangle = 0\big\}$. Hence, we get a CR-distribution $\operatorname{Lin}_{\mathbb{K}}(q, V_q) \subset \operatorname{T}_q \operatorname{S} V$, i.e., a contact structure on the absolute. The bundle $\pi_1 : \operatorname{Gr}^1_{\mathbb{C}}(k, V) \to \operatorname{Gr}^1_{\mathbb{C}}(1, V)$ is the grassmannization of this distribution ■

The case of $d > 1$ deals with a 'degenerate' part of the absolute in the grassmannian. The bundle $\pi_d : \operatorname{Gr}^d_{\mathbb{K}}(k, V) \to \operatorname{Gr}^d_{\mathbb{K}}(d, V)$ is no longer the grassmannization of the tangent bundle of $\operatorname{Gr}^d_{\mathbb{K}}(d, V)$ (considering, say, the real case). Nevertheless, the bundle indicates a distinguished distribution whose geometric nature would be interesting to dwell on.

3. Product, metric, and conformal structures

Consider a fibre $\pi_d^{-1}(q)$ of the bundle $\pi_d : \operatorname{Gr}^d_{\mathbb{K}}(k, V) \to \operatorname{Gr}^d_{\mathbb{K}}(d, V)$. Let $t_1, t_2 : V_q \to V_q$ be two footless tangent vectors to this fibre, where $V_q := q^\perp/q$. The footless vectors, the composition with projectors (i.e., the possibility to observe a vector from a point), and the *product* $t_1^* t_2$ constitute the **main** geometric structure on $\pi_d^{-1}(q)$ (t_1^* stands for the adjoint to t_1). Let us see what can we derive from the main structure.

Take $d = 0$, $p \in \operatorname{Gr}^0_{\mathbb{K}}(k, V)$, and $t_1, t_2 \in \operatorname{T}_p \operatorname{Gr}_{\mathbb{K}}(k, V) = \operatorname{Lin}_{\mathbb{K}}(p, p^\perp) \subset \operatorname{Lin}_{\mathbb{K}}(V, V)$. Then the equality

$$\langle t_1, t_2\rangle := \operatorname{tr}(t_1^* t_2)$$

defines the (hermitian) *metric* on $\operatorname{T}_p \operatorname{Gr}_{\mathbb{K}}(k, V)$. Many examples with $k = 1$ were dealt with in [AGr1]. One can introduce a more subtle structure by considering the coefficients $\operatorname{char}(t_1^* t_2) \in \mathbb{K}^{k-d}$ of the characteristic polynomial of $t_1^* t_2$. In this way, we obtain [AGr2] geometric characteristics of the geodesic determined by a tangent vector t such as the invariant $\det(t^* t)/\operatorname{tr}^{k-d}(t^* t)$.

Actually, we do not need the hermitian metric to measure distance. Let $p_1, p_2 \in \operatorname{Gr}^0_{\mathbb{K}}(k, V)$ and let $t \in \operatorname{Lin}_{\mathbb{K}}(p_1, p_1^\perp) \subset \operatorname{Lin}_{\mathbb{K}}(V, V)$ be a nonnull tangent vector at p_1. We observe t at p_2 and then observe the result back at p_1 thus obtaining a new tangent vector t' at p_1. The change suffered by t' reflects how p_1, p_2, and t are related. For example, take $k=1$. If t is tangent to the projective line joining p_1 and p_2, then[1] $t'=\operatorname{ta}^2(p_1,p_2)t$. If t is orthogonal to the projective line, then $t'=\operatorname{ta}(p_1,p_2)t$. Analogously, given points p_1,p_2,p_3 and a tangent vector t at p_1, a similar procedure of successive observations leads to a new tangent vector t' at p_1. Again, t' reflects the relation between p_1,p_2,p_3 and t. In particular, if p_1,p_2,p_3 lie in the riemannian part of a complex projective line, i.e., the triangle $\Delta(p_1,p_2,p_3)$ is $\mathbb{C}$-plane, and t is tangent to this line, then $t'=t_{12}^2t_{23}^2t_{31}^2\exp\big(2i\operatorname{Area}\Delta(p_1,p_2,p_3)\big)t$, where $t_{ij}:=\operatorname{ta}(p_i,p_j)$.

The variations involving the product are endless. For example, the product $t_1^*t_2$ of $t_i\in\operatorname{Lin}_\mathbb{K}(p_i,p_i^\perp)\subset\operatorname{Lin}_\mathbb{K}(V,V)$, $i=1,2$, makes sense for distinct points $p_1,p_2\in\operatorname{Gr}_\mathbb{K}^0(k,V)$. In particular, the hermitian metric is defined for tangent vectors at different points. In the case of $k=1$, this can be interpreted as follows. We take the horizontal-vertical decomposition $t_2=h+v$ of t_2 [AGr1, Section 5] where h and v are respectively tangent and orthogonal to the projective line joining p_1 and p_2. Then we make separately the parallel displacement of h and v along the geodesic from p_2 to p_1 obtaining the vectors h' and v'. It follows from [AGr1, Corollary 5.7] that

$$\operatorname{tr}(t_1^*t_2)=\big\langle t_1,\operatorname{ta}(p_1,p_2)h'+\sqrt{\operatorname{ta}(p_1,p_2)}v'\big\rangle. \tag{3.1}$$

Thus, for $k=1$, the hermitian product of tangent vectors at different points can be interpreted in the terms of the usual hermitian structure and parallel displacement.

In the global picture composed of the pseudo-riemannian pieces in $\operatorname{Gr}_\mathbb{K}^0(k,V)$ and of the absolute, the main geometric structure ties everything together. Tangent vectors to a point in one piece are observable from the points in the other pieces and we can take the product of tangent vectors at points in different pieces. This global picture also involves the geometry on the absolute.

We always found it curious and even a little bit mysterious the relation between the real hyperbolic structure on the ball and the conformal structure on its boundary. In other words, why do the corresponding groups coincide? It follows some explanation.

Let $\mathbb{K}=\mathbb{R}$. By Remark 2.2, the bundle $\pi_1:\operatorname{Gr}_\mathbb{R}^1(2,V)\to\operatorname{Gr}_\mathbb{R}^1(1,V)$ is the projectivized tangent bundle of the absolute $\mathrm{S}\,V=\operatorname{Gr}_\mathbb{R}^1(1,V)\subset\mathbb{P}_\mathbb{R}V$. Every fibre $\pi_1^{-1}(q)\simeq\mathbb{P}_\mathbb{R}V_q$ carries the structure of a classic projective geometry. The bundle $\pi_1:\operatorname{Gr}_\mathbb{R}^1(2,V)\to\mathrm{S}\,V$ with fibres equipped with such geometry is the *conformal* structure on $\mathrm{S}\,V$.

3.2. Example. Take $\mathbb{K}=\mathbb{R}$ and signature $+\cdots+-$. The real hyperbolic space $\mathbb{H}_\mathbb{R}^{n-1}$ is the negative part of $\mathbb{P}_\mathbb{R}V$ and its ideal boundary, the absolute, is the sphere $\mathrm{S}\,V\simeq\mathbb{S}^{n-2}$. Every fibre $\mathbb{P}_\mathbb{R}V_q\simeq\mathbb{P}_\mathbb{R}^{n-3}$ of the conformal structure carries the positive definite constant curvature metric. The distances in a fibre are nothing but the angles of the standard[2] conformal structure on $\mathrm{S}\,V$ ■

For $k>2$, the geometry on the grassmannization $\operatorname{Gr}_\mathbb{R}^1(k,V)\to\operatorname{Gr}_\mathbb{R}^1(1,V)$ of the tangent bundle of the absolute (see Remark 2.2) is related to the case of $k=2$ in the same way as are related the grassmannian and the projective classic geometries.

Let $\mathbb{K}=\mathbb{C}$. By Remark 2.2, the bundle $\pi_1:\operatorname{Gr}_\mathbb{C}^1(2,V)\to\operatorname{Gr}_\mathbb{C}^1(1,V)$ is the projectivization of the CR-distribution on the absolute $\mathrm{S}\,V=\operatorname{Gr}_\mathbb{C}^1(1,V)\subset\mathbb{P}_\mathbb{C}V$. Every fibre $\pi_1^{-1}(q)\simeq\mathbb{P}_\mathbb{C}V_q$ carries the structure of a classic projective geometry. The bundle $\pi_1:\operatorname{Gr}_\mathbb{C}^1(2,V)\to\mathrm{S}\,V$ with fibres equipped with such geometry is the *conformal contact* structure on $\mathrm{S}\,V$.

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[1] When p_1 and p_2 live in a same riemannian piece of $\operatorname{Gr}_\mathbb{K}^0(1,V)$, the tance $\operatorname{ta}(p_1,p_2)$ provides the distance between p_1 and p_2 [AGr1, Section 3]. Otherwise, the tance provides the distance or angle between the basic geometrical objects corresponding to p_1 and p_2.

[2] If one wishes to deal with angles varying in $[0,2\pi]$, then the projectivization should be taken with respect to $\mathbb{R}^+$, from the very beginning.

3.3. Example. Take $\mathbb{K}=\mathbb{C}$ and signature $+\cdots+-$. The complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^{n-1}$ is the negative part of $\mathbb{P}_{\mathbb{C}}V$ and its ideal boundary, the absolute, is the sphere $\mathrm{S}\,V\simeq\mathbb{S}^{2n-3}$. Every fibre $\mathbb{P}_{\mathbb{R}}V_q\simeq\mathbb{P}_{\mathbb{C}}^{n-3}$ of the conformal contact structure carries the Fubini-Study metric. The distances in a fibre are the angles between complex directions $\blacksquare$

The CR-structure is sometimes taken as analogous to the conformal one. The above shows that they are of distinct nature. Of course, the CR-structure underlies the conformal contact one. For a trivial reason, they coincide when $n=3$.

Note that any classic projective geometry can play the role of conformal structure. In particular, the conformal structure can possess its own absolute and so on ...

3.4. Comments and questions. The algebraic formulae dealing with geometrical quantities work as well for points in distinct pieces of $\mathrm{Gr}_{\mathbb{K}}^0(k,V)$. In the global picture, a given formula uses to alter its geometrical sense when the points involved are taken in the other pieces. In this respect, it is interesting to understand if there is an explicit geometrical interpretation of the main structure in the terms of the usual (pseudo-)riemannian concepts for $k>2$.

The bundle $\pi_d:\mathrm{Gr}_{\mathbb{K}}^d(k,V)\to\mathrm{Gr}_{\mathbb{K}}^d(d,V)$ might admit a canonical connection. If so, what is its explicit description?

4. References

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